Residuated frames

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Reporting on work and ideas with
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A *residuated frame* is a structure $\mathbf{W} = (W, W', N, \circ, \backslash\!\!\backslash, /\!\!/)$ subject to the condition: for all $x, y \in W$ and $w \in W'$

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Notation:

$x \circ y = \{z \in W' : \circ(x, y, z)\}$
$x \backslash\!\!\backslash z = \{s \in W' : \backslash\!\!\backslash(x, z, s)\}$
$z /\!\!/ y = \{s \in W' : /\!\!/(z, x, s)\}$

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We can also add constants and other relations (see below), as well as properties such as associativity, commutativity, etc. We will be sloppy about assuming such constants of conditions.

Intuition

The intuition comes from algebra (residuated lattices), proof-theory (sequent calculus), and duality theory/relational semantics (Kripke frames).

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Kripke semantics: In a finite lattice we need the join-irreducibles W and the meet-irreducibles W' to describe the lattice, as well as the restriction N of the order relation between W and W'. In residuated lattices, $\circ$ reflects the monoid structure.
(To view Kripke frames as residuated frames we take $W' = W$, $N = \not\geq$, while $\circ$ is *partial meet*.)

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For

$$(u, a) \,/\!\!/\, x = \{(u[\_ \circ x], a)\} \text{ and } x \,\backslash\!\!\backslash\, (u, a) = \{(u[x \circ \_], a)\},$$

we have

$$\begin{aligned} x \circ y N(u, a) \quad & \text{iff } \vdash_{\mathbf{L}} u[x \circ y] \Rightarrow a \\ & \text{iff } \vdash_{\mathbf{L}} u[x \circ y] \Rightarrow a \\ & \text{iff } x N(u[\_ \circ y], a) \\ & \text{iff } y N(u[x \circ \_], a). \end{aligned}$$

Theorem. (Lattice frames and lattices) Let $\mathbf{L}$ a perfect lattice and $\mathbf{W} = (J, M, N)$. Then $\mathbf{L} = \mathbf{W}^+$ iff $J^\infty(\mathbf{L}) \subseteq J$ and $M^\infty(\mathbf{L}) \subseteq M$, where $j \; N \; m \Leftrightarrow f(j) \leq g(m)$ for some $f : J \to L$, $g : M \to L$.

Here $J^\infty(\mathbf{L})$ and $M^\infty(\mathbf{L})$ denote the *completely join irreducible* elements of $\mathbf{L}$, namely elements j such that $j = \bigvee X$ iff $j \in X$, and the *completely meet irreducible* elements. $\mathbf{L}$ is a *perfect* if every element is a join of completely join irreducible elements of $\mathbf{L}$ and a meet of completely meet irreducible elements.

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If $\mathbf{L}$ is a *perfect* residuated lattice, then $\mathbf{W}^\infty_\mathbf{L} = (J^\infty(\mathbf{L}), M^\infty(\mathbf{L}), \leq, \cdot)$ is a residuated frame for $w' \,/\!\!/\, w_2$ the set of all meet irreducibles above w'/w_2, and likewise for $w_1 \,\backslash\!\!\backslash\, w'$.

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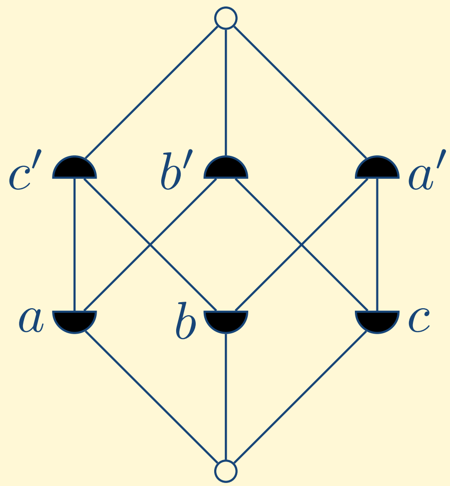
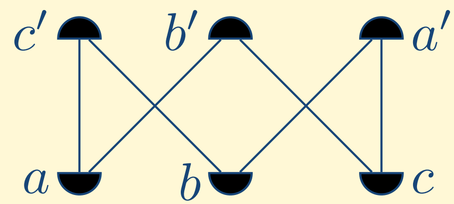
We can also form residuated frames by taking prime ideals, ((completely) prime) ideals, and relation *non-empty intersection*.

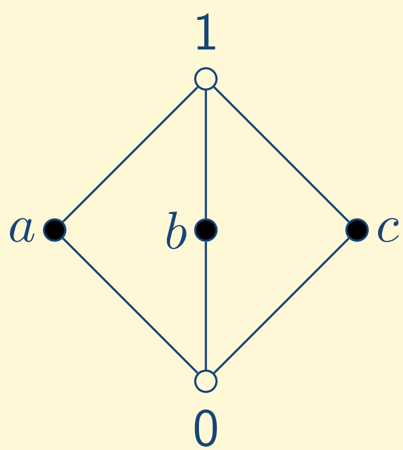
For $\mathbf{L}$ a residuated lattice, $\mathbf{W}_\mathbf{L}^{FI} = (\mathcal{F}(\mathbf{L}), \mathcal{I}(\mathbf{L}), N_{NEI}, \circ, \backslash\!\!\backslash, /\!\!/)$, the intermediate structure, aka the cannonical frame $\mathbf{A}_+$ of $\mathbf{A}$. Then $(\mathbf{A}_+)^+ = (\mathbf{W}_\mathbf{L}^{FI})^+$ is the canonical extension of $\mathbf{L}$.

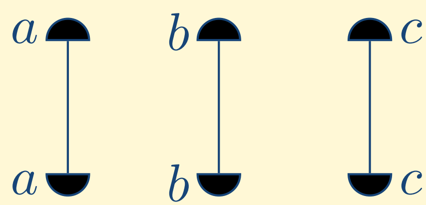
Contexts/polarities





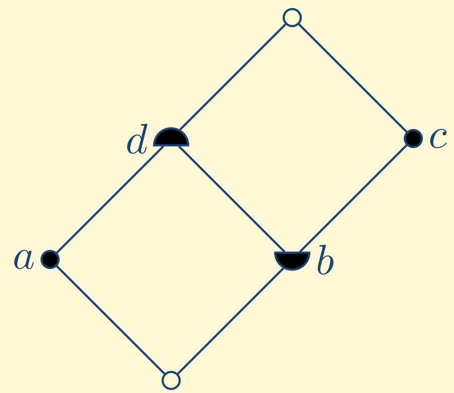
$\leq$	a'	b'	c'
a		$\times$	$\times$
b	$\times$		$\times$
c	$\times$	$\times$	

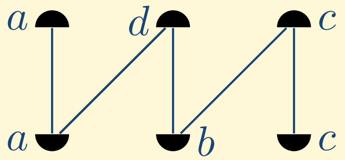




$\leq$	a'	b'	c'
a	$\times$		
b		$\times$	
c			$\times$

Dedekind-McNeille





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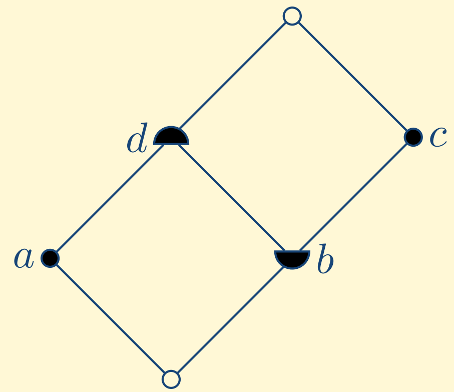
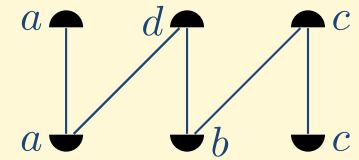
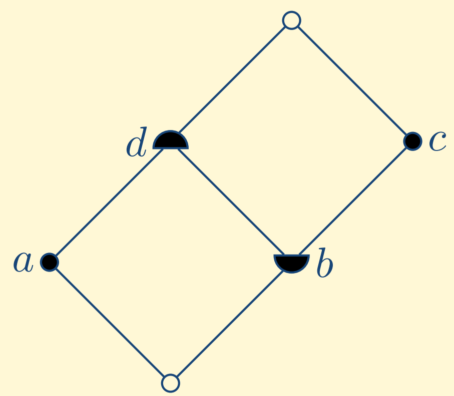
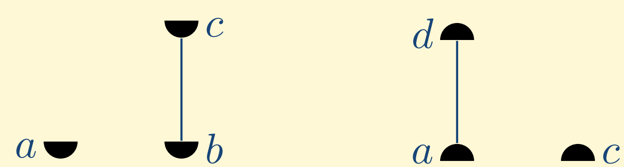




For every distributive lattice $M(\mathbf{L})$ is isomorphic to $J(\mathbf{L})$. Note $\uparrow a \cup \downarrow c = \uparrow b \cup \downarrow a = \uparrow c \cup \downarrow d = L$. *Splitting pairs*: $(a, c), (b, a), (c, d)$.





The dual algebra

For $X \subseteq W$ and $Y \subseteq W'$ we define

$$X^{\rhd} = \{b \in W' : x \; N \; b, \text{ for all } x \in X\}$$
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Lemma. If $\mathbf{W}$ is a residuated frame then the *dual algebra* $\mathbf{W}^+ = (\gamma[\mathcal{P}(W)], \cap, \cup_\gamma, \circ_\gamma, \backslash, /)$ is a complete residuated lattice.

$X \cup_\gamma Y = \gamma(X \cup Y)$
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$X \circ Y = \{z \in W : \circ(X, Y, z)\} = \{z \in W : \circ(x, y, z), \forall x \in X, y \in Y\}$
$X \backslash Y = \{z \in W : X \circ \{z\} \subseteq Y\}$
$Y / X = \{z \in W : \{z\} \circ X \subseteq Y\}$.

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$X \backslash Y = \{z \in W : X \circ \{z\} \subseteq Y\}$
$Y/X = \{z \in W : \{z\} \circ X \subseteq Y\}$.

Theorem If $\mathbf{W}$ satisfies (m)^p (*pointwise*), then it also satisfies (m)^s (*setwise*) namely $\mathbf{W}^+$ satisfies $x \cdot x \leq x$ ($xy \leq x \vee y$). ($\mathcal{N}_2$ eq.)

$$\frac{xNz \quad yNz}{x \circ yNz}\ (\mathrm{m})^p \qquad \frac{XNz \quad YNz}{X \circ YNz}\ (\mathrm{m})^s$$

GN

If we have a common subset B of W and W' that supports a (partial) algebra $\mathbf{B} = (B, \wedge, \vee, \cdot, \backslash, /, 1)$, then these are natural conditions inspired by the frame $\mathbf{W_L}$, for $a, b, c \in B$, $x, y \in W$, $z \in W'$. Often B generates $(W, \circ, 1)$ (and W' by actions from W); we call $(\mathbf{W}, \mathbf{B})$ a Gentzen frame.

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If we have a common subset B of W and W' that supports a (partial) algebra $\mathbf{B} = (B, \wedge, \vee, \cdot, \backslash, /, 1)$, then these are natural conditions inspired by the frame $\mathbf{W_L}$, for $a, b, c \in B$, $x, y \in W$, $z \in W'$. Often B generates $(W, \circ, 1)$ (and W' by actions from W); we call $(\mathbf{W}, \mathbf{B})$ a Gentzen frame.

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GN

If we have a common subset B of W and W' that supports a (partial) algebra $\mathbf{B} = (B, \wedge, \vee, \cdot, \backslash, /, 1)$, then these are natural conditions inspired by the frame $\mathbf{W_L}$, for $a, b, c \in B$, $x, y \in W$, $z \in W'$. Often B generates $(W, \circ, 1)$ (and W' by actions from W); we call $(\mathbf{W}, \mathbf{B})$ a Gentzen frame.

$$\frac{xNa \quad aNz}{xNz}\text{ (CUT)} \qquad \frac{}{aNa}\text{ (Id)}$$

$$\frac{aNz \quad bNz}{a \vee bNz}\text{ (}\vee\text{L)} \qquad \frac{xNa}{xNa \vee b}\text{ (}\vee\text{R}\ell\text{)} \qquad \frac{xNb}{xNa \vee b}\text{ (}\vee\text{R}r\text{)}$$

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1. $\mathbf{W}$ is a residuated frame
2. $\mathbf{B}$ is a (partial) algebra of the same type, $(\mathbf{B} = \mathbf{L}, \mathbf{Fm}, \mathbf{B})$
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For cut-free Genzten frames, we get only a *quasihomomorphism*.
$a \bullet_{\mathbf{B}} b \in \{a\}^{\lhd} \bullet_{\mathbf{W}^+} \{b\}^{\lhd} \subseteq \{a \bullet_{\mathbf{B}} b\}^{\lhd}$.

$$\frac{x \Rightarrow a \quad y \circ a \circ z \Rightarrow c}{y \circ x \circ z \Rightarrow c}\ (\mathrm{cut}) \qquad \frac{}{a \Rightarrow a}\ (\mathrm{Id})$$

$$\frac{y \circ a \circ z \Rightarrow c}{y \circ a \wedge b \circ z \Rightarrow c}\ (\wedge \mathrm{L}\ell) \quad \frac{y \circ b \circ z \Rightarrow c}{y \circ a \wedge b \circ z \Rightarrow c}\ (\wedge \mathrm{L}r) \quad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a \wedge b}\ (\wedge \mathrm{R})$$

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$$\frac{x \Rightarrow a \quad y \circ b \circ z \Rightarrow c}{y \circ x \circ (a \backslash b) \circ z \Rightarrow c}\ (\backslash \mathrm{L}) \qquad \frac{a \circ x \Rightarrow b}{x \Rightarrow a \backslash b}\ (\backslash \mathrm{R})$$

$$\frac{x \Rightarrow a \quad y \circ b \circ z \Rightarrow c}{y \circ (b/a) \circ x \circ z \Rightarrow c}\ (/\mathrm{L}) \qquad \frac{x \circ a \Rightarrow b}{x \Rightarrow b/a}\ (/\mathrm{R})$$

$$\frac{y \circ a \circ b \circ z \Rightarrow c}{y \circ a \cdot b \circ z \Rightarrow c}\ (\cdot \mathrm{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x \circ y \Rightarrow a \cdot b}\ (\cdot \mathrm{R})$$

$$\frac{y \circ z \Rightarrow a}{y \circ 1 \circ z \Rightarrow a}\ (1\mathrm{L}) \qquad \frac{}{\varepsilon \Rightarrow 1}\ (1\mathrm{R})$$

where $a, b, c \in Fm$, $x, y, z \in Fm^*$.

Frames and modules

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We call residuated frames for which these two simplifications apply *action residuated frames*.

Actions

Recall that given a monoid $\mathbf{W} = (W, \cdot, 1)$ and a set W', a map $* : W \times W' \to W'$ is called an *action* if it sarisfies:
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This allows us to link $\mathbf{W}$-sets and action residuated frames, as then an action residuated frame is equivalent to a bi-$\mathbf{W}$-set $(W', /\!\!/, \backslash\!\!\backslash)$ together with an arbitratry subset D of W' of designated elements.

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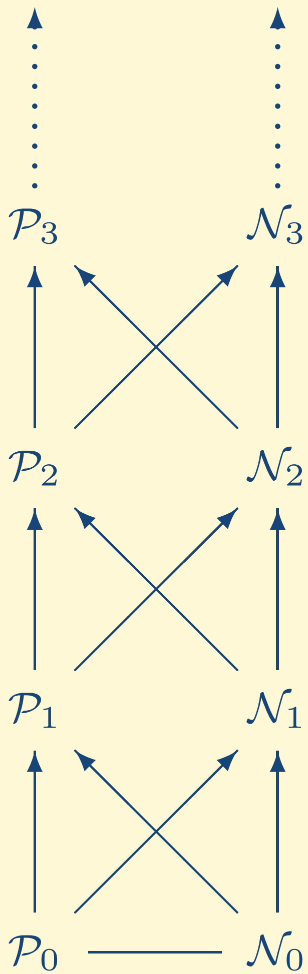
Formula hierarchy



- The sets $\mathcal{P}_n, \mathcal{N}_n$ of formulas are defined by:

(0) $\mathcal{P}_0 = \mathcal{N}_0 =$ the set of variables
(P1) $\mathcal{N}_n \subseteq \mathcal{P}_{n+1}$
(P2) $a, b \in \mathcal{P}_{n+1} \;\Rightarrow\; a \vee b, a \cdot b, 1 \in \mathcal{P}_{n+1}$
(N1) $\mathcal{P}_n \subseteq \mathcal{N}_{n+1}$
(N2) $a, b \in \mathcal{N}_{n+1} \;\Rightarrow\; a \wedge b \in \mathcal{N}_{n+1}$
(N3) $a \in \mathcal{P}_{n+1}, b \in \mathcal{N}_{n+1} \Rightarrow a \backslash b, b/a, 0 \in \mathcal{N}_{n+1}$

- $\mathcal{P}_{n+1} = \langle \mathcal{N}_n \rangle_{\bigvee, \prod}$; $\mathcal{N}_{n+1} = \langle \mathcal{P}_n \rangle_{\bigwedge, \mathcal{P}_{n+1} \backslash, /\mathcal{P}_{n+1}}$
- $\mathcal{P}_n \subseteq \mathcal{P}_{n+1}, \mathcal{N}_n \subseteq \mathcal{N}_{n+1}, \bigcup \mathcal{P}_n = \bigcup \mathcal{N}_n = Fm$
- $\mathcal{P}_1$-reduced: $\bigvee \prod p_i$
- $\mathcal{N}_1$-reduced: $\bigwedge (p_1 p_2 \cdots p_n \backslash r / q_1 q_2 \cdots q_m)$

$p_1 p_2 \cdots p_n q_1 q_2 \cdots q_m \leq r$

- Sequent: $a_1, a_2, \ldots, a_n \;\Rightarrow\; a_0 \; (a_i \in Fm)$

Submodules and nuclei

Given a semiring $\mathbf{P} = (P, \bigvee, \cdot, 1)$ and an onto homomorphism f to a semiring $\mathbf{S} = f[\mathbf{P}]$, we observe that f is residuated (with residual $f^* : \mathbf{S} \to \mathbf{P}$).

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If $P = N$ is the underlying set of a residuated lattice $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$ the two notions of nucelus coincide and $\mathbf{A}_\gamma = (A_\gamma, \wedge, \vee_\gamma, \cdot_\gamma, \backslash, /, \gamma(1))$ is also a residuated lattice.

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Residuated frames arise from studying submodules of $\mathcal{P}(\mathbf{W})$, where $\mathbf{W}$ is a monoid, namely nuclei on powersets (of monoids).

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Frames and modules

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Frames and display

Adding residuals

We define the system (plus associativity and exchage for simplicity).

$$\frac{x \Rightarrow a \quad y, a, z \Rightarrow c}{y, x, z \Rightarrow c}\ (\text{cut}) \qquad \frac{}{a \Rightarrow a}\ (\text{Id})$$

$$\frac{y, a, b, z \Rightarrow c}{y, a \cdot b, z \Rightarrow c}\ (\cdot\text{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x, y \Rightarrow a \cdot b}\ (\cdot\text{R})$$

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Is it conservative to extend the logic to one $\mathbf{L}_e$ with implication?

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Conservativity: if a sequent/inequality fails in the smaller logic (in a every commtative posemigroup), then it fails in the bigger logic (in a commutative residuated posemigroup).

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We can of course define a residuated frame $(Fm^*, Fm^* \times Fm, N, \circ, \backslash\!\backslash)$ based on this system.

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Algebraically. Show that every commutative posemigroup can be embedded into a residuated one. In other words we show something stronger: $\mathcal{S} = \mathsf{S}(\mathcal{R})$.

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There are a couple of ways to prove conservativity:

Proof-theoretically. Prove cut elimination and then use the subformula property.

Algebraically. Show that every commutative posemigroup can be embedded into a residuated one. In other words we show something stronger: $\mathcal{S} = \mathsf{S}(\mathcal{R})$.

Proof-theoretically (2): Via display logic.

Cut elimination via display

Consider the following display logic system $\mathbf{L}_e^\delta$. Here $x, y \in Fm^*$, $a, b \in Fm$ and z is of the form $x_1 > (x_2 > \ldots (x_n > a) \ldots)$.

$$\frac{x \Rightarrow a \quad a \Rightarrow z}{x \Rightarrow z} \text{ (cut)} \qquad \frac{}{a \Rightarrow a} \text{ (Id)} \qquad \frac{x \circ y \Rightarrow z}{\overline{\overline{y \Rightarrow x > z}}} \text{ (dis)}$$

$$\frac{a, b \Rightarrow z}{a \cdot b \Rightarrow z} \text{ ($\cdot$L)} \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x, y \Rightarrow a \cdot b} \text{ ($\cdot$R)}$$

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Yes! The two systems are mutually interpretable. (New sequents are *innocent*.) First every rule in $\mathbf{L}_e$ is derivable in $\delta\mathbf{L}_e$. (Using the display porperty.) Second, given a cut-free proof in $\mathbf{L}_e^\delta$ of a sequent free of $\to$ and $>$, we can convert it to a proof in cut-free $\mathbf{L}_e$.

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It is easy to see that if $\mathbf{L}$ satisfies contraction $x \leq x^2$, then so does $\mathbf{W}_{\mathbf{L}}$, and by a previous result so does $\mathbf{W}_{\mathbf{L}}^+$.

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If we have join in the language (next page) the same holds for mingle/expansion $x^2 \leq x$ ($xy \leq x \vee y$)

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If a connective gives an operator (order preserving; if we have join then it distributes) then it will be residuated at the completion and we can conservatively add its residual. Then one could chose to work in the bigger (and in some sense simpler) logic with no worries.

General principle: Let's be 'honest' about it and put the residuals at the frame level from the very beginning.

Distributive frames

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We add the rules:

$$\frac{u[x \owedge (y \owedge z)] \Rightarrow c}{u[(x \owedge y) \owedge z] \Rightarrow c} \;(\owedge a) \qquad \frac{u[x \owedge y] \Rightarrow c}{u[y \owedge x] \Rightarrow c} \;(\owedge e)$$

$$\frac{u[x] \Rightarrow c}{u[x \owedge y] \Rightarrow c} \;(\owedge i) \qquad \frac{u[x \owedge x] \Rightarrow c}{u[x] \Rightarrow c} \;(\owedge c)$$

And replace ($\wedge$L) by:

$$\frac{u[a \owedge b] \Rightarrow c}{u[a \wedge b] \Rightarrow c} \;(\wedge \mathsf{L})$$

Nuclei

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Given a residuated lattice expansion $\mathbf{L}' = (\mathbf{L}, \owedge)$, a *distributive nucleus* γ is $\cdot$-nucleus and $\owedge$-nucleus on $\mathbf{L}$ that satisfies $\gamma(x \owedge y) = \gamma(x) \wedge \gamma(y)$.

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Then $\bigcirc\!\!\!\!\wedge_\gamma = \wedge$ on L_γ and

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We aim for an embedding of distributive residuated lattices to Heyting residuated lattices.

Distributive frames

A *distributive residuated frame* is a structure $\mathbf{W} = (W, W', N, \circ, 1, \owedge, \backslash\!\backslash, /\!/, \hookrightarrow, \hookleftarrow)$ where W and W' are sets $N \subseteq W \times W'$,

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$$(x \circ y)\ N\ w \ \Leftrightarrow\ y\ N\ (x \backslash\!\backslash w) \ \Leftrightarrow\ x\ N\ (w /\!/ y)$$

$$(x \owedge y)\ N\ w \ \Leftrightarrow\ y\ N\ (x \hookrightarrow w) \ \Leftrightarrow\ x\ N\ (w \hookleftarrow y)$$

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Theorem. If $\mathbf{W}$ is a distributive frame, then γ_N is a distributive nucleus on $\mathcal{P}(W)$.

Corollary. If $\mathbf{W}$ is a distributive residuated frame then the *dual algebra* $\mathbf{W}^+$ is a distributive residuated lattice.

$$\frac{x N a \quad a N z}{x N z}\ (\mathsf{CUT}) \qquad \frac{}{a N a}\ (\mathsf{Id})$$

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$$\frac{x N a \quad b N z}{x \circ (a \backslash b) N z}\ (\backslash \mathsf{L}) \qquad \frac{a \circ x N b}{x N a \backslash b}\ (\backslash \mathsf{R})$$

$$\frac{x N a \quad b N z}{(b/a) \circ x N z}\ (/\mathsf{L}) \qquad \frac{x \circ a N b}{x N b/a}\ (/\mathsf{R})$$

$$\frac{a \circ b N z}{a \cdot b N z}\ (\cdot \mathsf{L}) \qquad \frac{x N a \quad y N b}{x \circ y N a \cdot b}\ (\cdot \mathsf{R}) \qquad \frac{\varepsilon N z}{1 N z}\ (1\mathsf{L}) \qquad \frac{}{\varepsilon N 1}\ (1\mathsf{R})$$

$$\frac{a \owedge b N z}{a \wedge b N z}\ (\wedge \mathsf{L}\ell) \qquad \frac{x N a \quad x N b}{x N a \wedge b}\ (\wedge \mathsf{R})$$

$$\frac{a N z \quad b N z}{a \vee b N z}\ (\vee \mathsf{L}) \qquad \frac{x N a}{x N a \vee b}\ (\vee \mathsf{R}\ell) \qquad \frac{x N b}{x N a \vee b}\ (\vee \mathsf{R}r)$$

Involutive FL

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Also, we get: $x\backslash y = (\sim x) + y$ and $y/x = y + (-x)$,

Toward BiFL

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Also, we get: $x \backslash y = (\sim x) + y$ and $y/x = y + (-x)$,

as well as: $x \mathbin{-\!\!\!\cdot} y = (\sim x) \cdot y$ and $y \mathbin{\dot{-}} x = y \cdot (-x)$.

Relativizing to InFL

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If we add a new type to negations $\sim x, -x : N_n \to P_n$, then we arrive at a new notion of sequent (multiple conclusion). The operations at the frame level corresponding to the negations are denoted by $\{\}^{\sim}$ and $\{\}^{-}$.

$$\frac{x \circ y \Rightarrow z}{y \Rightarrow x^{\sim} \circ z} \;(^{\sim}) \qquad \frac{x \circ y \Rightarrow z}{x \Rightarrow z \circ y^{-}} \;(^{-})$$

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If $a \in Fm$, we define $a^{\sim 0} = a$ and $a^{\sim(n+1)} = (a^{\sim n})^{\sim}$. A *negated formula* is of the form $a^{\sim n}$ or a^{-n}.
We set $a^{\sim -} = a = a^{-\sim}$.
We denote by Fm^i the free monoid over the set of negated formulas.
A *sequent* is of the form $x \Rightarrow y$ for $x, y \in Fm^i$.

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If we add a new type to negations $\sim x, -x : N_n \to P_n$, then we arrive at a new notion of sequent (multiple conclusion). The operations at the frame level corresponding to the negations are denoted by $\{\}^{\sim}$ and $\{\}^{-}$.

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If $a \in Fm$, we define $a^{\sim 0} = a$ and $a^{\sim(n+1)} = (a^{\sim n})^{\sim}$. A *negated formula* is of the form $a^{\sim n}$ or a^{-n}.
We set $a^{\sim -} = a = a^{- \sim}$.
We denote by Fm^i the free monoid over the set of negated formulas.
A *sequent* is of the form $x \Rightarrow y$ for $x, y \in Fm^i$.

For $x = a_1, \ldots, a_n$, we define

$$x^{\sim} = a_n^{\sim}, \ldots, a_1^{\sim} \text{ and } x^{-} = a_n^{-}, \ldots, a_1^{-}.$$

$x^{\sim -} = x = x^{- \sim}$, $(x \circ y)^{\sim} = y^{\sim} \circ x^{\sim}$, $(x \circ y)^{-} = y^{-} \circ x^{-}$.

InFL

$$\frac{x \Rightarrow a \quad a \Rightarrow z}{x \Rightarrow z}\ (\mathrm{CUT}) \qquad \frac{}{a \Rightarrow a}\ (\mathrm{Id})$$

$$\frac{x \Rightarrow a \quad b \Rightarrow z}{x \circ (a\backslash b) \Rightarrow z}\ (\backslash \mathrm{L}) \qquad \frac{a \circ x \Rightarrow b}{x \Rightarrow a\backslash b}\ (\backslash \mathrm{R})$$

$$\frac{x \Rightarrow a \quad b \Rightarrow z}{(b/a) \circ x \Rightarrow z}\ (/\mathrm{L}) \qquad \frac{x \circ a \Rightarrow b}{x \Rightarrow b/a}\ (/\mathrm{R})$$

$$\frac{a \circ b \Rightarrow z}{a \cdot b \Rightarrow z}\ (\cdot \mathrm{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x \circ y \Rightarrow a \cdot b}\ (\cdot \mathrm{R}) \qquad \frac{\varepsilon \Rightarrow z}{1 \Rightarrow z}\ (1\mathrm{L}) \qquad \frac{}{\varepsilon \Rightarrow 1}\ (1\mathrm{R})$$

$$\frac{a \Rightarrow z}{a \wedge b \Rightarrow z}\ (\wedge \mathrm{L}\ell) \qquad \frac{b \Rightarrow z}{a \wedge b \Rightarrow z}\ (\wedge \mathrm{L}r) \qquad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a \wedge b}\ (\wedge \mathrm{R})$$

$$\frac{a \Rightarrow z \quad b \Rightarrow z}{a \vee b \Rightarrow z}\ (\vee \mathrm{L}) \qquad \frac{x \Rightarrow a}{x \Rightarrow a \vee b}\ (\vee \mathrm{R}\ell) \qquad \frac{x \Rightarrow b}{x \Rightarrow a \vee b}\ (\vee \mathrm{R}r)$$

$$\frac{a^{ln} \Rightarrow z}{\sim a \Rightarrow z}\ (\sim \mathrm{L}) \qquad \frac{x \Rightarrow a^{\sim}}{x \Rightarrow \sim a}\ (\sim \mathrm{R})$$

$$\frac{a^{-} \Rightarrow z}{-a \Rightarrow z}\ (- \mathrm{L}) \qquad \frac{x \Rightarrow a^{-}}{x \Rightarrow -a}\ (- \mathrm{R})$$

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Involutive frames

An *involutive (residuated) frame* is a structure of the form $\mathbf{W} = (W, N, \circ, \varepsilon, {}^{\sim}, {}^{-})$, where

- $(W, \circ, \varepsilon)$ is a monoid
- $x^{\sim -} = x = x^{-\sim}$
- $(y^{\sim} \circ x^{\sim})^{-} = (y^{-} \circ x^{-})^{\sim}$ $[= x \oplus y]$
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$$X \circ Y \subseteq Z \;\Leftrightarrow\; Y \subseteq \sim(-Z \circ X) \;\Leftrightarrow\; X \subseteq -(Y \circ \sim Z)$$

Quasiembedding

Using the following rules of $\mathbf{InFL}$ we can prove the main theorem.

$$\frac{a^{\sim} N z}{\sim a N z}\ (\sim\mathrm{L}) \qquad \frac{x N a^{\sim}}{x N \sim a}\ (\sim\mathrm{R})$$

$$\frac{a^{-} N z}{-a N z}\ (-\mathrm{L}) \qquad \frac{x N a^{-}}{x N -a}\ (-\mathrm{R})$$

Theorem. For all $a \in B$, in an involutive Genzen frame $\sim\{a\}^{\lhd} = \{\sim a\}^{\lhd}$ and $-\{a\}^{\lhd} = \{-a\}^{\lhd}$.

BiFL

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More difficult than one residuated pair, as now there are non-innocent/bad sequents.

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Elements of W of the form $w < a$ and ε are called *proper*. For convenience we extend the multiplication of A to $a \in A \cup \{\varepsilon\}$ by $a \cdot \varepsilon = \varepsilon \cdot a = a$ (and $\varepsilon \to a = a$). Also, p, ε is simply p.

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We define the set W' to be given by the grammar $W' := P > A$, where P is the set of proper elements of W. We write a for $\varepsilon > a$.

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We define the (hyper)operation $\circ$ on proper elements by $p \circ \varepsilon = \varepsilon \circ p = p$ and $(w < a) \circ (w' < a') = \emptyset$. Then we 'extend' it to arbitrary elements by $(p, a) \circ (p', a') = (p \circ p'), (a \cdot a')$.

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Finally, for $x \in W$ and $a \in A$ we define $x^+[a]$ as follows by induction on the structure of x.
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Theorem Every FL^+-algebra can be embedded into a BiFL-algebra.

Adding structural rules

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Grishin(b): $x(y + z) \leq xy + z$ gives a stabilizing definition:
$(w < a) \circ (w' < a') = \{((w < a) \circ w') < a', ((w' < a') \circ w) < a\}$.

Applications

Frame applications

- DM-completion
- Perfect residuated lattices
- Completeness of the calculus
- Cut elimination
- Finite model property
- Finite embeddability property
- (Generalized super-)amalgamation property (Transferable injections, Congruence extension property)
- (Craig) Interpolation property
- Disjunction property
- Strong separation
- Stability under linear structural rules/equations over $\{\vee, \cdot, 1\}$.
- Densification
- Conservativity (via algebraic embeddings)

Examples of frames: FEP

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For

$$(u, a) \mathbin{/\!\!/} x = \{(u[\_ \cdot x], a)\} \text{ and } x \mathbin{\backslash\!\!\backslash} (u, a) = \{(u[x \cdot \_], a)\},$$

we have

$$\begin{aligned} x \cdot y N(u, a) \quad & \text{iff } u[x \cdot y] \leq a \\ & \text{iff } x N(u[\_ \cdot y], a) \\ & \text{iff } y N(u[x \cdot \_], a). \end{aligned}$$

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Lemma. Every equation over $\{\vee, \cdot, 1\}$ is equivalent to a conjunction of simple equations.

Proof For an equation ε over $\{\vee, \cdot, 1\}$ we distribute products over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j: monoid terms.

$s_1 \vee \cdots \vee s_m \leq t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leq s_1 \vee \cdots \vee s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \vee \cdots \vee t_n)$.

We proceed by example: $x^2 y \leq xy \vee yx$

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More generally, if x appears n-times on the LHS, we substitute $x_1 \vee \ldots \vee x_n$ for x, distribute and retain one representative term on the LHS (where all the x_i's occur).

Simple rules

Let $t_0, t_1, \ldots, t_n$ be monoid terms and let t_0 be linear. A *simple* rule is an expression of the form

$$\frac{t_1 \ N \ q \quad \cdots \quad t_n \ N \ q}{t_0 \ N \ q} \ (\mathsf{r})$$

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A Gentzen frame $(\mathbf{W}, \mathbf{B})$ *satisfies* (r) if for all $z \in W'$, and for all sequences $\bar{x}$ of elements of W matching the variables involved in $t_0, t_1, \ldots, t_n$, the conjunction of the conditions $t_i^{\mathbf{W}}(\bar{x}) \ N \ z$, for $i \in \{1, \ldots, n\}$, implies $t_0^{\mathbf{W}}(\bar{x}) \ N \ z$.

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In the context of $(\mathbf{W_{FL}}, \mathbf{Fm})$, $R(\varepsilon)$ takes the form

$$\frac{u[t_1] \ \Rightarrow \ a \quad \cdots \quad u[t_n] \ \Rightarrow \ a}{u[t_0] \ \Rightarrow \ a} \ (R(\varepsilon))$$

In that sense, we may view basic structural rules as simple rules.

Reduction to simple

Lemma. Every equation ε over $\{\vee, \cdot, 1\}$ is equivalent, relative to RL, to R(ε). More precisely, for every $\mathbf{A} \in$ RL, $\mathbf{A}$ satisfies ε iff $\mathbf{W_A}$ satisfies R(ε).

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$$\frac{x_1 \circ y \; N \; z \quad x_2 \circ y \; N \; z \quad y \circ x_1 \; N \; z \quad y \circ x_2 \; N \; z}{x_1 \circ x_2 \circ y \; N \; z} \; R(\varepsilon)$$

Simplicity preserved

Theorem. Let $(\mathbf{W}, \mathbf{B})$ be a cf Gentzen frame and let ε be a $\{\vee, \cdot, 1\}$-equation. Then $(\mathbf{W}, \mathbf{B})$ satisfies $\mathsf{R}(\varepsilon)$ iff $\mathbf{W}^+$ satisfies ε.

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Theorem. Every system obtained from $\mathbf{GL}$ by adding linear reducing rules (and also the equational theory of the corresponding variety) is decidable. (*reducing*: there is a complexity measure that decreases with upward applications of the rules).

Finite model property

For $\mathbf{W_{FL}}$, given $(x, z) \in W \times W'$ (if $z = (u, c)$, then $u(x) \Rightarrow c$ is a sequent), we define $(x, z)^{\uparrow}$ as the smallest subset of $W \times W'$ that contains (x, z) and is closed upwards with respect to the rules of $\mathbf{FL^f}$. Note that $(x, z)^{\uparrow}$ is finite.

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The new frame $\mathbf{W}'$ associated with $N' = N \cup ((y, v)^{\uparrow})^c$ is residuated and Gentzen.
Clearly, $(N')^c$ is finite, so it has a finite domain $Dom((N')^c)$ and codomain $Cod((N')^c)$.
For every $z \notin Cod((N')^c)$, $\{z\}^{\lhd} = W$. So, $\{\{z\}^{\lhd} : z \in W\}$ is finite and a basis for $\gamma_{N'}$. So, $\mathbf{W}'^{+}$ is finite.
Moreover, if $u(x) \Rightarrow c$ is not provable in $\mathbf{FL}$, then it is not valid in $\mathbf{W}'^{+}$.

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Corollary. The variety of residuated lattices is generated by its finite members. The same holds for the subvarieties corresponding to the above extensions.

FEP

A class of algebras $\mathcal{K}$ has the *finite embeddability property (FEP)* if for every $\mathbf{A} \in \mathcal{K}$, every finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ can be (partially) embedded in a finite $\mathbf{D} \in \mathcal{K}$.

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Theorem. Every variety of integral RL's axiomatized by equartions over $\{\vee, \cdot, 1\}$ has the FEP.

- $\mathbf{B}$ embeds in $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ via $\{\_\}^{\lhd} : \mathbf{B} \to \mathbf{W}^+$
- $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ is finite
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Corollary. These varieties are generated as quasivarieties by their finite members.

Corollary. The corresponding logics have the *strong finite model property*:
if $\Phi \not\vdash \psi$, for finite Φ, then there is a finite counter-model, namely there is $\mathbf{D} \in \mathcal{V}$ and a homomorphism $f : \mathbf{Fm} \to \mathbf{D}$, such that $f(\phi) = 1$, for all $\phi \in \Phi$, but $f(\psi) \neq 1$.

Amalgamation

A class $\mathcal{K}$ of similar algebras has the *amalgamation property* (AP), if for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, there is a $\mathbf{D} \in \mathcal{K}$ and embeddings $f' : \mathbf{B} \to \mathbf{D}$ and $g' : \mathbf{C} \to \mathbf{D}$ such that $f' \circ f = g' \circ g$.

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We will

- define γ (by giving an associated Galois connection) and $\mathbf{D}$,
- prove that $\mathbf{D} \in \mathsf{CRL_n}$,
- prove that $\mathbf{B}, \mathbf{C} \hookrightarrow_{\mathbf{A}} \mathbf{D}$.

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Equations

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Corollary. $\mathbf{D} \in \mathsf{CRL_n}$.

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Corollary. CRL$_n$ has the AP.

Gen. amalgamation

If we do not assume that f, g are injective, instead of

$$N = \langle (N_B \circ N_C) \cup (N_C \circ N_B) \rangle,$$

we take

$$N = \langle (N_B \circ f \circ g \circ N_C) \cup (N_C \circ g \circ f \circ N_B) \rangle.$$

Then we can prove AP, transferable injections, and transferable surjections and the congruence extension property all with a single argument.

Interpolation

Theorem. $\mathbf{FL_e}$ has the Craig interpolation property, i.e. if $\vdash_{\mathbf{FL_e}} \phi \to \psi$, then there is a χ such that

- $\vdash_{\mathbf{FL_e}} \phi \to \chi$ and $\vdash_{\mathbf{FL_e}} \chi \to \psi$
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Proof sketch. Define a frame with $W = Fm^*$, $W' = Fm^* \times Fm$ and $x \; N \; (u, d)$ iff for $X \cup Y = var(x, u, d)$, $B = Fm(X)$, $C = Fm(Y)$, and for all partitions $x = x_B \circ x_C$, $u = u_B \circ u_c$, with $x_B, u_B \in B^*$, $x_C, u_C \in C^*$

- if $d \in B$, then $\vdash_{\mathbf{FL_e}} u_C \circ x_C \;\Rightarrow\; a$ and $\vdash_{\mathbf{FL_e}} u_B \circ x_B \circ a \;\Rightarrow\; d$, for some $a \in B \cap C$ and
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Corollary. If $\vdash_{\mathbf{FL_e}} u \circ x \Rightarrow d$, then $u \circ x \; N \; d$. I.e., $\mathbf{FL_e}$ has the IP.

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The corresponding algebraic property is:
For $\mathbf{A} \in \mathcal{K}$, there is a $\mathbf{D} \in \mathcal{K}$ and an epimorphism $f : \mathbf{D} \to \mathbf{A}$ such that if $1 \leq_{\mathbf{D}} a \vee B$, then $1 \leq_{\mathbf{A}} f(a)$ or $1 \leq_{\mathbf{A}} f(b)$.

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This property holds for all subvarieties of CRL axiomatized with equations over $\{\vee, \cdot, 1\}$.

Assume that $\mathcal{K}$ is a sublanguage of $\mathcal{L}$ that contains $\backslash$. The system $\mathcal{K}\mathbf{FL}$ is defined to be the set of all rules from $\mathbf{FL}$ that involve $\mathcal{K}$-formulas and $\mathcal{K}$-solvable sequents.

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Given a structure $(\mathbf{W}, \mathbf{A}_{\mathcal{K}})$ and a meta-rule (r) of $\mathcal{K}\mathbf{FL}$, we define $(r)^{(\mathbf{W}, \mathbf{A}_{\mathcal{K}})}$. For example, $(\backslash\mathrm{L})^{(\mathbf{W}, \mathbf{A}_{\mathcal{K}})}$ is

$$\forall a, b, c \in \mathbf{A}, x \in W, uS_W, \text{ if } a \backslash_{\mathbf{A}} b \text{ is defined, then } x \; N \; a \text{ and } u[b] \; N \; c \text{ implies } u[x \circ (a \backslash b)] \; N \; c.$$

Strong separation: syst.

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Lemma A structure $(\mathbf{W}, \mathbf{A}_{\mathcal{K}})$ Gentzen frame iff the interpretation $(r)^{(\mathbf{W}, \mathbf{A}_{\mathcal{K}})}$ of every meta-rule (r) of $\mathcal{K}\mathbf{FL}$ holds.

Strong separation

Let $\mathcal{K}$ be a sublanguage of $\mathcal{L}$ that contains the connective $\backslash$ and let $B \cup \{c\}$ be a set of formulas over $\mathcal{K}$. Also, let $\mathbf{A}_{\mathcal{K}}$ be the partial subalgebra of $\mathbf{Fm}_{\mathcal{K}}$ of all subformulas of $B \cup \{c\}$. Consider the structure $(\mathbf{W}, \mathbf{A}_{\mathcal{K}})$, where W is the free monoid over $\mathbf{A}_{\mathcal{K}}$, $W' = S_W \times \mathbf{A}_{\mathcal{K}}$ and where $x \; N \; (u, a)$ iff $B \vdash_{\mathcal{K}\mathbf{HL}} \phi_{\mathcal{K}}(u(x) \Rightarrow a)$.

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Corollary. Let $\mathcal{K}$ be a sublanguage of $\mathcal{L}$ that contains the connective $\backslash$ and let $B \cup \{c\}$ be a set of formulas over $\mathcal{K}$. Then $(\mathbf{W}, \mathbf{A}_\mathcal{K})$ is a Gentzen frame.

Corollary If $B \cup \{c\}$ is a set of formulas over a sublanguage $\mathcal{K}$ of $\mathcal{L}$ that contains $\backslash$, then $B \vdash_{\mathbf{HL}} c$ iff $B \vdash_{\mathcal{K}-\mathbf{HL}} c$. In particular, the Hilbert system $\mathbf{HL}$ enjoys the separation property.

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$x_1 x_2 \wedge y \leq (x_1 \wedge y) \vee (x_2 \wedge y) \vee y x_1 \vee y x_2$

$$\frac{x_1 \wedge y \leq v \quad x_2 \wedge y \leq v \quad y x_1 \leq v \quad y x_2 \leq v}{x_1 x_2 \wedge y \leq v}$$

$$\frac{x_1 \owedge y \; N \; z \quad x_2 \owedge y \; N \; z \quad y \circ x_1 \; N \; z \quad y \circ x_2 \; N \; z}{x_1 \circ x_2 \owedge y \; N \; z} \; R(\varepsilon)$$

Structural rules

Given an equation ε of the form $t_0 \leq t_1 \vee \cdots \vee t_n$, where t_i are $\{\wedge, \cdot, 1\}$-terms we construct the rule $R(\varepsilon)$

$$\frac{u[t_1] \Rightarrow a \quad \cdots \quad u[t_n] \Rightarrow a}{u[t_0] \Rightarrow a} \; (R(\varepsilon))$$

where the t_i's are evaluated in $(W, \circ, \varepsilon)$. Such a rule is called *analytic* if all variables in t_0 are distinct.

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Theorem. If $(\mathbf{W}, \mathbf{B})$ is a Gentzen frame and ε an equation over $\{\wedge, \vee, \cdot, 1\}$, then $(\mathbf{W}, \mathbf{B})$ satisfies R(ε) iff $\mathbf{W}^+$ satisfies ε.

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Theorem. Every system obtained from $\mathbf{FL}$ by adding analytic rules has the cut elimination property.

FEP for DFL

Let $\mathcal{V}$ be a subvariety of DIRL axiomatized over $\{\vee, \wedge, \cdot, 1\}$. To establish the FEP for $\mathcal{V}$, for every $\mathbf{A}$ in $\mathcal{V}$ and $\mathbf{B}$ a finite partial subalgebra of $\mathbf{A}$, we construct an algebra $\mathbf{D} = \mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ such that

- $\mathbf{W}^+_{\mathbf{A},\mathbf{B}} \in \mathcal{V}$
- $\mathbf{B}$ embeds in $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$
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$\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ is defined by taking $(W, \circ, \owedge, 1)$ to be the $\{\cdot, \wedge, 1\}$-subreduct of $\mathbf{A}$ generated by B, $W' = S_W \times B$ and $x\ N\ (u, b)$ iff $u(x) \leq_{\mathbf{A}} b$.

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Theorem. Every subvariety of DIRL axiomatized over $\{\vee, \wedge, \cdot, 1\}$ has the FEP.