Display calculi in non-classical logics

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Hilbert's Program (around 1922)

- Proofs are the essence of mathematics—to establish a theorem.. present a proof!
- Historically, proofs were not the objects of mathematical investigations (unlike numbers, triangles. . .)
- Foundational crisis of mathematics (early 1900s)—formal development of the logical systems underlying mathematics
- In Hilbert's *Proof theory*: proofs are mathematical objects.

Hilbert calculus

- Mathematical investigation of proofs $\rightsquigarrow$ formal definition of proof
- Hilbert calculus fulfils this role.

A Hilbert calculus for propositional classical logic. Axiom schemata:

$$\begin{array}{ll} \text{Ax 1:} & A \to (B \to A) \\ \text{Ax 2:} & (A \to (B \to C)) \to ((A \to B) \to (A \to C)) \\ \text{Ax 3:} & (\neg A \to \neg B) \to ((\neg A \to B) \to A) \end{array}$$

and the rule of *modus ponens*:

$$\frac{A \qquad A \to B}{B}$$

Read $A \leftrightarrow B$ as $(A \to B) \land (B \to A)$. More axioms:

$$\text{Ax 4:} \quad A \lor B \leftrightarrow (\neg A \to B) \qquad\qquad \text{Ax 5:} \quad A \land B \leftrightarrow \neg(A \to \neg B)$$

Derivation of $A \to A$

Definition

A *formal proof* (derivation) of B is the finite sequence $C_1, C_2, \ldots, C_n \equiv B$ of formulae where each element C_j is an axiom instance or follows from two earlier elements by *modus ponens*.

1	$((A \to ((A \to A) \to A)) \to ((A \to (A \to A)) \to (A \to A)))$	Ax 2
2	$(A \to ((A \to A) \to A))$	Ax 1
3	$((A \to (A \to A)) \to (A \to A))$	MP: 1 and 2
4	$(A \to (A \to A))$	Ax 1
5	$A \to A$	MP: 3 and 4

Not easy to find! Proof has no clear structure (wrt $A \to A$)

Natural deduction and the sequent calculus

- Gentzen: proving consistency of arithmetic in weak extensions of finitistic reasoning.
- Hilbert calculus not convenient for studying the proofs (lack of structure). Gentzen introduces *Natural deduction* which formalises the way mathematicians reason.
- Gentzen introduced a proof-formalism with even more structure: the sequent calculus.
- Sequent calculus built from sequents $X \vdash Y$ where X, Y are lists/sets/multisets of formulae

Sequent calculus

sequent: $$\overbrace{A_1, A_2, \ldots, A_m}^{\textit{antecedent}} \underbrace{\quad\vdash\quad}_{\textit{turnstile}} \overbrace{B_1, B_2, \ldots, B_n}^{\textit{succedent}}$$

sequent calculus rule:
($\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_k$
are sequents)

$$\frac{\overbrace{\mathcal{S}_1 \qquad \ldots \qquad \mathcal{S}_k}^{k \geq 0 \textit{ premises}}}{\underbrace{\mathcal{S}}_{\textit{conclusion}}}$$

- Typically a rule for introducing each connective in the antecedent and succedent.
- A 0-premise rule is called an *initial sequent*

Definition (derivation)

A *derivation* in the sequent calculus is an initial sequent or a rule applied to derivations of the premise(s).

The sequent calculus $\mathcal{S}$Cp for classical logic *Cp*

$$\dfrac{}{p, X \vdash Y, p}\ \text{init} \qquad \dfrac{}{\bot, X \vdash Y}\ \bot\text{l}$$

$$\dfrac{X \vdash Y, A}{\neg A, X \vdash Y}\ \neg l \qquad \dfrac{A, X \vdash Y}{X \vdash Y, \neg A}\ \neg r$$

$$\dfrac{A, B, X \vdash Y}{A \wedge B, X \vdash Y}\ \wedge l \qquad \dfrac{X \vdash Y, A \qquad X \vdash Y, B}{X \vdash Y, A \wedge B}\ \wedge r$$

$$\dfrac{A, X \vdash Y \qquad B, X \vdash Y}{A \vee B, X \vdash Y}\ \vee l \qquad \dfrac{X \vdash Y, A, B}{X \vdash Y, A \vee B}\ \vee r$$

$$\dfrac{X \vdash Y, A \qquad B, X \vdash Y}{A \rightarrow B, X \vdash Y}\ \rightarrow l \qquad \dfrac{A, X \vdash Y, B}{X \vdash Y, A \rightarrow B}\ \rightarrow\text{r}$$

- Here X, Y are sets of formulae (possibly empty)
- There is a rule introducing each connective in the antecedent, succedent
- Aside: this calculus differs from Gentzen's calculus

Soundness and completeness of $\mathcal{S}Cp$ for Cp

Need to prove that $\mathcal{S}Cp$ is actually a sequent calculus for Cp.

Theorem

For every formula A we have: $\vdash A$ is derivable in $\mathcal{S}Cp \Leftrightarrow A \in Cp$.

($\Rightarrow$) direction is soundness.

($\Leftarrow$) direction is completeness.

Proof of completeness

Need to show: $A \in Cp \Rightarrow \vdash A$ derivable in $\mathcal{S}Cp$.

First show that $A, X \vdash Y, A$ is derivable (induction on size of A).

Show that every axiom of Cp is derivable (easy, below) and *modus ponens* can be simulated in $\mathcal{S}Cp$ (not clear)

$$
\begin{prooftree}
\AxiomC{$A, A \rightarrow (B \rightarrow C) \vdash C, A$}
\AxiomC{$B, A \vdash C, A$}
\AxiomC{$B, A \vdash C, B$}
\AxiomC{$r, B, A \vdash C$}
\BinaryInfC{$B \rightarrow C, B, A \vdash C$}
\BinaryInfC{$B, A, A \rightarrow (B \rightarrow C) \vdash C$}
\BinaryInfC{$A, A \rightarrow B, (A \rightarrow (B \rightarrow C)) \vdash C$}
\UnaryInfC{$A \rightarrow B, (A \rightarrow (B \rightarrow C)) \vdash (A \rightarrow C)$}
\UnaryInfC{$(A \rightarrow (B \rightarrow C)) \vdash (A \rightarrow B) \rightarrow (A \rightarrow C)$}
\UnaryInfC{$\vdash (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$}
\end{prooftree}
$$

How to simulate *modus ponens*

Gentzen's solution: to simulate *modus ponens* (below left) first add a new rule (below right) to $\mathcal{S}Cp$:

$$\frac{A \qquad A \to B}{B} \qquad\qquad \frac{X \vdash Y, A \qquad A, X \vdash Y}{X \vdash Y}\ cut$$

The following instance of the cut-rule illustrates the simulation of *modus ponens*.

$$\dfrac{\vdash A \qquad \dfrac{\vdash A \to B \qquad \dfrac{A \vdash A \qquad B \vdash B}{A \to B, A \vdash B}}{A \vdash B}}{\vdash B}\ cut$$

So: $A \in Cp \Rightarrow \vdash A$ derivable in $\mathcal{S}Cp + cut$!

Proof of soundness

Need to show: $\vdash A$ derivable in $\mathcal{S}Cp + cut \Rightarrow A \in \mathcal{S}Cp$.

We need to interpret $\mathcal{S}Cp + cut$ derivations in Cp.

For sequent S	$A_1, A_2, \ldots, A_m \vdash B_1, B_2, \ldots, B_n$
define translation $\tau(S)$	$A_1 \wedge A_2 \wedge \ldots \wedge A_m \rightarrow B_1 \vee B_2 \vee \ldots \vee B_n$

Comma on the left is conjunction, comma on the right is disjunction.

Translations of the intial sequents are theorems of Cp

$$p \wedge X \rightarrow Y \vee p \qquad\qquad \bot \wedge X \rightarrow Y$$

Show for each remaining rule ρ: if the translation of every premise is a theorem of Cp then so is the translation of the conclusion.

For $\dfrac{A, X \vdash B}{X \vdash A \rightarrow B}$ need to show: $\dfrac{A \wedge X \rightarrow B}{X \rightarrow (A \rightarrow B)}$

The cut-rule is undesirable in $\mathcal{S}Cp + cut$

We have shown

Theorem

For every formula A we have: $\vdash A$ is derivable in $\mathcal{S}Cp + cut$ $\Leftrightarrow$ $A \in Cp$.

- The *subformula property* states that every formua in a premise appears as a subformula of the conclusion.
- If all the rules of the calculus satisfy this property, the calculus is *analytic*
- Analyticity is crucial to using the calculus (for consistency, decidability. . .)
- $\mathcal{S}Cp + cut$ is *not* analytic because:

$$\frac{X \vdash Y, A \qquad A, X \vdash Y}{X \vdash Y}\; cut$$

- We want to show: $\vdash A$ is derivable in $\mathcal{S}Cp$ $\Leftrightarrow$ $A \in Cp$

Gentzen's *Hauptsatz* (main theorem): cut-elimination

Theorem

Suppose that δ is a derivation of $X \vdash Y$ in $\mathcal{S}Cp + cut$. Then there is a transformation to eliminate instances of the cut-rule from δ to obtain a derivation δ' of $X \vdash Y$ in $\mathcal{S}Cp$.

Since $\vdash A$ is derivable in $\mathcal{S}Cp + cut \Leftrightarrow A \in Cp$:

Theorem

For every formula A we have: $\vdash A$ is derivable in $\mathcal{S}Cp$ if and only if $A \in Cp$.

Applications: Consistency of classical logic

Consistency of classical logic is the statement that $A \wedge \neg A \notin Cp$.

Theorem

Classical logic is consistent.

Proof by contradiction. Suppose that $A \wedge \neg A \in Cp$. Then $A \wedge \neg A$ is derivable in $\mathcal{S}Cp$ (completeness). Let us try to derive it (read upwards from $\vdash A \wedge \neg A$):

$$\dfrac{\vdash A \qquad \dfrac{A \vdash}{\vdash \neg A}}{\vdash A \wedge \neg A}$$

So $\vdash A$ and $A \vdash$ are derivable. Thus $\vdash$ must be derivable in $\mathcal{S}Cp + cut$ (use cut) and hence in $\mathcal{S}Cp$ (by cut-elimination). This is impossible (why?) QED.

Theorem

Decidability of Cp.

Given a formula A, do backward proof search in $\mathcal{S}Cp$ on $\vdash A$. Since termination is guaranteed, we can decide if A is a theorem or not. QED.

Looking beyond the sequent calculus

- Aside from proofs of consistency, proof-theoretic methods enable us to extract other meta-logical results (decidability and complexity bounds, interpolation)
- Many more logics of interest than just first-order classical and intuitionistic logic
- How to give a proof-theory to these logics? Want analytic calculi with *modularity*
- In a modular calculus we can add rules corresponding to (suitable) axiomatic extensions and preserve analyticity.

Some nonclassical logics

- Consider a sequent $X \vdash Y$ built from lists X, Y (rather than sets or multisets) then $A, A, X \vdash Y$ and $A, X \vdash Y$ are no longer equivalent (without contraction). Also $A, B, X \vdash Y$ and $B, A, X \vdash Y$ are not equivalent (without exchange). Even more generally, $(A, B), X \vdash Y$ and $A, (B, X) \vdash Y$ are not equivalent (without associativity). The logics obtained by removing these properties are called *substructural logics*.
- An *intermediate logic* L is a set of formulae closed under *modus ponens* such that intuitionistic logic $Ip \subseteq L \subseteq Cp$.
- *Modal logics* extend classical language with modalities $\Box$ and $\Diamond$. The modalities were traditionally used to qualify statements like “it is *possible* that it will rain today". *Tense logics* include the temporal modalities $\blacklozenge$ and $\blacksquare$. Closed under *modus ponens* and *necessitation* rule ($A/\Box A$).

Sequent calculus inadequate for treating these logics (eg. no analytic sequent calculus for modal logic S5 despite analytic sequent calculus for S4)

Display Calculus

- Introduced as *Display Logic* (Belnap, 1982).
- Extends sequent calculus by introducing new structural connectives that interpret the logical connectives (enrich language)
- A *structure* is built from structural connectives and formulae.
- A display sequent: $X \vdash Y$ for structures X and Y
- Display property. A substructure in $X[U] \vdash Y$ *equi-derivable* (displayable) as $U \vdash W$ or $W \vdash U$ for some W.
- Key result. Belnap's general cut-elimination theorem applies when the rules of the calculus satisfy C1–C8 (*display conditions*)
- Display calculi have been presented for substructural logics, modal and poly-modal logics, tense logic, bunched logics, bi-intuitionistic logic. . .

Display calculi generalise the sequent calculus

Here is the sequent calculus $\mathcal{S}Cp$ once more:

$$\frac{}{p, X \vdash Y, p}\text{ init} \qquad \frac{}{\bot, X \vdash Y}\ \bot\text{l}$$

$$\frac{X \vdash Y, A}{\neg A, X \vdash Y}\ \neg l \qquad \frac{A, X \vdash Y}{X \vdash Y, \neg A}\ \neg r$$

$$\frac{A, B, X \vdash Y}{A \wedge B, X \vdash Y}\ \wedge l \qquad \frac{X \vdash Y, A \qquad X \vdash Y, B}{X \vdash Y, A \wedge B}\ \wedge r$$

$$\frac{A, X \vdash Y \qquad B, X \vdash Y}{A \vee B, X \vdash Y}\ \vee l \qquad \frac{X \vdash Y, A, B}{X \vdash Y, A \vee B}\ \vee r$$

$$\frac{X \vdash Y, A \qquad B, X \vdash Y}{A \to B, X \vdash Y}\ \to l \qquad \frac{A, X \vdash Y, B}{X \vdash Y, A \to B}\ \to\text{r}$$

Display calculi generalise the sequent calculus

Let's add a new structural connective $*$ for negation.

$$\frac{}{p, X \vdash Y, p}\ \text{init} \qquad \frac{}{\bot, X \vdash Y}\ \bot l$$

$$\frac{*A, X \vdash Y}{\neg A, X \vdash Y}\ \neg l \qquad \frac{X \vdash Y, *A}{X \vdash Y, \neg A}\ \neg r$$

$$\frac{A, B, X \vdash Y}{A \wedge B, X \vdash Y}\ \wedge l \qquad \frac{X \vdash Y, A \qquad X \vdash Y, B}{X \vdash Y, A \wedge B}\ \wedge r$$

$$\frac{A, X \vdash Y \qquad B, X \vdash Y}{A \vee B, X \vdash Y}\ \vee l \qquad \frac{X \vdash Y, A, B}{X \vdash Y, A \vee B}\ \vee r$$

$$\frac{X \vdash Y, A \qquad B, X \vdash Y}{A \rightarrow B, X \vdash Y}\ \rightarrow l \qquad \frac{A, X \vdash Y, B}{X \vdash Y, A \rightarrow B}\ \rightarrow r$$

Add the display rules

The addition of the following rules permit the display property:

Definition (display property)

The calculus has the display property if for any sequent $X \vdash Y$ containing a substructure U, there is a sequent $U \vdash W$ or $W \vdash U$ for some W such that

$$\frac{X \vdash Y}{\overline{U \vdash W}} \qquad \text{or} \qquad \frac{X \vdash Y}{\overline{W \vdash U}}$$

We say that U is *displayed* in the lower sequent.

$$\frac{X, Y \vdash Z}{\overline{X \vdash Z, *Y}} \qquad \frac{X, Y \vdash Z}{\overline{Y \vdash *X, Z}} \qquad \frac{X \vdash Y, Z}{\overline{X, *Z \vdash Y}}$$

$$\frac{X \vdash Y, Z}{\overline{*Y, X \vdash Z}} \qquad \frac{*X \vdash Y}{\overline{*Y \vdash X}} \qquad \frac{X \vdash *Y}{\overline{Y \vdash *X}}$$

$$\frac{**X \vdash Y}{\overline{X \vdash Y}} \qquad \frac{X \vdash **Y}{\overline{X \vdash Y}} \qquad \frac{X \vdash \bullet Y}{\overline{\bullet X \vdash Y}}$$

Using the display rules

Examples:

$$
\begin{array}{c}
*(A, *B) \vdash *(C, D) \\
\hline\hline
**(C, D) \vdash A, *B \\
\hline\hline
*A, **(C, D) \vdash *B \\
\hline\hline
B \vdash *(*A, **(C, D))
\end{array}
$$

B is displayed

$$
\begin{array}{c}
*(A, *B) \vdash *(C, D) \\
\hline\hline
C, D \vdash **(A, *B) \\
\hline\hline
D \vdash *C, **(A, *B)
\end{array}
$$

D is displayed

Specify the properties of the structural connectives

We want weakening, contraction, exchange, associativity.

Here **I** is a structural constant for the empty list.

$$\frac{X \vdash Z}{\mathbf{I}, X \vdash Z} \qquad \frac{X \vdash Z}{X \vdash \mathbf{I}, Z} \qquad \frac{\mathbf{I} \vdash Y}{*\mathbf{I} \vdash Y}$$

$$\frac{X \vdash \mathbf{I}}{X \vdash *\mathbf{I}} \qquad \frac{X \vdash Z}{Y, X \vdash Z} \qquad \frac{X \vdash Z}{X, Y \vdash Z}$$

$$\frac{X, Y \vdash Z}{Y, X \vdash Z} \qquad \frac{Z \vdash X, Y}{Z \vdash Y, X} \qquad \frac{X, X \vdash Z}{X \vdash Z}$$

$$\frac{Z \vdash X, X}{Z \vdash X} \qquad \frac{X_1, (X_2, X_3) \vdash Z}{(X_1, X_2), X_3 \vdash Z} \qquad \frac{Z \vdash X_1, (X_2, X_3)}{Z \vdash (X_1, X_2), X_3}$$

Display calculi generalise the sequent calculus

The presence of the display rules permit the following rewriting of the rules:

$$\frac{}{p \vdash p}\ \text{init} \qquad \frac{}{\bot \vdash \mathbf{I}}\ \bot\text{l}$$

$$\frac{*A \vdash Y}{\neg A \vdash Y}\ \neg l \qquad \frac{X \vdash *A}{X \vdash \neg A}\ \neg r$$

$$\frac{A, B \vdash Y}{A \wedge B \vdash Y}\ \wedge l \qquad \frac{X \vdash A \qquad X \vdash B}{X \vdash A \wedge B}\ \wedge r$$

$$\frac{A \vdash Y \qquad B \vdash Y}{A \vee B \vdash Y}\ \vee l \qquad \frac{X \vdash A, B}{X \vdash A \vee B}\ \vee r$$

$$\frac{X \vdash A \qquad B \vdash Y}{A \to B \vdash *X, Y}\ \to l \qquad \frac{A, X \vdash B}{X \vdash A \to B}\ \to\text{r}$$

The formulae are called principal formulae. The X, Y are *context* variables.

Sequent calculus to display calculus

From a *procedural* point of view, we obtained the display calculus δCp for Cp from the sequent calculus by

1. Addition of a structural connective $*$ for negation
2. Addition of the display rules to yield the display property
3. Additional structural rules for exchange, weakening, contraction etc.
4. Rewriting the logical rules so the principal formulae in the conclusion are all of the antecedent or succedent

Before we consider how to construct a display calculus utilising the properties of the logic, let us introduce Belnap's general cut-elimination theorem. . .

Belnap's general cut-elimination theorem

Belnap showed that *any* display calculus satisfying the *display conditions* has cut-elimination. The display conditions C1–C8 are syntactic conditions on the rules of the calculus.

Theorem

A display calculus that satisfies the Display Conditions C2–C8 has cut-elimination. If C1 is satisfied, then the calculus has the subformula property.

Proof 'follows' Gentzen's cut-elimination, uses display property.

Only C8 is non-trivial to verify.

Verifying C1–C8 is trivial for rules built only from structures (*structural rules*) since C8 does not apply!

$$\frac{X \vdash Z}{\overline{\mathbf{I}, X \vdash Z}} \qquad \frac{X \vdash Z}{\overline{X \vdash \mathbf{I}, Z}} \qquad \frac{\mathbf{I} \vdash Y}{\overline{*\mathbf{I} \vdash Y}}$$

$$\frac{X \vdash \mathbf{I}}{\overline{X \vdash *\mathbf{I}}} \qquad \frac{X \vdash Z}{Y, X \vdash Z} \qquad \frac{X \vdash Z}{X, Y \vdash Z}$$

Display conditions

Some sample rules:

$$\frac{X \vdash A \qquad B \vdash Y}{A \to B \vdash *X, Y}\ (\to l) \qquad \frac{A, X \vdash B}{X \vdash A \to B}\ (\to r) \qquad \frac{A \wedge B \vdash X}{A \vdash X}$$

(C1) Each schematic formula variable occurring in a premise of a rule ρ is a sub-formula of some schematic formula variable in the conclusion of ρ.

(C2) A parameter is an occurrence of a schematic structure variable in the rule schema. Occurrences of the identical structure variable are said to be *congruent* to one another (really a definition)

(C3) Each parameter is congruent to at most one structure variable in the conclusion. Ie. no two structure variables in the conclusion are congruent to each other.

Display conditions

Some sample rules:

$$\frac{X \vdash A \qquad B \vdash Y}{A \rightarrow B \vdash *X, Y}\ (\rightarrow\text{l}) \qquad \frac{A, X \vdash B}{X \vdash A \rightarrow B}\ (\rightarrow\text{r}) \qquad \frac{X \vdash Y}{X, X \vdash Y}$$

(C1) Each schematic formula variable occurring in a premise of a rule ρ is a sub-formula of some schematic formula variable in the conclusion of ρ.

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Display conditions

Some sample rules:

$$\frac{X \vdash A \qquad B \vdash Y}{A \to B \vdash *X, Y}\ (\to l) \qquad \frac{A, X \vdash B}{X \vdash A \to B}\ (\to r) \qquad \frac{X, Y \vdash Z}{X \vdash Z, Y}$$

(C4) Congruent parameters are all either a-part or s-part structures.

(C5) A schematic formula variable in the conclusion of an inference rule ρ is either the entire antecedent or the entire succedent. This formula is called a *principal formula* of ρ.

(C6/7) Each inference rule is closed under simultaneous substitution of arbitrary structures for congruent parameters.

Display conditions

Some sample rules:

$$\frac{X \vdash A \qquad B \vdash Y}{A \rightarrow B \vdash *X, Y}\ (\rightarrow l) \qquad \frac{A, X \vdash B}{X \vdash A \rightarrow B}\ (\rightarrow r) \qquad \frac{A, B, X \vdash Y}{A \wedge B, X \vdash Y}$$

(C4) Congruent parameters are all either a-part or s-part structures.

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C6/7) Each inference rule is closed under simultaneous substitution of arbitrary structures for congruent parameters.

Display conditions

Some sample rules:

$$\frac{X \vdash A \qquad B \vdash Y}{A \to B \vdash *X, Y}\ (\to\text{l}) \qquad \frac{A, X \vdash B}{X \vdash A \to B}\ (\to\text{r}) \qquad \frac{X, Y \vdash Z}{\overline{Y \vdash *X, Z}}$$

(C8) If there are inference rules ρ and σ with respective conclusions $X \vdash A$ and $A \vdash Y$ with formula A principal in both inferences (in the sense of C5) and if *cut* is applied to yield $X \vdash Y$, then $X \vdash Y$ is identical to either $X \vdash A$ or $A \vdash Y$; or it is possible to pass from the premises of ρ and σ to $X \vdash Y$ by means of inferences falling under *cut* where the cut-formula is always a proper sub-formula of A.

$$\frac{\dfrac{A, X \vdash B}{X \vdash A \to B}\ (\to\text{r}) \qquad \dfrac{X' \vdash A \qquad B \vdash Y'}{A \to B \vdash *X', Y'}\ (\to\text{l})}{X \vdash *X', Y'}\ \text{cut} \qquad \dfrac{\dfrac{X' \vdash A \qquad \dfrac{A, X \vdash B \qquad B \vdash Y'}{A, X \vdash Y'}\ \text{cut}}{X', X \vdash Y'}\ \text{cut}}{X \vdash *X', Y'}\ \text{dr}$$

Another look at constructing display calculi

Some questions that arise when constructing a display calculus include:

- For which logics can we give a display calculus?
- How do we know which structural connectives to add?
- How to choose the display rules to ensure display property?

Extending the display calculus via structural rules is convenient because the conditions for cut-elimination are easy to check (because C8 is not applicable)

- Suppose we have a display calculus for the logic L. For which extensions of L can we obtain structural rule extensions?

- Obtain from the sequent calculus $\mathcal{S}Cp$ for classical logic by removing assumptions (on the structural connective comma) of commutativity, contraction and weakening in a sequent $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$
- Or define algebraically.

A structure $A = \langle A, \leq, \leq, \rightarrow, \leftarrow, \wedge, \otimes, 1, \top, \succ\!\!-, -\!\!\prec, \vee, \oplus, 0, \bot \rangle$ is a Bi-FL algebra (short for Bi-Lambek algebra) if:

1. $\langle A, \leq, \vee, \wedge, \top, \bot \rangle$ is a lattice with least element $\bot = \top \succ\!\!- \top = \top -\!\!\prec \top$ and greatest element $\top = \bot \rightarrow \bot = \bot \leftarrow \bot$.
2. (a) $\langle A, \otimes, 1 \rangle$ is a groupoid with identity $1 \in A$ (b) $\langle A, \otimes, 0 \rangle$ is a groupoid with co-identity $0 \in A$.
3. (a) $z \otimes (x \vee y) \otimes w = (z \otimes x \otimes w) \vee (z \otimes y \otimes w)$ for every $x, y, z \in A$
(b) $z \oplus (x \wedge y) \oplus w = (z \oplus x \oplus w) \wedge (z \oplus y \oplus w)$ for every $x, y, z, w \in A$
4. (a) $x \otimes y \leq z$ iff $x \leq z \leftarrow y$ iff $y \leq x \rightarrow z$, for every $x, y, z \in A$
(b) $z \leq x \oplus y$ iff $x \succ\!\!- z \leq y$ iff $z -\!\!\prec y \leq x$, for every $x, y, z \in A$.

The residuation properties (in red) crucial for constructing the display calculus.

Residuated pairs for Bi-Lambek logic

Recall the residuation properties. For every $x, y, z \in A$:

$$
\begin{array}{ccccc}
x \otimes y \le z & \Leftrightarrow & x \le z \leftarrow y & \Leftrightarrow & y \le x \rightarrow z \\
z \le x \oplus y & \Leftrightarrow & x \succ z \le y & \Leftrightarrow & z \prec y \le x
\end{array}
$$

Assign the following structural connectives to the logical connectives:

$$
\begin{array}{ccccc}
x \overbrace{\otimes}^{,} y \le z & \Leftrightarrow & x \le z \overbrace{\leftarrow}^{<} y & \Leftrightarrow & y \le x \overbrace{\rightarrow}^{>} z \\
z \le x \underbrace{\oplus}_{,} y & \Leftrightarrow & x \underbrace{\succ}_{>} z \le y & \Leftrightarrow & z \underbrace{\prec}_{<} y \le x
\end{array}
$$

This gives us the following *rewrite* rules.

$$
\frac{A, B \vdash Y}{A \otimes B \vdash Y}\ \otimes l \qquad \frac{X \vdash A, B}{X \vdash A \oplus B}\ \oplus r \qquad \frac{X \vdash \Phi}{X \vdash 0}
$$

$$
\frac{A < B \vdash Y}{A \prec B \vdash Y}\ \prec l \qquad \frac{X \vdash A < B}{X \vdash A \leftarrow B}\ \leftarrow r \qquad \frac{\Phi \vdash X}{1 \vdash X}
$$

$$
\frac{A > B \vdash X}{A \succ B \vdash Y}\ \succ l \qquad \frac{X \vdash A > B}{X \vdash A \rightarrow B}\ \rightarrow r
$$

Adding the display rules

$$x \overbrace{\otimes}^{,} y \leq z \quad\Leftrightarrow\quad x \leq z \overbrace{\leftarrow}^{<} y \quad\Leftrightarrow\quad y \leq x \overbrace{\rightarrow}^{>} z$$

$$z \leq x \underbrace{\oplus}_{,} y \quad\Leftrightarrow\quad x \underbrace{\succ}_{>} z \leq y \quad\Leftrightarrow\quad z \underbrace{\prec}_{<} y \leq x$$

This gives us the following *rewrite* rules.

$$\frac{A, B \vdash Y}{A \otimes B \vdash Y}\otimes l \qquad \frac{X \vdash A, B}{X \vdash A \oplus B}\oplus r \qquad \frac{X \vdash \Phi}{X \vdash 0}$$

$$\frac{A < B \vdash Y}{A \prec B \vdash Y}\prec l \qquad \frac{X \vdash A < B}{X \vdash A \leftarrow B}\leftarrow r \qquad \frac{\Phi \vdash X}{1 \vdash X}$$

$$\frac{A > B \vdash X}{A \succ B \vdash Y}\succ l \qquad \frac{X \vdash A > B}{X \vdash A \rightarrow B}\rightarrow r$$

And the following display rules:

$$\begin{array}{c} X, Y \vdash Z \\ \hline\hline X \vdash Z < Y \\ \hline\hline Y \vdash X > Z \end{array} \qquad \begin{array}{c} Z \vdash X, Y \\ \hline\hline X > Z \vdash Y \\ \hline\hline Z < Y \vdash X \end{array}$$

Computing the *decoding* rules

$$\frac{A, B \vdash Y}{A \otimes B \vdash Y} \otimes l \qquad \frac{X \vdash A, B}{X \vdash A \oplus B} \oplus r \qquad \frac{X \vdash \Phi}{X \vdash 0}$$

$$\frac{A < B \vdash Y}{A \prec B \vdash Y} \prec l \qquad \frac{X \vdash A < B}{X \vdash A \leftarrow B} \leftarrow r \qquad \frac{\Phi \vdash X}{1 \vdash X}$$

$$\frac{A > B \vdash X}{A \succ B \vdash Y} \succ l \qquad \frac{X \vdash A > B}{X \vdash A \rightarrow B} \rightarrow r$$

Here are the missing *decoding* rules (Goré, 1998)

$$\frac{X \vdash A \qquad Y \vdash B}{X, Y \vdash A \otimes B} \otimes r \qquad \frac{A \vdash X \qquad B \vdash Y}{A \oplus B \vdash X, Y} \oplus l \qquad 0 \vdash \Phi$$

$$\frac{X \vdash A \qquad B \vdash Y}{X < Y \vdash A \prec B} \prec r \qquad \frac{A \vdash X \qquad Y \vdash B}{A \leftarrow B \vdash X < Y} \leftarrow l \qquad \Phi \vdash 1$$

$$\frac{A \vdash X \qquad Y \vdash B}{X > Y \vdash A \succ B} \succ r \qquad \frac{X \vdash A \qquad B \vdash Y}{A \rightarrow B \vdash X > Y} \rightarrow l$$

Rewrite rules are invertible

Constructing the decoding rules is systematic (but not obvious, reasoning not shown here) and enforces:

Lemma

Every rewrite rule is invertible.

For example, consider the rewrite rule and decoding rule for $\succ\!\!-$:

$$\frac{A > B \vdash Y}{A \succ\!\!- B \vdash Y} \succ\!\!-\mathrm{l} \qquad \frac{A \vdash X \qquad Y \vdash B}{X > Y \vdash A \succ\!\!- B} \succ\!\!-\mathrm{r}$$

Here is the derivation witnessing invertibiltiy of $\succ\!\!-$l.

$$\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A > B \vdash A \succ\!\!- B} \succ\!\!-\mathrm{r} \qquad A \succ\!\!- B \vdash Y}{A > B \vdash Y} \mathit{cut}$$

Constructing a display calculus: summary

- The residuation property tells us which connectives are interpreted as a structural connective in which position
- The residuation property then gives the display rules
- Add remaining introduction rules (decoding rules).
- axioms for weakening, contraction etc. are converted to structural rules. (to be shown)
- The construction is focussed on the logical connectives that are residuated. The other connectives in the language (lattice connectives) do not introduce new structural connectives.

$$\frac{\mathbf{I} \vdash X}{\top \vdash X}\ \top l \qquad \frac{X \vdash \mathbf{I}}{X \vdash \bot}\ \bot r$$

$$\frac{A \circ B \vdash X}{A \wedge B \vdash X}\ \wedge l \qquad \frac{X \vdash A \qquad X \vdash B}{X \vdash A \wedge B}\ \wedge r$$

$$\frac{A \vdash X \qquad B \vdash X}{A \vee B \vdash X}\ \vee l \qquad \frac{X \vdash A \bullet B}{X \vdash A \vee B}\ \vee r$$

Interpreting sequents

Define the interpretation functions l and r from structures into Bi-Lambek formulae.

$$
\begin{aligned}
l(A) &= A & r(A) &= A \\
l(\mathbf{I}) &= \top & r(\mathbf{I}) &= \bot \\
l(\Phi) &= 1 & r(\Phi) &= 0 \\
l(X, Y) &= l(X) \otimes l(Y) & r(X, Y) &= l(X) \oplus r(Y) \\
l(X > Y) &= l(X) \succ l(Y) & r(X > Y) &= r(X) \rightarrow r(Y) \\
l(X < Y) &= l(X) \prec l(Y) & r(X > Y) &= r(X) \leftarrow r(Y)
\end{aligned}
$$

A sequent $X \vdash Y$ is interpreted as $l(X) \leq r(Y)$.

Adding structural rules

Some structural rules are straightforward to determine.

$$\frac{X \vdash Y}{X \vdash Y, Z} \qquad \frac{X \vdash Y}{X, Z \vdash Y} \qquad \frac{X \vdash Y, Z}{X \vdash Z, Y}$$

$$\frac{X, Z \vdash Y}{Z, X \vdash Y} \qquad \frac{X \vdash Y, Y}{X \vdash Y} \qquad \frac{X, X \vdash Y}{X \vdash Y}$$

$$\frac{X \vdash (Y, Z), U}{X \vdash Y, (Z, U)} \qquad \frac{(X, Y), Z \vdash U}{X, (Y, Z) \vdash U}$$

Structural rules for the additive unit 1 and the multiplicative structural connectives:

$$\begin{array}{c} \mathbf{I}, X \vdash Y \\ \hline\hline X \vdash Y \\ \hline\hline X, \mathbf{I} \vdash Y \end{array} \qquad \begin{array}{c} X \vdash Y, \mathbf{I} \\ \hline\hline X \vdash Y \\ \hline\hline X \vdash \mathbf{I}, Y \end{array}$$

Structural rule extensions of display calculi: a general recipe

$$\begin{tikzcd} \text{Logic } L \arrow[r, leftrightarrow] \arrow[d, "\text{suitable axiomatic extension}"] & \text{base display calculus } \mathcal{C} \arrow[d, dashed, "\textit{structural}\text{ rule extension}"] \\ L + \{A_1, \ldots, A_n\} \arrow[r, dashed, leftrightarrow] & \mathcal{C} + \rho_1 + \ldots + \rho_m \end{tikzcd}$$

- Generalises method for obtaining hypersequent structural rules from axioms (Agata, Nick, Kaz)
- The approach is language and logic independent; purely syntactic conditions on the base calculus
- Extends Kracht's theorem on primitive tense formulae.

Obtaining a structural rule from a Hilbert axiom

δ*BiFL* is a display calculus for Bi-Lambek logic satisfying C1–C8. Let us obtain the structural rule extension of δ*BiFL* for the logic *BiFL* $+ (p \to 0) \oplus ((p \to 0) \to 0)$.

STEP 1. Start with the axiom (below left) and apply all possible *invertible* rules backwards (below right).

stop here: $\to$*l not invertible*

$$\dfrac{}{\mathbf{I} \vdash (p \to 0) \oplus ((p \to 0) \to 0)} \qquad \begin{array}{c l} \mathbf{I} < ((p \to 0) > \Phi) \vdash p > \Phi & \\ \hline & \mathit{drs}, 0\mathrm{r} \\ \mathbf{I} < ((p \to 0) > 0) \vdash p > 0 & \\ \hline\hline & \to\mathrm{r} \\ \mathbf{I} < ((p \to 0) > 0) \vdash p \to 0 & \\ \hline\hline & \mathit{drs} \\ \mathbf{I} < (p \to 0) \vdash (p \to 0) > 0 & \\ \hline\hline & \to\mathrm{r} \\ \mathbf{I} < (p \to 0) \vdash (p \to 0) \to 0 & \\ \hline\hline & \mathit{drs} \\ \mathbf{I} \vdash (p \to 0), ((p \to 0) \to 0) & \\ \hline\hline & \oplus r \\ \mathbf{I} \vdash (p \to 0) \oplus ((p \to 0) \to 0) & \end{array}$$

So it suffices to introduce a structural rule equivalent to $\mathbf{I} < ((p \to 0) > \Phi) \vdash p > \Phi$.

STEP 2. Apply Ackermann's Lemma.

Lemma

The following rules are pairwise equivalent

$$\boxed{\frac{\mathcal{S}}{X \vdash A}\rho_1 \quad \frac{\mathcal{S} \quad A \vdash \mathcal{L}}{X \vdash \mathcal{L}}\rho_2} \qquad \boxed{\frac{\mathcal{S}}{A \vdash X}\delta_1 \quad \frac{\mathcal{S} \quad \mathcal{L} \vdash A}{\mathcal{L} \vdash X}\delta_2}$$

where $\mathcal{S}$ is a set of sequents, $\mathcal{L}$ is a fresh schematic structure variable, and A is a tense formula.

$$\frac{}{\mathbf{I} < ((p \to 0) > \Phi) \vdash p > \Phi} \overset{\text{d.p.}}{\Leftrightarrow} \frac{}{p \to 0 \vdash (\mathbf{I} < (p > \Phi)) > \Phi} \overset{\text{lem}}{\Leftrightarrow} \frac{\mathcal{L} \vdash p \to 0}{\mathcal{L} \vdash (\mathbf{I} < (p > \Phi)) > \Phi}$$

$$\overset{\text{d.p.}}{\Leftrightarrow} \frac{\mathcal{L} \vdash p \to 0}{p \vdash (\mathbf{I} < (\mathcal{L} > \Phi)) > \Phi} \overset{\text{lem}}{\Leftrightarrow} \frac{\mathcal{L} \vdash p \to 0 \qquad \mathcal{M} \vdash p}{\mathcal{M} \vdash (\mathbf{I} < (\mathcal{L} > \Phi)) > \Phi}$$

Stop when there are no more formulae in the conclusion

STEP 3. Apply all possible invertible rules backwards.

$$\frac{\mathcal{L} \vdash p \to 0 \qquad \mathcal{M} \vdash p}{\mathcal{M} \vdash (\mathbf{I} < (\mathcal{L} > \Phi)) > \Phi} \quad \Leftrightarrow \quad \frac{\dfrac{\mathcal{L} \vdash p > \Phi}{\mathcal{L} \vdash p > 0} \qquad \mathcal{M} \vdash p}{\mathcal{M} \vdash (\mathbf{I} < (\mathcal{L} > \Phi)) > \Phi}$$

The following rule is not a structural rule.

$$\frac{\mathcal{L} \vdash p > \Phi \qquad \mathcal{M} \vdash p}{\mathcal{M} \vdash (\mathbf{I} < (\mathcal{L} > \Phi)) > \Phi}\rho$$

By Belnap's general cut-elimination theorem, $\delta Kt + \rho$ has cut-elimination. However it does not have the subformula property.

STEP 4. Apply all possible cuts (and verify termination)

$$\frac{\mathcal{L} \vdash p > \Phi \qquad \mathcal{M} \vdash p}{\mathcal{M} \vdash (\mathbf{I} < (\mathcal{L} > \Phi)) > \Phi}\rho \quad \overset{\text{d.p.}}{\Leftrightarrow} \quad \frac{p \vdash \mathcal{L} > \Phi \qquad \mathcal{M} \vdash p}{\mathcal{M} \vdash (\mathbf{I} < (\mathcal{L} > \Phi)) > \Phi}\rho$$

$$\Leftrightarrow \quad \frac{\mathcal{M} \vdash \mathcal{L} > \Phi}{\mathcal{M} \vdash (\mathbf{I} < (\mathcal{L} > \Phi)) > \Phi}\rho'$$

One direction is cut, the other direction is non-trivial.

We conclude:

δ*BiFLt* $+ \rho'$ is a calculus for *BiFL* $+ (p \to 0) \oplus ((p \to 0) \to 0)$ with cut-elimination and subformula property.

Summary of the recipe

(1) Invertible rules (2) Ackermann's lemma (3) invertible rules (4) all possible cuts

Only certain axioms can be handled

- Because we cannot decompose all connectives in the axiom
  - (i) we can handle a subformula $p \to q$ in negative position (Ackermann's lemma will take it to a positive position where $\to$ is invertible).
  - (ii) but not a subformula $A \to q$ in negative position, where A contains an $p \to q$ in negative position.
- And even if we can, Step (4) 'cutting step' should terminate in a structural rule. Eg. the following is problematic:

$$\frac{p, p \vdash \mathcal{L} > \Phi \qquad \mathcal{M} \vdash p, p}{\mathcal{M} \vdash (\mathbf{I} < (\mathcal{L} > \Phi)) > \Phi}\rho$$

Nevertheless we can capture a large class of axioms.
More invertible rules, more axioms! — eg. hypersequent, display calculus for intermediate logics

Definition. Amenable calculus

Let $\mathcal{C}$ be a display calculus satisfying C1–C8. l and r are functions from structures into formulae s.t. $\mathsf{l}(A) = \mathsf{r}(A) = A$. Also:

(i) $X \vdash \mathsf{l}(X)$ and $\mathsf{r}(X) \vdash X$ are derivable.

(ii) $X \vdash Y$ derivable implies $\mathsf{l}(X) \vdash \mathsf{r}(Y)$ is derivable.

There is a structure constant $\mathbf{I}$ such that the following are admissible:

$$\frac{\mathbf{I} \vdash X}{Y \vdash X}\ \mathbf{I}l \qquad\qquad \frac{X \vdash \mathbf{I}}{X \vdash Y}\ \mathbf{I}r$$

There are associative and commutative binary logical connectives $\vee, \wedge$ in $\mathcal{C}$ such that

$(a)_\vee$ $A \vdash X$ and $B \vdash X$ implies $\vee(A, B) \vdash X$

$(b)_\vee$ $X \vdash A$ implies $X \vdash \vee(A, B)$ for any formula B.

$(a)_\wedge$ $X \vdash A$ and $X \vdash B$ implies $X \vdash \wedge(A, B)$

$(b)_\wedge$ $A \vdash X$ implies $\wedge(A, B) \vdash X$ for any formula B.

Axioms to structural rules

A *proper* structural rule satisfies Belnap's conditions for cut-elimination C1-C8.

Theorem

Let $\mathcal{C}$ be an amenable calculus for L and suppose that axiom A is acyclic. Then there is a proper structural rule extension for L $+$ A.

What about the other direction? Which extensions of *L* can be written as structural rule extensions of $\mathcal{C}$?

Proper structural rules to axioms

Definition

An amenable calculus $\mathcal{C}$ is *well-behaved* if:

(i) the calculus contains an antecedent structural connective $\circ$ and a succedent structural connective $\bullet$ such that:

$$\mathsf{l}(X \circ Y) = \mathsf{l}(X) \wedge \mathsf{l}(Y) \qquad \mathsf{r}(X \bullet Y) = \mathsf{r}(X) \vee \mathsf{r}(Y)$$

Here $\wedge$ and $\vee$ are the connectives in the definition of amenable calculus (not necessarily conjunction, disjunction).

(ii) The following 'identity-like' rules are admissible:

$$\frac{X \vdash U \bullet \mathbf{I}}{X \vdash U} \qquad \frac{U \circ \mathbf{I} \vdash Y}{U \vdash Y}$$

(iii) In the sequent $X \vdash U \bullet V$ (resp. $U \circ V \vdash Y$) it is the case that V is a s-part (a-part) structure.

A characterisation of proper structural rules

Theorem

Let $\mathcal{C}$ is an amenable well-behaved calculus for the logic L, and let L′ be an axiomatic extension of L. Then there is a proper structural rule extension of $\mathcal{C}$ for L′ iff L′ is an extension of L by acyclic axioms.

Summary I

- The display calculus generalises the sequent calculus by the addition of new structural connectives.
- Display rules yield the display property.
- The display property is used to prove Belnap's general cut-elimination theorem.
- Residuation property central to choosing structural connectives, display rules.
- the display calculus is one of several proof-frameworks proposed to address the (lack of) analytic sequent calculi for logics of interests. Some other frameworks include hypersequents, nested sequents, labelled sequents.

Summary II

- In some frameworks such as the calculus of structures, we can operate 'inside' formulae (deep inference). The display calculus (below right) seems to mimic some notion of deep inference.

$$\frac{\vdash \Box B}{\vdash \Box(B \lor B')} \qquad \dfrac{\dfrac{\dfrac{\mathbf{I} \vdash \bullet B}{\bullet\mathbf{I} \vdash B}}{\bullet\mathbf{I} \vdash B, B'}}{\mathbf{I} \vdash \bullet(B, B')}$$

- Recent work used a display calculus as the starting point for an analytic calculus for Full intuitionistic linear logic (MILL extended with $\oplus$). A (deep inference) nested sequent calculus is then constructed to obtain complexity, conservativity results (Clouston *et al.*, 2013).
- Recall the display calculus is for a fully residuated logic. What if we want a fragment (FL, intuitionistic, modal logic) of the full Bi-FL, bi-intuitionistic, tense logic? Conservativity of Bi-L for L required.

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The slides can be found at <www.logic.at/revantha>