# The multiverse of naive set theory

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# Outline

# Warning/apologies

- I'm not an expert in forcing, I'm not a set theorist
- This is a rather programmatic speculative talk
- I want to propose a future line of promising research, no real result yet

# Outline

# Basic idea

Extensional naïve set theory is the theory with as non-logical axiom the full abstraction scheme $\mathcal{FA}$

$$\forall y(y \in \{x \mid A(x)\} \equiv A(y))$$

together with extensionality $\mathcal{E}$

$$\forall x \forall y(\forall z(z \in x \equiv z \in y) \supset x = y)$$

closed under a consequence relation **L** that does not trivialize the axioms.

Here I'll use **LP**, because we have a finitistic non-triviality proof (due to Greg Restall)

This theory is very weak (no Modus Ponens)

# Advantages of naive set theory

- Only the basics of what a collection theory intuitively is
- No artificial distinction between sets and classes
- Self reference (self-membership) is not a priori excluded
- Paradoxical sets can be formalized

# Outline

# Independence results

- An important domain of research in modern set theory are independence results
- Independence results: relative consistency proofs
- If ZFC is consistent then ZFC+CH is consistent and ZFC+$\neg$CH is consistent
- If ZF is consistent then ZF+C is consistent and ZF+$\neg$C is consistent
- if ZFC is consistent then ZFC+$V = L$ is consistent and ZFC+$\neg V = L$ is consistent

# Forcing

- The first independence result was due to Gödel, constructing an inner model of the set theoretic universe
- In 1963 Cohen invented an ingenious technique to construct models which *force* formulas to be true
- Later boolean valued models have been used to construct such models (by Dana Scott, Robert M. Solovay, and Petr Vopěnka)
- Forcing has led to a deep insight in the models of set theory

# Outline

# Multiverse view

- Opposes **against the platonistic universe view.** An absolute universe of set exists in which all set theoretic formulas are decided. This universe is all there is and we cannot get out of it.
- Instead he claims that many universes exist which are interrelated. All of them are real.
- CH is not true or false. It is true in some existing universes and false in others.
- Throwing away one of the universes (as unreal) would deny the deep understanding we have obtained of its structure.

# Multiverse axioms

1. **Realizability Principle.** For any universe $V$ if $W$ is a model of set theory and definable or interpreted in $V$, then $W$ is a universe
2. **Forcing Extension Principle.** For any universe $V$ and any forcing notion $\mathbb{P}$ in $V$, there is a forcing extension $V[G]$ where $G \subseteq \mathbb{P}$ is a $V$-generic filter.
3. **Reflection Axiom.** For every universe $V$, there is a much taller universe $W$ with an ordinal $\theta$ for which $V \precsim W_\theta \prec W$

# Multiverse axioms

1. **Countability Principle.** Every universe is coutable from the perspective of another, *better* universe.
2. **Well-foundedness Mirage.** Every universe $V$ is ill-founded from the perspective of another, better universe.
3. **Reverse embedding.** For every universe $V$ and every embedding $j : V \to M$ in $V$, there is a universe $W$ and embedding $h$ such that

$$W \xrightarrow{h} V \xrightarrow{j} M$$

4. **Absorption into L.** Every universe $V$ is a countable transitive model in another universe $W$ satisfying $V = L$.

# Modal logic of forcing

- $\Diamond A$ means: $A$ can be forced, i.e. $A$ is true in some forcing extension
- $\Box A$ means: $A$ is true in every forcing extension
- The logic underlying forcing (what is provable in ZFC in term of forcing relations) is **S4.2**, i.e. **S4** + $\Diamond\Box A \supset \Box\Diamond A$
- A Kripke frame for this logic (accessibility relation can be a directed pre-order, a convergent pre-order, or finite pre-lattice) contains universes as possible worlds.
- A Kripke frame thus corresponds to a possible multiverse.

# Multiverse view: discussion

- I think it can be explained in terms of a 'The Truman Show'-metaphor.
- It seems unnecessary to go as metaphysical as Hamkin; as he presents it, it is a lot worse than modal realism (Lewis) or plain platonism. Maybe a anti-realist interpretation?
- There is something weird about the metaphysics of the multiverse. Multimultiverse of multiverses?
- How can some universes be better than others if they all exist? Maybe epistemic interpretation?

# Outline

# Opening up the multiverse (1)

- Hamkins quantifies over universes, which are not sets, using set theoretic meta-language (which stands outside of all formalism).
- Where do we do this meta-mathematics?
- Why not inside the object language?

# Opening up the multiverse (2)

- Hamkins does not mention how weak one can go in order for something to still be a real set theoretic universe.
- Verifying at least ZFC, or only ZF, $ZF^{-}$, Second order arithmetic?
- Why not all the way up to the most intuitive concept of what a collection is?
- Comprehension and extensionality as the only necessary requirements for a universe to count as set theoretic.

# Opening up the multiverse (3)

- Why not also consider non-classical set theoretic universes as part of the multiverse?
- These universes can be studied just like the classical ones.
- Behind the multiverse view lies a credo that one should not restrict oneself to a certain universe.
- The classical viewpoint is restricting.

# Preferred universes and the expressive power of naïve set theory

- Naïve set theory is very weak and has some mathematically uninteresting models
- Better universes as disambiguating elements of inconsistent sets
- The universal set of a universe (if there is one) is inconsistent (its powerset is a member).
- There may be a 'better' universe which has all consistent members in common with the inconsistent universe, but it's an extension in which the universe of the first is also a consistent set
- Gradually solving solvable inconsistencies going up in the hierarchy
- Ideally try to go up as high as possible

# Adaptive logic

- Underlying logic of the "as consistent as possible"
- Provides dynamic proof theory

# Outline

# Conclusion

We may conclude that

- Going naive and non-classical is natural for the multiversalist
- Going multiversalistic can give the naive-set-theorist the strength necessary to do actual useful mathematics

# The end

Thank you!