Nick Galatos University of Denver ngalatos@du.edu Reporting on work and ideas with Agata, Kaz, Peter, Paolo, Revantha, Rosta

March, 2014

Nick Galatos, Prague workshop, March, 2014

Residuated frames Residuated frames Intuition Ex: sequent calculi Ex: filters and ideals Contexts/polarities Dedekind-McNeille The dual algebra GN Gentzen frames FL Frames and modules Frames and display Distributive frames

Involutive FL

BiFL

	Residuated frames
	Intuition
	Ex: sequent calculi
A residuated frame is a structure $\mathbf{W} = (W, W', N, \circ, \mathbb{N}, \mathbb{N})$ subject to	Ex: filters and ideals
	Contexts/polarities
the condition: for all $x, y \in W$ and $w \in W'$	Dedekind-McNeille
	The dual algebra
$() \mathbf{N} \mathbf{T} () \mathbf{T} () \mathbf{T} \mathbf{T} \mathbf{T} () \mathbf{T} \mathbf{T} \mathbf{T} \mathbf{T} () \mathbf{T} \mathbf{T} \mathbf{T} () \mathbf{T} \mathbf{T} \mathbf{T} \mathbf{T} () \mathbf{T} \mathbf{T} \mathbf{T} () \mathbf{T} \mathbf{T} \mathbf{T} \mathbf{T} () \mathbf{T} \mathbf{T} \mathbf{T} \mathbf{T} \mathbf{T} \mathbf{T} \mathbf{T} \mathbf{T}$	GN
$(x \circ y) \land w \Leftrightarrow y \land (x \setminus w) \Leftrightarrow x \land (w \not y)$	Gentzen frames
	FL
	Frames and modules
	Frames and display
	Distributive frames
	Involutive FL
	BiFL
	Applications

Residuated frames

	Residuated frames
	Intuition
	Ex: sequent calculi
) subject to	Ex: filters and ideals
, 5	Contexts/polarities
	Dedekind-McNeille
	The dual algebra
<u>۱</u>	GN
)	Gentzen frames
	FL
	Frames and modules
	Frames and display
	Distributive frames
	Involutive FL
	BiFL
	Applications

Residuated frames

A residuated frame is a structure $\mathbf{W} = (W, W', N, \circ, \mathbb{N}, \mathbb{N})$ subject to the condition: for all $x, y \in W$ and $w \in W'$

 $(x \circ y) N w \Leftrightarrow y N (x \setminus w) \Leftrightarrow x N (w / y)$

 $\begin{array}{ll} W, W' \text{ are sets,} & N \subseteq W \times W' \text{ is a relation,} \\ \circ \subseteq W^3 & \backslash \!\! \backslash \subseteq W \times W' \times W' & /\!\!/ \subseteq W' \times W \times W' \end{array}$

A residuated frame is a structure $\mathbf{W} = (W, W', N, \circ, \mathbb{N}, \mathbb{N})$ subject to the condition: for all $x, y \in W$ and $w \in W'$

$$(x \circ y) N w \Leftrightarrow y N (x \setminus w) \Leftrightarrow x N (w / y)$$

 $\begin{array}{ll} W,W' \text{ are sets,} & N \subseteq W \times W' \text{ is a relation,} \\ \circ \subseteq W^3 & \backslash\!\!\backslash \subseteq W \times W' \times W' & /\!\!/ \subseteq W' \times W \times W' \end{array}$

Notation:

 $x \circ y = \{z \in W' : \circ(x, y, z)\} \\ x \ \ z = \{s \in W' : \ \ (x, z, s)\} \\ z \ \ y = \{s \in W' : \ \ (z, x, s)\}$

XNY means xNy, for all $x \in X$ and $y \in Y$.

Residuated frames Residuated frames Intuition Ex: sequent calculi Ex: filters and ideals Contexts/polarities Dedekind-McNeille The dual algebra GN Gentzen frames FL Frames and modules Frames and display Distributive frames Involutive FL BiFL Applications

A residuated frame is a structure $\mathbf{W} = (W, W', N, \circ, \mathbb{N}, \mathbb{N})$ subject to the condition: for all $x, y \in W$ and $w \in W'$

 $(x \circ y) \ N \ w \ \Leftrightarrow \ y \ N \ (x \setminus w) \ \Leftrightarrow \ x \ N \ (w \not| y)$

 $\begin{array}{ll} W,W' \text{ are sets,} & N \subseteq W \times W' \text{ is a relation,} \\ \circ \subseteq W^3 & \backslash\!\!\backslash \subseteq W \times W' \times W' & /\!\!/ \subseteq W' \times W \times W' \end{array}$

Notation:

 $x \circ y = \{z \in W' : \circ(x, y, z)\} \\ x \ \ z = \{s \in W' : \ \ (x, z, s)\} \\ z \ \ y = \{s \in W' : \ \ (z, x, s)\}$

XNY means xNy, for all $x \in X$ and $y \in Y$.

We can also add constants and other relations (see below), as well as properties such as associativity, commutativity, etc.

Residuated frames
Residuated frames
Intuition
Ex: sequent calculi
Ex: filters and ideals
Contexts/polarities
Dedekind-McNeille
The dual algebra
GN
Gentzen frames
FL
Frames and modules
Frames and display
Distributive frames
Involutive FL
BiFL
Applications

A residuated frame is a structure $\mathbf{W} = (W, W', N, \circ, \mathbb{N}, \mathbb{N})$ subject to the condition: for all $x, y \in W$ and $w \in W'$

 $(x \circ y) N w \Leftrightarrow y N (x \setminus w) \Leftrightarrow x N (w / y)$

 $\begin{array}{ll} W,W' \text{ are sets,} & N \subseteq W \times W' \text{ is a relation,} \\ \circ \subseteq W^3 & \backslash\!\!\backslash \subseteq W \times W' \times W' & /\!\!/ \subseteq W' \times W \times W' \end{array}$

Notation:

 $x \circ y = \{z \in W' : \circ(x, y, z)\} \\ x \ \ z = \{s \in W' : \ \ (x, z, s)\} \\ z \ \ y = \{s \in W' : \ \ (z, x, s)\}$

XNY means xNy, for all $x \in X$ and $y \in Y$.

We can also add constants and other relations (see below), as well as properties such as associativity, commutativity, etc. We will be sloppy about assuming such constants of conditions.

Residuated frames Residuated frames Intuition Ex: sequent calculi Ex: filters and ideals Contexts/polarities Dedekind-McNeille The dual algebra GN Gentzen frames FL Frames and modules Frames and display Distributive frames Involutive FL BiFL Applications

Residuated frames

Residuated frames

Intuition

Ex: sequent calculi Ex: filters and ideals Contexts/polarities Dedekind-McNeille

Dedekind-ivicivelile

The dual algebra

Gentzen frames

GN

FL

Frames and modules

Frames and display

Distributive frames

Involutive FL

BiFL

Applications

The intuition comes from algebra (residuated lattices), proof-theory (sequent calculus), and duality theory/relational semantics (Kripke frames).

Residuated frames

Residuated frames Intuition Ex: sequent calculi

Ex: sequent calcul Ex: filters and ideals Contexts/polarities Dedekind-McNeille The dual algebra GN Gentzen frames FL Frames and modules Frames and display

Distributive frames

BiFL

Applications

The intuition comes from algebra (residuated lattices), proof-theory (sequent calculus), and duality theory/relational semantics (Kripke frames).

Algebra: The sets W and W' are both the underlying set, N is the order relation and \circ is multiplication.

If L is a RL, $\mathbf{W}_{\mathbf{L}} = (L, L, \leq, \cdot, \{1\}, \backslash, /)$ is a residuated frame.

Residuated frames Residuated frames

Intuition

Ex: sequent calculi Ex: filters and ideals Contexts/polarities Dedekind-McNeille The dual algebra GN Gentzen frames FL Frames and modules Frames and display Distributive frames Involutive FL BiFL Applications

The intuition comes from algebra (residuated lattices), proof-theory (sequent calculus), and duality theory/relational semantics (Kripke frames).

Algebra: The sets W and W' are both the underlying set, N is the order relation and \circ is multiplication.

If L is a RL, $\mathbf{W}_{\mathbf{L}} = (L, L, \leq, \cdot, \{1\}, \backslash, /)$ is a residuated frame.

Proof theory (single conclusion sequents): W plays the role of the LHS of sequents, W' the role of RHS of sequents and N the relation of provability of the sequent formed. In sequents, \circ is the comma on the LHS.

Residuated frames

Intuition

Ex: sequent calculi Ex: filters and ideals Contexts/polarities Dedekind-McNeille The dual algebra GN Gentzen frames FL Frames and modules Frames and display Distributive frames Involutive FL BiFL Applications

The intuition comes from algebra (residuated lattices), proof-theory (sequent calculus), and duality theory/relational semantics (Kripke frames).

Algebra: The sets W and W' are both the underlying set, N is the order relation and \circ is multiplication.

If L is a RL, $\mathbf{W}_{\mathbf{L}} = (L, L, \leq, \cdot, \{1\}, \backslash, /)$ is a residuated frame.

Proof theory (single conclusion sequents): W plays the role of the LHS of sequents, W' the role of RHS of sequents and N the relation of provability of the sequent formed. In sequents, \circ is the comma on the LHS.

Kripke semantics: In a finite lattice we need the join-irreducibles W and the meet-irreducibles W' to describe the lattice, as well as the restriction N of the order relation between W and W'. In residuated lattices, \circ reflects the monoid structure.

(To view Kripke frames as residuated frames we take W' = W, $N = \geq$, while \circ is *partial meet*.)

Consider the a sequent caculus L (single conclusion sequents).

We define the frame $\mathbf{W}_{\mathbf{L}}$, where

Intuition

Ex: sequent calculi

Ex: filters and ideals Contexts/polarities Dedekind-McNeille The dual algebra GN

Gentzen frames

Frames and modules

Frames and display

Distributive frames

Involutive FL

BiFL

FL

Consider the a sequent caculus L (single conclusion sequents).

We define the frame $\mathbf{W}_{\mathbf{L}}$, where

• (W, \circ, ε) is the free monoid over the set Fm of all formulas

Residuated frames Residuated frames

Intuition

Ex: sequent calculi

Ex: filters and ideals Contexts/polarities Dedekind-McNeille The dual algebra GN Gentzen frames FL

Frames and modules

Frames and display

Distributive frames

Involutive FL

BiFL

Consider the a sequent caculus L (single conclusion sequents).

We define the frame $\mathbf{W}_{\mathbf{L}}$, where

(W, \circ, ε) is the free monoid over the set Fm of all formulas $W' = S_W \times Fm$, where S_W is the set of all *unary linear* polynomials $u[x] = y \circ x \circ z$ of W, and Residuated frames Residuated frames Intuition

Ex: sequent calculi Ex: filters and ideals

Contexts/polarities Dedekind-McNeille The dual algebra GN Gentzen frames FL

Frames and modules

Frames and display

Distributive frames

Involutive FL

BiFL

Consider the a sequent caculus L (single conclusion sequents).

We define the frame $\mathbf{W}_{\mathbf{L}}$, where

 $(W, \circ, \varepsilon) \text{ is the free monoid over the set } Fm \text{ of all formulas}$ $W' = S_W \times Fm, \text{ where } S_W \text{ is the set of all unary linear}$ $polynomials u[x] = y \circ x \circ z \text{ of } W, \text{ and}$ $x N(u, a) \text{ iff } \vdash_{\mathbf{L}} u[x] \Rightarrow a.$

Residuated frames Intuition

Residuated frames

Ex: sequent calculi Ex: filters and ideals Contexts/polarities Dedekind-McNeille The dual algebra GN Gentzen frames FL Frames and modules Frames and display Distributive frames

Involutive FL

BiFL

Consider the a sequent caculus L (single conclusion sequents).

We define the frame $\mathbf{W}_{\mathbf{L}}$, where

 $\begin{array}{l} \blacksquare \quad (W,\circ,\varepsilon) \text{ is the free monoid over the set } Fm \text{ of all formulas} \\ \blacksquare \quad W' = S_W \times Fm, \text{ where } S_W \text{ is the set of all unary linear} \\ polynomials \ u[x] = y \circ x \circ z \text{ of } W, \text{ and} \\ \blacksquare \quad x \ N \ (u,a) \text{ iff } \vdash_{\mathbf{L}} u[x] \Rightarrow a. \end{array}$

For

we have

$$\begin{array}{ll} x \circ y N(u,a) & \text{iff} \vdash_{\mathbf{L}} u[x \circ y] \Rightarrow a \\ & \text{iff} \vdash_{\mathbf{L}} u[x \circ y] \Rightarrow a \\ & \text{iff} x N(u[_\circ y],a) \\ & \text{iff} y N(u[x \circ _],a). \end{array}$$

Residuated frames Residuated frames Intuition Ex: sequent calculi Ex: filters and ideals Contexts/polarities Dedekind-McNeille The dual algebra GN Gentzen frames FL Frames and modules Frames and display Distributive frames Involutive FL **BiFL** Applications

Theorem. (Lattice frames and lattices) Let L a perfect lattice and $\mathbf{W} = (J, M, N)$. Then $\mathbf{L} = \mathbf{W}^+$ iff $J^{\infty}(\mathbf{L}) \subseteq J$ and $M^{\infty}(\mathbf{L}) \subseteq M$, where $j \ N \ m \Leftrightarrow f(j) \leq g(m)$ for some $f: J \to L, \ g: M \to L$. Here $J^{\infty}(\mathbf{L})$ and $M^{\infty}(\mathbf{L})$ denote the *completely join irreducible* elements of L, namely elements j such that $j = \bigvee X$ iff $j \in X$, and the *completely meet irreducible* elements. L is a *perfect* if every element is a join of completely join irreducible elements of L and a meet of completely meet irreducible elements.

Residuated frames Residuated frames Intuition Ex: sequent calculi Ex: filters and ideals Contexts/polarities Dedekind-McNeille The dual algebra GN Gentzen frames FL Frames and modules Frames and display Distributive frames Involutive FL **BiFL** Applications

Theorem. (Lattice frames and lattices) Let L a perfect lattice and $\mathbf{W} = (J, M, N)$. Then $\mathbf{L} = \mathbf{W}^+$ iff $J^{\infty}(\mathbf{L}) \subseteq J$ and $M^{\infty}(\mathbf{L}) \subseteq M$, where $j \ N \ m \Leftrightarrow f(j) \leq g(m)$ for some $f : J \to L, \ g : M \to L$. Here $J^{\infty}(\mathbf{L})$ and $M^{\infty}(\mathbf{L})$ denote the *completely join irreducible* elements of L, namely elements j such that $j = \bigvee X$ iff $j \in X$, and the *completely meet irreducible* elements. L is a *perfect* if every element is a join of completely join irreducible elements.

If L is a *perfect* residuated lattice, then $\mathbf{W}_{\mathbf{L}}^{\infty} = (J^{\infty}(\mathbf{L}), M^{\infty}(\mathbf{L}), \leq, \cdot)$ is a residuated frame for $w' \not| w_2$ the set of all meet irreducibles above w'/w_2 , and likewise for $w_1 \setminus w'$. Residuated frames

Ex: sequent calculi

Intuition

Theorem. (Lattice frames and lattices) Let L a perfect lattice and $\mathbf{W} = (J, M, N)$. Then $\mathbf{L} = \mathbf{W}^+$ iff $J^{\infty}(\mathbf{L}) \subseteq J$ and $M^{\infty}(\mathbf{L}) \subseteq M$, where $j \ N \ m \Leftrightarrow f(j) \leq g(m)$ for some $f : J \to L, \ g : M \to L$. Here $J^{\infty}(\mathbf{L})$ and $M^{\infty}(\mathbf{L})$ denote the *completely join irreducible* elements of L, namely elements j such that $j = \bigvee X$ iff $j \in X$, and the *completely meet irreducible* elements. L is a *perfect* if every element is a join of completely join irreducible elements of L and a meet of completely meet irreducible elements.

If L is a *perfect* residuated lattice, then $\mathbf{W}_{\mathbf{L}}^{\infty} = (J^{\infty}(\mathbf{L}), M^{\infty}(\mathbf{L}), \leq, \cdot)$ is a residuated frame for $w' \not| w_2$ the set of all meet irreducibles above w'/w_2 , and likewise for $w_1 \setminus w'$. Then $\mathbf{L} \cong (\mathbf{W}_{\mathbf{L}}^{\infty})^+$.

Theorem. (Lattice frames and lattices) Let L a perfect lattice and $\mathbf{W} = (J, M, N)$. Then $\mathbf{L} = \mathbf{W}^+$ iff $J^{\infty}(\mathbf{L}) \subseteq J$ and $M^{\infty}(\mathbf{L}) \subseteq M$, where $j \ N \ m \Leftrightarrow f(j) \leq g(m)$ for some $f : J \to L, \ g : M \to L$. Here $J^{\infty}(\mathbf{L})$ and $M^{\infty}(\mathbf{L})$ denote the *completely join irreducible* elements of L, namely elements j such that $j = \bigvee X$ iff $j \in X$, and the *completely meet irreducible* elements. L is a *perfect* if every element is a join of completely join irreducible elements of L and a meet of completely meet irreducible elements.

If L is a *perfect* residuated lattice, then $\mathbf{W}_{\mathbf{L}}^{\infty} = (J^{\infty}(\mathbf{L}), M^{\infty}(\mathbf{L}), \leq, \cdot)$ is a residuated frame for $w' \not| w_2$ the set of all meet irreducibles above w'/w_2 , and likewise for $w_1 \setminus w'$. Then $\mathbf{L} \cong (\mathbf{W}_{\mathbf{L}}^{\infty})^+$.

We can also form residuated frames by taking prime ideals, ((completely) prime) ideals, and relation *non-empty intersection*.

Residuated frames		
Residuated frames		
Intuition		
Ex: sequent calculi		
Ex: filters and ideals		
Contexts/polarities		
Dedekind-McNeille		
The dual algebra		
GN		
Gentzen frames		
FL		
Frames and modules		
Frames and display		
Distributive frames		
Involutive FL		
BiFL		
Applications		

Theorem. (Lattice frames and lattices) Let L a perfect lattice and $\mathbf{W} = (J, M, N)$. Then $\mathbf{L} = \mathbf{W}^+$ iff $J^{\infty}(\mathbf{L}) \subseteq J$ and $M^{\infty}(\mathbf{L}) \subseteq M$, where $j \ N \ m \Leftrightarrow f(j) \leq g(m)$ for some $f: J \to L, \ g: M \to L$. Here $J^{\infty}(\mathbf{L})$ and $M^{\infty}(\mathbf{L})$ denote the *completely join irreducible* elements of L, namely elements j such that $j = \bigvee X$ iff $j \in X$, and the *completely meet irreducible* elements. L is a *perfect* if every element is a join of completely join irreducible elements of L and a meet of completely meet irreducible elements.

If L is a *perfect* residuated lattice, then $\mathbf{W}_{\mathbf{L}}^{\infty} = (J^{\infty}(\mathbf{L}), M^{\infty}(\mathbf{L}), \leq, \cdot)$ is a residuated frame for $w' \not| w_2$ the set of all meet irreducibles above w'/w_2 , and likewise for $w_1 \setminus w'$. Then $\mathbf{L} \cong (\mathbf{W}_{\mathbf{L}}^{\infty})^+$.

We can also form residuated frames by taking prime ideals, ((completely) prime) ideals, and relation *non-empty intersection*.

For L a residuated lattice, $\mathbf{W}_{\mathbf{L}}^{FI} = (\mathcal{F}(\mathbf{L}), \mathcal{I}(\mathbf{L}), N_{NEI}, \circ, \mathbb{N}, \mathbb{N})$, the intermediate structure, aka the cannonical frame \mathbf{A}_+ of \mathbf{A} . Then $(\mathbf{A}_+)^+ = (\mathbf{W}_{\mathbf{L}}^{FI})^+$ is the canonical extension of L.

Residuated frames Residuated frames Intuition Ex: sequent calculi Ex: filters and ideals Contexts/polarities Dedekind-McNeille The dual algebra GN Gentzen frames FL Frames and modules Frames and display Distributive frames Involutive FL **BiFL** Applications

Contexts/polarities



Dedekind-McNeille



Dedekind-McNeille



For every distributive lattice $M(\mathbf{L})$ is isomorphic to $J(\mathbf{L})$. Note $\uparrow a \cup \downarrow c = \uparrow b \cup \downarrow a = \uparrow c \cup \downarrow d = L$. *Splitting pairs*: (a, c), (b, a), (c, d).



	Residuated frames
_	Residuated frames
	Intuition
	Ex: sequent calculi
	Ex: filters and ideals
	Contexts/polarities
	Dedekind-McNeille
	The dual algebra
	GN
	Gentzen frames
	FL
	Frames and modules
	Frames and display
	Distributive frames
	Involutive FL
	BiFL
	Applications

For $X \subseteq W$ and $Y \subseteq W'$ we define

$$X^{\triangleright} = \{ b \in W' : x \ N \ b, \text{ for all } x \in X \}$$
$$Y^{\triangleleft} = \{ a \in W : a \ N \ y, \text{ for all } y \in Y \}$$

Residuated frames Residuated frames Intuition Ex: sequent calculi Ex: filters and ideals Contexts/polarities Dedekind-McNeille The dual algebra GN Gentzen frames FL Frames and modules Frames and display Distributive frames Involutive FL BiFL

Intuition Ex: sequent calculi For $X \subseteq W$ and $Y \subseteq W'$ we define Ex: filters and ideals Contexts/polarities Dedekind-McNeille $X^{\triangleright} = \{ b \in W' : x N b, \text{ for all } x \in X \}$ The dual algebra $Y^{\triangleleft} = \{ a \in W : a \ N \ y, \text{ for all } y \in Y \}$ GN Gentzen frames FL The map $\gamma_N : \mathcal{P}(W) \to \mathcal{P}(W), \ \gamma_N(X) = X^{\rhd \triangleleft}$, is a closure Frames and modules Frames and display operator. Distributive frames Involutive FL BiFL Applications

Residuated frames Residuated frames

For $X \subseteq W$ and $Y \subseteq W'$ we define $X^{\triangleright} = \{ b \in W' : x N b, \text{ for all } x \in X \}$ $Y^{\triangleleft} = \{ a \in W : a \ N \ y, \text{ for all } y \in Y \}$ The map $\gamma_N: \mathcal{P}(W) \to \mathcal{P}(W), \ \gamma_N(X) = X^{\rhd \triangleleft}$, is a closure operator. Lemma. If W is a residuated frame then the dual algebra $\mathbf{W}^+ = (\gamma[\mathcal{P}(W)], \cap, \cup_{\gamma}, \circ_{\gamma}, \backslash, /)$ is a complete residuated lattice. $X \cup_{\gamma} Y = \gamma(X \cup Y)$ $X \circ_{\gamma} Y = \gamma(X \circ Y)$ $X \circ Y = \{ z \in W : \circ(X, Y, z) \} = \{ z \in W : \circ(x, y, z), \forall x \in X, y \in Y \}$ $X \setminus Y = \{ z \in W : X \circ \{ z \} \subseteq Y \}$ $Y/X = \{z \in W : \{z\} \circ X \subseteq Y\}.$

Residuated frames Residuated frames Intuition Ex: sequent calculi Ex: filters and ideals Contexts/polarities Dedekind-McNeille The dual algebra GN Gentzen frames FL Frames and modules Frames and display Distributive frames Involutive FL **BiFL** Applications

For $X \subseteq W$ and $Y \subseteq W'$ we define $X^{\triangleright} = \{ b \in W' : x N b, \text{ for all } x \in X \}$ $Y^{\triangleleft} = \{ a \in W : a \ N \ y, \text{ for all } y \in Y \}$ The map $\gamma_N: \mathcal{P}(W) \to \mathcal{P}(W), \ \gamma_N(X) = X^{\rhd \triangleleft}$, is a closure operator. Lemma. If W is a residuated frame then the dual algebra $\mathbf{W}^+ = (\gamma[\mathcal{P}(W)], \cap, \cup_{\gamma}, \circ_{\gamma}, \backslash, /)$ is a complete residuated lattice. $X \cup_{\gamma} Y = \gamma(X \cup Y)$ $X \circ_{\gamma} Y = \gamma(X \circ Y)$ $X \circ Y = \{ z \in W : \circ(X, Y, z) \} = \{ z \in W : \circ(x, y, z), \forall x \in X, y \in Y \}$ $X \setminus Y = \{ z \in W : X \circ \{ z \} \subseteq Y \}$ $Y/X = \{z \in W : \{z\} \circ X \subseteq Y\}.$

Theorem If W satisfies $(m)^p$ (*pointwise*), then it also satisfies $(m)^s$ (*setwise*) namely W⁺ satisfies $x \cdot x \leq x$ ($xy \leq x \lor y$). (\mathcal{N}_2 eq.)

$$\frac{xNz \quad yNz}{x \circ yNz} (\mathsf{m})^p \qquad \frac{XNz \quad YNz}{X \circ YNz} (\mathsf{m})^s$$

Residuated frames

Ex: sequent calculi

The dual algebra

Gentzen frames

Frames and modules

Frames and display

Distributive frames

Involutive FL

Applications

BiFL

Ex: filters and ideals Contexts/polarities Dedekind-McNeille

Intuition

GN

FL

If we have a common subset B of W and W' that supports a (partial) algebra $\mathbf{B} = (B, \land, \lor, \lor, \backslash, /, 1)$, then these are natural conditions inspired by the frame $\mathbf{W}_{\mathbf{L}}$, for $a, b, c \in B$, $x, y \in W$, $z \in W'$. Often B generates $(W, \circ, 1)$ (and W' by actions from W); we call (\mathbf{W}, \mathbf{B}) a Gentzen frame.

Residuated frames Residuated frames Intuition Ex: sequent calculi Ex: filters and ideals Contexts/polarities Dedekind-McNeille The dual algebra GN

Gentzen frames FL

Frames and modules

Frames and display

Distributive frames

Involutive FL

BiFL

If we have a common subset B of W and W' that supports a (partial) algebra $\mathbf{B} = (B, \wedge, \vee, \cdot, \backslash, /, 1)$, then these are natural conditions inspired by the frame $\mathbf{W}_{\mathbf{L}}$, for $a, b, c \in B$, $x, y \in W$, $z \in W'$. Often B generates $(W, \circ, 1)$ (and W' by actions from W); we call (\mathbf{W}, \mathbf{B}) a Gentzen frame.

$$\frac{xNa \quad aNz}{xNz}$$
 (CUT) $\qquad \frac{1}{aNa}$ (Id)

Residuated frames
Residuated frames
Intuition
Ex: sequent calculi
Ex: filters and ideals
Contexts/polarities
Dedekind-McNeille
The dual algebra
GN
Gentzen frames
FL
FL Frames and modules
FL Frames and modules Frames and display
FL Frames and modules Frames and display Distributive frames
FL Frames and modules Frames and display Distributive frames Involutive FL
FL Frames and modules Frames and display Distributive frames Involutive FL BiFL

If we have a common subset B of W and W' that supports a (partial) algebra $\mathbf{B} = (B, \wedge, \vee, \cdot, \backslash, /, 1)$, then these are natural conditions inspired by the frame $\mathbf{W}_{\mathbf{L}}$, for $a, b, c \in B$, $x, y \in W$, $z \in W'$. Often B generates $(W, \circ, 1)$ (and W' by actions from W); we call (\mathbf{W}, \mathbf{B}) a Gentzen frame.

 $\frac{xNa \quad aNz}{xNz} \text{ (CUT)} \quad \frac{}{aNa} \text{ (Id)}$ $\frac{aNz \quad bNz}{a \lor bNz} \text{ (\lor L)} \quad \frac{xNa}{xNa \lor b} \text{ (\lor R\ell)} \quad \frac{xNb}{xNa \lor b} \text{ (\lor Rr)}$

Residuated frames

Intuition

If we have a common subset B of W and W' that supports a (partial) algebra $\mathbf{B} = (B, \wedge, \vee, \cdot, \backslash, /, 1)$, then these are natural conditions inspired by the frame $\mathbf{W}_{\mathbf{L}}$, for $a, b, c \in B$, $x, y \in W$, $z \in W'$. Often B generates $(W, \circ, 1)$ (and W' by actions from W); we call (\mathbf{W}, \mathbf{B}) a Gentzen frame.

 $\frac{xNa \quad aNz}{xNz} (CUT) \quad \frac{}{aNa} (Id)$ $\frac{aNz \quad bNz}{a \lor bNz} (\lor L) \quad \frac{xNa}{xNa \lor b} (\lor R\ell) \quad \frac{xNb}{xNa \lor b} (\lor Rr)$ $\frac{aNz}{a \land bNz} (\land L\ell) \quad \frac{bNz}{a \land bNz} (\land Lr) \quad \frac{xNa \quad xNb}{xNa \land b} (\land R)$

Residuated frames

If we have a common subset B of W and W' that supports a (partial) algebra $\mathbf{B} = (B, \wedge, \vee, \cdot, \backslash, /, 1)$, then these are natural conditions inspired by the frame $\mathbf{W}_{\mathbf{L}}$, for $a, b, c \in B$, $x, y \in W$, $z \in W'$. Often B generates $(W, \circ, 1)$ (and W' by actions from W); we call (\mathbf{W}, \mathbf{B}) a Gentzen frame.

 $\frac{xNa \quad aNz}{xNz} (CUT) \quad \overline{aNa} (Id)$ $\frac{aNz \quad bNz}{a \lor bNz} (\lor L) \quad \frac{xNa}{xNa \lor b} (\lor R\ell) \quad \frac{xNb}{xNa \lor b} (\lor Rr)$ $\frac{aNz}{a \land bNz} (\land L\ell) \quad \frac{bNz}{a \land bNz} (\land Lr) \quad \frac{xNa \quad xNb}{xNa \land b} (\land R)$ $\frac{a \circ bNz}{a \cdot bNz} (\cdot L) \quad \frac{xNa \quad yNb}{x \circ yNa \cdot b} (\cdot R) \quad \frac{\varepsilon Nz}{1Nz} (1L) \quad \overline{\varepsilon N1} (1R)$

Residuated frames Intuition Ex: sequent calculi Ex: filters and ideals Contexts/polarities Dedekind-McNeille The dual algebra GN Gentzen frames FL Frames and modules Frames and display **Distributive frames** Involutive FL BiFL Applications

Residuated frames

If we have a common subset B of W and W' that supports a (partial) algebra $\mathbf{B} = (B, \wedge, \vee, \cdot, \backslash, /, 1)$, then these are natural conditions inspired by the frame $\mathbf{W}_{\mathbf{L}}$, for $a, b, c \in B$, $x, y \in W$, $z \in W'$. Often B generates $(W, \circ, 1)$ (and W' by actions from W); we call (\mathbf{W}, \mathbf{B}) a Gentzen frame.

 $\frac{xNa \quad aNz}{xNz} (CUT) \quad \overline{aNa} (Id)$ $\frac{aNz \quad bNz}{a \lor bNz} (\lor L) \quad \frac{xNa}{xNa \lor b} (\lor R\ell) \quad \frac{xNb}{xNa \lor b} (\lor Rr)$ $\frac{aNz}{a \land bNz} (\land L\ell) \quad \frac{bNz}{a \land bNz} (\land Lr) \quad \frac{xNa \quad xNb}{xNa \land b} (\land R)$ $\frac{a \circ bNz}{a \cdot bNz} (\cdot L) \quad \frac{xNa \quad yNb}{x \circ yNa \cdot b} (\cdot R) \quad \frac{\varepsilon Nz}{1Nz} (1L) \quad \overline{\varepsilon N1} (1R)$ $\frac{xNa \quad bNz}{a \backslash bNx \lor z} (\land L) \quad \frac{xNa \lor b}{xNa \backslash b} (\land R)$

Residuated frames
Residuated frames
Intuition
Ex: sequent calculi
Ex: filters and ideals
Contexts/polarities
Dedekind-McNeille
The dual algebra
GN
Gentzen frames
FL
Frames and modules
Frames and modules Frames and display
Frames and modules Frames and display Distributive frames
Frames and modules Frames and display Distributive frames Involutive FL
Frames and modules Frames and display Distributive frames Involutive FL BiFL
Frames and modules Frames and display Distributive frames Involutive FL BiFL

If we have a common subset B of W and W' that supports a (partial) algebra $\mathbf{B} = (B, \wedge, \vee, \cdot, \backslash, /, 1)$, then these are natural conditions inspired by the frame $\mathbf{W}_{\mathbf{L}}$, for $a, b, c \in B$, $x, y \in W$, $z \in W'$. Often B generates $(W, \circ, 1)$ (and W' by actions from W); we call (\mathbf{W}, \mathbf{B}) a Gentzen frame.

 $\frac{xNa}{xNz} \frac{aNz}{aNz}$ (CUT) $\frac{aNa}{aNa}$ (Id) $\frac{aNz \quad bNz}{a \lor bNz} (\lor \mathsf{L}) \qquad \frac{xNa}{xNa \lor b} (\lor \mathsf{R}\ell) \qquad \frac{xNb}{xNa \lor b} (\lor \mathsf{R}r)$ $\frac{aNz}{a \wedge bNz} (\wedge L\ell) \qquad \frac{bNz}{a \wedge bNz} (\wedge Lr) \qquad \frac{xNa \quad xNb}{xNa \wedge b} (\wedge R)$ $\frac{a \circ bNz}{a \cdot bNz} (\cdot L) \qquad \frac{xNa \quad yNb}{x \circ yNa \cdot b} (\cdot R) \qquad \frac{\varepsilon Nz}{1Nz} (1L) \qquad \frac{\varepsilon N1}{\varepsilon N1} (1R)$ $\frac{xNa \quad bNz}{a \backslash bNx \ \| z} \ (\backslash \mathsf{L}) \qquad \frac{xNa \ \| b}{xNa \backslash b} \ (\backslash \mathsf{R})$ $\frac{xNa}{b/aNz} \frac{bNz}{x} (/L) \qquad \frac{xNb // a}{xNb/a} (/R)$

Residuated frames Residuated frames Intuition Ex: sequent calculi Ex: filters and ideals Contexts/polarities Dedekind-McNeille The dual algebra GN Gentzen frames FL Frames and modules Frames and display **Distributive frames** Involutive FL BiFL Applications

Nick Galatos, Prague workshop, March, 2014

	Gentzen frames	
		Residuated frames
		Intuition
		Ex: sequent calculi
The following properties hold for $\mathbf{W_L}, \mathbf{W_{FL}}$ and $\mathbf{W_{A,B}}$:		Ex: filters and ideals
		Contexts/polarities
		Dedekind-McNeille
1.	W is a residuated frame	The dual algebra
~		GN
2.	B is a (partial) algebra of the same type, $(B = L, Fm, B)$	Gentzen frames
2	D represented $(W - c)$ (as a repricted)	FL
3.	B generates (W, \circ, ε) (as a monoid)	Frames and modules
Λ	W' contains a conv of $B(h \leftrightarrow (id h))$	Frames and display
т.	\mathcal{U} contains a copy of D ($\mathcal{U} \leftrightarrow (\mathcal{U}, \mathcal{U})$)	Distributivo framos
5	N satisfies GN for all $a, b \in B$ $x, y \in W$ $z \in W'$	
0.	$1, \text{ substables carry for all } a, o \in D, a, g \in W, s \in W$	Involutive FL
		BiFL
		Applications
Gentzen frames

The following properties hold for W_L , W_{FL} and $W_{A,B}$:

- 1. W is a residuated frame
- 2. **B** is a (partial) algebra of the same type, $(\mathbf{B} = \mathbf{L}, \mathbf{Fm}, \mathbf{B})$
- 3. B generates (W, \circ, ε) (as a monoid)
- 4. W' contains a copy of B ($b \leftrightarrow (id, b)$)
- 5. N satisfies **GN**, for all $a, b \in B$, $x, y \in W$, $z \in W'$.

We call such pairs (\mathbf{W}, \mathbf{B}) Gentzen frames.

A cut-free Gentzen frame is not assumed to satisfy the (CUT)-rule.

Residuated frames
Residuated frames
Intuition
Ex: sequent calculi
Ex: filters and ideals
Contexts/polarities
Dedekind-McNeille
The dual algebra
GN
Gentzen frames
FL
FL Frames and modules
FL Frames and modules Frames and display
FL Frames and modules Frames and display Distributive frames
FL Frames and modules Frames and display Distributive frames Involutive FL
FL Frames and modules Frames and display Distributive frames Involutive FL BiFL

Gentzen frames

The following properties hold for W_L , W_{FL} and $W_{A,B}$:

- 1. W is a residuated frame
- 2. **B** is a (partial) algebra of the same type, $(\mathbf{B} = \mathbf{L}, \mathbf{Fm}, \mathbf{B})$
- 3. B generates (W, \circ, ε) (as a monoid)
- 4. W' contains a copy of B ($b \leftrightarrow (id, b)$)
- 5. N satisfies **GN**, for all $a, b \in B$, $x, y \in W$, $z \in W'$.

We call such pairs (\mathbf{W}, \mathbf{B}) Gentzen frames.

A cut-free Gentzen frame is not assumed to satisfy the (CUT)-rule.

Theorem. Given a Gentzen frame (\mathbf{W}, \mathbf{B}) , the map $\{\}^{\triangleleft} : \mathbf{B} \to \mathbf{W}^+, b \mapsto \{b\}^{\triangleleft}$ is a (partial) homomorphism. (Namely, if $a, b \in B$ and $a \bullet b \in B$ (\bullet is a connective) then $\{a \bullet_{\mathbf{B}} b\}^{\triangleleft} = \{a\}^{\triangleleft} \bullet_{\mathbf{W}^+} \{b\}^{\triangleleft}$).

	Residuated frames
_	Residuated frames
	Intuition
	Ex: sequent calculi
	Ex: filters and ideals
	Contexts/polarities
	Dedekind-McNeille
	The dual algebra
	GN
	Gentzen frames
	FL
	Frames and modules
	Frames and display
	Distributive frames
	Involutive FL
	BiFL

Gentzen frames

The following properties hold for W_L , W_{FL} and $W_{A,B}$:

- 1. W is a residuated frame
- 2. **B** is a (partial) algebra of the same type, $(\mathbf{B} = \mathbf{L}, \mathbf{Fm}, \mathbf{B})$
- 3. B generates (W, \circ, ε) (as a monoid)
- 4. W' contains a copy of B ($b \leftrightarrow (id, b)$)
- 5. N satisfies **GN**, for all $a, b \in B$, $x, y \in W$, $z \in W'$.

We call such pairs (\mathbf{W}, \mathbf{B}) Gentzen frames.

A cut-free Gentzen frame is not assumed to satisfy the (CUT)-rule.

Theorem. Given a Gentzen frame (\mathbf{W}, \mathbf{B}) , the map $\{\}^{\triangleleft} : \mathbf{B} \to \mathbf{W}^+, \ b \mapsto \{b\}^{\triangleleft}$ is a (partial) homomorphism. (Namely, if $a, b \in B$ and $a \bullet b \in B$ (\bullet is a connective) then $\{a \bullet_{\mathbf{B}} b\}^{\triangleleft} = \{a\}^{\triangleleft} \bullet_{\mathbf{W}^+} \{b\}^{\triangleleft}$). For cut-free Genzten frames, we get only a *quasihomomorphism*. $a \bullet_{\mathbf{B}} b \in \{a\}^{\triangleleft} \bullet_{\mathbf{W}^+} \{b\}^{\triangleleft} \subseteq \{a \bullet_{\mathbf{B}} b\}^{\triangleleft}$.

	Residuated frames
_	Residuated frames
	Intuition
	Ex: sequent calculi
	Ex: filters and ideals
	Contexts/polarities
	Dedekind-McNeille
	The dual algebra
	GN
	Gentzen frames
	FL
	Frames and modules
	Frames and display
	Distributive frames
	Involutive FL
	BiFL

FL

Residuated frames Residuated frames Intuition

Ex: sequent calculi

Ex: filters and ideals

$$\frac{x \Rightarrow a \quad y \circ a \circ z \Rightarrow c}{y \circ x \circ z \Rightarrow c} \text{ (cut) } \qquad \frac{a \Rightarrow a}{a \Rightarrow a} \text{ (Id)}$$

$$\frac{y \circ a \circ z \Rightarrow c}{y \circ a \wedge b \circ z \Rightarrow c} (\wedge L\ell) \quad \frac{y \circ b \circ z \Rightarrow c}{y \circ a \wedge b \circ z \Rightarrow c} (\wedge Lr) \quad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a \wedge b} (\wedge R)$$

$$\frac{y \circ a \circ z \Rightarrow c \quad y \circ b \circ z \Rightarrow c}{y \circ a \lor b \circ z \Rightarrow c} (\lor \mathsf{L}) \quad \frac{x \Rightarrow a}{x \Rightarrow a \lor b} (\lor \mathsf{R}\ell) \quad \frac{x \Rightarrow b}{x \Rightarrow a \lor b} (\lor \mathsf{R}r)$$

$$\frac{x \Rightarrow a \quad y \circ b \circ z \Rightarrow c}{y \circ x \circ (a \setminus b) \circ z \Rightarrow c} (\setminus \mathsf{L}) \qquad \frac{a \circ x \Rightarrow b}{x \Rightarrow a \setminus b} (\setminus \mathsf{R})$$

$$\frac{x \Rightarrow a \quad y \circ b \circ z \Rightarrow c}{y \circ (b/a) \circ x \circ z \Rightarrow c} (/L) \qquad \frac{x \circ a \Rightarrow b}{x \Rightarrow b/a} (/R)$$

$$\frac{y \circ a \circ b \circ z \Rightarrow c}{y \circ a \cdot b \circ z \Rightarrow c} (\cdot L) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x \circ y \Rightarrow a \cdot b} (\cdot R)$$
$$\frac{y \circ z \Rightarrow a}{y \circ 1 \circ z \Rightarrow a} (1L) \qquad \frac{\varepsilon \Rightarrow 1}{\varepsilon \Rightarrow 1} (1R)$$

where $a, b, c \in Fm$, $x, y, z \in Fm^*$.

Contexts/polarities Dedekind-McNeille The dual algebra GN Gentzen frames FL Frames and modules Frames and display Distributive frames Involutive FL

Applications

BiFL

Residuated frames

Frames and modules

Designated elements Actions Bi-modules Formula hierarchy Submodules and nuclei Frames and modules Densification

Frames and display

Distributive frames

Involutive FL

BiFL

Applications

Frames and modules

In a residuated frame we can replace the relation N by a subset D of

W' in an interdefinable way by: (in the spirit of AAL, the positive

cone of a residuated lattice, hyperframes $D = \vdash$)

Residuated frames

Frames and modules Designated elements

Actions Bi-modules

Formula hierarchy Submodules and nuclei Frames and modules Densification

Frames and display

Distributive frames

Involutive FL

BiFL

Applications

In a residuated frame we can replace the relation N by a subset D of W' in an interdefinable way by: (in the spirit of AAL, the positive cone of a residuated lattice, hyperframes $D = \vdash$)

$$D_N = \{\varepsilon\}^{\triangleright} \text{ and } x N_D z \Leftrightarrow x \setminus \!\!\! \setminus z \subseteq D.$$

Designated elements Actions Bi-modules Formula hierarchy Submodules and nuclei Frames and modules Densification Frames and display

Residuated frames

Frames and modules

Distributive frames

Involutive FL

BiFL

Applications

Residuated frames Frames and modules Designated elements Actions In a residuated frame we can replace the relation N by a subset D of **Bi-modules** W' in an interdefinable way by: (in the spirit of AAL, the positive Formula hierarchy Submodules and nuclei Frames and modules Densification Frames and display Distributive frames Involutive FL BiFL Applications

cone of a residuated lattice, hyperframes $D = \vdash$)

The nuclearity contition for N becomes

 $y \setminus (x \setminus z) \subseteq D$ iff $(x \circ y) \setminus z \subseteq D$.

 $D_N = \{\varepsilon\}^{\triangleright} \text{ and } x N_D z \Leftrightarrow x \setminus \!\!\! \setminus z \subseteq D.$

In a residuated frame we can replace the relation N by a subset D of W' in an interdefinable way by: (in the spirit of AAL, the positive cone of a residuated lattice, hyperframes $D = \vdash$)

 $D_N = \{\varepsilon\}^{\triangleright} \text{ and } x N_D z \Leftrightarrow x \setminus \!\!\! \setminus z \subseteq D.$

The nuclearity contition for N becomes $y \upharpoonright (x \upharpoonright z) \subseteq D$ iff $(x \circ y) \upharpoonright z \subseteq D$.

In all residuated frames we have $(y \setminus (x \setminus z))^{\triangleleft} = ((x \circ y) \setminus z)^{\triangleleft}$, but often we actually have $y \setminus (x \setminus z) = (x \circ y) \setminus z$. For those residuated frames the condition for D is automatically satified.

Frames and modulesDesignated elementsActionsBi-modulesFormula hierarchySubmodules and nucleiFrames and modulesDensificationFrames and displayDistributive framesInvolutive FL

Residuated frames

BiFL

Applications

In a residuated frame we can replace the relation N by a subset D of W' in an interdefinable way by: (in the spirit of AAL, the positive cone of a residuated lattice, hyperframes $D = \vdash$)

 $D_N = \{\varepsilon\}^{\triangleright} \text{ and } x N_D z \Leftrightarrow x \setminus \!\!\! \setminus z \subseteq D.$

The nuclearity contition for N becomes $y \upharpoonright (x \upharpoonright z) \subseteq D$ iff $(x \circ y) \upharpoonright z \subseteq D$.

In all residuated frames we have $(y \setminus (x \setminus z))^{\triangleleft} = ((x \circ y) \setminus z)^{\triangleleft}$, but often we actually have $y \setminus (x \setminus z) = (x \circ y) \setminus z$. For those residuated frames the condition for D is automatically satified.

If $x \circ y$, $x \setminus z$ and $z \not| x$ are singletons (instead of sets), as it happens with most applications, then $\subseteq D$ above can be replaced by $\in D$.

Frames and modules
Designated elements
Actions
Bi-modules
Formula hierarchy
Submodules and nuclei
Frames and modules
Densification
Frames and display
Distributive frames
Involutive FL
BiFL

Residuated frames

Applications

In a residuated frame we can replace the relation N by a subset D of W' in an interdefinable way by: (in the spirit of AAL, the positive cone of a residuated lattice, hyperframes $D = \vdash$)

 $D_N = \{\varepsilon\}^{\triangleright} \text{ and } x N_D z \Leftrightarrow x \setminus \!\!\! \setminus z \subseteq D.$

The nuclearity contition for N becomes $y \upharpoonright (x \upharpoonright z) \subseteq D$ iff $(x \circ y) \upharpoonright z \subseteq D$.

In all residuated frames we have $(y \setminus (x \setminus z))^{\triangleleft} = ((x \circ y) \setminus z)^{\triangleleft}$, but often we actually have $y \setminus (x \setminus z) = (x \circ y) \setminus z$. For those residuated frames the condition for D is automatically satified.

If $x \circ y$, $x \setminus z$ and $z \not| x$ are singletons (instead of sets), as it happens with most applications, then $\subseteq D$ above can be replaced by $\in D$.

We call residuated frames for which these two simplifications apply *action residuated frames*.

Frames and modules
Designated elements
Actions
Bi-modules
Formula hierarchy
Submodules and nuclei
Frames and modules
Densification
Frames and display
Distributive frames
Involutive FL
BiFL
Applications

Residuated frames

Frames and modules

Designated elements

Actions

Bi-modules

Formula hierarchy Submodules and nuclei Frames and modules Densification

Frames and display

Distributive frames

Involutive FL

BiFL

Applications

Recall that given a monoid $\mathbf{W} = (W, \cdot, 1)$ and a set W', a map $*: W \times W' \to W'$ is called an *action* if it sarisfies: 1 * z = x and $(x \cdot y) * z = x * (y * z)$.

Residuated frames

Frames and modules

Designated elements

Actions

Bi-modules

Formula hierarchy Submodules and nuclei Frames and modules Densification

Frames and display

Distributive frames

Involutive FL

BiFL

Applications

Recall that given a monoid $\mathbf{W} = (W, \cdot, 1)$ and a set W', a map $*: W \times W' \to W'$ is called an *action* if it sarisfies: 1 * z = x and $(x \cdot y) * z = x * (y * z)$.

Then we say that (W', *) is an **W**-set.

Residuated frames

Frames and modulesDesignated elementsActionsBi-modulesFormula hierarchySubmodules and nucleiFrames and modulesDensificationFrames and displayDistributive framesInvolutive FLBiFLApplications

Recall that given a monoid $\mathbf{W} = (W, \cdot, 1)$ and a set W', a map $*: W \times W' \to W'$ is called an *action* if it sarisfies: 1 * z = x and $(x \cdot y) * z = x * (y * z)$.

Then we say that (W', *) is an **W**-set.

If we also have another map $\star : W' \times W \to W'$ such that $z \star 1 = z$, $(z \star y) \star x = z \star (yx)$ and $x \star (z \star y) = (x \star z) \star y$, then we say that (W', \star, \star) is an bi-W-set.

Residuated frames

Frames and modules
Designated elements
Actions
Bi-modules
Formula hierarchy
Submodules and nuclei
Frames and modules
Densification
Frames and display
Distributive frames
Involutive FL
BiFL
Applications

Recall that given a monoid $\mathbf{W} = (W, \cdot, 1)$ and a set W', a map $*: W \times W' \to W'$ is called an *action* if it sarisfies: 1 * z = x and $(x \cdot y) * z = x * (y * z)$.

Then we say that (W', *) is an **W**-set.

If we also have another map $\star : W' \times W \to W'$ such that $z \star 1 = z$, $(z \star y) \star x = z \star (yx)$ and $x \star (z \star y) = (x \star z) \star y$, then we say that $(W', *, \star)$ is an bi-W-set.

This allows us to link W-sets and action residuated frames, as then an action residuated frame is equivalent to a bi-W-set $(W', //, \mathbb{N})$ together with an arbitratry subset D of W' of designated elements.

Bi-modules	Residuated frames
Let's assume that $P = N$ is the underlying set of a residuated lattice	Frames and modules Designated elements Actions
Let's assume that I — IV is the underlying set of a residuated lattice.	Bi-modules Formula hierarchy Submodules and nuclei Frames and modules
	Densification Frames and display
	Distributive frames Involutive FL
	Applications

Residuated frames

Frames and modules

Designated elements Actions

Bi-modules

Formula hierarchy Submodules and nuclei Frames and modules Densification

Frames and display

Distributive frames

Involutive FL

BiFL

Applications

Let's assume that P = N is the underlying set of a residuated lattice.

$$\begin{array}{ll} \bullet & x \cdot 1 = x = 1 \cdot x, \ (xy)z = x(yz) \\ \bullet & x(y \lor z) = xy \lor xz \ \text{and} \ (y \lor z)x = yx \lor zx \end{array}$$

Residuated frames

Frames and modules
Designated elements

Bi-modules

Actions

Formula hierarchy Submodules and nuclei Frames and modules Densification

Frames and display

Distributive frames

Involutive FL

BiFL

Applications

Let's assume that P = N is the underlying set of a residuated lattice.

$$\begin{array}{ll} \bullet & x \cdot 1 = x = 1 \cdot x, \ (xy)z = x(yz) \\ \bullet & x(y \lor z) = xy \lor xz \ \text{and} \ (y \lor z)x = yx \lor zx \end{array}$$

So, $(P, \lor, \cdot, 1)$ is a semiring. [In the complete case, a quantale.]

Residuated frames

Frames and modulesDesignated elementsActionsBi-modulesFormula hierarchySubmodules and nucleiFrames and modulesDensificationFrames and displayDistributive framesInvolutive FLBiFLApplications

Let's assume that P = N is the underlying set of a residuated lattice.

$$\begin{array}{ll} \bullet & x \cdot 1 = x = 1 \cdot x, \ (xy)z = x(yz) \\ \bullet & x(y \lor z) = xy \lor xz \ \text{and} \ (y \lor z)x = yx \lor zx \end{array}$$

So, $(P, \lor, \cdot, 1)$ is a semiring. [In the complete case, a quantale.]

Residuated frames

Frames and modulesDesignated elementsActionsBi-modulesFormula hierarchySubmodules and nucleiFrames and modulesDensificationFrames and displayDistributive framesInvolutive FLBiFLApplications

Let's assume that P = N is the underlying set of a residuated lattice.

$$\begin{array}{ll} \bullet & x \cdot 1 = x = 1 \cdot x, \ (xy)z = x(yz) \\ \bullet & x(y \lor z) = xy \lor xz \ \text{and} \ (y \lor z)x = yx \lor zx \end{array}$$

So, $(P, \lor, \cdot, 1)$ is a semiring. [In the complete case, a quantale.]

1\
$$x = x = x/1$$

 $(yz) \setminus x = z \setminus (y \setminus x) \text{ and } x/(zy) = (x/y)/z$
 $x \setminus (y/z) = (x \setminus y)/z$
 $x \setminus (y \wedge z) = (x \setminus y) \wedge (x \setminus z) \text{ and } (y \wedge z)/x = (y/x) \wedge (z/x)$
 $(y \vee z) \setminus x = (y \setminus x) \wedge (z \setminus x) \text{ and } x/(y \vee z) = (x/y) \wedge (x/z)$

So, $(P, \lor, \cdot, 1)$ acts on both sides on (N, \land) by $p \star n = n/p$ and $n \star p = p \backslash n$. Thus, $((N, \land), \star)$ becomes a $(P, \lor, \cdot, 1)$ -bimodule.

Residuated frames

Frames and modulesDesignated elementsActionsBi-modulesFormula hierarchySubmodules and nucleiFrames and modulesDensificationFrames and displayDistributive framesInvolutive FLBiFLApplications

Let's assume that P = N is the underlying set of a residuated lattice.

$$\begin{array}{ll} \bullet & x \cdot 1 = x = 1 \cdot x, \ (xy)z = x(yz) \\ \bullet & x(y \lor z) = xy \lor xz \ \text{and} \ (y \lor z)x = yx \lor zx \end{array}$$

So, $(P, \lor, \cdot, 1)$ is a semiring. [In the complete case, a quantale.]

$$\begin{array}{l} 1 \setminus x = x = x/1 \\ (yz) \setminus x = z \setminus (y \setminus x) \text{ and } x/(zy) = (x/y)/z \\ x \setminus (y/z) = (x \setminus y)/z \\ x \setminus (y \wedge z) = (x \setminus y) \wedge (x \setminus z) \text{ and } (y \wedge z)/x = (y/x) \wedge (z/x) \\ (y \vee z) \setminus x = (y \setminus x) \wedge (z \setminus x) \text{ and } x/(y \vee z) = (x/y) \wedge (x/z) \end{array}$$

So, $(P, \lor, \cdot, 1)$ acts on both sides on (N, \land) by $p \star n = n/p$ and $n \star p = p \backslash n$. Thus, $((N, \land), \star)$ becomes a $(P, \lor, \cdot, 1)$ -bimodule. This split matches the notion of *polarity*.

Residuated frames

Frames and modulesDesignated elementsActionsBi-modulesFormula hierarchySubmodules and nucleiFrames and modulesDensificationFrames and displayDistributive framesInvolutive FLBiFLApplications

Let's assume that P = N is the underlying set of a residuated lattice.

$$\begin{array}{ll} \bullet & x \cdot 1 = x = 1 \cdot x, \ (xy)z = x(yz) \\ \bullet & x(y \lor z) = xy \lor xz \ \text{and} \ (y \lor z)x = yx \lor zx \end{array}$$

So, $(P, \lor, \cdot, 1)$ is a semiring. [In the complete case, a quantale.]

$$1 \setminus x = x = x/1$$

$$(yz) \setminus x = z \setminus (y \setminus x) \text{ and } x/(zy) = (x/y)/z$$

$$x \setminus (y/z) = (x \setminus y)/z$$

$$x \setminus (y \wedge z) = (x \setminus y) \wedge (x \setminus z) \text{ and } (y \wedge z)/x = (y/x) \wedge (z/x)$$

$$(y \vee z) \setminus x = (y \setminus x) \wedge (z \setminus x) \text{ and } x/(y \vee z) = (x/y) \wedge (x/z)$$

So, $(P, \lor, \cdot, 1)$ acts on both sides on (N, \land) by $p \star n = n/p$ and $n \star p = p \backslash n$. Thus, $((N, \land), \star)$ becomes a $(P, \lor, \cdot, 1)$ -bimodule. This split matches the notion of *polarity*. It also extends to \bigvee , \bigwedge .

Residuated frames

Frames and modulesDesignated elementsActionsBi-modulesFormula hierarchySubmodules and nucleiFrames and modulesDensificationFrames and displayDistributive framesInvolutive FLBiFLApplications

Let's assume that P = N is the underlying set of a residuated lattice.

$$\begin{array}{ll} \bullet & x \cdot 1 = x = 1 \cdot x, \ (xy)z = x(yz) \\ \bullet & x(y \lor z) = xy \lor xz \ \text{and} \ (y \lor z)x = yx \lor zx \end{array}$$

So, $(P, \lor, \cdot, 1)$ is a semiring. [In the complete case, a quantale.]

$$\begin{array}{l} 1 \setminus x = x = x/1 \\ (yz) \setminus x = z \setminus (y \setminus x) \text{ and } x/(zy) = (x/y)/z \\ x \setminus (y/z) = (x \setminus y)/z \\ x \setminus (y \wedge z) = (x \setminus y) \wedge (x \setminus z) \text{ and } (y \wedge z)/x = (y/x) \wedge (z/x) \\ (y \vee z) \setminus x = (y \setminus x) \wedge (z \setminus x) \text{ and } x/(y \vee z) = (x/y) \wedge (x/z) \end{array}$$

So, $(P, \lor, \cdot, 1)$ acts on both sides on (N, \land) by $p \star n = n/p$ and $n \star p = p \backslash n$. Thus, $((N, \land), \star)$ becomes a $(P, \lor, \cdot, 1)$ -bimodule. This split matches the notion of *polarity*. It also extends to \bigvee , \bigwedge .

The bimodule can be viewed as a two-sorted algebra $(P, \lor, \cdot, 1, N, \land, \backslash, /).$

Residuated frames

Frames and modulesDesignated elementsActionsBi-modulesFormula hierarchySubmodules and nucleiFrames and modulesDensificationFrames and displayDistributive framesInvolutive FLBiFLApplications

Let's assume that P = N is the underlying set of a residuated lattice.

$$\begin{array}{ll} \bullet & x \cdot 1 = x = 1 \cdot x, \ (xy)z = x(yz) \\ \bullet & x(y \lor z) = xy \lor xz \ \text{and} \ (y \lor z)x = yx \lor zx \end{array}$$

So, $(P, \lor, \cdot, 1)$ is a semiring. [In the complete case, a quantale.]

$$\begin{array}{l} 1 \setminus x = x = x/1 \\ (yz) \setminus x = z \setminus (y \setminus x) \text{ and } x/(zy) = (x/y)/z \\ x \setminus (y/z) = (x \setminus y)/z \\ x \setminus (y \wedge z) = (x \setminus y) \wedge (x \setminus z) \text{ and } (y \wedge z)/x = (y/x) \wedge (z/x) \\ (y \vee z) \setminus x = (y \setminus x) \wedge (z \setminus x) \text{ and } x/(y \vee z) = (x/y) \wedge (x/z) \end{array}$$

So, $(P, \lor, \cdot, 1)$ acts on both sides on (N, \land) by $p \star n = n/p$ and $n \star p = p \backslash n$. Thus, $((N, \land), \star)$ becomes a $(P, \lor, \cdot, 1)$ -bimodule. This split matches the notion of *polarity*. It also extends to \bigvee , \bigwedge .

The bimodule can be viewed as a two-sorted algebra $(P, \lor, \cdot, 1, N, \land, \backslash, /).$

The absolutely free algebra for P = N generated by $P_0 = N_0 = Var$ (the set of propositional variables) gives the set of all formulas.

Residuated frames

Frames and modulesDesignated elementsActionsBi-modulesFormula hierarchySubmodules and nucleiFrames and modulesDensificationFrames and displayDistributive framesInvolutive FLBiFLApplications

Let's assume that P = N is the underlying set of a residuated lattice.

$$\begin{array}{ll} \bullet & x \cdot 1 = x = 1 \cdot x, \ (xy)z = x(yz) \\ \bullet & x(y \lor z) = xy \lor xz \ \text{and} \ (y \lor z)x = yx \lor zx \end{array}$$

So, $(P, \lor, \cdot, 1)$ is a semiring. [In the complete case, a quantale.]

$$\begin{array}{l} 1 \setminus x = x = x/1 \\ (yz) \setminus x = z \setminus (y \setminus x) \text{ and } x/(zy) = (x/y)/z \\ x \setminus (y/z) = (x \setminus y)/z \\ x \setminus (y \wedge z) = (x \setminus y) \wedge (x \setminus z) \text{ and } (y \wedge z)/x = (y/x) \wedge (z/x) \\ (y \vee z) \setminus x = (y \setminus x) \wedge (z \setminus x) \text{ and } x/(y \vee z) = (x/y) \wedge (x/z) \end{array}$$

So, $(P, \lor, \cdot, 1)$ acts on both sides on (N, \land) by $p \star n = n/p$ and $n \star p = p \backslash n$. Thus, $((N, \land), \star)$ becomes a $(P, \lor, \cdot, 1)$ -bimodule. This split matches the notion of *polarity*. It also extends to \bigvee , \bigwedge .

The bimodule can be viewed as a two-sorted algebra $(P, \lor, \cdot, 1, N, \land, \backslash, /).$

The absolutely free algebra for P = N generated by $P_0 = N_0 = Var$ (the set of propositional variables) gives the set of all formulas. The steps of the generation process yield the *substructural hierarchy*.

Nick Galatos, Prague workshop, March, 2014

Formula hierarchy

Residuated frames



Given a semiring $\mathbf{P} = (P, \bigvee, \cdot, 1)$ and an onto homomorphism f to a semiring $\mathbf{S} = f[\mathbf{P}]$, we observe that f is residuated (with residual $f^* : \mathbf{S} \to \mathbf{P}$).

Frames and modulesDesignated elementsActionsBi-modulesFormula hierarchySubmodules and nucleiFrames and modulesDensificationFrames and displayDistributive framesInvolutive FLBiFL

Applications

Given a semiring $\mathbf{P} = (P, \bigvee, \cdot, 1)$ and an onto homomorphism f to a

semiring S = f[P], we observe that f is residuated (with residual

satisfies $\gamma(x) \cdot \gamma(y) \leq \gamma(x \cdot y)$; such a map on **P** is called *nucleus*.

 $f^*: \mathbf{S} \to \mathbf{P}$). Note that $\gamma = f^* \circ f$ is a closure operator that

Residuated frames

Frames and modules Designated elements Actions Bi-modules Formula hierarchy Submodules and nuclei Frames and modules Densification Frames and display Distributive frames Involutive FL BiFL Applications

Nick Galatos, Prague workshop, March, 2014

Residuated frames

Frames and modulesDesignated elementsActionsBi-modulesFormula hierarchySubmodules and nucleiFrames and modulesDensificationFrames and displayDistributive framesInvolutive FLBiFLApplications

Given a semiring $\mathbf{P} = (P, \bigvee, \cdot, 1)$ and an onto homomorphism f to a semiring $\mathbf{S} = f[\mathbf{P}]$, we observe that f is residuated (with residual $f^* : \mathbf{S} \to \mathbf{P}$). Note that $\gamma = f^* \circ f$ is a closure operator that satisfies $\gamma(x) \cdot \gamma(y) \leq \gamma(x \cdot y)$; such a map on \mathbf{P} is called *nucleus*. For $P_{\gamma} = \{\gamma(x) : x \in P\}, \bigvee_{\gamma} x_i = \gamma(\bigvee x_i)$ and $x \cdot_{\gamma} y = \gamma(x \cdot y)$, the algebra $\mathbf{P}_{\gamma} = (P_{\gamma}, \bigvee_{\gamma}, \cdot_{\gamma}, \gamma(1))$ is a semiring and γ becomes a homomprhism from \mathbf{P} to \mathbf{P}_{γ} .

Given a semiring $\mathbf{P} = (P, \bigvee, \cdot, 1)$ and an onto homomorphism f to a

semiring S = f[P], we observe that f is residuated (with residual

satisfies $\gamma(x) \cdot \gamma(y) \leq \gamma(x \cdot y)$; such a map on **P** is called *nucleus*.

 $f^*: \mathbf{S} \to \mathbf{P}$). Note that $\gamma = f^* \circ f$ is a closure operator that

algebra $\mathbf{P}_{\gamma} = (P_{\gamma}, \bigvee_{\gamma}, \cdot_{\gamma}, \gamma(1))$ is a semiring and γ becomes a

homomprhism from $\vec{\mathbf{P}}$ to \mathbf{P}_{γ} . Nuclei are, up to isomorphism, the

Residuated frames

Frames and modules Designated elements Actions **Bi-modules** Formula hierarchy Submodules and nuclei Frames and modules Densification Frames and display For $P_{\gamma} = \{\gamma(x) : x \in P\}$, $\bigvee_{\gamma} x_i = \gamma(\bigvee x_i)$ and $x \cdot_{\gamma} y = \gamma(x \cdot y)$, the Distributive frames Involutive FL **BiFL** Applications

Nick Galatos, Prague workshop, March, 2014

onto homorphisms.

Residuated frames

Frames and modulesDesignated elementsActionsBi-modulesFormula hierarchySubmodules and nucleiFrames and modulesDensificationFrames and displayDistributive framesInvolutive FLBiFLApplications

Given a semiring $\mathbf{P} = (P, \bigvee, \cdot, 1)$ and an onto homomorphism f to a semiring $\mathbf{S} = f[\mathbf{P}]$, we observe that f is residuated (with residual $f^* : \mathbf{S} \to \mathbf{P}$). Note that $\gamma = f^* \circ f$ is a closure operator that satisfies $\gamma(x) \cdot \gamma(y) \leq \gamma(x \cdot y)$; such a map on \mathbf{P} is called *nucleus*. For $P_{\gamma} = \{\gamma(x) : x \in P\}, \bigvee_{\gamma} x_i = \gamma(\bigvee x_i)$ and $x \cdot_{\gamma} y = \gamma(x \cdot y)$, the algebra $\mathbf{P}_{\gamma} = (P_{\gamma}, \bigvee_{\gamma}, \cdot_{\gamma}, \gamma(1))$ is a semiring and γ becomes a homomprhism from \mathbf{P} to \mathbf{P}_{γ} . Nuclei are, up to isomorphism, the onto homorphisms.

Given a $(P, \bigvee, \cdot, 1)$ -bimodule $((N, \bigwedge), \setminus, /)$, each *sub-bimodule* is defined by a \bigwedge -closed subset that is also closed under the actions.

Residuated frames

Frames and modulesDesignated elementsActionsBi-modulesFormula hierarchySubmodules and nucleiFrames and modulesDensificationFrames and displayDistributive framesInvolutive FLBiFLApplications

Given a semiring $\mathbf{P} = (P, \bigvee, \cdot, 1)$ and an onto homomorphism f to a semiring $\mathbf{S} = f[\mathbf{P}]$, we observe that f is residuated (with residual $f^* : \mathbf{S} \to \mathbf{P}$). Note that $\gamma = f^* \circ f$ is a closure operator that satisfies $\gamma(x) \cdot \gamma(y) \leq \gamma(x \cdot y)$; such a map on \mathbf{P} is called *nucleus*. For $P_{\gamma} = \{\gamma(x) : x \in P\}, \bigvee_{\gamma} x_i = \gamma(\bigvee x_i) \text{ and } x \cdot_{\gamma} y = \gamma(x \cdot y)$, the algebra $\mathbf{P}_{\gamma} = (P_{\gamma}, \bigvee_{\gamma}, \cdot_{\gamma}, \gamma(1))$ is a semiring and γ becomes a homomprhism from \mathbf{P} to \mathbf{P}_{γ} . Nuclei are, up to isomorphism, the onto homorphisms.

Given a $(P, \bigvee, \cdot, 1)$ -bimodule $((N, \bigwedge), \setminus, /)$, each *sub-bimodule* is defined by a \bigwedge -closed subset that is also closed under the actions. Namely, it is defined by a *nucleus*: a closure operator γ on N such that $p \in P, n \in N$ implies $p \setminus n, n/p \in N$.

Residuated frames

Frames and modulesDesignated elementsActionsBi-modulesFormula hierarchySubmodules and nucleiFrames and modulesDensificationFrames and displayDistributive framesInvolutive FLBiFLApplications

Given a semiring $\mathbf{P} = (P, \bigvee, \cdot, 1)$ and an onto homomorphism f to a semiring $\mathbf{S} = f[\mathbf{P}]$, we observe that f is residuated (with residual $f^* : \mathbf{S} \to \mathbf{P}$). Note that $\gamma = f^* \circ f$ is a closure operator that satisfies $\gamma(x) \cdot \gamma(y) \leq \gamma(x \cdot y)$; such a map on \mathbf{P} is called *nucleus*. For $P_{\gamma} = \{\gamma(x) : x \in P\}, \bigvee_{\gamma} x_i = \gamma(\bigvee x_i) \text{ and } x \cdot_{\gamma} y = \gamma(x \cdot y)$, the algebra $\mathbf{P}_{\gamma} = (P_{\gamma}, \bigvee_{\gamma}, \cdot_{\gamma}, \gamma(1))$ is a semiring and γ becomes a homomprhism from \mathbf{P} to \mathbf{P}_{γ} . Nuclei are, up to isomorphism, the onto homorphisms.

Given a $(P, \bigvee, \cdot, 1)$ -bimodule $((N, \bigwedge), \setminus, /)$, each *sub-bimodule* is defined by a \bigwedge -closed subset that is also closed under the actions. Namely, it is defined by a *nucleus*: a closure operator γ on N such that $p \in P, n \in N$ implies $p \setminus n, n/p \in N$.

If P = N is the underlying set of a residuated lattice $\mathbf{A} = (A, \land, \lor, \cdot, \backslash, /, 1)$ the two notions of nucelus coincide and $\mathbf{A}_{\gamma} = (A_{\gamma}, \land, \lor_{\gamma}, \cdot_{\gamma}, \backslash, /, \gamma(1))$ is also a residuated lattice.

Residuated frames

Frames and modulesDesignated elementsActionsBi-modulesFormula hierarchySubmodules and nucleiFrames and modulesDensificationFrames and displayDistributive framesInvolutive FLBiFLApplications

Given a semiring $\mathbf{P} = (P, \bigvee, \cdot, 1)$ and an onto homomorphism f to a semiring $\mathbf{S} = f[\mathbf{P}]$, we observe that f is residuated (with residual $f^* : \mathbf{S} \to \mathbf{P}$). Note that $\gamma = f^* \circ f$ is a closure operator that satisfies $\gamma(x) \cdot \gamma(y) \leq \gamma(x \cdot y)$; such a map on \mathbf{P} is called *nucleus*. For $P_{\gamma} = \{\gamma(x) : x \in P\}, \bigvee_{\gamma} x_i = \gamma(\bigvee x_i) \text{ and } x \cdot_{\gamma} y = \gamma(x \cdot y)$, the algebra $\mathbf{P}_{\gamma} = (P_{\gamma}, \bigvee_{\gamma}, \cdot_{\gamma}, \gamma(1))$ is a semiring and γ becomes a homomprhism from \mathbf{P} to \mathbf{P}_{γ} . Nuclei are, up to isomorphism, the onto homorphisms.

Given a $(P, \bigvee, \cdot, 1)$ -bimodule $((N, \bigwedge), \setminus, /)$, each *sub-bimodule* is defined by a \bigwedge -closed subset that is also closed under the actions. Namely, it is defined by a *nucleus*: a closure operator γ on N such that $p \in P, n \in N$ implies $p \setminus n, n/p \in N$.

If P = N is the underlying set of a residuated lattice $\mathbf{A} = (A, \land, \lor, \cdot, \backslash, /, 1)$ the two notions of nucelus coincide and $\mathbf{A}_{\gamma} = (A_{\gamma}, \land, \lor_{\gamma}, \lor_{\gamma}, \backslash, /, \gamma(1))$ is also a residuated lattice.

Residuated frames arise from studying submodules of $\mathcal{P}(\mathbf{W})$, where \mathbf{W} is a monoid, namely nuclei on powersets (of monoids).

Frames and modules

Note that $(\mathcal{P}(W), \bigcup, \circ)$ is a complete semiring and $(\mathcal{P}(W), \bigcap)$ is a

Residuated frames

Frames and modules
Designated elements
Actions
Bi-modules
Formula hierarchy
Submodules and nuclei
Frames and modules
Densification
Frames and display
Distributive frames
Involutive FL
BiFL

module over it, via \setminus .

Frames and modules

Residuated frames

Frames and modules
Designated elements
Actions
Bi-modules
Formula hierarchy
Submodules and nuclei
Frames and modules
Densification
Frames and display
Distributive frames
Involutive FL
BiFL
Applications

Note that $(\mathcal{P}(W), \bigcup, \circ)$ is a complete semiring and $(\mathcal{P}(W), \bigcap)$ is a module over it, via \backslash .

Given a W-set (W', \mathbb{N}) , we have that $(\mathcal{P}(W'), \bigcup)^{\partial}$ is also a module over $(\mathcal{P}(W), \bigcup, \circ)$ with lifted action $X \mathbb{N} Z = \{x \mathbb{N} z : x \in X, z \in Z\}$:
Residuated frames

Frames and modules
Designated elements
Actions
Bi-modules
Formula hierarchy
Submodules and nuclei
Frames and modules
Densification
Frames and display
Distributive frames
Involutive FL
BiFL
Applications

Note that $(\mathcal{P}(W), \bigcup, \circ)$ is a complete semiring and $(\mathcal{P}(W), \bigcap)$ is a module over it, via \backslash .

Given a W-set (W', \mathbb{N}) , we have that $(\mathcal{P}(W'), \bigcup)^{\partial}$ is also a module over $(\mathcal{P}(W), \bigcup, \circ)$ with lifted action $X \mathbb{N} Z = \{x \mathbb{N} z : x \in X, z \in Z\}$: $\bigcup X_i \mathbb{N} \bigcup^{\partial} Z_j = \bigcup^{\partial} (X_i \mathbb{N} Z_j)$ and $Y \mathbb{N} (X \mathbb{N} Z) = (X \circ Y) \mathbb{N} Z$.

Residuated frames

Frames and modules
Designated elements
Actions
Bi-modules
Formula hierarchy
Submodules and nuclei
Frames and modules
Densification
Frames and display
Distributive frames
Involutive FL
BiFL
Applications

Note that $(\mathcal{P}(W), \bigcup, \circ)$ is a complete semiring and $(\mathcal{P}(W), \bigcap)$ is a module over it, via \backslash .

Given a W-set (W', \mathbb{N}) , we have that $(\mathcal{P}(W'), \bigcup)^{\partial}$ is also a module over $(\mathcal{P}(W), \bigcup, \circ)$ with lifted action $X \mathbb{N} Z = \{x \mathbb{N} z : x \in X, z \in Z\}$: $\bigcup X_i \mathbb{N} \bigcup^{\partial} Z_j = \bigcup^{\partial} (X_i \mathbb{N} Z_j)$ and $Y \mathbb{N} (X \mathbb{N} Z) = (X \circ Y) \mathbb{N} Z$. Note that the map $\triangleleft : (\mathcal{P}(W') \bigcup)^{\partial} \to (\mathcal{P}(W), \bigcap)$ is a module morphism, namely $(\bigcup^{\partial} Z_i)^{\triangleleft} = \bigcap Z_i^{\triangleleft}$ and $(X \mathbb{N} Z)^{\triangleleft} = X \setminus Z^{\triangleleft}$.

Residuated frames

Frames and modules
Designated elements
Actions
Bi-modules
Formula hierarchy
Submodules and nuclei
Frames and modules
Densification
Frames and display
Distributive frames
Involutive FL
BiFL
Applications

Note that $(\mathcal{P}(W), \bigcup, \circ)$ is a complete semiring and $(\mathcal{P}(W), \bigcap)$ is a module over it, via \backslash .

Given a W-set (W', \mathbb{N}) , we have that $(\mathcal{P}(W'), \bigcup)^{\partial}$ is also a module over $(\mathcal{P}(W), \bigcup, \circ)$ with lifted action $X \mathbb{N} Z = \{x \mathbb{N} z : x \in X, z \in Z\}$: $\bigcup X_i \mathbb{N} \bigcup^{\partial} Z_j = \bigcup^{\partial} (X_i \mathbb{N} Z_j)$ and $Y \mathbb{N} (X \mathbb{N} Z) = (X \circ Y) \mathbb{N} Z$. Note that the map $\triangleleft : (\mathcal{P}(W') \bigcup)^{\partial} \to (\mathcal{P}(W), \bigcap)$ is a module morphism, namely $(\bigcup^{\partial} Z_i)^{\triangleleft} = \bigcap Z_i^{\triangleleft}$ and $(X \mathbb{N} Z)^{\triangleleft} = X \mathbb{N} Z^{\triangleleft}$. The image of this module morphism is exactly the dual algebra.

Residuated frames

_	
	Frames and modules
	Designated elements
	Actions
	Bi-modules
	Formula hierarchy
	Submodules and nuclei
	Frames and modules
	Densification
	Frames and display
	Distributive frames
	Involutive FL
	BiFL
	Applications

Note that $(\mathcal{P}(W), \bigcup, \circ)$ is a complete semiring and $(\mathcal{P}(W), \bigcap)$ is a module over it, via \backslash .

Given a W-set (W', \mathbb{N}) , we have that $(\mathcal{P}(W'), \bigcup)^{\partial}$ is also a module over $(\mathcal{P}(W), \bigcup, \circ)$ with lifted action $X \mathbb{N} Z = \{x \mathbb{N} z : x \in X, z \in Z\}$: $\bigcup X_i \mathbb{N} \bigcup^{\partial} Z_j = \bigcup^{\partial} (X_i \mathbb{N} Z_j)$ and $Y \mathbb{N} (X \mathbb{N} Z) = (X \circ Y) \mathbb{N} Z$. Note that the map $\triangleleft : (\mathcal{P}(W') \bigcup)^{\partial} \to (\mathcal{P}(W), \bigcap)$ is a module morphism, namely $(\bigcup^{\partial} Z_i)^{\triangleleft} = \bigcap Z_i^{\triangleleft}$ and $(X \mathbb{N} Z)^{\triangleleft} = X \mathbb{N} Z^{\triangleleft}$. The image of this module morphism is exactly the dual algebra.

Note that for $Z \subseteq W'$, we have $Z^{\triangleleft} = \bigcap_{z \in Z} \{z\}^{\triangleleft}$. The sets $\{z\}^{\triangleleft}$ are called *basic* and every closed set is an intersection of basic closed.

Residuated frames

Frames and modules
Designated elements
Actions
Bi-modules
Formula hierarchy
Submodules and nuclei
Frames and modules
Densification
Frames and display
Distributive frames
Involutive FL
BiFL
Applications

Note that $(\mathcal{P}(W), \bigcup, \circ)$ is a complete semiring and $(\mathcal{P}(W), \bigcap)$ is a module over it, via \backslash .

Given a W-set (W', \mathbb{N}) , we have that $(\mathcal{P}(W'), \bigcup)^{\partial}$ is also a module over $(\mathcal{P}(W), \bigcup, \circ)$ with lifted action $X \mathbb{N} Z = \{x \mathbb{N} z : x \in X, z \in Z\}$: $\bigcup X_i \mathbb{N} \bigcup^{\partial} Z_j = \bigcup^{\partial} (X_i \mathbb{N} Z_j)$ and $Y \mathbb{N} (X \mathbb{N} Z) = (X \circ Y) \mathbb{N} Z$. Note that the map $\triangleleft : (\mathcal{P}(W') \bigcup)^{\partial} \to (\mathcal{P}(W), \bigcap)$ is a module morphism, namely $(\bigcup^{\partial} Z_i)^{\triangleleft} = \bigcap Z_i^{\triangleleft}$ and $(X \mathbb{N} Z)^{\triangleleft} = X \mathbb{N} Z^{\triangleleft}$. The image of this module morphism is exactly the dual algebra.

Note that for $Z \subseteq W'$, we have $Z^{\triangleleft} = \bigcap_{z \in Z} \{z\}^{\triangleleft}$. The sets $\{z\}^{\triangleleft}$ are called *basic* and every closed set is an intersection of basic closed.

It is important to chose the W-set W' wisely.

Residuated frames

_	
	Frames and modules
	Designated elements
	Actions
	Bi-modules
	Formula hierarchy
	Submodules and nuclei
	Frames and modules
	Densification
	Frames and display
	Distributive frames
	Involutive FL
	BiFL
	Applications

Note that $(\mathcal{P}(W), \bigcup, \circ)$ is a complete semiring and $(\mathcal{P}(W), \bigcap)$ is a module over it, via \backslash .

Given a W-set (W', \mathbb{N}) , we have that $(\mathcal{P}(W'), \bigcup)^{\partial}$ is also a module over $(\mathcal{P}(W), \bigcup, \circ)$ with lifted action $X \mathbb{N} Z = \{x \mathbb{N} z : x \in X, z \in Z\}$: $\bigcup X_i \mathbb{N} \bigcup^{\partial} Z_j = \bigcup^{\partial} (X_i \mathbb{N} Z_j)$ and $Y \mathbb{N} (X \mathbb{N} Z) = (X \circ Y) \mathbb{N} Z$. Note that the map $\triangleleft : (\mathcal{P}(W') \bigcup)^{\partial} \to (\mathcal{P}(W), \bigcap)$ is a module morphism, namely $(\bigcup^{\partial} Z_i)^{\triangleleft} = \bigcap Z_i^{\triangleleft}$ and $(X \mathbb{N} Z)^{\triangleleft} = X \mathbb{N} Z^{\triangleleft}$. The image of this module morphism is exactly the dual algebra.

Note that for $Z \subseteq W'$, we have $Z^{\triangleleft} = \bigcap_{z \in Z} \{z\}^{\triangleleft}$. The sets $\{z\}^{\triangleleft}$ are called *basic* and every closed set is an intersection of basic closed.

It is important to chose the W-set W' wisely. Otherwise the module $\mathcal{P}(W')$ will either be too far or too close to the dual algebra.

Residuated frames

_	
	Frames and modules
	Designated elements
	Actions
	Bi-modules
	Formula hierarchy
	Submodules and nuclei
	Frames and modules
	Densification
	Frames and display
	Distributive frames
	Involutive FL
	BiFL
	Applications

Note that $(\mathcal{P}(W), \bigcup, \circ)$ is a complete semiring and $(\mathcal{P}(W), \bigcap)$ is a module over it, via \backslash .

Given a W-set (W', \mathbb{N}) , we have that $(\mathcal{P}(W'), \bigcup)^{\partial}$ is also a module over $(\mathcal{P}(W), \bigcup, \circ)$ with lifted action $X \mathbb{N} Z = \{x \mathbb{N} z : x \in X, z \in Z\}$: $\bigcup X_i \mathbb{N} \bigcup^{\partial} Z_j = \bigcup^{\partial} (X_i \mathbb{N} Z_j)$ and $Y \mathbb{N} (X \mathbb{N} Z) = (X \circ Y) \mathbb{N} Z$. Note that the map $\triangleleft : (\mathcal{P}(W') \bigcup)^{\partial} \to (\mathcal{P}(W), \bigcap)$ is a module morphism, namely $(\bigcup^{\partial} Z_i)^{\triangleleft} = \bigcap Z_i^{\triangleleft}$ and $(X \mathbb{N} Z)^{\triangleleft} = X \mathbb{N} Z^{\triangleleft}$. The image of this module morphism is exactly the dual algebra.

Note that for $Z \subseteq W'$, we have $Z^{\triangleleft} = \bigcap_{z \in Z} \{z\}^{\triangleleft}$. The sets $\{z\}^{\triangleleft}$ are called *basic* and every closed set is an intersection of basic closed.

It is important to chose the W-set W' wisely. Otherwise the module $\mathcal{P}(W')$ will either be too far or too close to the dual algebra. We want W to have a natural description,

Residuated frames

_	
	Frames and modules
	Designated elements
	Actions
	Bi-modules
	Formula hierarchy
	Submodules and nuclei
	Frames and modules
	Densification
	Frames and display
	Distributive frames
	Involutive FL
	BiFL
	Applications

Note that $(\mathcal{P}(W), \bigcup, \circ)$ is a complete semiring and $(\mathcal{P}(W), \bigcap)$ is a module over it, via \backslash .

Given a W-set (W', \mathbb{N}) , we have that $(\mathcal{P}(W'), \bigcup)^{\partial}$ is also a module over $(\mathcal{P}(W), \bigcup, \circ)$ with lifted action $X \mathbb{N} Z = \{x \mathbb{N} z : x \in X, z \in Z\}$: $\bigcup X_i \mathbb{N} \bigcup^{\partial} Z_j = \bigcup^{\partial} (X_i \mathbb{N} Z_j)$ and $Y \mathbb{N} (X \mathbb{N} Z) = (X \circ Y) \mathbb{N} Z$. Note that the map $\triangleleft : (\mathcal{P}(W') \bigcup)^{\partial} \to (\mathcal{P}(W), \bigcap)$ is a module morphism, namely $(\bigcup^{\partial} Z_i)^{\triangleleft} = \bigcap Z_i^{\triangleleft}$ and $(X \mathbb{N} Z)^{\triangleleft} = X \mathbb{N} Z^{\triangleleft}$. The image of this module morphism is exactly the dual algebra.

Note that for $Z \subseteq W'$, we have $Z^{\triangleleft} = \bigcap_{z \in Z} \{z\}^{\triangleleft}$. The sets $\{z\}^{\triangleleft}$ are called *basic* and every closed set is an intersection of basic closed.

It is important to chose the W-set W' wisely. Otherwise the module $\mathcal{P}(W')$ will either be too far or too close to the dual algebra. We want W to have a natural description, but we don't want it to have unnecessary elements.

Residuated frames

-	
	Frames and modules
	Designated elements
	Actions
	Bi-modules
	Formula hierarchy
	Submodules and nuclei
	Frames and modules
	Densification
	Frames and display
	Distributive frames
	Involutive FL
	BiFL
	Applications

Note that $(\mathcal{P}(W), \bigcup, \circ)$ is a complete semiring and $(\mathcal{P}(W), \bigcap)$ is a module over it, via \backslash .

Given a W-set (W', \mathbb{N}) , we have that $(\mathcal{P}(W'), \bigcup)^{\partial}$ is also a module over $(\mathcal{P}(W), \bigcup, \circ)$ with lifted action $X \mathbb{N} Z = \{x \mathbb{N} z : x \in X, z \in Z\}$: $\bigcup X_i \mathbb{N} \bigcup^{\partial} Z_j = \bigcup^{\partial} (X_i \mathbb{N} Z_j)$ and $Y \mathbb{N} (X \mathbb{N} Z) = (X \circ Y) \mathbb{N} Z$. Note that the map $\triangleleft : (\mathcal{P}(W') \bigcup)^{\partial} \to (\mathcal{P}(W), \bigcap)$ is a module morphism, namely $(\bigcup^{\partial} Z_i)^{\triangleleft} = \bigcap Z_i^{\triangleleft}$ and $(X \mathbb{N} Z)^{\triangleleft} = X \mathbb{N} Z^{\triangleleft}$. The image of this module morphism is exactly the dual algebra.

Note that for $Z \subseteq W'$, we have $Z^{\triangleleft} = \bigcap_{z \in Z} \{z\}^{\triangleleft}$. The sets $\{z\}^{\triangleleft}$ are called *basic* and every closed set is an intersection of basic closed.

It is important to chose the W-set W' wisely. Otherwise the module $\mathcal{P}(W')$ will either be too far or too close to the dual algebra. We want W to have a natural description, but we don't want it to have unnecessary elements. So, we want it to be minimal, as given by the basic closed sets (no two should be equal), but the action should support this.

Residuated frames

Frames and modules Designated elements Actions Bi-modules Formula hierarchy Submodules and nuclei Frames and modules Densification Frames and display Distributive frames

Involutive FL

BiFL



Residuated frames

Frames and modulesDesignated elementsActionsBi-modulesFormula hierarchySubmodules and nucleiFrames and modulesDensificationFrames and displayDistributive framesInvolutive FLBiFLApplications

Given an FL_e -chain **A** with a gap g < h, extend it to one where this is no longer a gap (namely there is a new point p with g).

We need to embed in an FL_e chain the partial algebra $\mathbf{A} \cup \{p\}$, where g .

Residuated frames

Frames and modules
Designated elements
Actions
Bi-modules
Formula hierarchy
Submodules and nuclei
Frames and modules
Densification
Densineation
Frames and display
Frames and display Distributive frames
Frames and display Distributive frames Involutive FL
Frames and display Distributive frames Involutive FL BiFL
Frames and display Distributive frames Involutive FL BiFL Applications

Given an FL_e -chain **A** with a gap g < h, extend it to one where this is no longer a gap (namely there is a new point p with g).

We need to embed in an FL_e chain the partial algebra $\mathbf{A} \cup \{p\}$, where g .

It suffices to construct a residuated frame $W_{A,p}$ from this data such that $(W_{A,p}, A \cup \{p\})$ is a Gentzen frame and $W_{A,p}^+$ is linear.

Residuated frames

Frames and modules
Designated elements
Actions
Bi-modules
Formula hierarchy
Submodules and nuclei
Frames and modules
Densification
Frames and display
Distributive frames
Involutive FL
BiFL
Applications

Given an FL_e -chain **A** with a gap g < h, extend it to one where this is no longer a gap (namely there is a new point p with g).

We need to embed in an FL_e chain the partial algebra $\mathbf{A} \cup \{p\}$, where g .

It suffices to construct a residuated frame $W_{A,p}$ from this data such that $(W_{A,p}, A \cup \{p\})$ is a Gentzen frame and $W_{A,p}^+$ is linear.

If we take W' to be unnecessarily big, checking that $\mathbf{W}_{\mathbf{A},p}^+$ is linear takes some effort.

Residuated frames

Frames and modules
Designated elements
Actions
Bi-modules
Formula hierarchy
Submodules and nuclei
Frames and modules
Densification
Frames and display
Distributive frames
Involutive FL
BiFL

Given an FL_e -chain **A** with a gap g < h, extend it to one where this is no longer a gap (namely there is a new point p with g).

We need to embed in an FL_e chain the partial algebra $\mathbf{A} \cup \{p\}$, where g .

It suffices to construct a residuated frame $W_{A,p}$ from this data such that $(W_{A,p}, A \cup \{p\})$ is a Gentzen frame and $W_{A,p}^+$ is linear.

If we take W' to be unnecessarily big, checking that $\mathbf{W}_{\mathbf{A},p}^+$ is linear takes some effort. If we take it to be just right, checking linearity is easy.

Residuated frames

Frames and modules

Frames and display

Adding residuals

Conservativity Cut elimination via

display

Via algebraic

completions

With disjunction

Distributive frames

Involutive FL

BiFL

Applications

Frames and display

We define the system (plus associativity and exchage for simplicity).

$$\frac{x \Rightarrow a \quad y, a, z \Rightarrow c}{y, x, z \Rightarrow c} \text{ (cut) } \qquad \frac{a \Rightarrow a}{a \Rightarrow a} \text{ (Id)}$$

$$\frac{y, a, b, z \Rightarrow c}{y, a \cdot b, z \Rightarrow c} (\cdot \mathsf{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x, y \Rightarrow a \cdot b} (\cdot \mathsf{R})$$

Residuated frames

Frames and modules

Frames and display

Adding residuals

Conservativity Cut elimination via display Via algebraic completions With disjunction

Distributive frames

Involutive FL

BiFL

We define the system (plus associativity and exchage for simplicity).

 $\frac{x \Rightarrow a \quad y, a, z \Rightarrow c}{y, x, z \Rightarrow c} \text{ (cut) } \qquad \frac{a \Rightarrow a}{a \Rightarrow a} \text{ (Id)}$

$$\frac{y, a, b, z \Rightarrow c}{y, a \cdot b, z \Rightarrow c} (\cdot \mathsf{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x, y \Rightarrow a \cdot b} (\cdot \mathsf{R})$$

This logic is complete with respect to commutative posemigroups; $\mathbf{L} = (L, \leq, \cdot)$ where multiplication preserves the order. Frames and modulesFrames and displayAdding residualsConservativityCut elimination via
displayVia algebraic
completionsWith disjunctionDistributive framesInvolutive FLBiFLApplications

Residuated frames

We define the system (plus associativity and exchage for simplicity).

 $\frac{x \Rightarrow a \quad y, a, z \Rightarrow c}{y, x, z \Rightarrow c} \text{ (cut) } \qquad \frac{a \Rightarrow a}{a \Rightarrow a} \text{ (Id)}$

$$\frac{y, a, b, z \Rightarrow c}{y, a \cdot b, z \Rightarrow c} (\cdot \mathsf{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x, y \Rightarrow a \cdot b} (\cdot \mathsf{R})$$

This logic is complete with respect to commutative posemigroups; $\mathbf{L} = (L, \leq, \cdot)$ where multiplication preserves the order.

Is it conservative to extend the logic to one L_e with implication?

$$\frac{x \Rightarrow a \quad y, b, z \Rightarrow c}{y, x, a \to b, z \Rightarrow a} (\to \mathsf{L}) \qquad \frac{a, x \Rightarrow b}{x \Rightarrow a \to b} (\to \mathsf{R})$$

Conservativity: if a sequent/inequality fails in the smaller logic (in a every commtative posemigroup), then it fails in the bigger logic (in a commutative residuated posemigroup).

Frames and modulesFrames and displayAdding residualsConservativityCut elimination via
displayVia algebraic
completionsWith disjunctionDistributive framesInvolutive FLBiFL
Applications

Residuated frames

We define the system (plus associativity and exchage for simplicity).

 $\frac{x \Rightarrow a \quad y, a, z \Rightarrow c}{y, x, z \Rightarrow c} \text{ (cut) } \qquad \frac{a \Rightarrow a}{a \Rightarrow a} \text{ (Id)}$

$$\frac{y, a, b, z \Rightarrow c}{y, a \cdot b, z \Rightarrow c} (\cdot \mathsf{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x, y \Rightarrow a \cdot b} (\cdot \mathsf{R})$$

This logic is complete with respect to commutative posemigroups; $\mathbf{L} = (L, \leq, \cdot)$ where multiplication preserves the order.

Is it conservative to extend the logic to one L_e with implication?

$$\frac{x \Rightarrow a \quad y, b, z \Rightarrow c}{y, x, a \to b, z \Rightarrow a} (\to \mathsf{L}) \qquad \frac{a, x \Rightarrow b}{x \Rightarrow a \to b} (\to \mathsf{R})$$

Conservativity: if a sequent/inequality fails in the smaller logic (in a every commtative posemigroup), then it fails in the bigger logic (in a commutative residuated posemigroup).

We can of course define a residuated frame $(Fm^*, Fm^* \times Fm, N, \circ, \mathbb{N})$ based on this system.

Nick Galatos, Prague workshop, March, 2014

Residuated frames

Frames and modules

Frames and display

Adding residuals Conservativity

Let S denotes all commutive posemigroups and \mathcal{R} all the posemigroup reducts of the residuated ones.

Frames and modules

Frames and display

Adding residuals

Conservativity

Cut elimination via display

Via algebraic completions

With disjunction

Distributive frames

Involutive FL

BiFL

Let S denotes all commutive posemigroups and \mathcal{R} all the posemigroup reducts of the residuated ones. Then $S(\mathcal{R})$ (all subreducts) forms an (order) quasivariety.

Frames and modules

Frames and display

Adding residuals

Conservativity

Cut elimination via display

Via algebraic

completions

With disjunction

Distributive frames

Involutive FL

BiFL

Residuated frames

Frames and modules

Frames and display

Adding residuals

Conservativity

Cut elimination via display Via algebraic completions

With disjunction

Distributive frames

Involutive FL

BiFL

Applications

Let S denotes all commutive posemigroups and \mathcal{R} all the posemigroup reducts of the residuatred ones. Then $S(\mathcal{R})$ (all subreducts) forms an (order) quasivariety. Conservativity states that $S = H(S(\mathcal{R}))$. In other words every posemigroup is a homomorphic image of one that can be embedded to a residuated posemigroup.

Let S denotes all commutive posemigroups and \mathcal{R} all the posemigroup reducts of the residuatred ones. Then $S(\mathcal{R})$ (all subreducts) forms an (order) quasivariety. Conservativity states that $S = H(S(\mathcal{R}))$. In other words every posemigroup is a homomorphic image of one that can be embedded to a residuated posemigroup.

There are a couple of ways to prove conservativity:

Frames and modulesFrames and displayAdding residualsConservativityCut elimination via
displayVia algebraic
completionsWith disjunctionDistributive framesInvolutive FLBiFLApplications

Residuated frames

Let S denotes all commutive posemigroups and \mathcal{R} all the posemigroup reducts of the residuatred ones. Then $S(\mathcal{R})$ (all subreducts) forms an (order) quasivariety. Conservativity states that $S = H(S(\mathcal{R}))$. In other words every posemigroup is a homomorphic image of one that can be embedded to a residuated posemigroup.

There are a couple of ways to prove conservativity:

Proof-theoretically. Prove cut elimination and then use the subformula property.

Frames and modules
Frames and display
Adding residuals
Conservativity
Cut elimination via display
Via algebraic
completions
With disjunction
Distributive frames
Involutive FL
BiFL
Applications

Residuated frames

Let S denotes all commutive posemigroups and \mathcal{R} all the posemigroup reducts of the residuatred ones. Then $S(\mathcal{R})$ (all subreducts) forms an (order) quasivariety. Conservativity states that $S = H(S(\mathcal{R}))$. In other words every posemigroup is a homomorphic image of one that can be embedded to a residuated posemigroup.

There are a couple of ways to prove conservativity:

Proof-theoretically. Prove cut elimination and then use the subformula property.

Algebraically. Show that every commutative posemigroup can be embedded into a residuated one. In other words we show something stronger: S = S(R).

Residuated frames
Frames and modules
Frames and display
Adding residuals
Conservativity
Cut elimination via display
Via algebraic completions
With disjunction
Distributive frames
Involutive FL
BiFL

Let S denotes all commutive posemigroups and \mathcal{R} all the posemigroup reducts of the residuatred ones. Then $S(\mathcal{R})$ (all subreducts) forms an (order) quasivariety. Conservativity states that $S = H(S(\mathcal{R}))$. In other words every posemigroup is a homomorphic image of one that can be embedded to a residuated posemigroup.

There are a couple of ways to prove conservativity:

Proof-theoretically. Prove cut elimination and then use the subformula property.

Algebraically. Show that every commutative posemigroup can be embedded into a residuated one. In other words we show something stronger: S = S(R).

Proof-theoretically (2): Via display logic.

Residuated frames Frames and modules Frames and display Adding residuals Conservativity

Cut elimination via display Via algebraic completions With disjunction Distributive frames

Involutive FL

BiFL

Consider the following display logic system
$$\mathbf{L}_e^{\delta}$$
. Here $x, y \in Fm^*$, $a, b \in Fm$ and z is of the form $x_1 > (x_2 > \dots (x_n > a) \dots)$.

 $\frac{x \Rightarrow a \quad a \Rightarrow z}{x \Rightarrow z} \text{ (cut)} \qquad \overline{a \Rightarrow a} \text{ (Id)} \qquad \frac{x \circ y \Rightarrow z}{\overline{y \Rightarrow x > z}} \text{ (dis)}$ $\frac{a, b \Rightarrow z}{a \cdot b \Rightarrow z} \text{ (\cdotL)} \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x, y \Rightarrow a \cdot b} \text{ (\cdotR)}$ $\frac{x \Rightarrow a \quad b \Rightarrow z}{a \rightarrow b \Rightarrow x > z} \text{ (\rightarrowL)} \qquad \frac{x \Rightarrow a > b}{x \Rightarrow a \rightarrow b} \text{ (\rightarrowR)}$

Frames and modules

Frames and display Adding residuals Conservativity Cut elimination via

display Via algebraic

completions With disjunction

Involutive FL

Applications

BiFL

Distributive frames

Residuated frames

Frames and modules

Frames and display Adding residuals

Distributive frames

Involutive FL

Applications

BiFL

Conservativity Cut elimination via

display Via algebraic completions With disjunction

Residuated frames – 24 / 62

Consider the following display logic system
$$\mathbf{L}_e^{\delta}$$
. Here $x, y \in Fm^*$, $a, b \in Fm$ and z is of the form $x_1 > (x_2 > \dots (x_n > a) \dots)$.

$$\frac{x \Rightarrow a \quad a \Rightarrow z}{x \Rightarrow z} \text{ (cut)} \qquad \overline{a \Rightarrow a} \text{ (Id)} \qquad \frac{x \circ y \Rightarrow z}{\overline{y \Rightarrow x > z}} \text{ (dis)}$$

$$\frac{a, b \Rightarrow z}{a \cdot b \Rightarrow z} \text{ (\cdotL)} \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x, y \Rightarrow a \cdot b} \text{ (\cdotR)}$$

$$\frac{x \Rightarrow a \quad b \Rightarrow z}{a \to b \Rightarrow x > z} \text{ (\rightarrowL)} \qquad \frac{x \Rightarrow a > b}{x \Rightarrow a \to b} \text{ (\rightarrowR)}$$

We could build a residuated frame $(W, W', \Rightarrow , \{,\}, >)$.

Consider the following display logic system \mathbf{L}_e^{δ} . Here $x, y \in Fm^*$, $a, b \in Fm$ and z is of the form $x_1 > (x_2 > \dots (x_n > a) \dots)$.

 $\frac{x \Rightarrow a \quad a \Rightarrow z}{x \Rightarrow z} \text{ (cut)} \qquad \frac{a \Rightarrow a}{a \Rightarrow a} \text{ (Id)} \qquad \frac{x \circ y \Rightarrow z}{y \Rightarrow x > z} \text{ (dis)}$

$$\frac{a, b \Rightarrow z}{a \cdot b \Rightarrow z} (\cdot \mathsf{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x, y \Rightarrow a \cdot b} (\cdot \mathsf{R})$$

$$\frac{x \Rightarrow a \quad b \Rightarrow z}{a \rightarrow b \Rightarrow x > z} (\rightarrow \mathsf{L}) \qquad \frac{x \Rightarrow a > b}{x \Rightarrow a \rightarrow b} (\rightarrow \mathsf{R})$$

We could build a residuated frame $(W, W', \Rightarrow , \{,\}, >)$. Then the system \mathbf{L}_e^{δ} has cut elimination 'for free' (by Belnap's conditions) and the subformula property (and decidability). So \mathbf{L}_e^{δ} is conservative over its \rightarrow -free fragment. **Residuated frames**

Frames and modulesFrames and displayAdding residualsConservativityCut elimination viadisplayVia algebraiccompletionsWith disjunctionDistributive framesInvolutive FLBiFL

Consider the following display logic system \mathbf{L}_e^{δ} . Here $x, y \in Fm^*$, $a, b \in Fm$ and z is of the form $x_1 > (x_2 > \dots (x_n > a) \dots)$.

 $\frac{x \Rightarrow a \quad a \Rightarrow z}{x \Rightarrow z} \text{ (cut)} \qquad \frac{a \Rightarrow a}{a \Rightarrow a} \text{ (Id)} \qquad \frac{x \circ y \Rightarrow z}{y \Rightarrow x > z} \text{ (dis)}$

$$\frac{a, b \Rightarrow z}{a \cdot b \Rightarrow z} (\cdot L) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x, y \Rightarrow a \cdot b} (\cdot R)$$

$$\frac{x \Rightarrow a \quad b \Rightarrow z}{a \rightarrow b \Rightarrow x > z} (\rightarrow \mathsf{L}) \qquad \frac{x \Rightarrow a > b}{x \Rightarrow a \rightarrow b} (\rightarrow \mathsf{R})$$

We could build a residuated frame $(W, W', \Rightarrow , \{,\}, >)$. Then the system \mathbf{L}_e^{δ} has cut elimination 'for free' (by Belnap's conditions) and the subformula property (and decidability). So \mathbf{L}_e^{δ} is conservative over its \rightarrow -free fragment. But is that the same as the fragment of cut-free \mathbf{L}_e ? If so, we have conservativity.

Residuated frames

Frames and modules
Frames and display
Adding residuals
Conservativity
Cut elimination via display
Via algebraic
completions
With disjunction
Distributive frames
Involutive FL
BiFL
Applications

Consider the following display logic system \mathbf{L}_e^{δ} . Here $x, y \in Fm^*$, $a, b \in Fm$ and z is of the form $x_1 > (x_2 > \dots (x_n > a) \dots)$.

 $\frac{x \Rightarrow a \quad a \Rightarrow z}{x \Rightarrow z} \text{ (cut)} \qquad \frac{a \Rightarrow a}{a \Rightarrow a} \text{ (Id)} \qquad \frac{x \circ y \Rightarrow z}{y \Rightarrow x > z} \text{ (dis)}$

$$\frac{a, b \Rightarrow z}{a \cdot b \Rightarrow z} (\cdot L) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x, y \Rightarrow a \cdot b} (\cdot R)$$

$$\frac{x \Rightarrow a \quad b \Rightarrow z}{a \rightarrow b \Rightarrow x > z} (\rightarrow \mathsf{L}) \qquad \frac{x \Rightarrow a > b}{x \Rightarrow a \rightarrow b} (\rightarrow \mathsf{R})$$

We could build a residuated frame $(W, W', \Rightarrow , \{,\}, >)$. Then the system \mathbf{L}_{e}^{δ} has cut elimination 'for free' (by Belnap's

Nick Galatos, Prague workshop, March, 2014

conditions) and the subformula property (and decidability). So \mathbf{L}_e^{δ} is conservative over its \rightarrow -free fragment. But is that the same as the fragment of cut-free \mathbf{L}_e ? If so, we have conservativity.

Yes! The two systems are mutually interpretable. (New sequents are *innocent*.) First every rule in \mathbf{L}_e is derivable in $\delta \mathbf{L}_e$. (Using the display porperty.) Second, given a cut-free proof in \mathbf{L}_e^{δ} of a sequent free of \rightarrow and >, we can convert it to a proof in cut-free \mathbf{L}_e .

Residuated frames

Frames and modules

Frames and display

Adding residuals

Conservativity

Cut elimination via

display Via algebraic

completions

With disjunction

Distributive frames

Involutive FL

BiFL

Residuated frames

Frames and modules

Frames and display

Adding residuals

Conservativity Cut elimination via

display

Via algebraic

completions With disjunction

Distributive frames

Involutive FL

BiFL

Applications

Let \mathbf{L} be a commutative posemigroup.

Let \mathbf{L} be a commutative posemigroup.

We consider the frame $\mathbf{W}_{L}^{+} = (L, L \times L, N, \cdot, \mathbb{N})$, where $x \mathbb{N}(y, z) = (yx, z)$ and x N(y, z) iff $yx \leq z$.

Residuated frames

Frames and modules

Frames and display

Adding residuals

Conservativity

Cut elimination via display

Via algebraic

completions

With disjunction

Distributive frames

Involutive FL

BiFL

Let \mathbf{L} be a commutative posemigroup.

We consider the frame $\mathbf{W}_{L}^{+} = (L, L \times L, N, \cdot, \mathbb{N})$, where $x \mathbb{N}(y, z) = (yx, z)$ and x N(y, z) iff $yx \leq z$.

We identify (1, z) with z. Then $y \setminus z = (y, z)$, or rather $\{(y, z)\}^{\triangleleft}$, is a 'formal' residual.

Residuated framesFrames and modulesFrames and displayAdding residualsConservativityCut elimination viadisplayVia algebraiccompletionsWith disjunctionDistributive framesInvolutive FLBiFLApplications

Let \mathbf{L} be a commutative posemigroup.

We consider the frame $\mathbf{W}_{L}^{+} = (L, L \times L, N, \cdot, \mathbb{N})$, where $x \mathbb{N}(y, z) = (yx, z)$ and x N(y, z) iff $yx \leq z$.

We identify (1, z) with z. Then $y \setminus z = (y, z)$, or rather $\{(y, z)\}^{\triangleleft}$, is a 'formal' residual.

Then $\mathbf{W}_{\mathbf{L}}^+$ is a commutative residuated posemigroup into which \mathbf{L} embeds.

Residuated frames
Frames and modules
Frames and display
Adding residuals
Conservativity
Cut elimination via
display
Via algebraic
completions
With disjunction
Distributive frames
Involutive FL
BiFL

Let \mathbf{L} be a commutative posemigroup.

We consider the frame $\mathbf{W}_{L}^{+} = (L, L \times L, N, \cdot, \mathbb{N})$, where $x \mathbb{N}(y, z) = (yx, z)$ and x N(y, z) iff $yx \leq z$.

We identify (1, z) with z. Then $y \setminus z = (y, z)$, or rather $\{(y, z)\}^{\triangleleft}$, is a 'formal' residual.

Then $\mathbf{W}_{\mathbf{L}}^+$ is a commutative residuated posemigroup into which \mathbf{L} embeds.

It is easy to see that if L satisfies contraction $x \leq x^2$, then so does W_L , and by a previous result so does W_L^+ .

 $\frac{x \circ xNz}{xNz}$ (c)^p

Residuated frames
Frames and modules
Frames and display
Adding residuals
Conservativity
Cut elimination via
display
Via algebraic
completions
With disjunction
Distributive frames
Involutive FL
BiFL
Via algebraic completions

Let \mathbf{L} be a commutative posemigroup.

We consider the frame $\mathbf{W}_{L}^{+} = (L, L \times L, N, \cdot, \mathbb{N})$, where $x \mathbb{N}(y, z) = (yx, z)$ and x N(y, z) iff $yx \leq z$.

We identify (1, z) with z. Then $y \setminus z = (y, z)$, or rather $\{(y, z)\}^{\triangleleft}$, is a 'formal' residual.

Then $\mathbf{W}_{\mathbf{L}}^+$ is a commutative residuated posemigroup into which \mathbf{L} embeds.

It is easy to see that if L satisfies contraction $x \leq x^2$, then so does W_L , and by a previous result so does W_L^+ .

$$\frac{x \circ xNz}{xNz}$$
 (c)^p

If we have join in the language (next page) the same holds for mingle/expansion $x^2 \le x$ ($xy \le x \lor y$)

$$rac{xNz \quad yNz}{x \circ yNz} \, \left(\mathsf{m}
ight)^p$$

We could now consider posemigroups with joins and ask if it is conservative to add residuals.

Residuated frames

Frames and modules

Frames and display

Adding residuals

Conservativity Cut elimination via

display

Via algebraic

completions

With disjunction

Distributive frames

Involutive FL

BiFL

We could now consider posemigroups with joins and ask if it is

 $x(y \lor z) = zy \lor xz$. So we take *join posemigroups* (semirings without

conservative to add residuals. The answer is 'no', as we get

Residuated frames

Frames and modules

Frames and display

Adding residuals

Conservativity Cut elimination via

display

Via algebraic

completions

With disjunction

Distributive frames

Involutive FL

BiFL

Applications

1).

We could now consider posemigroups with joins and ask if it is conservative to add residuals. The answer is 'no', as we get $x(y \lor z) = zy \lor xz$. So we take *join posemigroups* (semirings without 1).

We have completeness for the calculus extended with:

$$\frac{y, a, z \Rightarrow c \quad y, b, z \Rightarrow c}{y, a \lor b, z \Rightarrow c} (\lor \mathsf{L}) \quad \frac{x \Rightarrow a}{x \Rightarrow a \lor b} (\lor \mathsf{R}\ell) \quad \frac{x \Rightarrow b}{x \Rightarrow a \lor b} (\lor \mathsf{R}r)$$

and in display style

$$\frac{a \Rightarrow z \quad b \Rightarrow z}{a \lor b \Rightarrow z} (\lor \mathsf{L}) \quad \frac{x \Rightarrow a}{x \Rightarrow a \lor b} (\lor \mathsf{R}\ell) \quad \frac{x \Rightarrow b}{x \Rightarrow a \lor b} (\lor \mathsf{R}r)$$

Residuated frames

Frames and modules

Frames and display

Adding residuals

Conservativity Cut elimination via

display

Via algebraic

completions

With disjunction

Distributive frames

Involutive FL

BiFL

We could now consider posemigroups with joins and ask if it is conservative to add residuals. The answer is 'no', as we get $x(y \lor z) = zy \lor xz$. So we take *join posemigroups* (semirings without 1).

We have completeness for the calculus extended with:

$$\frac{y, a, z \Rightarrow c \quad y, b, z \Rightarrow c}{y, a \lor b, z \Rightarrow c} (\lor \mathsf{L}) \quad \frac{x \Rightarrow a}{x \Rightarrow a \lor b} (\lor \mathsf{R}\ell) \quad \frac{x \Rightarrow b}{x \Rightarrow a \lor b} (\lor \mathsf{R}r)$$

and in display style

$$\frac{a \Rightarrow z \quad b \Rightarrow z}{a \lor b \Rightarrow z} (\lor \mathsf{L}) \quad \frac{x \Rightarrow a}{x \Rightarrow a \lor b} (\lor \mathsf{R}\ell) \quad \frac{x \Rightarrow b}{x \Rightarrow a \lor b} (\lor \mathsf{R}r)$$

If a connective gives an operator (order preserving; if we have join then it distributes) then it will be residuated at the completion and we can conservatively add its residual.

We could now consider posemigroups with joins and ask if it is conservative to add residuals. The answer is 'no', as we get $x(y \lor z) = zy \lor xz$. So we take *join posemigroups* (semirings without 1).

We have completeness for the calculus extended with:

$$\frac{y, a, z \Rightarrow c \quad y, b, z \Rightarrow c}{y, a \lor b, z \Rightarrow c} (\lor \mathsf{L}) \quad \frac{x \Rightarrow a}{x \Rightarrow a \lor b} (\lor \mathsf{R}\ell) \quad \frac{x \Rightarrow b}{x \Rightarrow a \lor b} (\lor \mathsf{R}r)$$

and in display style

$$\frac{a \Rightarrow z \quad b \Rightarrow z}{a \lor b \Rightarrow z} (\lor \mathsf{L}) \quad \frac{x \Rightarrow a}{x \Rightarrow a \lor b} (\lor \mathsf{R}\ell) \quad \frac{x \Rightarrow b}{x \Rightarrow a \lor b} (\lor \mathsf{R}r)$$

If a connective gives an operator (order preserving; if we have join then it distributes) then it will be residuated at the completion and we can conservatively add its residual. Then one could chose to work in the bigger (and in some sense simpler) logic with no worries.

Residuated framesFrames and modulesFrames and displayAdding residualsConservativityCut elimination viadisplayVia algebraiccompletionsWith disjunctionDistributive framesInvolutive FLBiFLApplications

We could now consider posemigroups with joins and ask if it is conservative to add residuals. The answer is 'no', as we get $x(y \lor z) = zy \lor xz$. So we take *join posemigroups* (semirings without 1).

We have completeness for the calculus extended with:

$$\frac{y, a, z \Rightarrow c \quad y, b, z \Rightarrow c}{y, a \lor b, z \Rightarrow c} (\lor \mathsf{L}) \quad \frac{x \Rightarrow a}{x \Rightarrow a \lor b} (\lor \mathsf{R}\ell) \quad \frac{x \Rightarrow b}{x \Rightarrow a \lor b} (\lor \mathsf{R}r)$$

and in display style

$$\frac{a \Rightarrow z \quad b \Rightarrow z}{a \lor b \Rightarrow z} (\lor \mathsf{L}) \quad \frac{x \Rightarrow a}{x \Rightarrow a \lor b} (\lor \mathsf{R}\ell) \quad \frac{x \Rightarrow b}{x \Rightarrow a \lor b} (\lor \mathsf{R}r)$$

If a connective gives an operator (order preserving; if we have join then it distributes) then it will be residuated at the completion and we can conservatively add its residual. Then one could chose to work in the bigger (and in some sense simpler) logic with no worries.

General principle: Let's be 'honest' about it and put the residuals at the frame level from the very beginning.

Frames and modules

Frames and display

Adding residuals

Conservativity Cut elimination via

display Via algebraic

completions

With disjunction

Distributive frames

Involutive FL

Applications

BiFL

Residuated frames

Frames and modules

Frames and display

Distributive frames

DFL

Nuclei

Distributive frames

DGN

Involutive FL

BiFL

Applications

Distributive frames

DFL

Residuated frames

Frames and modules

Frames and display

Distributive frames

DFL

Nuclei

Distributive frames

DGN

Involutive FL

BiFL

Applications

Sequents of DFL have as LHS the elements of $(Fm^{\gamma}, \circ, \varepsilon, \bigcirc)$, the

free (monoid) algebra over Fm.

DFL

Sequents of **DFL** have as LHS the elements of $(Fm^{\gamma}, \circ, \varepsilon, \bigotimes)$, the

Also, u is a unary linear polynomial over this signature.

Residuated frames

Frames and modules

Frames and display

Distributive frames

DFL

Nuclei

Distributive frames

DGN

Involutive FL

BiFL

Applications

Nick Galatos, Prague workshop, March, 2014

free (monoid) algebra over Fm.

DFL

Residuated frames

Frames and modules

Frames and display

Distributive frames

DFL

DGN

Nuclei

Distributive frames

Involutive FL

BiFL

Applications

Sequents of **DFL** have as LHS the elements of $(Fm^{\gamma}, \circ, \varepsilon, \bigcirc)$, the free (monoid) algebra over Fm.

Also, u is a unary linear polynomial over this signature.

We add the rules:

$$\frac{u[x \bigotimes (y \bigotimes z)] \Rightarrow c}{u[(x \bigotimes y) \bigotimes z] \Rightarrow c} (\bigotimes a) \qquad \frac{u[x \bigotimes y] \Rightarrow c}{u[y \bigotimes x] \Rightarrow c} (\bigotimes e)$$
$$\frac{u[x] \Rightarrow c}{u[x \bigotimes y] \Rightarrow c} (\bigotimes i) \qquad \frac{u[x \bigotimes x] \Rightarrow c}{u[y \bigotimes x] \Rightarrow c} (\bigotimes c)$$

$$\frac{u[x \bigotimes y] \Rightarrow c}{u[x \bigotimes y] \Rightarrow c} (\bigotimes i) \qquad \frac{u[x \bigotimes x] \Rightarrow c}{u[x] \Rightarrow c} (\bigotimes$$

And replace (\wedge L) by:

$$\frac{u[a \bigotimes b] \Rightarrow c}{u[a \land b] \Rightarrow c} (\land \mathsf{L})$$

Residuated frames

Frames and modules

Frames and display

Distributive frames

DFL Nuclei

Distributive frames

DGN

Involutive FL

BiFL

Applications

Recall that $\wedge : N_n \times N_n \to N_n$.

Residuated frames

Frames and modules

Frames and display

Distributive frames

DFL Nuclei

Distributive frames DGN

Involutive FL

BiFL

Applications

new type, then we arrive at a new notion of sequent. The operation at the frame level corresponding to \wedge is denoted by \bigcirc . We obtain *distributive sequents* and the calculus **DFL**.

Recall that $\wedge : N_n \times N_n \to N_n$. If we add $\wedge : P_n \times P_n \to P_n$ as a

Residuated frames

Frames and modules

Frames and display

Distributive frames

DFL Nuclei

Nuc

Distributive frames **DGN**

Involutive FL

BiFL

Applications

Recall that $\wedge : N_n \times N_n \to N_n$. If we add $\wedge : P_n \times P_n \to P_n$ as a new type, then we arrive at a new notion of sequent. The operation at the frame level corresponding to \wedge is denoted by \bigcirc . We obtain *distributive sequents* and the calculus **DFL**.

Given a residuated lattice expansion $\mathbf{L}' = (\mathbf{L}, \bigotimes)$, a *distributive nucleus* γ is \cdot -nucleus and \bigotimes -nucleus on \mathbf{L} that satisfies $\gamma(x \bigotimes y) = \gamma(x) \land \gamma(y)$.

Residuated frames

Frames and modules

Frames and display

Distributive frames

DFL Nuclei

Distributive frames

DGN

Involutive FL

BiFL

Applications

Recall that $\wedge : N_n \times N_n \to N_n$. If we add $\wedge : P_n \times P_n \to P_n$ as a new type, then we arrive at a new notion of sequent. The operation at the frame level corresponding to \wedge is denoted by \bigcirc . We obtain *distributive sequents* and the calculus **DFL**.

Given a residuated lattice expansion $\mathbf{L}' = (\mathbf{L}, \bigotimes)$, a *distributive nucleus* γ is \cdot -nucleus and \bigotimes -nucleus on \mathbf{L} that satisfies $\gamma(x \bigotimes y) = \gamma(x) \land \gamma(y)$.

Then $\bigotimes_{\gamma} = \wedge$ on L_{γ} and

 $\mathbf{L}_{\gamma} = (L_{\gamma}, \wedge, \vee_{\gamma}, \cdot_{\gamma}, \backslash, /, \gamma(1))$

is a distributive residuated lattice.

Residuated frames

Frames and modules

Frames and display

Distributive frames

DFL Nuclei

Distributive frames

Involutive FL

BiFL

DGN

Applications

Recall that $\wedge : N_n \times N_n \to N_n$. If we add $\wedge : P_n \times P_n \to P_n$ as a new type, then we arrive at a new notion of sequent. The operation at the frame level corresponding to \wedge is denoted by \bigcirc . We obtain *distributive sequents* and the calculus **DFL**.

Given a residuated lattice expansion $\mathbf{L}' = (\mathbf{L}, \bigotimes)$, a *distributive nucleus* γ is \cdot -nucleus and \bigotimes -nucleus on \mathbf{L} that satisfies $\gamma(x \bigotimes y) = \gamma(x) \land \gamma(y)$.

Then $\bigotimes_{\gamma} = \wedge$ on L_{γ} and

 $\mathbf{L}_{\gamma} = (L_{\gamma}, \wedge, \vee_{\gamma}, \cdot_{\gamma}, \backslash, /, \gamma(1))$

is a distributive residuated lattice.

Note that distributive residuated lattices are double semirings.

Residuated frames

Frames and modules

Frames and display

Distributive frames

DFL Nuclei

DGN

Distributive frames

Involutive FL

BiFL

Applications

Recall that $\wedge : N_n \times N_n \to N_n$. If we add $\wedge : P_n \times P_n \to P_n$ as a new type, then we arrive at a new notion of sequent. The operation at the frame level corresponding to \wedge is denoted by \otimes . We obtain *distributive sequents* and the calculus **DFL**.

Given a residuated lattice expansion $\mathbf{L}' = (\mathbf{L}, \bigotimes)$, a *distributive nucleus* γ is \cdot -nucleus and \bigotimes -nucleus on \mathbf{L} that satisfies $\gamma(x \bigotimes y) = \gamma(x) \land \gamma(y)$.

Then $\bigotimes_{\gamma} = \wedge$ on L_{γ} and

 $\mathbf{L}_{\gamma} = (L_{\gamma}, \wedge, \vee_{\gamma}, \cdot_{\gamma}, \backslash, /, \gamma(1))$

is a distributive residuated lattice.

Note that distributive residuated lattices are double semirings.

We aim for an embedding of distributive residuated lattices to Heyting residuated lattices.

Distributive frames

Residuated frames

Frames and modules

Frames and display

Distributive frames

DFL

Nuclei

Distributive frames

Involutive FL

BiFL

DGN

Applications

 $N \subseteq W \times W'$,

A *distributive residuated frame* is a structure

 $\mathbf{W} = (W, W', N, \circ, 1, \bigcirc, \backslash\!\!\backslash, /\!\!/, \hookrightarrow, \hookleftarrow)$ where W and W' are sets

Distributive frames

A *distributive residuated frame* is a structure $\mathbf{W} = (W, W', N, \circ, 1, \bigcirc, \backslash\!\!\backslash, /\!\!/, \hookrightarrow, \hookleftarrow)$ where W and W' are sets $N \subseteq W \times W', (W, \circ, 1)$ is a monoid and for all $x, y \in W, w \in W'$ $(x \circ y) N w \Leftrightarrow y N (x \setminus w) \Leftrightarrow x N (w / y)$ **BiFL**

 $(x \bigotimes y) N w \Leftrightarrow y N (x \hookrightarrow w) \Leftrightarrow x N (w \hookleftarrow y)$

$$\frac{x \bigotimes (y \bigotimes w) Nz}{(x \bigotimes y) \bigotimes w Nz} (\bigotimes a) \qquad \frac{x \bigotimes y Nz}{y \bigotimes x Nz} (\bigotimes e)$$

$$\frac{xNz}{x \bigotimes yNz} (\bigotimes i) \qquad \frac{x \bigotimes xNz}{xNz} (\bigotimes c)$$

Residuated frames

Frames and modules

Distributive frames

A distributive residuated frame is a structure $\mathbf{W} = (W, W', N, \circ, 1, \bigodot, \backslash\!\!\!/, \hookrightarrow, \hookleftarrow) \text{ where } W \text{ and } W' \text{ are sets } N \subseteq W \times W', (W, \circ, 1) \text{ is a monoid and for all } x, y \in W, w \in W'$

$$(x \circ y) \ N \ w \ \Leftrightarrow \ y \ N \ (x \setminus w) \ \Leftrightarrow \ x \ N \ (w \not| \! / y)$$

$$(x \bigotimes y) \ N \ w \ \Leftrightarrow \ y \ N \ (x \hookrightarrow w) \ \Leftrightarrow \ x \ N \ (w \hookleftarrow y)$$

$$\frac{x \bigotimes (y \bigotimes w) Nz}{(x \bigotimes y) \bigotimes w Nz} (\bigotimes a) \qquad \frac{x \bigotimes y Nz}{y \bigotimes x Nz} (\bigotimes e)$$

$$\frac{xNz}{\overline{x \otimes yNz}} (\bigotimes i) \qquad \frac{x \bigotimes xNz}{xNz} (\bigotimes c)$$

Theorem. If W is a distributive frame, then γ_N is a distributive nucleus on $\mathcal{P}(W)$.

Corollary. If W is a distributive residuated frame then the *dual* algebra W^+ is a distributive residuated lattice.

Residuated frames

Frames and modules

Frames and display

Distributive frames

Distributive frames

Involutive FL

Applications

DFL Nuclei

DGN

BiFL

DGN

Residuated frames

	Frames and module
	Frames and display
xNa aNz (CUT) (14)	Distributive frames
$\frac{1}{xNz}$ (COT) $\frac{1}{aNa}$ (Id)	DFL
	Nuclei
$x \bigotimes (y \bigotimes w) N z$ () $x \bigotimes y N z$ ()	Distributive frames
$\frac{\overline{(\otimes a)}}{\overline{(\otimes a)}} \xrightarrow{(\otimes a)} \overline{(\otimes a)} \xrightarrow{(\otimes a)} \overline{(\otimes e)}$	
$(x \bigotimes y) \bigotimes w N z$ $y \bigotimes x N z$	Involutive FL
$x N \sim x \wedge x N \sim$	BiFL
$= \frac{x \cdot x \cdot z}{(\bigtriangleup i)} \qquad \frac{x \cdot (\Im x \cdot x \cdot z)}{(\bigtriangleup c)}$	Applications
$x \bigotimes yNz$ ($\bigcirc \lor$) xNz ($\bigcirc \lor$)	
$\frac{x N a b N z}{(1)} \frac{a \circ x N b}{(1)} (R)$	
$x \circ (a \setminus b) Nz$ (\-) $x Na \setminus b$ (\.)	
$\frac{xNa bNz}{x (1)} \frac{x \circ aNb}{x (R)}$	
$(b/a) \circ xNz \xrightarrow{(/L)} xNb/a \xrightarrow{(/N)}$	
$a \circ bNz$ (1) $xNa yNb$ (D) εNz (1) (1D)	
$\frac{a + c + w}{a + b + N z}$ (·L) $\frac{c + w}{a + b}$ (·R) $\frac{c + w}{1 + w}$ (IL) $\frac{c + w}{c + w}$ (IR)	
$a \cdot b = x \circ y = a \cdot b = 1 = x \circ y = a \cdot b = 1$	
$a \bigotimes bNz$ $x Na x Nb$	
$\frac{d + Q + 2 + 2}{d + 2} (\wedge L\ell) = \frac{d + 2 + 2 + 2}{2} (\wedge R)$	
$a \wedge bNz$, $xNa \wedge b$, ,	
$aNz bNz (\dots) xNa (\dots) xNb (\dots)$	
$\frac{dree}{dree} (VL) = \frac{dree}{dree} (VR\ell) = \frac{dree}{dree} (VR\ell) $	
$u \vee 0 I v Z \qquad \qquad x I v u \vee 0 \qquad \qquad x I v u \vee 0$	

Residuated frames

Frames and modules

Frames and display

Distributive frames

Involutive FL

Toward BiFL Relativizing to InFL InFL Involutive frames Quasiembedding

BiFL

Applications

Involutive FL

Residuated frames

Frames and modules

Frames and display

Distributive frames

Involutive FL

Toward BiFL

Relativizing to InFL

InFL

Involutive frames

Quasiembedding

BiFL

Applications

 $z \le x + y \Leftrightarrow z \dashv y \le x \Leftrightarrow x \dashv z \le y$

We want to add another residuated pair:

Residuated frames

Frames and modules

Frames and display

Distributive frames

Involutive FL

Toward BiFL

Relativizing to InFL InFL

Involutive frames

Quasiembedding

BiFL

Applications

We want to add another residuated pair:

 $z \leq x + y \Leftrightarrow z \vdash y \leq x \Leftrightarrow x \vdash z \leq y$

This can/is done easily if the residuated lattice **A** is *involutive*: for some $0 \in A$, $\sim -x = x = -\backslash x$, where $\sim x = x - 0$ and -x = 0/x.

We want to add another residuated pair:

 $z \leq x + y \Leftrightarrow z \vdash y \leq x \Leftrightarrow x \vdash z \leq y$

This can/is done easily if the residuated lattice A is *involutive*: for some $0 \in A$, $\sim -x = x = -\backslash x$, where $\sim x = x - 0$ and -x = 0/x.

Then we can define $x + y := \sim (-y \cdot -x) = -(\sim y \cdot \sim x)$.

Residuated framesFrames and modulesFrames and displayDistributive framesInvolutive FLToward BiFLRelativizing to InFLInFLInvolutive framesQuasiembeddingBiFL

We want to add another residuated pair:

 $z \leq x + y \Leftrightarrow z \vdash y \leq x \Leftrightarrow x \vdash z \leq y$

This can/is done easily if the residuated lattice A is *involutive*: for some $0 \in A$, $\sim -x = x = -\backslash x$, where $\sim x = x - 0$ and -x = 0/x.

Then we can define $x + y := \sim (-y \cdot -x) = -(\sim y \cdot \sim x)$.

Also, we get: $x \setminus y = (\sim x) + y$ and y/x = y + (-x),

Frames and modulesFrames and displayDistributive framesInvolutive FLToward BiFLRelativizing to InFLInFLInvolutive framesQuasiembeddingBiFL

Residuated frames

We want to add another residuated pair:

 $z \leq x + y \Leftrightarrow z \vdash y \leq x \Leftrightarrow x \vdash z \leq y$

This can/is done easily if the residuated lattice **A** is *involutive*: for some $0 \in A$, $\sim -x = x = -\backslash x$, where $\sim x = x - 0$ and -x = 0/x.

Then we can define $x + y := \sim (-y \cdot -x) = -(\sim y \cdot \sim x)$.

Also, we get: $x \setminus y = (\sim x) + y$ and y/x = y + (-x),

as well as: $x - y = (\sim x) \cdot y$ and $y - x = y \cdot (-x)$.

Residuated frames

Frames and modules

Frames and display

Distributive frames

Involutive FL

Toward BiFL

Relativizing to InFL InFL Involutive frames

Quasiembedding

BiFL

Recall	that	0	is	of ty	be N_n	hence	$\sim x,$	-x:	P_n	$\rightarrow N_n$.	
--------	------	---	----	-------	----------	-------	-----------	-----	-------	---------------------	--

Recall that 0 is of type N_n , hence $\sim x, -x: P_n \to N_n$.

If we add a new type to negations $\sim x, -x : N_n \to P_n$, then we arrive at a new notion of sequent (multiple conclusion). The operations at the frame level corresponding to the negations are denoted by $\{\}^{\sim}$ and $\{\}^{-}$.

$$\frac{x \circ y \Rightarrow z}{\overline{y \Rightarrow x^{\sim} \circ z}} (\sim) \qquad \frac{x \circ y \Rightarrow z}{\overline{x \Rightarrow z \circ y^{-}}} (\sim)$$

Recall that 0 is of type N_n , hence $\sim x, -x: P_n \to N_n$.

If we add a new type to negations $\sim x, -x : N_n \to P_n$, then we arrive at a new notion of sequent (multiple conclusion). The operations at the frame level corresponding to the negations are denoted by $\{\}^{\sim}$ and $\{\}^{-}$.

$$\frac{x \circ y \Rightarrow z}{y \Rightarrow x^{\sim} \circ z} (^{\sim}) \qquad \frac{x \circ y \Rightarrow z}{x \Rightarrow z \circ y^{-}} (^{-})$$

If $a \in Fm$, we define $a^{\sim 0} = a$ and $a^{\sim (n+1)} = (a^{\sim n})^{\sim}$. A negated formula is of the form $a^{\sim n}$ or a^{-n} . We set $a^{\sim -} = a = a^{-\sim}$. We denote by Fm^i the free monoid over the set of negated formulas. A sequent is of the form $x \Rightarrow y$ for $x, y \in Fm^i$.

Residuated frames
Frames and modules
Frames and display
Distributive frames
Involutive FL
Toward BiFL
Relativizing to InFL
InFL
Involutive frames
Quasiembedding
BiFL

Recall that 0 is of type N_n , hence $\sim x, -x: P_n \to N_n$.

If we add a new type to negations $\sim x, -x : N_n \to P_n$, then we arrive at a new notion of sequent (multiple conclusion). The operations at the frame level corresponding to the negations are denoted by $\{\}^{\sim}$ and $\{\}^{-}$.

$$\frac{x \circ y \Rightarrow z}{y \Rightarrow x^{\sim} \circ z} (^{\sim}) \qquad \frac{x \circ y \Rightarrow z}{x \Rightarrow z \circ y^{-}} (^{-})$$

If $a \in Fm$, we define $a^{\sim 0} = a$ and $a^{\sim (n+1)} = (a^{\sim n})^{\sim}$. A negated formula is of the form $a^{\sim n}$ or a^{-n} . We set $a^{\sim -} = a = a^{-\sim}$. We denote by Fm^i the free monoid over the set of negated formulas. A sequent is of the form $x \Rightarrow y$ for $x, y \in Fm^i$.

For $x = a_1, \ldots, a_n$, we define

$$x^{\sim} = a_n^{\sim}, \ldots, a_1^{\sim}$$
 and $x^- = a_n^-, \ldots, a_1^-$.

 $x^{\sim -} = x = x^{-\sim}$, $(x \circ y)^{\sim} = y^{\sim} \circ x^{\sim}$, $(x \circ y)^{-} = y^{-} \circ x^{-}$.

InFL

Residuated frames

Frames and modules

Frames and display

Distributive frames

Involutive FL

Toward BiFL

Relativizing to InFL

InFL

Involutive frames Quasiembedding

BiFL

Applications

$$\frac{x \Rightarrow a \quad b \Rightarrow z}{x \circ (a \setminus b) \Rightarrow z} (\setminus L) \qquad \frac{a \circ x \Rightarrow b}{x \Rightarrow a \setminus b} (\setminus R)$$

$$\frac{x \Rightarrow a \quad b \Rightarrow z}{(b / a) \circ x \Rightarrow z} (/L) \qquad \frac{x \circ a \Rightarrow b}{x \Rightarrow b / a} (/R)$$

$$\frac{a \circ b \Rightarrow z}{a \cdot b \Rightarrow z} (\cdot L) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x \circ y \Rightarrow a \cdot b} (\cdot R) \qquad \frac{z \Rightarrow z}{1 \Rightarrow z} (1L) \qquad \frac{z \Rightarrow 1}{z \Rightarrow 1} (1R)$$

$$\frac{a \Rightarrow z}{a \wedge b \Rightarrow z} (\wedge L\ell) \qquad \frac{b \Rightarrow z}{a \wedge b \Rightarrow z} (\wedge Lr) \qquad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a \wedge b} (\wedge R)$$

$$\frac{a \Rightarrow z \quad b \Rightarrow z}{a \vee b \Rightarrow z} (\vee L) \qquad \frac{x \Rightarrow a}{x \Rightarrow a \vee b} (\vee R\ell) \qquad \frac{x \Rightarrow b}{x \Rightarrow a \vee b} (\vee Rr)$$

$$\frac{a^{\ln} \Rightarrow z}{-a \Rightarrow z} (-L) \qquad \frac{x \Rightarrow a^{-}}{x \Rightarrow -a} (-R)$$

$$\frac{x \circ y \Rightarrow z}{y \Rightarrow x^{-} \circ z} (\sim) \qquad \frac{x \circ y \Rightarrow z}{x \Rightarrow z \circ y^{-}} (-)$$

 $\frac{x \Rightarrow a \quad a \Rightarrow z}{x \Rightarrow z}$ (CUT) $\qquad \frac{a \Rightarrow a}{a \Rightarrow a}$ (Id)

Residuated frames
 Frames and modules
Frames and display
Distributive fremes
Distributive frames
Involutive FL
Toward BiFL
Relativizing to InFL
InFL
Involutive frames
Quasiembedding
BiFL

Applications



$$\blacksquare (W, \circ, \varepsilon) \text{ is a monoid}$$

$$x^{\sim -} = x = x^{-\sim}$$

Residuated frames
Frames and modules
Frames and display
Distributive frames
Involutive FL
Toward BiFL
Relativizing to InFL
InFL
Involutive frames
Quasiembedding
BiFL
Applications

An *involutive (residuated) frame* is a structure of the form $\mathbf{W} = (W, N, \circ, \varepsilon, \sim, -)$, where

$$\blacksquare \quad (W, \circ, \varepsilon) \text{ is a monoid}$$

$$x^{\sim -} = x = x^{-\sim}$$

$$(y^{\sim} \circ x^{\sim})^{-} = (y^{-} \circ x^{-})^{\sim} [= x \oplus y]$$

 $\blacksquare \quad x \circ y \ N \ z \text{ iff } y \ N \ x^{\sim} \oplus z \text{ iff } x \ N \ z \oplus y^{-}, \text{ for all } x, y, z \in W$

If L is an involutive FL-algebra, then $\mathbf{W}_{\mathbf{L}} = (L, \leq, \cdot, 1, \sim, -)$ is an involutive frame.

	Frai
) <i>frame</i> is a structure of the form	Dist
where	Invo
	Том
	Rela
	InF

An involutive (residuated $\mathbf{W} = (W, N, \circ, \varepsilon, \sim, -)$, v

$$\blacksquare \quad (W, \circ, \varepsilon) \text{ is a monoid}$$

$$x^{\sim -} = x = x^{-\sim}$$

$$(y^{\sim} \circ x^{\sim})^{-} = (y^{-} \circ x^{-})^{\sim} [= x \oplus y]$$

If L is an involutive FL-algebra, then $\mathbf{W}_{\mathbf{L}} = (L, \leq, \cdot, 1, \sim, -)$ is an involutive frame.

On the dual algebra we define $-Y := Y^{\triangleright -} = Y^{-\triangleleft}$ and $\sim Y = Y^{\triangleright} \sim = Y^{\sim} \triangleleft$ for $Y \subseteq W$.

An *involutive (residuated) frame* is a structure of the form $\mathbf{W} = (W, N, \circ, \varepsilon, \sim, -)$, where

$$\blacksquare \quad (W, \circ, \varepsilon) \text{ is a monoid}$$

$$x^{\sim -} = x = x^{-\sim}$$

$$(y^{\sim} \circ x^{\sim})^{-} = (y^{-} \circ x^{-})^{\sim} [= x \oplus y]$$

•
$$x \circ y \ N \ z \text{ iff } y \ N \ x^{\sim} \oplus z \text{ iff } x \ N \ z \oplus y^{-}$$
, for all $x, y, z \in W$

If L is an involutive FL-algebra, then $\mathbf{W}_{\mathbf{L}} = (L, \leq, \cdot, 1, \sim, -)$ is an involutive frame.

On the dual algebra we define $-Y := Y^{\triangleright} - = Y^{\neg}$ and $\sim Y = Y^{\triangleright} - = Y^{\sim}$ for $Y \subseteq W$. Then we have

$$X \circ Y \subseteq Z \iff Y \subseteq \sim (-Z \circ X) \iff X \subseteq -(Y \circ \sim Z)$$
Quasiembedding

Using the following rules of InFL we can prove the main theorem.



Frames and modulesFrames and displayDistributive framesInvolutive FLToward BiFLRelativizing to InFLInFLInvolutive framesQuasiembeddingBiFLApplications

Residuated frames

Theorem. For all $a \in B$, in an involutive Genzen frame $\sim \{a\}^{\triangleleft} = \{\sim a\}^{\triangleleft}$ and $-\{a\}^{\triangleleft} = \{-a\}^{\triangleleft}$.

Residuated frames

Frames and modules

Frames and display

Distributive frames

Involutive FL

BiFL

No involution The frame (sets) The frame (ops) Adding structural rules

Applications

BiFL

We wonder if expansions of FL-algebras with a dual operator +,

 $x + (y \wedge z) = (x + y) \wedge (x + z)$, can be conservatively extended with

Residuated frames

Frames and modules

Frames and display

Distributive frames

Involutive FL

BiFL No involution

The frame (sets) The frame (ops) Adding structural rules

Applications

dual implication.

We wonder if expansions of FL-algebras with a dual operator +, $x + (y \wedge z) = (x + y) \wedge (x + z)$, can be conservatively extended with dual implication.

In other words is there any interference between a residuated and a dually residuated pair/triple?

Frames and modulesFrames and displayDistributive framesInvolutive FLBiFLNo involutionThe frame (sets)The frame (ops)Adding structural rules

Residuated frames

We wonder if expansions of FL-algebras with a dual operator +, $x + (y \wedge z) = (x + y) \wedge (x + z)$, can be conservatively extended with dual implication.

In other words is there any interference between a residuated and a dually residuated pair/triple?

Note that there was no interference when the two pairs were both residuated.

Frames and modulesFrames and displayDistributive framesInvolutive FLBiFLNo involutionThe frame (sets)The frame (ops)Adding structural rules

Residuated frames

We wonder if expansions of FL-algebras with a dual operator +, $x + (y \wedge z) = (x + y) \wedge (x + z)$, can be conservatively extended with dual implication.

In other words is there any interference between a residuated and a dually residuated pair/triple?

Note that there was no interference when the two pairs were both residuated.

Can an FL⁺ algebra be embedded into a bi-FL algebra?

Frames and modules		
Frames and display		
Distributive frames		
Involutive FL		
BiFL		
No involution		
The frame (sets)		
The frame (ops)		
Adding structural rules		

Residuated frames

We wonder if expansions of FL-algebras with a dual operator +, $x + (y \wedge z) = (x + y) \wedge (x + z)$, can be conservatively extended with dual implication.

In other words is there any interference between a residuated and a dually residuated pair/triple?

Note that there was no interference when the two pairs were both residuated.

Can an FL⁺ algebra be embedded into a bi-FL algebra?

Is there a cut-free calculus for FL^+ .

Frames and modules		
Frames and display		
Distributive frames		
Involutive FL		
BiFL		
No involution		
The frame (sets)		
The frame (ops)		

Residuated frames

Residuated frames

Frames and modules

Frames and display

Distributive frames

Involutive FL

BiFL No involution

The frame (sets) The frame (ops) Adding structural rules

Applications

We wonder if expansions of FL-algebras with a dual operator +, $x + (y \wedge z) = (x + y) \wedge (x + z)$, can be conservatively extended with dual implication.

In other words is there any interference between a residuated and a dually residuated pair/triple?

Note that there was no interference when the two pairs were both residuated.

Can an FL⁺ algebra be embedded into a bi-FL algebra?

Is there a cut-free calculus for FL^+ .

More difficult than one residuated pair, as now there are non-innocent/bad sequents.

Given a (commutative) FL^+ -algebra A we will define a residuated

Residuated frames

Frames and modules

Frames and display

Distributive frames

Involutive FL

BiFL

No involution

The frame (sets)

The frame (ops)

Adding structural rules

Applications

frame $(W, W', N, \circ, //, \oplus, \mathbb{N})$.

Residuated frames

Frames and modules

Frames and display

Distributive frames

Involutive FL

BiFL

No involution

The frame (sets)

The frame (ops)

Adding structural rules

Applications

We define the set W by the following grammar:

frame $(W, W', N, \circ, //, \oplus, \mathbb{N})$.

Given a (commutative) FL^+ -algebra A we will define a residuated

 $W := W, A \mid W < A \mid \varepsilon$

Residuated frames

Frames and modules

Frames and display

Distributive frames

Involutive FL

BiFL

No involution

The frame (sets)

The frame (ops)

Adding structural rules

Applications

Given a (commutative) FL^+ -algebra **A** we will define a residuated frame $(W, W', N, \circ, //, \oplus, \mathbb{N})$.

We define the set W by the following grammar:

 $W := W, A \mid W < A \mid \varepsilon$

Elements of W of the form w < a and ε are called *proper*. For convenience we extend the multiplication of A to $a \in A \cup \{\varepsilon\}$ by $a \cdot \varepsilon = \varepsilon \cdot a = a$ (and $\varepsilon \to a = a$). Also, p, ε is simply p.

Residuated frames

Frames and modules

Frames and display

Distributive frames

Involutive FL

BiFL

No involution

The frame (sets)

The frame (ops)

Adding structural rules

Applications

Given a (commutative) FL^+ -algebra **A** we will define a residuated frame $(W, W', N, \circ, //, \oplus, \mathbb{N})$.

We define the set W by the following grammar:

 $W := W, A \mid W < A \mid \varepsilon$

Elements of W of the form w < a and ε are called *proper*. For convenience we extend the multiplication of A to $a \in A \cup \{\varepsilon\}$ by $a \cdot \varepsilon = \varepsilon \cdot a = a$ (and $\varepsilon \to a = a$). Also, p, ε is simply p.

Then every element of W is of the form p, a, where p is proper and $a \in A \cup \{\varepsilon\}$.

Residuated frames

Frames and modules

Frames and display

Distributive frames

Involutive FL

BiFL

No involution

The frame (sets)

The frame (ops) Adding structural rules

Applications

Given a (commutative) FL^+ -algebra **A** we will define a residuated frame $(W, W', N, \circ, //, \oplus, \mathbb{N})$.

We define the set W by the following grammar:

 $W := W, A \mid W < A \mid \varepsilon$

Elements of W of the form w < a and ε are called *proper*. For convenience we extend the multiplication of A to $a \in A \cup \{\varepsilon\}$ by $a \cdot \varepsilon = \varepsilon \cdot a = a$ (and $\varepsilon \to a = a$). Also, p, ε is simply p.

Then every element of W is of the form p, a, where p is proper and $a \in A \cup \{\varepsilon\}$.

We define the set W' to be given by the grammar W' := P > A, where P is the set of proper elements of W. We write a for $\varepsilon > a$. We define the (hyper)operation \circ on proper elements by $p \circ \varepsilon = \varepsilon \circ p = p$ and $(w < a) \circ (w' < a') = \emptyset$. Then we 'extend' it to arbitrary elements by $(p, a) \circ (p', a') = (p \circ p'), (a \cdot a')$.

Residuated framesFrames and modulesFrames and displayDistributive framesInvolutive FLBiFLNo involutionThe frame (sets)The frame (ops)Adding structural rules

Residuated frames

Frames and modules

Frames and display

Distributive frames

Involutive FL

BiFL

No involution

The frame (sets)

The frame (ops)

Adding structural rules

Applications

We define the (hyper)operation \circ on proper elements by $p \circ \varepsilon = \varepsilon \circ p = p$ and $(w < a) \circ (w' < a') = \emptyset$. Then we 'extend' it to arbitrary elements by $(p, a) \circ (p', a') = (p \circ p'), (a \cdot a')$.

We define \oplus on W' by $(p > a) \oplus (p' > a') = a + a'$ if $p = p' = \varepsilon$; and \emptyset otherwise.

We define the (hyper)operation \circ on proper elements by $p \circ \varepsilon = \varepsilon \circ p = p$ and $(w < a) \circ (w' < a') = \emptyset$. Then we 'extend' it to arbitrary elements by $(p, a) \circ (p', a') = (p \circ p'), (a \cdot a')$.

We define \oplus on W' by $(p > a) \oplus (p' > a') = a + a'$ if $p = p' = \varepsilon$; and \emptyset otherwise.

Also, we define $(p,a) \setminus (p' > a) = (p \circ p') > (a \to a')$,

Residuated framesFrames and modulesFrames and displayDistributive framesInvolutive FLBiFLNo involutionThe frame (sets)The frame (ops)Adding structural rulesApplications

We define the (hyper)operation \circ on proper elements by $p \circ \varepsilon = \varepsilon \circ p = p$ and $(w < a) \circ (w' < a') = \emptyset$. Then we 'extend' it to arbitrary elements by $(p, a) \circ (p', a') = (p \circ p'), (a \cdot a')$.

We define \oplus on W' by $(p > a) \oplus (p' > a') = a + a'$ if $p = p' = \varepsilon$; and \emptyset otherwise.

Also, we define $(p, a) \setminus (p' > a) = (p \circ p') > (a \to a')$,

and (p, a) / (p' > a') = (p, a) < a', if $p' = \varepsilon$; and \emptyset otherwise.



We define the (hyper)operation \circ on proper elements by $p \circ \varepsilon = \varepsilon \circ p = p$ and $(w < a) \circ (w' < a') = \emptyset$. Then we 'extend' it to arbitrary elements by $(p, a) \circ (p', a') = (p \circ p'), (a \cdot a')$.

We define \oplus on W' by $(p > a) \oplus (p' > a') = a + a'$ if $p = p' = \varepsilon$; and \emptyset otherwise.

Also, we define $(p, a) \setminus (p' > a) = (p \circ p') > (a \rightarrow a')$,

and (p, a) / (p' > a') = (p, a) < a', if $p' = \varepsilon$; and \emptyset otherwise.

Finally, for $x \in W$ and $a \in A$ we define $x^+[a]$ as follows by induction on the structure of x.

 $\begin{aligned} & (\varepsilon)^{+}[a] := a, \\ & (x,b)^{+}[a] := x^{+}[b \to a], \\ & (x < b)^{+}[a] := x^{+}[b + a]. \end{aligned}$

We define the (hyper)operation \circ on proper elements by $p \circ \varepsilon = \varepsilon \circ p = p$ and $(w < a) \circ (w' < a') = \emptyset$. Then we 'extend' it to arbitrary elements by $(p, a) \circ (p', a') = (p \circ p'), (a \cdot a')$.

We define \oplus on W' by $(p > a) \oplus (p' > a') = a + a'$ if $p = p' = \varepsilon$; and \emptyset otherwise.

Also, we define $(p, a) \setminus (p' > a) = (p \circ p') > (a \to a')$,

and (p, a) / (p' > a') = (p, a) < a', if $p' = \varepsilon$; and \emptyset otherwise.

Finally, for $x \in W$ and $a \in A$ we define $x^+[a]$ as follows by induction on the structure of x.

 $\begin{aligned} & (\varepsilon)^{+}[a] := a, \\ & (x,b)^{+}[a] := x^{+}[b \to a], \\ & (x < b)^{+}[a] := x^{+}[b + a]. \end{aligned}$

We also define the set of designated elements of W' to be $D = \{x > a : 1 \le x^+[a]\}$ and (p, a) N (p' > a) iff $1 \le (p \circ p')^+[a \to a'].$

We define the (hyper)operation \circ on proper elements by $p \circ \varepsilon = \varepsilon \circ p = p$ and $(w < a) \circ (w' < a') = \emptyset$. Then we 'extend' it to arbitrary elements by $(p, a) \circ (p', a') = (p \circ p'), (a \cdot a')$.

We define \oplus on W' by $(p > a) \oplus (p' > a') = a + a'$ if $p = p' = \varepsilon$; and \emptyset otherwise.

Also, we define $(p, a) \setminus (p' > a) = (p \circ p') > (a \to a')$,

and (p, a) / (p' > a') = (p, a) < a', if $p' = \varepsilon$; and \emptyset otherwise.

Finally, for $x \in W$ and $a \in A$ we define $x^+[a]$ as follows by induction on the structure of x.

 $\begin{aligned} & (\varepsilon)^{+}[a] := a, \\ & (x,b)^{+}[a] := x^{+}[b \to a], \\ & (x < b)^{+}[a] := x^{+}[b + a]. \end{aligned}$

We also define the set of designated elements of W' to be $D = \{x > a : 1 \le x^+[a]\}$ and (p, a) N (p' > a) iff $1 \le (p \circ p')^+[a \to a'].$

Theorem Every FL⁺-algebra can be embedded into a BiFL-algebra.

Adding structural rules

Residuated frames

Frames and modules

Frames and display

Distributive frames

Involutive FL

BiFL

No involution

The frame (sets)

The frame (ops)

Adding structural rules

Applications

Question: If A satisfies mingle $x \le x \cdot x$, $x + x \le x$ then does $\mathbf{W}_{\mathbf{A}}^+$ also satisfy it?

Nick Galatos, Prague workshop, March, 2014

Adding structural rules

Question: If A satisfies mingle $x \le x \cdot x$, $x + x \le x$ then does $\mathbf{W}_{\mathbf{A}}^+$ also satisfy it?

$$\frac{p, aNz \quad p', a'Nz}{(p, a) \circ (p', a')Nz} \qquad \frac{xNp > a \quad xNp' > a'}{xN(p > a) \oplus (p' > a')}$$

Residuated framesFrames and modulesFrames and displayDistributive framesDistributive framesInvolutive FLBiFLNo involutionThe frame (sets)The frame (ops)Adding structural rulesApplications

Question: If A satisfies mingle $x \le x \cdot x$, $x + x \le x$ then does $\mathbf{W}_{\mathbf{A}}^+$ also satisfy it?

$$\frac{p, aNz \quad p', a'Nz}{(p, a) \circ (p', a')Nz} \qquad \frac{xNp > a \quad xNp' > a'}{xN(p > a) \oplus (p' > a')}$$

Solution: Modify the frame $\mathbf{W}_{\mathbf{A}}$. The above conditions holds iff

$$\frac{p, aNz \quad a'Nz}{(p,a) \circ a'Nz} \qquad \frac{xNa \quad xNa'}{xNa \oplus a'}$$

	Residuated frames
_	Frames and modules
	Frames and display
	Distributive frames
	DIFI
	No involution
	The frame (sets)
	The frame (ops)
	Adding structural rules
	Applications

Question: If A satisfies mingle $x \le x \cdot x$, $x + x \le x$ then does $\mathbf{W}_{\mathbf{A}}^+$ also satisfy it?

$$\frac{p, aNz \quad p', a'Nz}{(p, a) \circ (p', a')Nz} \qquad \frac{xNp > a \quad xNp' > a'}{xN(p > a) \oplus (p' > a')}$$

Solution: Modify the frame $\mathbf{W}_{\mathbf{A}}$. The above conditions holds iff

$$\frac{p, aNz \quad a'Nz}{(p,a) \circ a'Nz} \qquad \frac{xNa \quad xNa'}{xNa \oplus a'}$$

 $\begin{array}{l} \text{Grishin(b):} \ x(y+z) \leq xy+z \ \text{gives a stabilizing definition:} \\ (w < a) \circ (w' < a') = \{((w < a) \circ w') < a', ((w' < a') \circ w) < a\}. \end{array}$

Residuated frames

Frames and modules

Frames and display

Distributive frames

Involutive FL

BiFL

Applications

Frame applications Examples of frames: FEP Simple equations Simple rules Reduction to simple Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

Frame applications

Residuated frames Frames and modules Frames and display **Distributive frames** Involutive FL **BiFL** Applications Frame applications Examples of frames: **FEP** Simple equations Simple rules Reduction to simple Simplicity preserved **FMP FEP** Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

DM-completion

- Perfect residuated lattices
- Completeness of the calculus
- Cut elimination
- Finite model property
- Finite embeddability property
- (Generalized super-)amalgamation property (Transferable injections, Congruence extension property)
- (Craig) Interpolation property
- Disjunction property
- Strong separation
- Stability under linear structural rules/equations over $\{\lor, \cdot, 1\}$.
- Densification
- Conservativity (via algebraic embeddings)

Let \mathbf{A} be a residuated lattice and \mathbf{B} a partial subalgebra of \mathbf{A} .

We define the frame $\mathbf{W}_{\mathbf{A},\mathbf{B}}$, where

Residuated frames

Let \mathbf{A} be a residuated lattice and \mathbf{B} a partial subalgebra of \mathbf{A} .

 $(W, \cdot, 1)$ to be the submonoid of A generated by B,

We define the frame $W_{A,B}$, where

Residuated frames Frames and modules Frames and display Distributive frames Involutive FL BiFL Applications Frame applications Examples of frames: FEP Simple equations Simple rules Reduction to simple Simplicity preserved **FMP FEP** Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

Let \mathbf{A} be a residuated lattice and \mathbf{B} a partial subalgebra of \mathbf{A} .

 $(W, \cdot, 1)$ to be the submonoid of A generated by B,

 $W' = S_B \times B$, where S_W is the set of all *unary linear*

We define the frame $W_{A,B}$, where

polynomials $u[x] = y \circ x \circ z$ of $(W, \cdot, 1)$, and

Residuated frames Frames and modules Frames and display **Distributive frames** Involutive FL BiFL Applications Frame applications Examples of frames: FEP Simple equations Simple rules Reduction to simple Simplicity preserved **FMP FEP** Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

Let \mathbf{A} be a residuated lattice and \mathbf{B} a partial subalgebra of \mathbf{A} .

 $(W, \cdot, 1)$ to be the submonoid of A generated by B,

 $W' = S_B \times B$, where S_W is the set of all *unary linear*

We define the frame $W_{A,B}$, where

x N(u, b) by $u[x] \leq_{\mathbf{A}} b$.

polynomials $u[x] = y \circ x \circ z$ of $(W, \cdot, 1)$, and

Residuated frames Frames and modules Frames and display **Distributive frames** Involutive FL **BiFL** Applications Frame applications Examples of frames: FEP Simple equations Simple rules Reduction to simple Simplicity preserved **FMP FEP** Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

Let A be a residuated lattice and B a partial subalgebra of A.

We define the frame $\mathbf{W}_{\mathbf{A},\mathbf{B}}$, where

 $\begin{array}{ll} & (W,\cdot,1) \text{ to be the submonoid of } \mathbf{A} \text{ generated by } B, \\ & W' = S_B \times B, \text{ where } S_W \text{ is the set of all unary linear } \\ & polynomials \ u[x] = y \circ x \circ z \text{ of } (W,\cdot,1), \text{ and} \\ & \quad x \ N \ (u,b) \text{ by } u[x] \leq_{\mathbf{A}} b. \end{array}$

For

$$(u,a) /\!\!/ x = \{(u[_\cdot x],a)\} \text{ and } x \setminus\!\!\! \setminus (u,a) = \{(u[x \cdot _],a)\},$$

we have

$$\begin{aligned} x \cdot y N(u, a) & \quad \text{iff } u[x \cdot y] \leq a \\ & \quad \text{iff } x N(u[_ \cdot y], a) \\ & \quad \text{iff } y N(u[x \cdot _], a) \end{aligned}$$

Residuated frames
Frames and modules
Frames and display
Distributive frames
Involutive FL
BiFl
Applications
Frame applications
Examples of frames:
FEP Simple equations
Simple equations
Reduction to simple
Simplicity preserved
EMD
FFP
Maehara frame
Equations
Gen, amalgamation
Interpolation
Disjunction property
Strong separation: svst.
Strong separation
Equations for DFL
Structural rules
FEP for DFL

An equation is called *simple* if it is of the form $t_0 \leq t_1 \vee \cdots \vee t_n$,

where t_i are $\{\cdot, 1\}$ -terms and t_0 is linear.

13	Residuated frames
	Frames and modules
	Frames and display
	Distributive frames
	Involutive FL
	BiFL
	Applications
	Frame applications Examples of frames: FEP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara trame
	Equations
	Gen. amaigamation
	Disjunction property
	Strong separation: syst.
	Strong separation
	Equations for DFL
	Structural rules
	FEP for DFL

Residuated frames
 Frames and modules
Frames and display
Distributive frames
Involutive FL
BiFL
Applications
Frame applications
Examples of frames: FEP
Simple equations
Simple rules
Reduction to simple
Simplicity preserved
FMP
FEP
Amalgamation
Maehara frame
Equations
Gen. amalgamation
Interpolation
Disjunction property
Strong separation: syst.
Strong separation
Equations for DFL
Structural rules
FEP for DFL

An equation is called *simple* if it is of the form $t_0 \le t_1 \lor \cdots \lor t_n$, where t_i are $\{\cdot, 1\}$ -terms and t_0 is linear.

Lemma. Every equation over $\{\lor, \cdot, 1\}$ is equivalent to a conjunction of simple equations.

Residuated frames Frames and modules Frames and display Distributive frames Involutive FL BiFL Applications Frame applications Examples of frames: **FEP** Simple equations Simple rules Reduction to simple Simplicity preserved **FMP FEP** Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

An equation is called *simple* if it is of the form $t_0 \leq t_1 \vee \cdots \vee t_n$, where t_i are $\{\cdot, 1\}$ -terms and t_0 is linear.

Lemma. Every equation over $\{\lor, \cdot, 1\}$ is equivalent to a conjunction of simple equations.

Proof For an equation ε over $\{\vee, \cdot, 1\}$ we distribute products over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j : monoid terms.

5	Residuated frames
	Frames and modules
	Frames and display
	Distributive frames
	Involutive FL
n	BiFL
[]	Applications
	Frame applications
	Examples of frames: FEP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
	Equations
	Gen. amalgamation
	Interpolation
	Disjunction property
	Strong separation: syst.
	Strong separation
	Equations for DFL
	Structural rules
	FEP for DFL

An equation is called *simple* if it is of the form $t_0 \le t_1 \lor \cdots \lor t_n$, where t_i are $\{\cdot, 1\}$ -terms and t_0 is linear.

Lemma. Every equation over $\{\lor, \cdot, 1\}$ is equivalent to a conjunction of simple equations.

Proof For an equation ε over $\{\vee, \cdot, 1\}$ we distribute products over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j : monoid terms.

 $s_1 \lor \cdots \lor s_m \leq t_1 \lor \cdots \lor t_n$ and $t_1 \lor \cdots \lor t_n \leq s_1 \lor \cdots \lor s_m$.

5	Residuated frames
	Frames and modules
	Frames and display
	Distributive frames
	Involutive FL
-	BiFL
n	Applications
	Frame applications
	Examples of frames: FEP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
	Equations
	Gen. amalgamation
	Interpolation
	Disjunction property
	Strong separation: syst.
	Strong separation
	Equations for DFL
	Structural rules
	FEP for DFL

An equation is called *simple* if it is of the form $t_0 \le t_1 \lor \cdots \lor t_n$, where t_i are $\{\cdot, 1\}$ -terms and t_0 is linear.

Lemma. Every equation over $\{\lor, \cdot, 1\}$ is equivalent to a conjunction of simple equations.

Proof For an equation ε over $\{\vee, \cdot, 1\}$ we distribute products over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j : monoid terms.

 $s_1 \lor \cdots \lor s_m \leq t_1 \lor \cdots \lor t_n$ and $t_1 \lor \cdots \lor t_n \leq s_1 \lor \cdots \lor s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \lor \cdots \lor t_n).$
5	Residuated frames
	Frames and modules
	Frames and display
	Distributive frames
	Involutive FL
	BiFL
n	Applications
	Frame applications Examples of frames:
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
	Equations
	Gen. amalgamation
	Interpolation
	Disjunction property
	Strong separation: syst.
	Strong separation
	Equations for DFL
	Structural rules
	FEP for DFL

An equation is called *simple* if it is of the form $t_0 \le t_1 \lor \cdots \lor t_n$, where t_i are $\{\cdot, 1\}$ -terms and t_0 is linear.

Lemma. Every equation over $\{\vee, \cdot, 1\}$ is equivalent to a conjunction of simple equations.

Proof For an equation ε over $\{\vee, \cdot, 1\}$ we distribute products over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j : monoid terms.

 $s_1 \lor \cdots \lor s_m \leq t_1 \lor \cdots \lor t_n$ and $t_1 \lor \cdots \lor t_n \leq s_1 \lor \cdots \lor s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \lor \cdots \lor t_n).$

We proceed by example: $x^2y \le xy \lor yx$

6	Residuated frames
	Frames and modules
	Frames and display
	Distributive frames
	Involutive FL
	BiFL
1	Applications
	Frame applications
	Examples of frames: FEP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
	Equations
	Gen. amalgamation
	Interpolation
	Disjunction property
	Strong separation: syst.
	Strong separation
	Equations for DFL
	Structural rules
	FEP for DFL

An equation is called *simple* if it is of the form $t_0 \le t_1 \lor \cdots \lor t_n$, where t_i are $\{\cdot, 1\}$ -terms and t_0 is linear.

Lemma. Every equation over $\{\lor, \cdot, 1\}$ is equivalent to a conjunction of simple equations.

Proof For an equation ε over $\{\vee, \cdot, 1\}$ we distribute products over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j : monoid terms.

 $s_1 \lor \cdots \lor s_m \leq t_1 \lor \cdots \lor t_n$ and $t_1 \lor \cdots \lor t_n \leq s_1 \lor \cdots \lor s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \lor \cdots \lor t_n).$

We proceed by example: $x^2y \leq xy \lor yx$

 $(x_1 \lor x_2)^2 y \le (x_1 \lor x_2) y \lor y(x_1 \lor x_2)$

5	Residuated frames
	Frames and modules
	Frames and display
	Distributive frames
	Involutive FL
	BiFL
1	Applications
	Frame applications
	Examples of frames: FEP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
	Equations
	Gen. amalgamation
	Interpolation
	Disjunction property
	Strong separation: syst.
	Strong separation
	Equations for DFL
	Structural rules
	FEP for DFL

An equation is called *simple* if it is of the form $t_0 \le t_1 \lor \cdots \lor t_n$, where t_i are $\{\cdot, 1\}$ -terms and t_0 is linear.

Lemma. Every equation over $\{\vee, \cdot, 1\}$ is equivalent to a conjunction of simple equations.

Proof For an equation ε over $\{\vee, \cdot, 1\}$ we distribute products over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j : monoid terms.

 $s_1 \lor \cdots \lor s_m \le t_1 \lor \cdots \lor t_n$ and $t_1 \lor \cdots \lor t_n \le s_1 \lor \cdots \lor s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \lor \cdots \lor t_n).$

We proceed by example: $x^2y \le xy \lor yx$

 $(x_1 \lor x_2)^2 y \le (x_1 \lor x_2) y \lor y(x_1 \lor x_2)$ $x_1^2 y \lor x_1 x_2 y \lor x_2 x_1 y \lor x_2^2 y \le x_1 y \lor x_2 y \lor y x_1 \lor y x_2$

5	Residuated frames
	Frames and modules
	Frames and display
	Distributive frames
	Involutive FL
_	BiFL
า	Applications
	Frame applications
	Examples of frames: FEP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
	Equations
	Gen. amalgamation
	Interpolation
	Disjunction property
	Strong separation: syst.
	Strong separation
	Equations for DFL
	Structural rules
	FEP for DFL

An equation is called *simple* if it is of the form $t_0 \le t_1 \lor \cdots \lor t_n$, where t_i are $\{\cdot, 1\}$ -terms and t_0 is linear.

Lemma. Every equation over $\{\lor, \cdot, 1\}$ is equivalent to a conjunction of simple equations.

Proof For an equation ε over $\{\vee, \cdot, 1\}$ we distribute products over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j : monoid terms.

 $s_1 \lor \cdots \lor s_m \le t_1 \lor \cdots \lor t_n$ and $t_1 \lor \cdots \lor t_n \le s_1 \lor \cdots \lor s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \lor \cdots \lor t_n).$

We proceed by example: $x^2y \leq xy \lor yx$ $(x_1 \lor x_2)^2 y \leq (x_1 \lor x_2)y \lor y(x_1 \lor x_2)$ $x_1^2 y \lor x_1 x_2 y \lor x_2 x_1 y \lor x_2^2 y \leq x_1 y \lor x_2 y \lor y x_1 \lor y x_2$ $x_1 x_2 y \leq x_1 y \lor x_2 y \lor y x_1 \lor y x_2$

	Residuated frames
	Frames and modules
	Frames and display
	Distributive frames
	Involutive FL
	BiFL
1	Applications
	Frame applications
	Examples of frames: FEP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
	Equations
	Gen. amalgamation
	Interpolation
	Disjunction property
	Strong separation: syst.
	Strong separation
	Equations for DFL
	Structural rules

An equation is called *simple* if it is of the form $t_0 \le t_1 \lor \cdots \lor t_n$, where t_i are $\{\cdot, 1\}$ -terms and t_0 is linear.

Lemma. Every equation over $\{\lor, \cdot, 1\}$ is equivalent to a conjunction of simple equations.

Proof For an equation ε over $\{\vee, \cdot, 1\}$ we distribute products over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j : monoid terms.

 $s_1 \vee \cdots \vee s_m \leq t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leq s_1 \vee \cdots \vee s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \lor \cdots \lor t_n).$

We proceed by example: $x^2y \leq xy \lor yx$

 $(x_1 \lor x_2)^2 y \le (x_1 \lor x_2) y \lor y(x_1 \lor x_2)$

 $x_1^2 y \lor x_1 x_2 y \lor x_2 x_1 y \lor x_2^2 y \le x_1 y \lor x_2 y \lor y x_1 \lor y x_2$

 $x_1x_2y \le x_1y \lor x_2y \lor yx_1 \lor yx_2$

More generally, if x appears n-times on the LHS, we substitute $x_1 \vee \ldots \vee x_n$ for x, distribute and retain one representative term on the LHS (where all the x_i 's occur).

Let t_0, t_1, \ldots, t_n be monoid terms and let t_0 be linear. A *simple* rule is an expression of the form

$$rac{t_1\;N\;q\;\cdots\;\;t_n\;N\;q}{t_0\;N\;q}$$
 (r

where q is a variable not occurring in t_0, t_1, \ldots, t_n .

Frames and modulesFrames and displayDistributive framesInvolutive FLBiFLApplicationsFrame applicationsExamples of frames:FEPSimple equationsSimple rulesReduction to simpleSimplicity preservedFMPFEPAmalgamationMaehara frameEquationsGen. amalgamationInterpolationDisjunction propertyStrong separation: syst.Strong separationEquations for DFLCircute and a law	Frames and modulesFrames and displayDistributive framesInvolutive FLBiFLApplicationsFrame applicationsExamples of frames: FEPSimple equationsSimple rulesReduction to simpleSimplicity preservedFMPFEPAmalgamationMaehara frameEquationsGen. amalgamationInterpolationDisjunction propertyStrong separation: syst.Strong separationEquations for DFLStructural rulesFEP for DFL	Residuated frames
Frames and displayDistributive framesInvolutive FLBiFLApplicationsFrame applicationsExamples of frames:FEPSimple equationsSimple rulesReduction to simpleSimplicity preservedFMPFEPAmalgamationMaehara frameEquationsGen. amalgamationInterpolationDisjunction propertyStrong separation: syst.Strong separationEquations for DFLCircute and a large	Frames and displayDistributive framesInvolutive FLBiFLApplicationsFrame applicationsExamples of frames:FEPSimple equationsSimple rulesReduction to simpleSimplicity preservedFMPFEPAmalgamationMaehara frameEquationsGen. amalgamationInterpolationDisjunction propertyStrong separation: syst.Strong separationEquations for DFLStructural rulesFEP for DFL	Frames and modules
Distributive frames Involutive FL BiFL Applications Frame applications Examples of frames: FEP Simple equations Simple rules Reduction to simple Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL	Distributive frames Involutive FL BiFL Applications Frame applications Examples of frames: FEP Simple equations Simple rules Reduction to simple Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL	Frames and display
Involutive FL BiFL Applications Frame applications Examples of frames: FEP Simple equations Simple rules Reduction to simple Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL	Involutive FL BiFL Applications Frame applications Examples of frames: FEP Simple equations Simple rules Reduction to simple Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL	Distributive frames
BiFL Applications Frame applications Examples of frames: FEP Simple equations Simple rules Reduction to simple Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Circuit and a back	BiFLApplicationsFrame applicationsExamples of frames:FEPSimple equationsSimple rulesReduction to simpleSimplicity preservedFMPFEPAmalgamationMaehara frameEquationsGen. amalgamationInterpolationDisjunction propertyStrong separation: syst.Strong separationEquations for DFLStructural rulesFEP for DFL	Involutive FL
ApplicationsFrame applicationsExamples of frames:FEPSimple equationsSimple rulesReduction to simpleSimplicity preservedFMPFEPAmalgamationMaehara frameEquationsGen. amalgamationInterpolationDisjunction propertyStrong separation: syst.Strong separationEquations for DFLCircuit and a back	ApplicationsFrame applicationsExamples of frames:FEPSimple equationsSimple rulesReduction to simpleSimplicity preservedFMPFEPAmalgamationMaehara frameEquationsGen. amalgamationInterpolationDisjunction propertyStrong separation: syst.Strong separationEquations for DFLStructural rulesFEP for DFL	BiFL
Frame applications Examples of frames: FEP Simple equations Simple rules Reduction to simple Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL	Frame applications Examples of frames: FEP Simple equations Simple rules Reduction to simple Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL	Applications
Examples of frames: FEP Simple equations Simple rules Reduction to simple Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL	Examples of frames: FEP Simple equations Simple rules Reduction to simple Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL	Frame applications
Simple equations Simple rules Reduction to simple Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL	Simple equations Simple rules Reduction to simple Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL	Examples of frames: FEP
Simple rules Reduction to simple Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL	Simple rules Reduction to simple Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL	Simple equations
Reduction to simple Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL	Reduction to simple Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL	Simple rules
Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL	Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL	Reduction to simple
FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL	Simplicity preserved
FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL	FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL	FMP
Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL	Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL	FEP
Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL	Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL	Amalgamation
Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL	Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL	Maehara frame
Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL	Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL	Equations
Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL	Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL	Gen. amalgamation
Disjunction property Strong separation: syst. Strong separation Equations for DFL	Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL	Interpolation
Strong separation: syst. Strong separation Equations for DFL	Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL	Disjunction property
Strong separation Equations for DFL	Strong separation Equations for DFL Structural rules FEP for DFL	Strong separation: syst.
Equations for DFL	Equations for DFL Structural rules FEP for DFL	Strong separation
Circuit all the	Structural rules FEP for DFL	Equations for DFL
Structural rules	FEP for DFL	Structural rules
FEP for DFL		FEP for DFL

Let t_0, t_1, \ldots, t_n be monoid terms and let t_0 be linear. A *simple* rule is an expression of the form

$$rac{t_1 \; N \; q \; \cdots \; t_n \; N \; q}{t_0 \; N \; q}$$
 (r

where q is a variable not occurring in t_0, t_1, \ldots, t_n .

A Gentzen frame (\mathbf{W}, \mathbf{B}) satisfies (r) if for all $z \in W'$, and for all sequences \bar{x} of elements of W matching the variables involved in t_0, t_1, \ldots, t_n , the conjunction of the conditions $t_i^{\mathbf{W}}(\bar{x}) N z$, for $i \in \{1, \ldots, n\}$, implies $t_0^{\mathbf{W}}(\bar{x}) N z$.

Frames and modules Frames and display **Distributive frames** Involutive FL BiFL Applications Frame applications Examples of frames: FFP Simple equations Simple rules Reduction to simple Simplicity preserved **FMP FEP** Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

Residuated frames

Let t_0, t_1, \ldots, t_n be monoid terms and let t_0 be linear. A *simple* rule is an expression of the form

$$rac{t_1 \; N \; q \; \cdots \; t_n \; N \; q}{t_0 \; N \; q}$$
 (r

where q is a variable not occurring in t_0, t_1, \ldots, t_n .

A Gentzen frame (\mathbf{W}, \mathbf{B}) satisfies (r) if for all $z \in W'$, and for all sequences \bar{x} of elements of W matching the variables involved in t_0, t_1, \ldots, t_n , the conjunction of the conditions $t_i^{\mathbf{W}}(\bar{x}) N z$, for $i \in \{1, \ldots, n\}$, implies $t_0^{\mathbf{W}}(\bar{x}) N z$.

Given a simple equation $t_0 \leq t_1 \vee \cdots \vee t_n$, we write $R(\varepsilon)$ for (r).

Residuated frames Frames and modules Frames and display **Distributive frames** Involutive FL BiFL Applications Frame applications Examples of frames: FFP Simple equations Simple rules Reduction to simple Simplicity preserved **FMP FEP** Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

Let t_0, t_1, \ldots, t_n be monoid terms and let t_0 be linear. A *simple* rule is an expression of the form

$$rac{t_1 \; N \; q \; \cdots \; t_n \; N \; q}{t_0 \; N \; q}$$
 (r

where q is a variable not occurring in t_0, t_1, \ldots, t_n .

A Gentzen frame (\mathbf{W}, \mathbf{B}) satisfies (r) if for all $z \in W'$, and for all sequences \bar{x} of elements of W matching the variables involved in t_0, t_1, \ldots, t_n , the conjunction of the conditions $t_i^{\mathbf{W}}(\bar{x}) N z$, for $i \in \{1, \ldots, n\}$, implies $t_0^{\mathbf{W}}(\bar{x}) N z$.

Given a simple equation $t_0 \leq t_1 \vee \cdots \vee t_n$, we write $R(\varepsilon)$ for (r). In the context of $(\mathbf{W_{FL}}, \mathbf{Fm})$, $R(\varepsilon)$ takes the form

$$\frac{u[t_1] \Rightarrow a \cdots u[t_n] \Rightarrow a}{u[t_0] \Rightarrow a} (R(\varepsilon))$$

Residuated frames

Frames and modules

Frames and display

Distributive frames

Involutive FL

BiFL

Applications Frame applications Examples of frames: FFP Simple equations Simple rules Reduction to simple Simplicity preserved **FMP FEP** Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL

Structural rules FEP for DFL

In that sense, we may view basic structural rules as simple rules.

	Residuated frames
	Frames and modules
	Frames and display
-,	Distributive frames
	Involutive FL
	BiFL
	Applications
	Frame applications
	Examples of frames: FFP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
	Equations
	Gen. amalgamation
	Interpolation
	Disjunction property
	Strong separation: syst.
	Strong separation
	Equations for DFL
	Structural rules
	FEP for DFL

Lemma. Every equation ε over $\{\forall, \cdot, 1\}$ is equivalent, relative to RL to $R(\varepsilon)$. More precisely, for every $A \in RL$, A satisfies ε iff W_A satisfies $R(\varepsilon)$.

	Residuated frames
	Frames and modules
	Frames and display
-,	Distributive frames
	Involutive FL
	BiFL
	Applications
	Frame applications
	Examples of frames:
	FEP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
	Equations
	A CONTRACTOR OF
	Gen. amalgamation
	Gen. amalgamation Interpolation
	Gen. amalgamation Interpolation Disjunction property
	Gen. amalgamation Interpolation Disjunction property Strong separation: syst.
	Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation
	Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL
	Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules
	Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL
	Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

Lemma. Every equation ε over $\{\forall, \cdot, 1\}$ is equivalent, relative to RL to $R(\varepsilon)$. More precisely, for every $A \in RL$, A satisfies ε iff W_A satisfies $R(\varepsilon)$.

Proof. We continue the example.

 $x_1x_2y \le x_1y \lor x_2y \lor yx_1 \lor yx_2$

	Residuated frames
	Frames and modules
	Frames and display
-,	Distributive frames
	Involutive FL
	BiFL
	Applications
	Frame applications
	Examples of frames: FEP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Amalgamation Maehara frame
	Amalgamation Maehara frame Equations
	Amalgamation Maehara frame Equations Gen. amalgamation
	Amalgamation Maehara frame Equations Gen. amalgamation Interpolation
	Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property
	Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst.
	Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation
	Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL
	Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules
	Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL
	Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

Lemma. Every equation ε over $\{\forall, \cdot, 1\}$ is equivalent, relative to RL to $R(\varepsilon)$. More precisely, for every $A \in RL$, A satisfies ε iff W_A satisfies $R(\varepsilon)$.

Proof. We continue the example.

 $x_1x_2y \le x_1y \lor x_2y \lor yx_1 \lor yx_2$

$$\frac{x_1y \le v \quad x_2y \le v \quad yx_1 \le v \quad yx_2 \le v}{x_1x_2y \le v}$$

•	Residuated frames
	Frames and modules
	Frames and display
-,	Distributive frames
	Involutive FL
	BiFL
	Applications
	Frame applications
	Examples of frames: FFP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	Simplicity preserved FMP
	Simplicity preserved FMP FEP
	Simplicity preserved FMP FEP Amalgamation
	Simplicity preserved FMP FEP Amalgamation Maehara frame
	Simplicity preserved FMP FEP Amalgamation Maehara frame Equations
	Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation
	Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation
	Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property
	Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst.
	Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation
	 Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL
	Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules
	Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

Lemma. Every equation ε over $\{\vee, \cdot, 1\}$ is equivalent, relative to RL to $R(\varepsilon)$. More precisely, for every $A \in RL$, A satisfies ε iff W_A satisfies $R(\varepsilon)$.

Proof. We continue the example.

$$x_1x_2y \le x_1y \lor x_2y \lor yx_1 \lor yx_2$$

$$\frac{x_1y \le v \quad x_2y \le v \quad yx_1 \le v \quad yx_2 \le v}{x_1x_2y \le v}$$

$$\frac{x_1 \circ y \ N \ z \quad x_2 \circ y \ N \ z \quad y \circ x_1 \ N \ z \quad y \circ x_2 \ N \ z}{x_1 \circ x_2 \circ y \ N \ z} \ R(\varepsilon)$$

Simplicity preserved

Theorem. Let (\mathbf{W}, \mathbf{B}) be a cf Gentzen frame and let ε be a $\{\lor, \cdot, 1\}$ -equation. Then (\mathbf{W}, \mathbf{B}) satisfies $\mathsf{R}(\varepsilon)$ iff \mathbf{W}^+ satisfies ε .

Residuated frames

Frames and modules

Frames and display

Distributive frames

Frame applications Examples of frames:

Simple equations Simple rules

Amalgamation Maehara frame Equations

Interpolation

Gen. amalgamation

Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

Reduction to simple Simplicity preserved

Involutive FL

Applications

BiFL

FEP

FMP FEP

Simplicity preserved

Theorem. Let (\mathbf{W}, \mathbf{B}) be a cf Gentzen frame and let ε be a $\{\lor, \cdot, 1\}$ -equation. Then (\mathbf{W}, \mathbf{B}) satisfies $R(\varepsilon)$ iff \mathbf{W}^+ satisfies ε . **Theorem.** Every system obtained from **GL** by adding linear rules has the cut elimination property.

Distributive frames Involutive FL BiFL Applications Frame applications Examples of frames: FEP Simple equations Simple rules Reduction to simple Simplicity preserved FMP **FEP** Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

Residuated frames

Frames and modules

Frames and display

Simplicity preserved

Theorem. Let (\mathbf{W}, \mathbf{B}) be a cf Gentzen frame and let ε be a $\{\lor, \cdot, 1\}$ -equation. Then (\mathbf{W}, \mathbf{B}) satisfies $R(\varepsilon)$ iff \mathbf{W}^+ satisfies ε . **Theorem.** Every system obtained from **GL** by adding linear rules has the cut elimination property.

Theorem. Every system obtained from **GL** by adding linear reducing rules (and also the equational theory of the corresponding variety) is decidable. (*reducing*: there is a complexity measure that decreases with upward applications of the rules).

	Residuated frames
_	Frames and modules
	Frames and display
	Distributive frames
	Involutive FL
	BiFL
	Applications
	Frame applications
	Examples of frames: FEP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FMP FEP
	FMP FEP Amalgamation
	FMP FEP Amalgamation Maehara frame
	FMP FEP Amalgamation Maehara frame Equations
	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation
	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation
	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property
	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst.
	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation
	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL
	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules
	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

J	Residuated frames
	Frames and modules
	Frames and display
is a	Distributive frames
at	Involutive FL
	BiFI
	Applications
	Frame applications
	Examples of frames: FEP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	omplicity preserved
	FMP
	FMP FEP
	FMP FEP Amalgamation
	FMP FEP Amalgamation Maehara frame
	FMP FEP Amalgamation Maehara frame Equations
	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation
	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation
	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property
	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst
	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst Strong separation
	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst Strong separation Equations for DFL
	FMPFEPAmalgamationMaehara frameEquationsGen. amalgamationInterpolationDisjunction propertyStrong separation: systStrong separationEquations for DFLStructural rules
	FMPFEPAmalgamationMaehara frameEquationsGen. amalgamationInterpolationDisjunction propertyStrong separation: systStrong separationEquations for DFLStructural rulesFEP for DFL
	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst Strong separation Equations for DFL Structural rules FEP for DFL

For $\mathbf{W_{FL}}$, given $(x, z) \in W \times W'$ (if z = (u, c), then $u(x) \Rightarrow c$ is a sequent), we define $(x, z)^{\uparrow}$ as the smallest subset of $W \times W'$ that contains (x, z) and is closed upwards with respect to the rules of $\mathbf{FL^{f}}$. Note that $(x, z)^{\uparrow}$ is finite.

J	Residuated frames
	Frames and modules
	Frames and display
c is a	Distributive frames
nat	Involutive FL
f	BiFL
1	Applications
	Frame applications
	Examples of frames: FFP
uated	Simple equations
	Simple rules
	Reduction to simple
d	Simplicity preserved
	FMP
	FEP
inite	Amalgamation
	Maehara frame
	Maehara frame Equations
d in	Maenara frame Equations Gen. amalgamation
d in	Maenara frame Equations Gen. amalgamation Interpolation
d in	Maenara frame Equations Gen. amalgamation Interpolation Disjunction property
d in	Maenara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst.
d in	Maenara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation
d in	Maenara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL
d in	Maenara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules
d in	Maenara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL
d in	Maenara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

For $\mathbf{W_{FL}}$, given $(x, z) \in W \times W'$ (if z = (u, c), then $u(x) \Rightarrow c$ is a sequent), we define $(x, z)^{\uparrow}$ as the smallest subset of $W \times W'$ that contains (x, z) and is closed upwards with respect to the rules of $\mathbf{FL}^{\mathbf{f}}$. Note that $(x, z)^{\uparrow}$ is finite.

The new frame \mathbf{W}' associated with $N' = N \cup ((y, v)^{\uparrow})^c$ is residuated and Gentzen. Clearly, $(N')^c$ is finite, so it has a finite domain $Dom((N')^c)$ and

codomain $Cod((N')^c)$. For every $z \notin Cod((N')^c)$, $\{z\}^{\triangleleft} = W$. So, $\{\{z\}^{\triangleleft} : z \in W\}$ is finite and a basis for $\gamma_{N'}$. So, $\mathbf{W'^+}$ is finite. Moreover, if $u(x) \Rightarrow c$ is not provable in **FL**, then it is not valid in $\mathbf{W'^+}$.

······································	Residuated frames
	Frames and modules
	Frames and display
e is a	Distributive frames
at	Involutive FL
:	BiFL
	Applications
	Frame applications
	Examples of frames:
lated	Simple equations
	Simple rules
	Reduction to simple
h	Simplicity preserved
u	
u	FMP
u	FMP FEP
nite	FMP FEP Amalgamation
nite	FMP FEP Amalgamation Maehara frame
nite	FMP FEP Amalgamation Maehara frame Equations
nite d in	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation
nite d in	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation
nite d in	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property
nite d in	FMPFEPAmalgamationMaehara frameEquationsGen. amalgamationInterpolationDisjunction propertyStrong separation: syst.
nite d in	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation
nite d in ame	FMPFEPAmalgamationMaehara frameEquationsGen. amalgamationInterpolationDisjunction propertyStrong separation: syst.Strong separationEquations for DFL
nite d in ame	FMPFEPAmalgamationMaehara frameEquationsGen. amalgamationInterpolationDisjunction propertyStrong separation: syst.Strong separationEquations for DFLStructural rules
nite d in ame	FMPFEPAmalgamationMaehara frameEquationsGen. amalgamationInterpolationDisjunction propertyStrong separation: syst.Strong separationEquations for DFLStructural rulesFEP for DFL

For $\mathbf{W_{FL}}$, given $(x, z) \in W \times W'$ (if z = (u, c), then $u(x) \Rightarrow c$ is a sequent), we define $(x, z)^{\uparrow}$ as the smallest subset of $W \times W'$ that contains (x, z) and is closed upwards with respect to the rules of $\mathbf{FL}^{\mathbf{f}}$. Note that $(x, z)^{\uparrow}$ is finite.

The new frame \mathbf{W}' associated with $N' = N \cup ((y, v)^{\uparrow})^c$ is residuated and Gentzen. Clearly, $(N')^c$ is finite, so it has a finite domain $Dom((N')^c)$ and codomain $Cod((N')^c)$. For every $z \notin Cod((N')^c)$, $\{z\}^{\triangleleft} = W$. So, $\{\{z\}^{\triangleleft} : z \in W\}$ is finite and a basis for $\gamma_{N'}$. So, \mathbf{W}'^+ is finite. Moreover, if $u(x) \Rightarrow c$ is not provable in **FL**, then it is not valid in \mathbf{W}'^+ .

Corollary. The system **FL** has the finite model property. The same holds for reducing extensions of **GL**.

Ly	Residuated frames
	Frames and modules
	Frames and display
is a	Distributive frames
t	Involutive FL
	BiFL
	Applications
	Frame applications
	Examples of frames: FEP
ted	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FMP FEP
ite	FMP FEP Amalgamation
ite	FMP FEP Amalgamation Maehara frame
ite	FMP FEP Amalgamation Maehara frame Equations
ite	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation
ite in	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation
ite in	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property
ite in	FMPFEPAmalgamationMaehara frameEquationsGen. amalgamationInterpolationDisjunction propertyStrong separation: syst.
ite in	FMPFEPAmalgamationMaehara frameEquationsGen. amalgamationInterpolationDisjunction propertyStrong separation: syst.Strong separation
ite in ne	FMPFEPAmalgamationMaehara frameEquationsGen. amalgamationInterpolationDisjunction propertyStrong separation: syst.Strong separationEquations for DFL
ite in ne	FMPFEPAmalgamationMaehara frameEquationsGen. amalgamationInterpolationDisjunction propertyStrong separation: syst.Strong separationEquations for DFLStructural rules
ite in ne	FMPFEPAmalgamationMaehara frameEquationsGen. amalgamationInterpolationDisjunction propertyStrong separation: syst.Strong separationEquations for DFLStructural rulesFEP for DFL
ite in ne	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

For W_{FL} , given $(x, z) \in W \times W'$ (if z = (u, c), then $u(x) \Rightarrow c$ is a sequent), we define $(x, z)^{\uparrow}$ as the smallest subset of $W \times W'$ that contains (x, z) and is closed upwards with respect to the rules of FL^{f} . Note that $(x, z)^{\uparrow}$ is finite.

The new frame \mathbf{W}' associated with $N' = N \cup ((y, v)^{\uparrow})^c$ is residuated and Gentzen. Clearly, $(N')^c$ is finite, so it has a finite domain $Dom((N')^c)$ and codomain $Cod((N')^c)$. For every $z \notin Cod((N')^c)$, $\{z\}^{\triangleleft} = W$. So, $\{\{z\}^{\triangleleft} : z \in W\}$ is finite and a basis for $\gamma_{N'}$. So, \mathbf{W}'^+ is finite. Moreover, if $u(x) \Rightarrow c$ is not provable in **FL**, then it is not valid in \mathbf{W}'^+ .

Corollary. The system \mathbf{FL} has the finite model property. The same holds for reducing extensions of \mathbf{GL} .

Corollary. The variety of residuated lattices is generated by its finite members. The same holds for the subvarieties corresponding to the above extensions.

FEP

Residuated frames

Frames and modules

Frames and display

Distributive frames

Involutive FL

BiFL

Applications

Frame applications Examples of frames: FEP

Simple equations Simple rules

Reduction to simple Simplicity preserved

FMP FEP

Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

A class of algebras \mathcal{K} has the *finite embeddability property (FEP)* if for every $\mathbf{A} \in \mathcal{K}$, every finite partial subalgebra \mathbf{B} of \mathbf{A} can be (partially) embedded in a finite $\mathbf{D} \in \mathcal{K}$.

FEP

Residuated frames

Frames and modules

Frames and display

Distributive frames

Involutive FL

BiFL

Applications

Frame applications Examples of frames: FEP Simple equations Simple rules Reduction to simple Simplicity preserved

FMP FEP

Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

A class of algebras \mathcal{K} has the *finite embeddability property (FEP)* if for every $\mathbf{A} \in \mathcal{K}$, every finite partial subalgebra \mathbf{B} of \mathbf{A} can be (partially) embedded in a finite $\mathbf{D} \in \mathcal{K}$.

Theorem. Every variety of integral RL's axiomatized by equartions over $\{\lor, \cdot, 1\}$ has the FEP.

- $\blacksquare \quad \mathbf{B} \text{ embeds in } \mathbf{W}^+_{\mathbf{A},\mathbf{B}} \text{ via } \{_\}^{\lhd}: \mathbf{B} \to \mathbf{W}^+$
- **\mathbf{W}_{\mathbf{A},\mathbf{B}}^+ is finite**

FEP

Residuated frames

Frames and modules

Frames and display

Distributive frames

Involutive FL

BiFL

Applications

Frame applications Examples of frames: FEP Simple equations Simple rules Reduction to simple Simplicity preserved FMP

FEP

Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

A class of algebras \mathcal{K} has the *finite embeddability property (FEP)* if for every $\mathbf{A} \in \mathcal{K}$, every finite partial subalgebra \mathbf{B} of \mathbf{A} can be (partially) embedded in a finite $\mathbf{D} \in \mathcal{K}$.

Theorem. Every variety of integral RL's axiomatized by equartions over $\{\lor, \cdot, 1\}$ has the FEP.

- $\blacksquare \quad \mathbf{B} \text{ embeds in } \mathbf{W}^+_{\mathbf{A},\mathbf{B}} \text{ via } \{_\}^{\lhd}: \mathbf{B} \to \mathbf{W}^+$
- **\mathbf{W}_{\mathbf{A},\mathbf{B}}^+ is finite**

Corollary. These varieties are generated as quasivarieties by their finite members.

Corollary. The corresponding logics have the *strong finite model* property: if $\Phi \not\vdash \psi$, for finite Φ , then there is a finite counter-model, namely there is $\mathbf{D} \in \mathcal{V}$ and a homomorphism $f : \mathbf{Fm} \to \mathbf{D}$, such that $f(\phi) = 1$, for all $\phi \in \Phi$, but $f(\psi) \neq 1$.

Residuated frames

Frames and	modul	es
------------	-------	----

Frames and display

Distributive frames

Involutive FL

BiFL

Applications Frame applications

Examples of frames: FEP

Simple equations

Simple rules Reduction to simple

Simplicity preserved

FMP FEP

Amalgamation

Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

A class \mathcal{K} of similar algebras has the *amalgamation property* (AP), if for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, there is a $\mathbf{D} \in \mathcal{K}$ and embeddings $f' : \mathbf{B} \to \mathbf{D}$ and $g' : \mathbf{C} \to \mathbf{D}$ such that $f' \circ f = g' \circ g$.

Nick Galatos, Prague workshop, March, 2014

Residuated frames
 Frames and modules
Frames and display
Distributive frames
Involutive FL
BiFL
Applications
Frame applications
Examples of frames: FEP
Simple equations
Simple rules
Reduction to simple
Simplicity preserved
FMP
FEP
Amalgamation
Maehara frame
Equations
Gen. amalgamation
Interpolation
Disjunction property
Strong separation: syst.
Strong separation: syst. Strong separation
Strong separation: syst. Strong separation Equations for DFL
Strong separation: syst. Strong separation Equations for DFL Structural rules
Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

A class \mathcal{K} of similar algebras has the *amalgamation property* (AP), if for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, there is a $\mathbf{D} \in \mathcal{K}$ and embeddings $f' : \mathbf{B} \to \mathbf{D}$ and $g' : \mathbf{C} \to \mathbf{D}$ such that $f' \circ f = g' \circ g$.

We will show that CRL_n has the AP, where $D \subseteq \mathcal{P}((B \cup C)^*)$.

Residuated frames
Frames and modules
Frames and display
Distributive frames
Involutive FL
BiFL
Applications
Frame applications
Examples of frames: FEP
Simple equations
Simple rules
Reduction to simple
Simplicity preserved
Simplicity preserved FMP
Simplicity preserved FMP FEP
Simplicity preserved FMP FEP Amalgamation
Simplicity preserved FMP FEP Amalgamation Maehara frame
Simplicity preserved FMP FEP Amalgamation Maehara frame Equations
Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation
Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation
Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property
 Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst.
 Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation
 Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL
 Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules

A class \mathcal{K} of similar algebras has the *amalgamation property* (AP), if for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, there is a $\mathbf{D} \in \mathcal{K}$ and embeddings $f' : \mathbf{B} \to \mathbf{D}$ and $g' : \mathbf{C} \to \mathbf{D}$ such that $f' \circ f = g' \circ g$.

We will show that CRL_n has the AP, where $D \subseteq \mathcal{P}((B \cup C)^*)$.

Note that $\mathcal{P}((B \cup C)^*)$ is a commutative residuated lattice.

	Residuated frames
_	Frames and modules
	Frames and display
	Distributive frames
	Involutive FL
	BiFL
	Applications
	Frame applications
	Examples of frames: FEP
	Simple equations
	Simple rules
	Reduction to simple
	Reduction to simple Simplicity preserved
	Reduction to simple Simplicity preserved FMP
	Reduction to simple Simplicity preserved FMP FEP
	Reduction to simple Simplicity preserved FMP FEP Amalgamation
	Reduction to simple Simplicity preserved FMP FEP Amalgamation Maehara frame
	Reduction to simple Simplicity preserved FMP FEP Amalgamation Maehara frame Equations
	Reduction to simpleSimplicity preservedFMPFEPAmalgamationMaehara frameEquationsGen. amalgamation
	Reduction to simpleSimplicity preservedFMPFEPAmalgamationMaehara frameEquationsGen. amalgamationInterpolation
	Reduction to simpleSimplicity preservedFMPFEPAmalgamationMaehara frameEquationsGen. amalgamationInterpolationDisjunction property
	Reduction to simpleSimplicity preservedFMPFEPAmalgamationMaehara frameEquationsGen. amalgamationInterpolationDisjunction propertyStrong separation: syst.
	Reduction to simpleSimplicity preservedFMPFEPAmalgamationMaehara frameEquationsGen. amalgamationInterpolationDisjunction propertyStrong separation: syst.Strong separation
	Reduction to simpleSimplicity preservedFMPFEPAmalgamationMaehara frameEquationsGen. amalgamationInterpolationDisjunction propertyStrong separation: syst.Strong separationEquations for DFL
	Reduction to simpleSimplicity preservedFMPFEPAmalgamationMaehara frameEquationsGen. amalgamationInterpolationDisjunction propertyStrong separation: syst.Strong separationEquations for DFLStructural rules
	Reduction to simple Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

A class \mathcal{K} of similar algebras has the *amalgamation property* (AP), if for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, there is a $\mathbf{D} \in \mathcal{K}$ and embeddings $f' : \mathbf{B} \to \mathbf{D}$ and $g' : \mathbf{C} \to \mathbf{D}$ such that $f' \circ f = g' \circ g$.

We will show that CRL_n has the AP, where $D \subseteq \mathcal{P}((B \cup C)^*)$.

Note that $\mathcal{P}((B \cup C)^*)$ is a commutative residuated lattice.

Actually, $D = \gamma [\mathcal{P}((B \cup C)^*)]$, for some closure operator γ .

A class \mathcal{K} of similar algebras has the *amalgamation property* (AP), if for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, there is a $\mathbf{D} \in \mathcal{K}$ and embeddings $f' : \mathbf{B} \to \mathbf{D}$ and $g' : \mathbf{C} \to \mathbf{D}$ such that $f' \circ f = g' \circ g$.

We will show that CRL_n has the AP, where $D \subseteq \mathcal{P}((B \cup C)^*)$.

Note that $\mathcal{P}((B \cup C)^*)$ is a commutative residuated lattice.

Actually, $D = \gamma[\mathcal{P}((B \cup C)^*)]$, for some closure operator γ .

We will

- define γ (by giving an associated Galois connection) and D,
- **prove that** $\mathbf{D} \in CRL_n$,
- $\blacksquare \quad \text{prove that } \mathbf{B}, \mathbf{C} \hookrightarrow_{\mathbf{A}} \mathbf{D}.$

Residuated frames
Frames and modules
Frames and display
Distributive frames
Involutive FL
BiFL
Applications
Frame applications
Examples of frames: FEP
Simple equations
Simple rules
Reduction to simple
· · · ·
Simplicity preserved
Simplicity preserved FMP
Simplicity preserved FMP FEP
Simplicity preserved FMP FEP Amalgamation
Simplicity preserved FMP FEP Amalgamation Maehara frame
Simplicity preserved FMP FEP Amalgamation Maehara frame Equations
Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation
Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation
Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property
Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst.
Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation
Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL
Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules
Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

Residuated frames Frames and modules Frames and display Distributive frames Involutive FL BiFL Applications Frame applications Examples of frames: FEP Simple equations Simple rules Reduction to simple Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

We define $W = (B \cup C)^*$,

Maehara frame Residuated frames Frames and modules Frames and display We define $W = (B \cup C)^*$, $W' = (B \cup C)^* \times (B \cup C)$ and Distributive frames Involutive FL BiFL Applications Frame applications Examples of frames: FEP Simple equations Simple rules Reduction to simple Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

	Frames and modules
	Frames and display
We define $W = (B \cup C)^*$, $W' = (B \cup C)^* \times (B \cup C)$ and	Distributive frames
$x \ N \ (u \ d)$ iff for all partitions $u \cdot x = w_{\mathcal{D}} \cdot w_{\mathcal{C}}$ with $w_{\mathcal{D}} \in B^*$ and	Involutive FL
$a \to (a, a) \text{ in for an particulars } a = a = a = a = a = a = a = a = a = a$	BiFL
$w_C \in C^+$	Applications
if $d \in B$ then $w_a \leq a$ and $w_B \cdot a \leq b d$ for some $a \in A$ and	Frame applications
$\blacksquare \text{if } a \subseteq D, \text{ then } w_U \subseteq U \text{ a diad } w_B a \subseteq B \text{ a, for some } a \subseteq H \text{ diad}$	Examples of frames:
If $d \in C$, then $w_B \leq_B a$ and $w_C \cdot a \leq_C d$, for some $a \in A$.	FEP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
	Equations
	Gen. amalgamation
	Interpolation
	Disjunction property
	Strong separation: syst.
	Strong separation
	Equations for DFL
	Structural rules
	FEP for DFL

Residuated frames

	Frames and modules
M_{0} define $W_{-} (D + C) * W_{-} (D + C) * \times (D + C)$ and	Frames and display
we define $W = (B \cup C)$, $W = (B \cup C) \times (B \cup C)$ and	Distributive frames
$x N(u, d)$ iff for all partitions $u \cdot x = w_B \cdot w_C$, with $w_B \in B^*$ and	Involutive FL
$C C^*$	BiFL
$w_C \in \mathbb{C}$	Applications
if $d \in B$, then $w_C \leq_C a$ and $w_B \cdot a \leq_B d$, for some $a \in A$ and	Frame applications
if $d \in C$, then $w_B \leq B a$ and $w_C \cdot a \leq C d$, for some $a \in A$.	Examples of frames: FEP
	Simple equations
Notational conventions:	Simple rules
Notational conventions.	Reduction to simple
For $d \in B \sqcup C$ we identify (ε, d) with d	Simplicity preserved
\square For $a \in D \cup O$, we identify (c, a) with a .	FMP
For $x \in B^*$, we write simply x for its interpretation in B .	FEP
	Amalgamation

Residuated frames

Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

We define $W = (B \cup C)^*$, $W' = (B \cup C)^* imes (B \cup C)$ and

 $x \ N \ (u,d)$ iff for all partitions $u \cdot x = w_B \cdot w_C$, with $w_B \in B^*$ and $w_C \in C^*$

- if $d \in B$, then $w_C \leq_C a$ and $w_B \cdot a \leq_B d$, for some $a \in A$ and
- If $d \in C$, then $w_B \leq_B a$ and $w_C \cdot a \leq_C d$, for some $a \in A$.

Notational conventions:

- For $d \in B \cup C$, we identify (ε, d) with d.
- For $x \in B^*$, we write simply x for its interpretation in **B**.

Lemma. $W_{B,C}^{A} = (W, W', N, \circ, \varepsilon, \rightsquigarrow)$ is a residuated frame (called the Maehara frame)

	Residuated frames
_	Frames and modules
	Frames and display
	Distributive frames
	Involutive FL
	BiFL
	Applications
	Frame applications
	Examples of frames:
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FMP FEP
	FMP FEP Amalgamation
	FMP FEP Amalgamation Maehara frame
	FMP FEP Amalgamation Maehara frame Equations
	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation
	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation
	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property
	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst.
	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation
	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL
	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules
	FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

We define $W = (B \cup C)^*$, $W' = (B \cup C)^* \times (B \cup C)$ and

 $x \ N \ (u, d)$ iff for all partitions $u \cdot x = w_B \cdot w_C$, with $w_B \in B^*$ and $w_C \in C^*$

- if $d \in B$, then $w_C \leq_C a$ and $w_B \cdot a \leq_B d$, for some $a \in A$ and
- If $d \in C$, then $w_B \leq_B a$ and $w_C \cdot a \leq_C d$, for some $a \in A$.

Notational conventions:

- For $d \in B \cup C$, we identify (ε, d) with d.
- For $x \in B^*$, we write simply x for its interpretation in **B**.

Lemma. $\mathbf{W}_{\mathbf{B},\mathbf{C}}^{\mathbf{A}} = (W, W', N, \circ, \varepsilon, \rightsquigarrow)$ is a residuated frame (called the Maehara frame)

Proof. We have $x \circ y N(u, b)$, for $b \in B$, iff for all partitions $x = x_B \cdot x_C$, $y = y_B \cdot y_C$ and $u = u_B \cdot u_C$, with $x_B, y_B, u_B \in B^*$ and $x_C, y_C, u_C \in C^*$, there exists $a \in A$ such that $u_C \cdot x_C \cdot y_C N_C a$ and $u_b \cdot x_B \cdot y_B \cdot a N_B b$.

	Residuated frames
_	Frames and modules
	Frames and display
	Distributive frames
	Involutive FL
	BiFL
	Applications
	Frame applications
	Examples of frames: FEP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
	Equations
	Gen. amalgamation
	Interpolation
	Disjunction property
	Strong separation: syst.
	Strong separation
	Equations for DFL
	Structural rules
	FEP for DFL

We define $W = (B \cup C)^*$, $W' = (B \cup C)^* imes (B \cup C)$ and

 $x \ N \ (u,d)$ iff for all partitions $u \cdot x = w_B \cdot w_C$, with $w_B \in B^*$ and $w_C \in C^*$

- if $d \in B$, then $w_C \leq_C a$ and $w_B \cdot a \leq_B d$, for some $a \in A$ and
- if $d \in C$, then $w_B \leq_B a$ and $w_C \cdot a \leq_C d$, for some $a \in A$.

Notational conventions:

- For $d \in B \cup C$, we identify (ε, d) with d.
- For $x \in B^*$, we write simply x for its interpretation in **B**.

Lemma. $W_{B,C}^{A} = (W, W', N, \circ, \varepsilon, \rightsquigarrow)$ is a residuated frame (called the Maehara frame)

Proof. We have $x \circ y N(u, b)$, for $b \in B$, iff for all partitions $x = x_B \cdot x_C$, $y = y_B \cdot y_C$ and $u = u_B \cdot u_C$, with $x_B, y_B, u_B \in B^*$ and $x_C, y_C, u_C \in C^*$, there exists $a \in A$ such that $u_C \cdot x_C \cdot y_C N_C a$ and $u_b \cdot x_B \cdot y_B \cdot a N_B b$. This statement is equivalent to $x N(u \circ y, b)$.

	Residuated frames
	Frames and modules
	Frames and display
	Distributive frames
	Involutive FL
	BiFL
	Applications
	Frame applications
	Examples of frames: FEP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
	Equations
	Gen. amalgamation
	Interpolation
	Disjunction property
	Strong separation: syst.
1	Strong separation
1	Equations for DFL
4	Structural rules
	FEP for DFL

We define $W = (B \cup C)^*$, $W' = (B \cup C)^* imes (B \cup C)$ and

 $x \ N \ (u,d)$ iff for all partitions $u \cdot x = w_B \cdot w_C$, with $w_B \in B^*$ and $w_C \in C^*$

- if $d \in B$, then $w_C \leq_C a$ and $w_B \cdot a \leq_B d$, for some $a \in A$ and
- If $d \in C$, then $w_B \leq_B a$ and $w_C \cdot a \leq_C d$, for some $a \in A$.

Notational conventions:

- For $d \in B \cup C$, we identify (ε, d) with d.
- For $x \in B^*$, we write simply x for its interpretation in **B**.

Lemma. $W_{B,C}^{A} = (W, W', N, \circ, \varepsilon, \rightsquigarrow)$ is a residuated frame (called the Maehara frame)

Proof. We have $x \circ y N(u, b)$, for $b \in B$, iff for all partitions $x = x_B \cdot x_C$, $y = y_B \cdot y_C$ and $u = u_B \cdot u_C$, with $x_B, y_B, u_B \in B^*$ and $x_C, y_C, u_C \in C^*$, there exists $a \in A$ such that $u_C \cdot x_C \cdot y_C N_C a$ and $u_b \cdot x_B \cdot y_B \cdot a N_B b$. This statement is equivalent to $x N(u \circ y, b)$.

Corollary. $\mathbf{D} = \mathcal{P}((B \cup C)^*)_{\gamma}$ is a commutative residuated lattice. **Lemma.** $\mathbf{W}_{\mathbf{B},\mathbf{C}}^{\mathbf{A}}$ is a Genzen frame. Equations for DFL

Structural rules

FEP for DFL

Residuated frames
Maehara frame

We define $W = (B \cup C)^*$, $W' = (B \cup C)^* \times (B \cup C)$ and

 $x \; N \; (u,d)$ iff for all partitions $u \cdot x = w_B \cdot w_C$, with $w_B \in B^*$ and $w_C \in C^*$

- if $d \in B$, then $w_C \leq_C a$ and $w_B \cdot a \leq_B d$, for some $a \in A$ and
- If $d \in C$, then $w_B \leq_B a$ and $w_C \cdot a \leq_C d$, for some $a \in A$.

Notational conventions:

- For $d \in B \cup C$, we identify (ε, d) with d.
- For $x \in B^*$, we write simply x for its interpretation in **B**.

Lemma. $W_{B,C}^{A} = (W, W', N, \circ, \varepsilon, \rightsquigarrow)$ is a residuated frame (called the Maehara frame)

Proof. We have $x \circ y N(u, b)$, for $b \in B$, iff for all partitions $x = x_B \cdot x_C, y = y_B \cdot y_C$ and $u = u_B \cdot u_C$, with $x_B, y_B, u_B \in B^*$ and $x_C, y_C, u_C \in C^*$, there exists $a \in A$ such that $u_C \cdot x_C \cdot y_C N_C a$ and $u_b \cdot x_B \cdot y_B \cdot a N_B b$. This statement is equivalent to $x N(u \circ y, b)$.

Corollary. $\mathbf{D} = \mathcal{P}((B \cup C)^*)_{\gamma}$ is a commutative residuated lattice. **Lemma.** $\mathbf{W}_{\mathbf{B},\mathbf{C}}^{\mathbf{A}}$ is a Genzen frame.

Corollary. B. C \hookrightarrow **D**.

Nick Galatos, Prague workshop, March, 2014

Residuated frames

Frames and modules

Frames and display

Distributive frames

Involutive FL

BiFL

Applications Frame applications

Examples of frames: FEP Simple equations Simple rules Reduction to simple

Simplicity preserved FMP FEP

Amalgamation

Maehara frame

Equations

Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

Recall that if (\mathbf{W}, \mathbf{S}) is a Gentzen frame and ε an equation over $\{\vee, \cdot, 1\}$, then (\mathbf{W}, \mathbf{S}) satisfies $\mathsf{R}(\varepsilon)$ iff \mathbf{W}^+ satisfies ε .

Residuated frames

Frames and modules

Frames and display

Distributive frames

Involutive FL

BiFL

Applications Frame applications

Examples of frames: FEP Simple equations Simple rules Reduction to simple Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation

Disjunction property Strong separation: syst.

Strong separation Equations for DFL Structural rules FEP for DFL

Recall that if (\mathbf{W}, \mathbf{S}) is a Gentzen frame and ε an equation over $\{\vee, \cdot, 1\}$, then (\mathbf{W}, \mathbf{S}) satisfies $\mathsf{R}(\varepsilon)$ iff \mathbf{W}^+ satisfies ε .

Moreover, $\mathsf{R}(x \leq x^n)$ is the condition $[x^{\circ n} = x \circ \cdots \circ x \text{ (}n \text{ times).}]$

 $\frac{x^{\circ n} N z}{x N z} (n)$

Recall that if (\mathbf{W}, \mathbf{S}) is a Gentzen frame and ε an equation over $\{\vee, \cdot, 1\}$, then (\mathbf{W}, \mathbf{S}) satisfies $R(\varepsilon)$ iff \mathbf{W}^+ satisfies ε .

Moreover, $\mathsf{R}(x \leq x^n)$ is the condition $[x^{\circ n} = x \circ \cdots \circ x \text{ (}n \text{ times).}]$

$$\frac{x^{\circ n} N z}{x N z} (n)$$

Lemma. The frame $W_{B,C}^{A}$ satisfies condition (n).

Residuated frames
Frames and modules
Frames and display
Distributive frames
Involutive FL
BiFL
Applications
Frame applications
Examples of frames: FEP
Simple equations
Simple rules
Reduction to simple
Simplicity preserved
FMP
FEP
Amalgamation
Maehara frame
Equations
Gen. amalgamation
Interpolation
Disjunction property
Strong separation: syst.
Strong separation
Equations for DFL Structural rules
EEP for DEI

Recall that if (\mathbf{W}, \mathbf{S}) is a Gentzen frame and ε an equation over $\{\vee, \cdot, 1\}$, then (\mathbf{W}, \mathbf{S}) satisfies $R(\varepsilon)$ iff \mathbf{W}^+ satisfies ε .

Moreover, $\mathsf{R}(x \leq x^n)$ is the condition $[x^{\circ n} = x \circ \cdots \circ x \text{ (}n \text{ times).}]$

$$\frac{x^{\circ n} N z}{x N z} (n)$$

Lemma. The frame $W_{B,C}^{A}$ satisfies condition (n).

Proof. For z = (u, b), to show that x N (u, b), let $x = x_B \circ x_C$ and $u = u_B \circ u_C$, where $x_B, u_B \in B^*$, $x_C, u_C \in C^*$.

Residuated frames

Equations for DFL Structural rules FEP for DFL

Recall that if (\mathbf{W}, \mathbf{S}) is a Gentzen frame and ε an equation over $\{\vee, \cdot, 1\}$, then (\mathbf{W}, \mathbf{S}) satisfies $\mathsf{R}(\varepsilon)$ iff \mathbf{W}^+ satisfies ε .

Moreover, $\mathsf{R}(x \le x^n)$ is the condition $[x^{\circ n} = x \circ \cdots \circ x \text{ (}n \text{ times).}]$

$$\frac{x^{\circ n} N z}{x N z} (n)$$

Lemma. The frame $W_{B,C}^{A}$ satisfies condition (n).

Proof. For z = (u, b), to show that $x \ N(u, b)$, let $x = x_B \circ x_C$ and $u = u_B \circ u_C$, where $x_B, u_B \in B^*$, $x_C, u_C \in C^*$. Since $x^{\circ n} \ N(u, b)$, there exists $i \in A$ such that $u_C \circ x_C^n \leq_C i$ and $u_B x_B^n i \leq_B b$. Since **B** and **C** satisfy $x \leq x^n$, we get $u_C \circ x_C \leq_C i$ and $u_B x_B i \leq_B b$. Consequently, $x \ N(u, b)$.

Residuated frames Frames and modules Frames and display Distributive frames Involutive FL BiFL Applications Frame applications Examples of frames: **FEP** Simple equations Simple rules Reduction to simple Simplicity preserved **FMP FEP** Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation

> Equations for DFL Structural rules FEP for DFL

Nick Galatos, Prague workshop, March, 2014

Recall that if (\mathbf{W}, \mathbf{S}) is a Gentzen frame and ε an equation over $\{\vee, \cdot, 1\}$, then (\mathbf{W}, \mathbf{S}) satisfies $\mathsf{R}(\varepsilon)$ iff \mathbf{W}^+ satisfies ε .

Moreover, $\mathsf{R}(x \leq x^n)$ is the condition $[x^{\circ n} = x \circ \cdots \circ x \text{ (}n \text{ times).}]$

$$\frac{x^{\circ n} N z}{x N z} (n)$$

Lemma. The frame $W_{B,C}^{A}$ satisfies condition (n).

Proof. For z = (u, b), to show that x N(u, b), let $x = x_B \circ x_C$ and $u = u_B \circ u_C$, where $x_B, u_B \in B^*$, $x_C, u_C \in C^*$. Since $x^{\circ n} N(u, b)$, there exists $i \in A$ such that $u_C \circ x_C^n \leq_C i$ and $u_B x_B^n i \leq_B b$. Since **B** and **C** satisfy $x \leq x^n$, we get $u_C \circ x_C \leq_C i$ and $u_B x_B i \leq_B b$ Consequently, x N(u, b).

Corollary. $\mathbf{D} \in CRL_n$.

Recall that if (\mathbf{W}, \mathbf{S}) is a Gentzen frame and ε an equation over $\{\vee, \cdot, 1\}$, then (\mathbf{W}, \mathbf{S}) satisfies $\mathsf{R}(\varepsilon)$ iff \mathbf{W}^+ satisfies ε .

Moreover, $\mathsf{R}(x \le x^n)$ is the condition $[x^{\circ n} = x \circ \cdots \circ x \text{ (}n \text{ times).}]$

$$\frac{x^{\circ n} N z}{x N z} (n)$$

Lemma. The frame $W_{B,C}^{A}$ satisfies condition (n).

Proof. For z = (u, b), to show that x N(u, b), let $x = x_B \circ x_C$ and $u = u_B \circ u_C$, where $x_B, u_B \in B^*$, $x_C, u_C \in C^*$. Since $x^{\circ n} N(u, b)$, there exists $i \in A$ such that $u_C \circ x_C^n \leq_C i$ and $u_B x_B^n i \leq_B b$. Since **B** and **C** satisfy $x \leq x^n$, we get $u_C \circ x_C \leq_C i$ and $u_B x_B i \leq_B b$. Consequently, x N(u, b).

Corollary. $D \in CRL_n$.

Corollary. CRL_n has the AP.

Distributive frames Involutive FL BiFL Applications Frame applications Examples of frames: **FEP** Simple equations Simple rules Reduction to simple Simplicity preserved **FMP FEP** Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation

Residuated frames

Frames and modules

Frames and display

Gen. amalgamation

If we do not assume that f, g are injective, instead of

 $N = \langle (N_B \circ N_C) \cup (N_C \circ N_B) \rangle,$

we take

$$N = \langle (N_B \circ f \circ g \circ N_C) \cup (N_C \circ g \circ f \circ N_B) \rangle.$$

Then we can prove AP, transferable injections, and transferable surjections and the congruence extension property all with a single argument.

Residuated frames Frames and modules Frames and display Distributive frames Involutive FL BiFL Applications Frame applications Examples of frames: FEP Simple equations Simple rules Reduction to simple Simplicity preserved **FMP FEP** Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

Residuated frames

Frames and modules

Frames and display

Distributive frames

Involutive FL

BiFL

Applications Frame applications Examples of frames: FEP Simple equations Simple rules Reduction to simple Simplicity preserved FMP FEP Amalgamation Maehara frame Equations

Gen. amalgamation Interpolation

Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules

FEP for DFL

Theorem. $\mathbf{FL}_{\mathbf{e}}$ has the Craig interpolation property, i.e. if $\vdash_{\mathbf{FL}_{\mathbf{e}}} \phi \rightarrow \psi$, then there is a χ such that

 $\vdash_{\mathbf{FL}_{\mathbf{e}}} \phi \to \chi \text{ and } \vdash_{\mathbf{FL}_{\mathbf{e}}} \chi \to \psi$ $\exists var(\chi) \subseteq var(\phi) \cap var(\psi).$

Nick Galatos, Prague workshop, March, 2014

Theorem. $\mathbf{FL}_{\mathbf{e}}$ has the Craig interpolation property, i.e. if $\vdash_{\mathbf{FL}_{\mathbf{e}}} \phi \to \psi$, then there is a χ such that $\models_{\mathbf{FL}_{\mathbf{e}}} \phi \to \chi$ and $\vdash_{\mathbf{FL}_{\mathbf{e}}} \chi \to \psi$ $= var(\chi) \subseteq var(\phi) \cap var(\psi)$.

Proof sketch. Define a frame with $W = Fm^*$, $W' = Fm^* \times Fm$ and $x \ N \ (u, d)$ iff for $X \cup Y = var(x, u, d)$, B = Fm(X), C = Fm(Y), and for all partitions $x = x_B \circ x_C$, $u = u_B \circ u_c$, with $x_B, u_B \in B^*$, $x_C, u_C \in C^*$ **a** if $d \in B$, then $\vdash_{\mathbf{FL}_{\mathbf{e}}} u_C \circ x_C \Rightarrow a$ and $\vdash_{\mathbf{FL}_{\mathbf{e}}} u_B \circ x_B \circ a \Rightarrow d$, for some $a \in B \cap C$ and **b** if $d \in C$, then $\vdash_{\mathbf{FL}_{\mathbf{e}}} u_B \circ x_B \Rightarrow a$ and $\vdash_{\mathbf{FL}_{\mathbf{e}}} u_C \circ x_C \circ a \Rightarrow d$, for some $a \in B \cap C$

Residuated frames Frames and modules Frames and display **Distributive frames** Involutive FL BiFL Applications Frame applications Examples of frames: **FEP** Simple equations Simple rules Reduction to simple Simplicity preserved **FMP FEP** Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

Theorem. $\mathbf{FL}_{\mathbf{e}}$ has the Craig interpolation property, i.e. if $\vdash_{\mathbf{FL}_{\mathbf{e}}} \phi \rightarrow \psi$, then there is a χ such that $\models_{\mathbf{FL}_{\mathbf{e}}} \phi \rightarrow \chi$ and $\vdash_{\mathbf{FL}_{\mathbf{e}}} \chi \rightarrow \psi$ $= var(\chi) \subseteq var(\phi) \cap var(\psi)$.

Proof sketch. Define a frame with $W = Fm^*$, $W' = Fm^* \times Fm$ and $x \ N \ (u, d)$ iff for $X \cup Y = var(x, u, d)$, B = Fm(X), C = Fm(Y), and for all partitions $x = x_B \circ x_C$, $u = u_B \circ u_c$, with $x_B, u_B \in B^*$, $x_C, u_C \in C^*$ **a** if $d \in B$, then $\vdash_{\mathbf{FL}_{\mathbf{e}}} u_C \circ x_C \Rightarrow a$ and $\vdash_{\mathbf{FL}_{\mathbf{e}}} u_B \circ x_B \circ a \Rightarrow d$, for some $a \in B \cap C$ and **b** if $d \in C$, then $\vdash_{\mathbf{FL}_{\mathbf{e}}} u_B \circ x_B \Rightarrow a$ and $\vdash_{\mathbf{FL}_{\mathbf{e}}} u_C \circ x_C \circ a \Rightarrow d$, for some $a \in B \cap C$

Theorem. If (\mathbf{W}, \mathbf{S}) is a cut-free Gentzen frame, then every sequent valid in \mathbf{W}^+ is also valid in (\mathbf{W}, \mathbf{S}) .

Residuated frames Frames and modules Frames and display **Distributive frames** Involutive FL BiFL Applications Frame applications Examples of frames: **FEP** Simple equations Simple rules Reduction to simple Simplicity preserved **FMP FEP** Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules

FEP for DFL

Theorem. $\mathbf{FL}_{\mathbf{e}}$ has the Craig interpolation property, i.e. if $\vdash_{\mathbf{FL}_{\mathbf{e}}} \phi \rightarrow \psi$, then there is a χ such that $\vdash_{\mathbf{FL}_{\mathbf{e}}} \phi \rightarrow \chi$ and $\vdash_{\mathbf{FL}_{\mathbf{e}}} \chi \rightarrow \psi$ $\quad var(\chi) \subseteq var(\phi) \cap var(\psi).$

Proof sketch. Define a frame with $W = Fm^*$, $W' = Fm^* \times Fm$ and $x \ N \ (u, d)$ iff for $X \cup Y = var(x, u, d)$, B = Fm(X), C = Fm(Y), and for all partitions $x = x_B \circ x_C$, $u = u_B \circ u_c$, with $x_B, u_B \in B^*$, $x_C, u_C \in C^*$ **a** if $d \in B$, then $\vdash_{\mathbf{FL}_{\mathbf{e}}} u_C \circ x_C \Rightarrow a$ and $\vdash_{\mathbf{FL}_{\mathbf{e}}} u_B \circ x_B \circ a \Rightarrow d$, for some $a \in B \cap C$ and **b** if $d \in C$, then $\vdash_{\mathbf{FL}_{\mathbf{e}}} u_B \circ x_B \Rightarrow a$ and $\vdash_{\mathbf{FL}_{\mathbf{e}}} u_C \circ x_C \circ a \Rightarrow d$, for some $a \in B \cap C$

Theorem. If (\mathbf{W}, \mathbf{S}) is a cut-free Gentzen frame, then every sequent valid in \mathbf{W}^+ is also valid in (\mathbf{W}, \mathbf{S}) .

Corollary. If $\vdash_{\mathbf{FL}_{\mathbf{e}}} u \circ x \Rightarrow d$, then $u \circ x N d$. I.e., $\mathbf{FL}_{\mathbf{e}}$ has the IP.

	Residuated frames
	Frames and modules
	Frames and display
Theorem. $\mathbf{FL}_{\mathbf{e}}$ has the Disjunction property, i.e. if $\vdash_{\mathbf{FL}_{\mathbf{e}}} \phi \lor \psi$,	Distributive frames
then $\vdash_{\mathbf{FL}_{\mathbf{e}}} \phi$ or $\vdash_{\mathbf{FL}_{\mathbf{e}}} \psi$.	Involutive FL
	BiFL
	Applications
	Frame applications Examples of frames: FEP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
	Equations
	Gen. amalgamation
	Interpolation
	Disjunction property
	Strong separation: syst.
	Strong separation
	Equations for DFL
	Structural rules
	FEP for DFL

y	Residuated frames
	Frames and modules
	Frames and display
	Distributive frames
	Involutive FL
	BiFL
	Applications
	Frame applications
	Examples of frames: FEP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
	Equations
	Gen. amalgamation
	Interpolation
	Disjunction property
	Strong separation: syst.
	Strong separation
	Equations for DFL
	Structural rules
	FEP for DFL

Theorem. $\mathbf{FL}_{\mathbf{e}}$ has the Disjunction property, i.e. if $\vdash_{\mathbf{FL}_{\mathbf{e}}} \phi \lor \psi$, then $\vdash_{\mathbf{FL}_{\mathbf{e}}} \phi$ or $\vdash_{\mathbf{FL}_{\mathbf{e}}} \psi$.

Proof sketch. Define a frame with $W = Fm^*$, $W' = Fm^* \times Fm \times Fm$ and $x \ N \ (u, a, b)$ iff

$$if u \circ x \neq \varepsilon, then \vdash_{\mathbf{FL}_{\mathbf{e}}} u, x \Rightarrow a \lor b$$

If $u \circ x = \varepsilon$, then $\vdash_{\mathbf{FL}_{e}} a$ or $\vdash_{\mathbf{FL}_{e}} b$.

)	Residuated frames
	Frames and modules
	Frames and display
	Distributive frames
	Involutive FL
	BiFL
	Applications
	Frame applications
	Examples of frames:
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
	Equations
	Gen. amalgamation
	Interpolation
	Disjunction property
	Strong separation: syst.
	Strong separation
	Equations for DFL
	Structural rules
	FEP for DFL

Theorem. FL_e has the Disjunction property, i.e. if $\vdash_{\mathbf{FL}_{e}} \phi \lor \psi$, then $\vdash_{\mathbf{FL}_{e}} \phi$ or $\vdash_{\mathbf{FL}_{e}} \psi$.

Proof sketch. Define a frame with $W = Fm^*$, $W' = Fm^* \times Fm \times Fm$ and $x \ N \ (u, a, b)$ iff if $u \circ x \neq \varepsilon$, then $\vdash_{\mathbf{FL}_{\mathbf{e}}} u, x \Rightarrow a \lor b$

if $u \circ x = \varepsilon$, then $\vdash_{\mathbf{FL}_{e}} a$ or $\vdash_{\mathbf{FL}_{e}} b$.

HW23. Work out the details.

- 5	Residuated frames
	Frames and modules
	Frames and display
	Distributive frames
	Involutive FL
	BiFL
	Applications
	Frame applications
	Examples of frames: FEP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
	Equations
h	Gen. amalgamation
	Interpolation
	Disjunction property
	Strong separation: syst.
	Strong separation
	Equations for DFL
	Structural rules
	FEP for DFL

Theorem. $\mathbf{FL}_{\mathbf{e}}$ has the Disjunction property, i.e. if $\vdash_{\mathbf{FL}_{\mathbf{e}}} \phi \lor \psi$, then $\vdash_{\mathbf{FL}_{\mathbf{e}}} \phi$ or $\vdash_{\mathbf{FL}_{\mathbf{e}}} \psi$.

Proof sketch. Define a frame with $W = Fm^*$, $W' = Fm^* \times Fm \times Fm$ and $x \ N \ (u, a, b)$ iff if $u \circ x \neq \varepsilon$, then $\vdash_{\mathbf{FL}_{\mathbf{e}}} u, x \Rightarrow a \lor b$ if $u \circ x = \varepsilon$, then $\vdash_{\mathbf{FL}_{\mathbf{e}}} a \text{ or } \vdash_{\mathbf{FL}_{\mathbf{e}}} b$.

HW23. Work out the details.

The corresponding algebraic property is: For $\mathbf{A} \in \mathcal{K}$, there is a $\mathbf{D} \in \mathcal{K}$ and an epimorphism $f : \mathbf{D} \to \mathbf{A}$ such that if $1 \leq_{\mathbf{D}} a \lor B$, then $1 \leq_{\mathbf{A}} f(a)$ or $1 \leq_{\mathbf{A}} f(b)$.

-5	Residuated frames
	Frames and modules
	Frames and display
	Distributive frames
	Involutive FL
	BiFL
	Applications
	Frame applications
	Examples of frames: FEP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
	Equations
h	Gen. amalgamation
	Interpolation
	Disjunction property
	Strong separation: syst.
	Strong separation
	Equations for DFL
	Structural rules
	FEP for DFL

Theorem. FL_e has the Disjunction property, i.e. if $\vdash_{\mathbf{FL}_{e}} \phi \lor \psi$, then $\vdash_{\mathbf{FL}_{e}} \phi$ or $\vdash_{\mathbf{FL}_{e}} \psi$.

Proof sketch. Define a frame with $W = Fm^*$, $W' = Fm^* \times Fm \times Fm$ and $x \ N \ (u, a, b)$ iff if $u \circ x \neq \varepsilon$, then $\vdash_{\mathbf{FL}_{\mathbf{e}}} u, x \Rightarrow a \lor b$ if $u \circ x = \varepsilon$, then $\vdash_{\mathbf{FL}_{\mathbf{e}}} a \text{ or } \vdash_{\mathbf{FL}_{\mathbf{e}}} b$.

HW23. Work out the details.

The corresponding algebraic property is: For $\mathbf{A} \in \mathcal{K}$, there is a $\mathbf{D} \in \mathcal{K}$ and an epimorphism $f : \mathbf{D} \to \mathbf{A}$ such that if $1 \leq_{\mathbf{D}} a \lor B$, then $1 \leq_{\mathbf{A}} f(a)$ or $1 \leq_{\mathbf{A}} f(b)$.

This property holds for all subvarieties of CRL axiomatized with equations over $\{\lor, \cdot, 1\}$.

Strong separation: syst.

Assume that \mathcal{K} is a sublanguage of \mathcal{L} that contains \setminus . The system $\mathcal{K}\mathbf{FL}$ is defined to be the set of all rules from \mathbf{FL} that involve \mathcal{K} -formulas and \mathcal{K} -solvable sequents.

Frames and modules

Frames and display

Distributive frames

Involutive FL

BiFL

Applications Frame applications Examples of frames: FEP Simple equations Simple rules Reduction to simple Simplicity preserved **FMP FEP** Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules

Nick Galatos, Prague workshop, March, 2014

FEP for DFL

Strong separation: syst.

Assume that \mathcal{K} is a sublanguage of \mathcal{L} that contains \backslash . The system $\mathcal{K}\mathbf{FL}$ is defined to be the set of all rules from \mathbf{FL} that involve \mathcal{K} -formulas and \mathcal{K} -solvable sequents.

Given a structure $(\mathbf{W}, \mathbf{A}_{\mathcal{K}})$ and a meta-rule (r) of $\mathcal{K}\mathbf{FL}$, we define $(r)^{(\mathbf{W}, \mathbf{A}_{\mathcal{K}})}$. For example, $(\backslash L)^{(\mathbf{W}, \mathbf{A}_{\mathcal{K}})}$ is

 $\forall a, b, c \in \mathbf{A}, x \in W, uS_W$, if $a \setminus_{\mathbf{A}} b$ is defined, then $x \ N \ a$ and $u[b] \ N \ c$ implies $u[x \circ (a \setminus b)] \ N \ c$.

Frames and modules Frames and display Distributive frames Involutive FL BiFL Applications Frame applications Examples of frames: **FEP** Simple equations Simple rules Reduction to simple Simplicity preserved **FMP FEP** Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

Residuated frames

Strong separation: syst.

Assume that \mathcal{K} is a sublanguage of \mathcal{L} that contains \backslash . The system $\mathcal{K}\mathbf{FL}$ is defined to be the set of all rules from \mathbf{FL} that involve \mathcal{K} -formulas and \mathcal{K} -solvable sequents.

Given a structure $(\mathbf{W}, \mathbf{A}_{\mathcal{K}})$ and a meta-rule (r) of $\mathcal{K}\mathbf{FL}$, we define $(r)^{(\mathbf{W}, \mathbf{A}_{\mathcal{K}})}$. For example, $(\backslash L)^{(\mathbf{W}, \mathbf{A}_{\mathcal{K}})}$ is

 $\forall a, b, c \in \mathbf{A}, x \in W, uS_W$, if $a \setminus_{\mathbf{A}} b$ is defined, then $x \ N \ a$ and $u[b] \ N \ c$ implies $u[x \circ (a \setminus b)] \ N \ c$.

Lemma A structure $(\mathbf{W}, \mathbf{A}_{\mathcal{K}})$ Gentzen frame iff the interpretation $(r)^{(\mathbf{W}, \mathbf{A}_{\mathcal{K}})}$ of every meta-rule (r) of $\mathcal{K}\mathbf{FL}$ holds.

Frames and display Distributive frames Involutive FL BiFL Applications Frame applications Examples of frames: **FEP** Simple equations Simple rules Reduction to simple Simplicity preserved **FMP FEP** Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

Residuated frames

Frames and modules

Strong separation

Let \mathcal{K} be a sublanguage of \mathcal{L} that contains the connective \setminus and let $B \cup \{c\}$ be a set of formulas over \mathcal{K} . Also, let $\mathbf{A}_{\mathcal{K}}$ be the partial subalgebra of $\mathbf{Fm}_{\mathcal{K}}$ of all subformulas of $B \cup \{c\}$. Consider the structure $(\mathbf{W}, \mathbf{A}_{\mathcal{K}})$, where W is the free monoid over $\mathbf{A}_{\mathcal{K}}$, $W' = S_W \times \mathbf{A}_{\mathcal{K}}$ and where $x \ N \ (u, a)$ iff $B \vdash_{\mathcal{K}\mathbf{HL}} \phi_{\mathcal{K}}(u(x) \Rightarrow a)$.

Residuated frames Frames and modules Frames and display Distributive frames Involutive FL BiFL Applications Frame applications Examples of frames: FFP Simple equations Simple rules Reduction to simple Simplicity preserved **FMP FEP** Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

Strong separation

Let \mathcal{K} be a sublanguage of \mathcal{L} that contains the connective \setminus and let $B \cup \{c\}$ be a set of formulas over \mathcal{K} . Also, let $\mathbf{A}_{\mathcal{K}}$ be the partial subalgebra of $\mathbf{Fm}_{\mathcal{K}}$ of all subformulas of $B \cup \{c\}$. Consider the structure $(\mathbf{W}, \mathbf{A}_{\mathcal{K}})$, where W is the free monoid over $\mathbf{A}_{\mathcal{K}}$, $W' = S_W \times \mathbf{A}_{\mathcal{K}}$ and where $x \ N(u, a)$ iff $B \vdash_{\mathcal{K}\mathbf{HL}} \phi_{\mathcal{K}}(u(x) \Rightarrow a)$.

Corollary. Let \mathcal{K} be a sublanguage of \mathcal{L} that contains the connective \setminus and let $B \cup \{c\}$ be a set of formulas over \mathcal{K} . Then $(\mathbf{W}, \mathbf{A}_{\mathcal{K}})$ is a Gentzen frame.

Residuated frames Frames and modules Frames and display Distributive frames Involutive FL **BiFL** Applications Frame applications Examples of frames: FFP Simple equations Simple rules Reduction to simple Simplicity preserved **FMP FEP** Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

Strong separation

Let \mathcal{K} be a sublanguage of \mathcal{L} that contains the connective \setminus and let $B \cup \{c\}$ be a set of formulas over \mathcal{K} . Also, let $\mathbf{A}_{\mathcal{K}}$ be the partial subalgebra of $\mathbf{Fm}_{\mathcal{K}}$ of all subformulas of $B \cup \{c\}$. Consider the structure $(\mathbf{W}, \mathbf{A}_{\mathcal{K}})$, where W is the free monoid over $\mathbf{A}_{\mathcal{K}}$, $W' = S_W \times \mathbf{A}_{\mathcal{K}}$ and where $x \ N \ (u, a)$ iff $B \vdash_{\mathcal{K}\mathbf{HL}} \phi_{\mathcal{K}}(u(x) \Rightarrow a)$.

Corollary. Let \mathcal{K} be a sublanguage of \mathcal{L} that contains the connective \setminus and let $B \cup \{c\}$ be a set of formulas over \mathcal{K} . Then $(\mathbf{W}, \mathbf{A}_{\mathcal{K}})$ is a Gentzen frame.

Corollary If $B \cup \{c\}$ is a set of formulas over a sublanguage \mathcal{K} of \mathcal{L} that contains \setminus , then $B \vdash_{\mathbf{HL}} c$ iff $B \vdash_{\mathcal{K}-\mathbf{HL}} c$. In particular, the Hilbert system **HL** enjoys the separation property.

Residuated frames Frames and modules Frames and display Distributive frames Involutive FL BiFL Applications Frame applications Examples of frames: FFP Simple equations Simple rules Reduction to simple Simplicity preserved **FMP FEP** Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

	Residuated frames
	Frames and modules
	Frames and display
Idea: Express equations over $\{\land,\lor,\cdot,1\}$ at the frame level.	Distributive frames
	Involutive FL
	BiFL
	Applications
	Frame applications
	Examples of frames: FEP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
	Equations
	Gen. amalgamation
	Interpolation
	Disjunction property
	Strong separation: syst.
	Strong separation
	Equations for DFL
	Structural rules
	FEP for DFL

	Residuated frames
	Frames and modules
	Frames and display
Idea: Express equations over $\{\land,\lor,\cdot,1\}$ at the frame level.	Distributive frames
	Involutive FL
For an equation ε over $\{\wedge, \lor, \cdot, 1\}$ we distribute products and meets	BiFL
over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. $s_i, t_j : \{\wedge, \cdot, 1\}$ -terms.	Applications
	Frame applications
	Examples of frames: FEP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
	Equations
	Gen. amalgamation
	Interpolation
	Disjunction property
	Strong separation: syst.
	Strong separation
	Equations for DFL
	Structural rules
	FEP for DFL

	Frames and modules
	Frames and display
Idea: Express equations over $\{\land,\lor,\cdot,1\}$ at the frame level.	Distributive frames
$\Box_{i} = \left\{ \left\{ \left\{ x \in \mathcal{X}_{i} \right\} \right\} = \left\{ \left\{ \left\{ x \in \mathcal{X}_{i} \right\} \right\} = \left\{ \left\{ x \in \mathcal{X}_{i} \right\} \right\} = \left\{ \left\{ x \in \mathcal{X}_{i} \right\} \right\} = \left\{ \left\{ x \in \mathcal{X}_{i} \right\} = \left\{ x \in \mathcal$	Involutive FL
For an equation ε over $\{\wedge, \vee, \cdot, 1\}$ we distribute products and meets	BiFL
over joins to get $s_1 \lor \cdots \lor s_m = t_1 \lor \cdots \lor t_n$. $s_i, t_j \colon \{\land, \cdot, 1\}$ -terms.	Applications
	Frame applications
$s_1 \vee \cdots \vee s_m \leq t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leq s_1 \vee \cdots \vee s_m$.	Examples of frames: FEP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
	Equations
	Gen. amalgamation
	Interpolation
	Disjunction property
	Strong separation: syst.
	Strong separation
	Equations for DFL
	Structural rules
	FEP for DFL

Residuated frames

	Frames and modules
	Frames and display
Idea: Express equations over $\{\land,\lor,\cdot,1\}$ at the frame level.	Distributive frames
	Involutive FL
For an equation ε over $\{\wedge, \lor, \cdot, 1\}$ we distribute products and meets	BiFL
over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. $s_i, t_j: \{\wedge, \cdot, 1\}$ -terms.	Applications
	Frame applications
$s_1 \lor \cdots \lor s_m \leq t_1 \lor \cdots \lor t_n$ and $t_1 \lor \cdots \lor t_n \leq s_1 \lor \cdots \lor s_m$.	Examples of frames: FEP
The first is equivalent to $\theta_{-}(z, z, t, y) = y(t, y)$	Simple equations
The first is equivalent to: $\&(s_j \leq t_1 \lor \cdots \lor t_n).$	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
	Equations
	Gen. amalgamation
	Interpolation
	Disjunction property
	Strong separation: syst.
	Strong separation
	Equations for DFL
	Structural rules
	FEP for DFL

Residuated frames

Idea: Express equations over $\{\wedge, \lor, \cdot, 1\}$ at the frame level.
For an equation ε over $\{\wedge, \lor, \cdot, 1\}$ we distribute products and meets over joins to get $s_1 \lor \cdots \lor s_m = t_1 \lor \cdots \lor t_n$. s_i, t_j : $\{\wedge, \cdot, 1\}$ -terms.
$s_1 \lor \cdots \lor s_m \leq t_1 \lor \cdots \lor t_n$ and $t_1 \lor \cdots \lor t_n \leq s_1 \lor \cdots \lor s_m$.
The first is equivalent to: $\&(s_j \leq t_1 \lor \cdots \lor t_n).$
We proceed by example: $x^2 \wedge y \leq (x \wedge y) \lor yx$

Involutive FL BiFL Applications Frame applications Examples of frames: FEP Simple equations Simple rules Reduction to simple Simplicity preserved FMP FEP Amalgamation Maehara frame Equations Gen. amalgamation Interpolation Disjunction property Strong separation: syst. Strong separation Equations for DFL Structural rules FEP for DFL

Residuated frames

Frames and modules

Frames and display

Distributive frames

	Frames and modules
	Frames and display
e frame level.	Distributive frames
	Involutive FL
te products and meets	BiFL
$s_i, t_j: \{\wedge, \cdot, 1\}$ -terms.	Applications
	Frame applications
$\leq s_1 \lor \cdots \lor s_m.$	Examples of frames: FEP
	Simple equations
(n).	Simple rules
	Reduction to simple
x	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
	Equations
	Gen. amalgamation
	Interpolation
	Disjunction property
	Strong separation: syst.
	Strong separation
	Equations for DFL
	Structural rules

Residuated frames

For an equation ε over $\{\wedge, \lor, \cdot, 1\}$ we distribute products and meets over joins to get $s_1 \lor \cdots \lor s_m = t_1 \lor \cdots \lor t_n$. s_i, t_j : $\{\wedge, \cdot, 1\}$ -terms

 $s_1 \lor \cdots \lor s_m \leq t_1 \lor \cdots \lor t_n$ and $t_1 \lor \cdots \lor t_n \leq s_1 \lor \cdots \lor s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \lor \cdots \lor t_n).$

We proceed by example: $x^2 \wedge y \leq (x \wedge y) \vee yx$

 $(x_1 \lor x_2)^2 \land y \le [(x_1 \lor x_2) \land y] \lor y(x_1 \lor x_2)$

FEP for DFL

	Frames and modules
	Frames and display
frame level.	Distributive frames
	Involutive FL
e products and meets	BiFL
s_i, t_j : $\{\land, \cdot, 1\}$ -terms.	Applications
-	Frame applications
$\leq s_1 \lor \cdots \lor s_m.$	Examples of frames: FEP
\ \	Simple equations
).	Simple rules
	Reduction to simple
c	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
$(y) \lor (x_2 \land y) \lor y x_1 \lor y x_2$	Equations
	Gen. amalgamation
	Interpolation
	Disjunction property
	Strong separation: syst.
	Strong separation
	Equations for DFL
	Structural rules
	FEP for DFL

Residuated frames

Idea: Express equations over $\{\land,\lor,\cdot,1\}$ at the frame level.

For an equation ε over $\{\wedge, \lor, \cdot, 1\}$ we distribute products and meets over joins to get $s_1 \lor \cdots \lor s_m = t_1 \lor \cdots \lor t_n$. $s_i, t_j \colon \{\wedge, \cdot, 1\}$ -terms

 $s_1 \lor \cdots \lor s_m \leq t_1 \lor \cdots \lor t_n$ and $t_1 \lor \cdots \lor t_n \leq s_1 \lor \cdots \lor s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \lor \cdots \lor t_n).$

We proceed by example: $x^2 \wedge y \leq (x \wedge y) \vee yx$

 $(x_1 \lor x_2)^2 \land y \le [(x_1 \lor x_2) \land y] \lor y(x_1 \lor x_2)$

	Frames and modules
$[A \to f] = \{A \to f\} = \{A \to f\}$	Frames and display
Idea: Express equations over $\{\land, \lor, \cdot, 1\}$ at the frame level.	Distributive frames
	Involutive FL
For an equation ε over $\{\wedge, \lor, \cdot, 1\}$ we distribute products and meets	BiFL
over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. $s_i, t_j: \{\wedge, \cdot, 1\}$ -terms.	Applications
	Frame applications
$s_1 \lor \cdots \lor s_m \leq t_1 \lor \cdots \lor t_n$ and $t_1 \lor \cdots \lor t_n \leq s_1 \lor \cdots \lor s_m$.	Examples of frames: FEP
The first is equivalent to $\theta_{-}(z, z, t, y, t, y, t, y)$	Simple equations
The first is equivalent to: $\alpha(s_j \leq t_1 \vee \cdots \vee t_n)$.	Simple rules
	Reduction to simple
We proceed by example: $x^2 \wedge y \leq (x \wedge y) \vee yx$	Simplicity preserved
	FMP
$(x_1 \lor x_2)^2 \land y \le [(x_1 \lor x_2) \land y] \lor y(x_1 \lor x_2)$	Amalgamation
	Maehara frame
$(r_1^2 \wedge u) \vee (r_1 r_2 \wedge u) \vee (r_2 r_1 \wedge u) \vee (r_2^2 \wedge u) < (r_1 \wedge u) \vee (r_2 \wedge u) \vee (ur_1 \vee ur_2)$	Equations
$(x_1, y_2) \land (x_1, x_2, y_2) \land (x_2, y_2) \land (x_2, y_2) \rightharpoonup (x_1, y_2) \land (x_2, y_2) \land y_2 \land$	Gen. amalgamation
$m = (m + \alpha) / (m + \alpha) / (m + \alpha) / \alpha = (m +$	Interpolation
$x_1x_2 \land y \ge (x_1 \land y) \lor (x_2 \land y) \lor yx_1 \lor yx_2$	Disjunction property

Strong separation: syst.

Strong separation Equations for DFL Structural rules FEP for DFL

Residuated frames

	Residuated frames
	Frames and modules
	Frames and display
ne level.	Distributive frames
1	Involutive FL
oducts and meets	BiFL
$j: \{\wedge, \cdot, 1\}$ -terms.	Applications
	Frame applications
$\vee \cdots \vee s_m$.	Examples of frames: FEP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
$(x_2 \land y) \lor y x_1 \lor y x_2$	Equations
· · · · · · · · · · · · · · · · · · ·	Gen. amalgamation
	Interpolation
	Disjunction property
	Strong separation: syst.
$c_2 \leq v$	Strong separation
	Equations for DFL
	Structural rules
	FEP for DFL

Idea: Express equations over $\{\wedge,\vee,\cdot,1\}$ at the frame level.

For an equation ε over $\{\wedge, \lor, \cdot, 1\}$ we distribute products and meets over joins to get $s_1 \lor \cdots \lor s_m = t_1 \lor \cdots \lor t_n$. s_i, t_j : $\{\wedge, \cdot, 1\}$ -terms

 $s_1 \lor \cdots \lor s_m \leq t_1 \lor \cdots \lor t_n$ and $t_1 \lor \cdots \lor t_n \leq s_1 \lor \cdots \lor s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \lor \cdots \lor t_n).$

We proceed by example: $x^2 \wedge y \leq (x \wedge y) \vee yx$

 $(x_1 \lor x_2)^2 \land y \le [(x_1 \lor x_2) \land y] \lor y(x_1 \lor x_2)$

 $(x_1^2 \wedge y) \lor (x_1 x_2 \wedge y) \lor (x_2 x_1 \wedge y) \lor (x_2^2 \wedge y) \le (x_1 \wedge y) \lor (x_2 \wedge y) \lor y x_1 \lor y x_2$ $x_1 x_2 \wedge y \le (x_1 \wedge y) \lor (x_2 \wedge y) \lor y x_1 \lor y x_2$

$$\frac{x_1 \wedge y \leq v \quad x_2 \wedge y \leq v \quad yx_1 \leq v \quad yx_2 \leq v}{x_1 x_2 \wedge y \leq v}$$

	Residuated frames
	Frames and modules
	Frames and display
vel.	Distributive frames
	Involutive FL
ts and meets	BiFL
$\wedge, \cdot, 1\}$ -terms.	Applications
	Frame applications
$\cdot \lor s_m$.	Examples of frames: FEP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
<u>`</u>	Maehara frame
$(y) \lor y x_1 \lor y x_2$	Equations
0, 0 - 0 -	Gen. amalgamation
	Interpolation
	Disjunction property
	Strong separation: syst.
v	Strong separation
_	Equations for DFL
	FEP TOP UFL

Idea: Express equations over
$$\{\land,\lor,\cdot,1\}$$
 at the frame level.

For an equation ε over $\{\wedge, \lor, \cdot, 1\}$ we distribute products and meets over joins to get $s_1 \lor \cdots \lor s_m = t_1 \lor \cdots \lor t_n$. s_i, t_j : $\{\wedge, \cdot, 1\}$ -terms

 $s_1 \lor \cdots \lor s_m \leq t_1 \lor \cdots \lor t_n$ and $t_1 \lor \cdots \lor t_n \leq s_1 \lor \cdots \lor s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \lor \cdots \lor t_n).$

We proceed by example: $x^2 \wedge y \leq (x \wedge y) \vee yx$

 $(x_1 \lor x_2)^2 \land y \le [(x_1 \lor x_2) \land y] \lor y(x_1 \lor x_2)$

 $(x_1^2 \wedge y) \lor (x_1 x_2 \wedge y) \lor (x_2 x_1 \wedge y) \lor (x_2^2 \wedge y) \le (x_1 \wedge y) \lor (x_2 \wedge y) \lor y x_1 \lor y x_2$ $x_1 x_2 \wedge y \le (x_1 \wedge y) \lor (x_2 \wedge y) \lor y x_1 \lor y x_2$

$$\frac{x_1 \wedge y \leq v \quad x_2 \wedge y \leq v \quad yx_1 \leq v \quad yx_2 \leq v}{x_1 x_2 \wedge y \leq v}$$

$$\frac{x_1 \bigotimes y \ N \ z \quad x_2 \bigotimes y \ N \ z \quad y \circ x_1 \ N \ z \quad y \circ x_2 \ N \ z}{x_1 \circ x_2 \bigotimes y \ N \ z} \ R(\varepsilon)$$

Structural rules

Residuated frames
 Frames and modules
Frames and display
Distributive frames
Involutive FL
BiFL
Applications
Frame applications
Examples of frames: FEP
Simple equations
Simple rules
Reduction to simple
Simplicity preserved
FMP
FEP
Amalgamation
Maehara frame
Equations
Gen. amalgamation
Interpolation
Disjunction property
Strong separation: syst.
Strong separation
Equations for DFL
Structural rules
FEP for DFL

Given an equation ε of the form $t_0 \leq t_1 \vee \cdots \vee t_n$, where t_i are $\{\wedge, \cdot, 1\}$ -terms we construct the rule $R(\varepsilon)$

$$\frac{u[t_1] \Rightarrow a \cdots u[t_n] \Rightarrow a}{u[t_0] \Rightarrow a} (R(\varepsilon))$$

where the t_i 's are evaluated in (W, \circ, ε) . Such a rule is called *analytic* if all variables in t_0 are distinct.

Structural rules

Given an equation ε of the form $t_0 \leq t_1 \vee \cdots \vee t_n$, where t_i are $\{\wedge, \cdot, 1\}$ -terms we construct the rule $R(\varepsilon)$

$$\frac{u[t_1] \Rightarrow a \cdots u[t_n] \Rightarrow a}{u[t_0] \Rightarrow a} (R(\varepsilon))$$

where the t_i 's are evaluated in (W, \circ, ε) . Such a rule is called *analytic* if all variables in t_0 are distinct.

Theorem. If (\mathbf{W}, \mathbf{B}) is a Gentzen frame and ε an equation over $\{\wedge, \lor, \cdot, 1\}$, then (\mathbf{W}, \mathbf{B}) satisfies $\mathsf{R}(\varepsilon)$ iff \mathbf{W}^+ satisfies ε .

(The linearity of the denominator of $R(\varepsilon)$ plays an important role in the proof.)

Residuated frames
Frames and modules
Frames and display
Distributive frames
Involutive FL
BiFL
Applications
Frame applications
Examples of frames: FEP
Simple equations
Simple rules
Reduction to simple
Simplicity preserved
FMP
FEP
Amalgamation
Maehara frame
Equations
Gen. amalgamation
Interpolation
Disjunction property
Strong separation: syst.
Strong separation
Equations for DFL
Structural rules
FEP for DFL
Structural rules

Given an equation ε of the form $t_0 \leq t_1 \vee \cdots \vee t_n$, where t_i are $\{\wedge, \cdot, 1\}$ -terms we construct the rule $R(\varepsilon)$

$$\frac{u[t_1] \Rightarrow a \cdots u[t_n] \Rightarrow a}{u[t_0] \Rightarrow a} (R(\varepsilon))$$

where the t_i 's are evaluated in (W, \circ, ε) . Such a rule is called *analytic* if all variables in t_0 are distinct.

Theorem. If (\mathbf{W}, \mathbf{B}) is a Gentzen frame and ε an equation over $\{\wedge, \lor, \cdot, 1\}$, then (\mathbf{W}, \mathbf{B}) satisfies $\mathsf{R}(\varepsilon)$ iff \mathbf{W}^+ satisfies ε .

(The linearity of the denominator of $R(\varepsilon)$ plays an important role in the proof.)

Theorem. Every system obtained from \mathbf{FL} by adding analytic rules has the cut elimination property.

	Residuated frames
_	Frames and modules
	Frames and display
	Distributive frames
	Involutive FL
	BiFL
	Applications
	Frame applications
	Examples of frames:
	FEP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
	Equations
	Gen. amalgamation
	Interpolation
	Disjunction property
	Strong separation: syst.
	Strong separation
	Equations for DFL
	Structural rules
	FEP for DFL

FEP for DFL

Let \mathcal{V} be a subvariety of DIRL axiomatized over $\{\lor, \land, \cdot, 1\}$. To establish the FEP for \mathcal{V} , for every \mathbf{A} in \mathcal{V} and \mathbf{B} a finite partial subalgebra of \mathbf{A} , we construct an algebra $\mathbf{D} = \mathbf{W}_{\mathbf{A},\mathbf{B}}^+$ such that

- $\blacksquare \quad \mathbf{W}_{\mathbf{A},\mathbf{B}}^+ \in \mathcal{V}$
- **B** embeds in $\mathbf{W}_{\mathbf{A},\mathbf{B}}^+$
- **W** $^+_{\mathbf{A},\mathbf{B}}$ is finite

Frames and modules
Frames and display
Distributive frames
Involutive FL
BiFL
Applications
Frame applications
Examples of frames: FEP
Simple equations
Simple rules
Reduction to simple
Simplicity preserved
FMP
FEP
Amalgamation
Maehara frame
Equations
Gen. amalgamation
Interpolation
Disjunction property
Strong separation: syst.
Strong separation
Equations for DFL
Structural rules
FEP for DFL

Residuated frames

Let \mathcal{V} be a subvariety of DIRL axiomatized over $\{\lor, \land, \cdot, 1\}$. To establish the FEP for \mathcal{V} , for every \mathbf{A} in \mathcal{V} and \mathbf{B} a finite partial subalgebra of \mathbf{A} , we construct an algebra $\mathbf{D} = \mathbf{W}_{\mathbf{A},\mathbf{B}}^+$ such that

- $\blacksquare \quad \mathbf{W}_{\mathbf{A},\mathbf{B}}^+ \in \mathcal{V}$
- **B** embeds in $\mathbf{W}_{\mathbf{A},\mathbf{B}}^+$
- $\blacksquare \quad \mathbf{W}^+_{\mathbf{A},\mathbf{B}} \text{ is finite}$

 $\mathbf{W}_{\mathbf{A},\mathbf{B}}^+$ is defined by taking $(W, \circ, \bigotimes, 1)$ to be the $\{\cdot, \wedge, 1\}$ -subreduct of \mathbf{A} generated by $B, W' = S_W \times B$ and x N(u, b) iff $u(x) \leq_{\mathbf{A}} b$.

FEP for DFL

Frames and modules
Frames and display
Distributive frames
Involutive FL
BiFL
Applications
Frame applications
Examples of frames:
FEP
Simple equations
Simple rules
Reduction to simple
Simplicity preserved
FMP
FEP
Amalgamation
Maehara frame
Equations
Gen. amalgamation
Interpolation
Disjunction property
Strong separation: syst.
Strong separation
Equations for DFL
Structural rules
FEP for DFL

Residuated frames

Let \mathcal{V} be a subvariety of DIRL axiomatized over $\{\lor, \land, \cdot, 1\}$. To establish the FEP for \mathcal{V} , for every \mathbf{A} in \mathcal{V} and \mathbf{B} a finite partial subalgebra of \mathbf{A} , we construct an algebra $\mathbf{D} = \mathbf{W}_{\mathbf{A},\mathbf{B}}^+$ such that

- $\blacksquare \quad \mathbf{W}_{\mathbf{A},\mathbf{B}}^+ \in \mathcal{V}$
- **B** embeds in $\mathbf{W}_{\mathbf{A},\mathbf{B}}^+$
- $\blacksquare \quad \mathbf{W}^+_{\mathbf{A},\mathbf{B}} \text{ is finite}$

 $\mathbf{W}_{\mathbf{A},\mathbf{B}}^+$ is defined by taking $(W, \circ, \bigotimes, 1)$ to be the $\{\cdot, \wedge, 1\}$ -subreduct of \mathbf{A} generated by $B, W' = S_W \times B$ and x N(u, b) iff $u(x) \leq_{\mathbf{A}} b$.

Theorem. Every subvariety of DIRL axiomatized over $\{\lor, \land, \cdot, 1\}$ has the FEP.

	Residuated frames
	Frames and modules
	Frames and display
	Distributive frames
	Involutive El
	BIFL
	Applications
	Frame applications
	Examples of frames:
	FEP
	Simple equations
	Simple rules
	Reduction to simple
	Simplicity preserved
	FMP
	FEP
	Amalgamation
	Maehara frame
	Equations
	Gen. amalgamation
	Interpolation
	Disjunction property
5	Strong separation: syst.
	Strong separation
	Equations for DFL
	Structural rules
	FEP for DFL