TOWARDS NONLINEAR QRPA DESCRIPTION OF STATES OF MULTIPHONON ORIGIN

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OUTLINE

- motivation by double beta decay
- simplistic model and its exact solution
- standard QRPA solution of the simplistic model
- nonlinear QRPA solution of the simplistic model
- conclusions

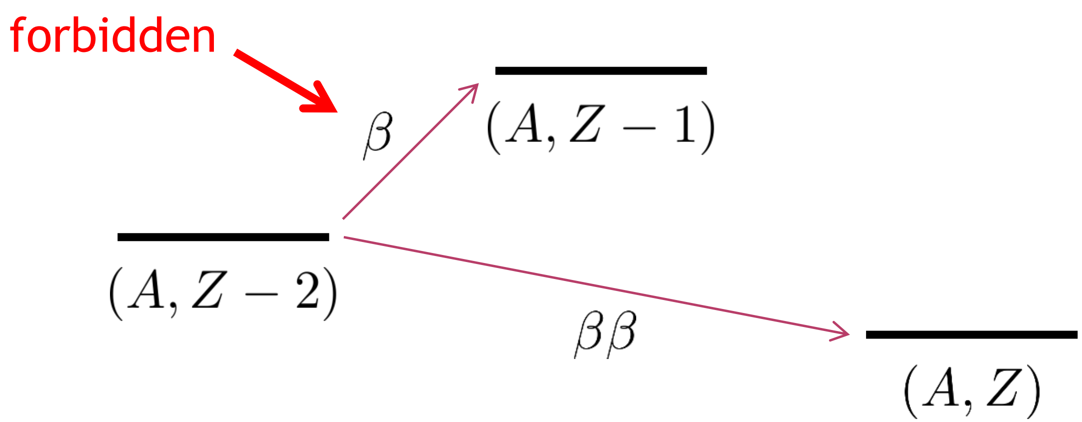
MOTIVATION

DOUBLE BETA DECAY

- nuclear matrix element must be calculated from nuclear structure model

$$\left(T_{1/2}^{\beta\beta}\right)^{-1} = g_A^4 G^{\beta\beta} |M^{\beta\beta}|^2 \left(\frac{\langle m_{\beta\beta}\rangle}{m_e}\right)^2$$

- it constitutes of a nuclear transition:

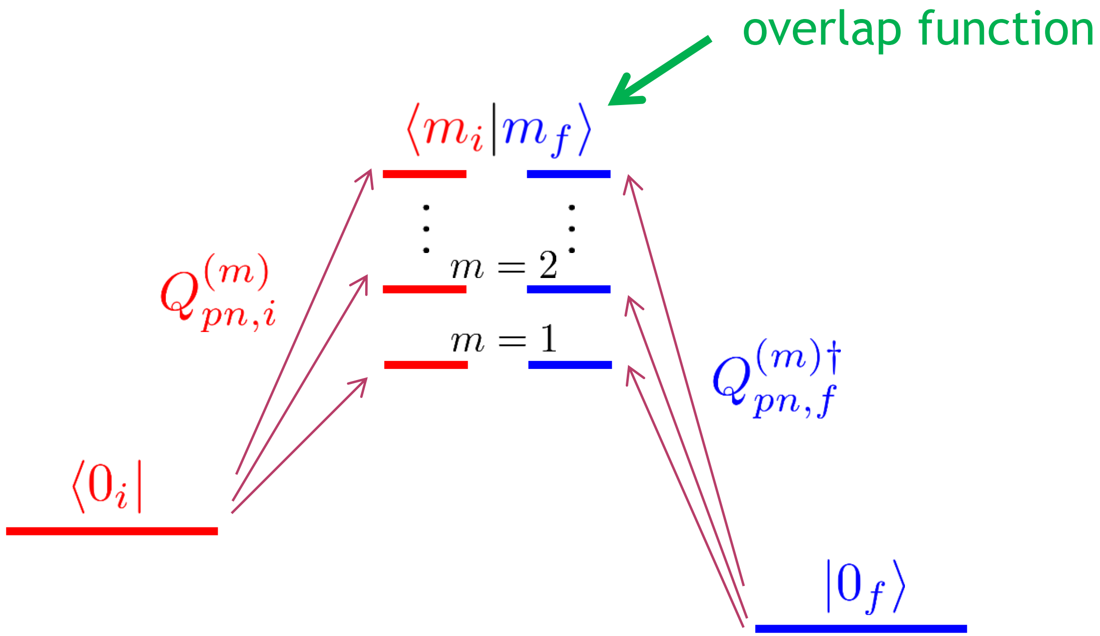


QRPA FOR DOUBLE BETA DECAY

Quasiparticle Random Phase Approximation

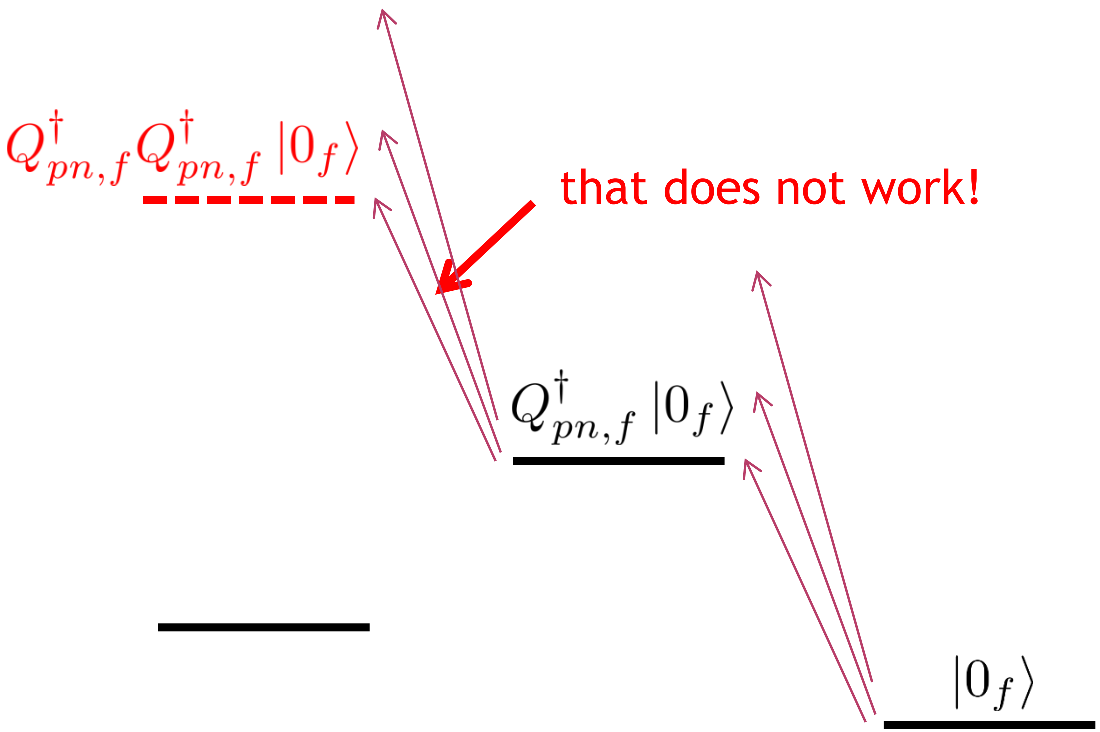
$$Q_{pn}^{(m)\dagger} = X^{(m)} A_{pn}^{JM\dagger} - Y^{(m)} \tilde{A}_{pn}^{JM}$$

$$A_{pn}^{JM} = [a_p a_n]^{JM}$$



QRPA DESCRIPTION OF MULTIPHONON STATES

- we can try to use result of the one-phonon state QRPA

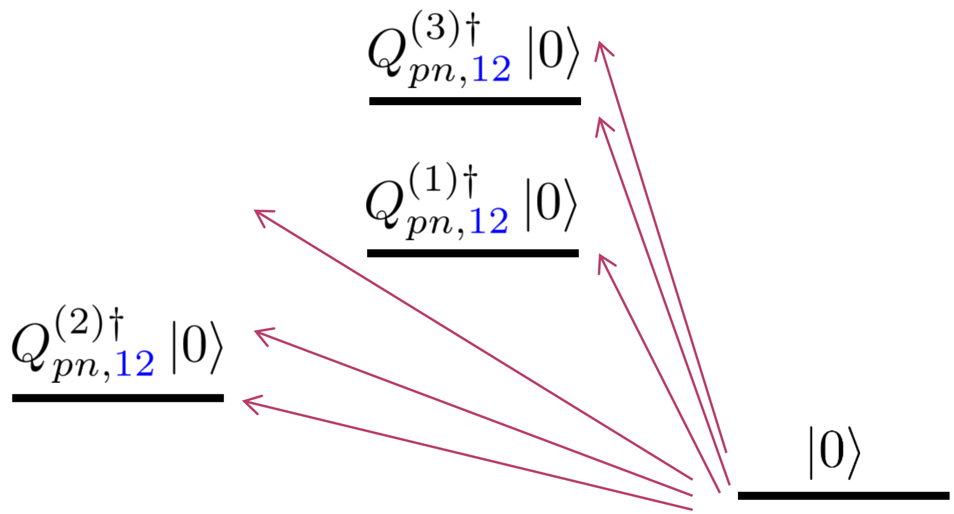


harmonic oscillator approach

it is as good approximation as well HO approximates a given hamiltonian

NONLINEAR QRPA DESCRIPTION OF MULTIPHONON STATES

- our goal is to formulate QRPA system which allows for simultaneous description of states of multiphonon origin



- for that the phonon operator must be nonlinear

$$Q^{(m)\dagger}_{pn,12} = X^{(m)} A^{JM\dagger}_{pn} - Y^{(m)} \tilde{A}^{JM}_{pn} + \dots$$

novel approach

nonlinear phonon operators have been used to better describe the *same-nucleus* excited states

MODEL

SCHEMATIC MODEL

exactly solvable model to test quality of our approach

Before formulating the realistic nonlinear QRPA approach we study

- ***pn*-Lipkin model** [Hirsch,Hess,Civitarese 1996]
  - it has the structure of the realistic hamiltonian
  - it is defined on a single *J*-shell with semidegeneracy Ω

$$H_F = \varepsilon C + \lambda_1 A^\dagger A + \lambda_2 (A^\dagger A^\dagger + AA)$$

where

$$C \equiv \sum_m a^\dagger_{pm} a_{pm} + \sum_m a^\dagger_{nm} a_{nm} \, , \qquad A^\dagger \equiv [a^\dagger_p a^\dagger_n]^{00}$$

- satisfying the algebra

$$[A, A^\dagger] = 1 - C/(2\Omega) \qquad [C, A^\dagger] = 2A^\dagger$$

- κ' parametrizes particle-particle interactions
- χ' parametrizes particle-hole interactions

$$\lambda_1 = 2[\chi'(u_p^2 v_n^2 + v_p^2 u_n^2) - \kappa'(u_p^2 u_n^2 + v_p^2 v_n^2)]$$
$$\lambda_2 = 2(\chi' + \kappa') u_p v_p u_n v_n$$

BOSON MAPPING

Marumori mapping

$$[A, A^\dagger] = 1 - C/(2\Omega) \quad \longrightarrow \quad [B, B^\dagger] = 1$$

$$H_F \quad \longrightarrow \quad H_{BM} = \sum_{i+j \leq \max} \alpha_{ij} B^i B^{\dagger j}$$

- **bosonic model**
  - It is excellent approximation for first 2Ω eigenstates
    - already for $\max = 4$,
    - for moderate values of κ', χ', Ω,
  - up to quadratic terms
    - good approximation only for $\Omega \to \infty$

$$H_B = (2\varepsilon + \lambda_1) B^\dagger B + \lambda_2 (B^\dagger B^\dagger + BB)$$

SIMPLISTIC MODEL

*is in fact equivalent to the **Harmonic Oscillator***

$$H_B = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 + \text{const.}$$

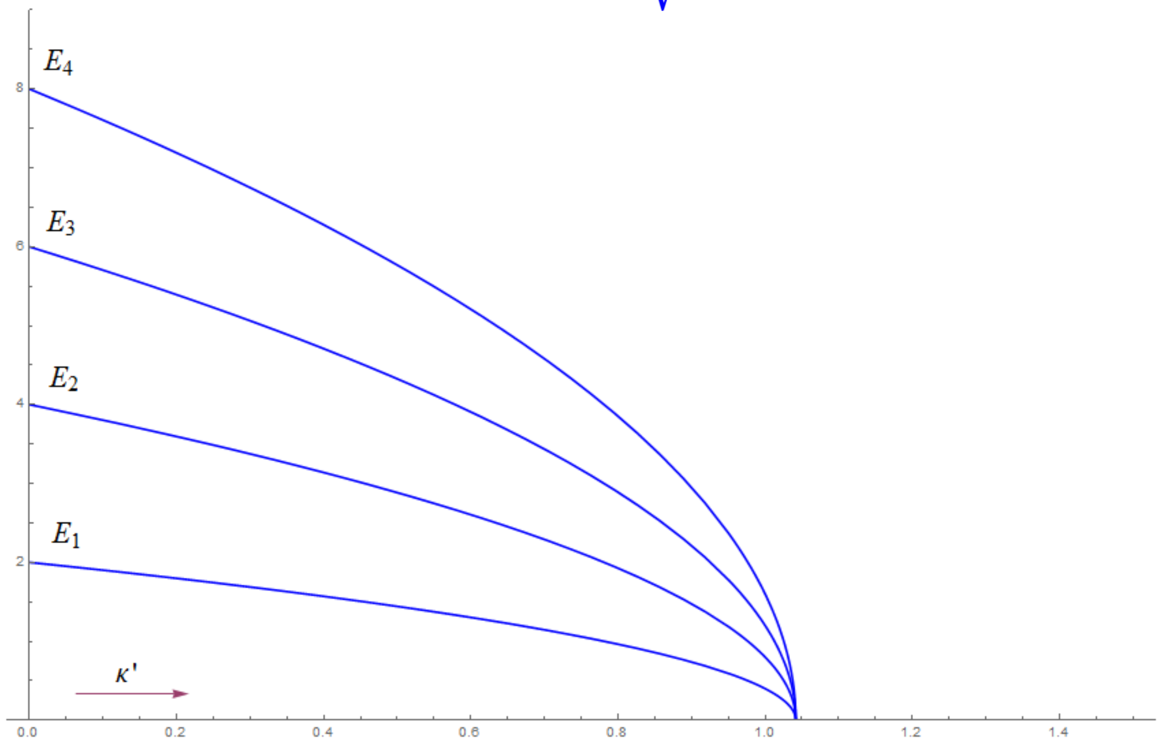
$$q = \frac{B^\dagger + B}{\sqrt{2}}, \quad p = \mathrm{i}\frac{B^\dagger - B}{\sqrt{2}}, \quad [q,p] = \mathrm{i}$$

$$\frac{1}{m} = (2\varepsilon + \lambda_1 - 2\lambda_2)$$

$$m\omega^2 = (2\varepsilon + \lambda_1 + 2\lambda_2)$$

- it has a text-book solution:

$$E_n - E_0 = nE, \quad E = \omega = \sqrt{(2\varepsilon + \lambda_1)^2 - 4\lambda_2^2}$$



STANDARD QRPA

STANDARD QRPA

- define a *linear* phonon operator

$$Q_1^\dagger = X_1 B^\dagger - Y_1 B$$

- set an Ansatz for the ground state

$$|0\rangle = \mathcal{N} \mathrm{e}^{d B^\dagger B^\dagger} |\rangle \ , \quad \mathcal{N}^2 = \sqrt{1 - 4d^2}$$

- annihilation condition determines the ground state parameter

$$Q_1 |0\rangle = 0 \implies d = \frac{1}{2} \frac{Y_1}{X_1}$$

STANDARD QRPA

RPA equation of motion

$$\begin{pmatrix} \mathcal{A}_1 & \mathcal{B}_1 \\ \mathcal{B}_1 & \mathcal{A}_1 \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = E \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}$$

$$\mathcal{A}_1 = \langle 0| [B, H_B, B^\dagger] |0\rangle = 2\varepsilon + \lambda_1 \,, \quad \mathcal{B}_1 = - \langle 0| [B, H_B, B] |0\rangle = 2\lambda_2$$

- RPA eq. has an analytic solution

$$\begin{aligned} E &= \sqrt{\mathcal{A}_1^2 - \mathcal{B}_1^2} \\ X_1 &= \frac{\mathcal{A}_1 + E}{\sqrt{(\mathcal{A}_1 + E)^2 - \mathcal{B}_1^2}} \\ Y_1 &= \frac{-\mathcal{B}_1}{\sqrt{(\mathcal{A}_1 + E)^2 - \mathcal{B}_1^2}} \end{aligned}$$

$$\begin{aligned} E &= \sqrt{(2\varepsilon + \lambda_1)^2 - 4\lambda_2^2} \\ d &= -\frac{(2\varepsilon + \lambda_1) - E}{4\lambda_2} \end{aligned}$$

Exact solution reproduced!

MULTIPHONON APPROACH

the simplest approach towards higher excited states

$$|1\rangle = Q_1^\dagger |0\rangle$$

$$|n\rangle = \frac{1}{\sqrt{n!}} Q_1^{\dagger n} |0\rangle$$

- in general, the multiphonon approach leads to equidistant spectrum

$$E_n - E_0 = nE$$

- it an exact solution as the Hamiltonian may be rewritten as HO

$$H_B = E Q_1^\dagger Q_1$$

NONLINEAR QRPA

QRPA FOR THE 2ND EXCITED STATE

- define a *nonlinear* phonon operator

$$Q_2^\dagger = X_2(B^\dagger B^\dagger + c_2) - Y_2(BB + c_2)$$

- keep the Ansatz for the ground state

$$|0\rangle = \mathcal{N} \mathrm{e}^{d_2 B^\dagger B^\dagger} \,|\rangle \;, \quad \mathcal{N}^2 = \sqrt{1 - 4d_2^2}$$

- RPA eq. gets little bit more complicated

$$\begin{pmatrix} \mathcal{A}_2 & \mathcal{B}_2 \\ \mathcal{B}_2 & \mathcal{A}_2 \end{pmatrix} \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = E_2 \begin{pmatrix} \mathcal{U}_2 & 0 \\ 0 & -\mathcal{U}_2 \end{pmatrix} \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}$$

$$\mathcal{A}_2 = \langle 0| \left[BB, H_B, B^\dagger B^\dagger\right] |0\rangle \;, \quad \mathcal{B}_2 = -\langle 0| \left[BB, H_B, BB\right] |0\rangle \;, \quad \mathcal{U}_2 = \langle 0| \left[BB, B^\dagger B^\dagger\right] |0\rangle$$

- annihilation condition

$$Q_2 \,|0\rangle = 0 \implies d_2, \; c_2$$

QRPA FOR THE 2ND EXCITED STATE

unexpected solution

- an analytic solution is available

$$
\begin{aligned}
d_2 &= \frac{1}{2}\sqrt[2]{\frac{Y_2}{X_2}} = -\frac{(2\varepsilon+\lambda_1)-E}{4\lambda_2} \\
c_2 &= \frac{(2d_2)}{(2d_2)^2-1} \\
E_2 &= 2\sqrt{(2\varepsilon+\lambda_1)^2-4\lambda_2^2}
\end{aligned}
$$

RPA ground state and exact solution reproduced!

QRPA FOR THE 3RD EXCITED STATE

$$Q_3^\dagger = X_3(B^{\dagger 3} + c_3 B^\dagger) - Y_3(B^3 + c_3 B)$$

$$|0\rangle = \mathcal{N} \mathrm{e}^{d_3 B^\dagger B^\dagger} |\rangle$$

- follow the same procedure

$$d_3 = \frac{1}{2}\sqrt[3]{\frac{Y_3}{X_3}} = -\frac{(2\varepsilon + \lambda_1) - E}{4\lambda_2}$$

$$c_3 = \frac{3(2d_3)}{(2d_3)^2 - 1}$$

$$E_3 = 3\sqrt{(2\varepsilon + \lambda_1)^2 - 4\lambda_2^2}$$

$$d_2 = \frac{1}{2}\sqrt[2]{\frac{Y_2}{X_2}} \quad d_1 = \frac{1}{2}\frac{Y_1}{X_1}$$

$$c_2 = \frac{(2d_2)}{(2d_2)^2 - 1}$$

RPA ground state and exact solution reproduced!

QRPA FOR EVERY EXCITED STATE

is there a phonon operator for any of excited states?

- searching for a *nonlinear* phonon operator

$$Q_n^\dagger = X_n(B^{\dagger n} + \ldots) - Y_n(B^n + \ldots)$$

- and keeping the Ansatz for the ground state

$$|0\rangle = \mathcal{N} \mathrm{e}^{dB^\dagger B^\dagger} |\rangle \ , \quad \mathcal{N}^2 = \sqrt{1 - 4d^2}$$

QRPA FOR EVERY EXCITED STATE

important observation

- remember, the multiphonon approach gives the exact solution!

$$|n\rangle = \frac{1}{\sqrt{n!}} Q_1^{\dagger n} |0\rangle = \frac{1}{\sqrt{n!}} \frac{1}{X_1^n} \mathcal{P}_n^\dagger |0\rangle$$

- where

$$\begin{aligned}
\mathcal{P}_n^\dagger &\stackrel{n=0}{=} 1 \\
&\stackrel{n=1}{=} B^\dagger \\
&\stackrel{n=2}{=} (B^{\dagger 2} + c) \\
&\stackrel{n=3}{=} (B^{\dagger 3} + 3cB^\dagger) \\
&\stackrel{n=4}{=} (B^{\dagger 4} + 6cB^{\dagger 2} + 3c^2) \\
&\stackrel{n=5}{=} (B^{\dagger 5} + 10cB^{\dagger 3} + 15c^2 B^\dagger) \\
&\stackrel{n=6}{=} (B^{\dagger 6} + 15cB^{\dagger 4} + 45c^2 B^{\dagger 2} + 15c^3) \\
&= \ldots
\end{aligned}$$

with

$$c \equiv -X_1 Y_1$$

the same operators used in definition of phonon operators (for $n=1$ to $n=3$)

QRPA FOR EVERY EXCITED STATE

is there a phonon operator for any of excited states?

- let's define the phonon operators as

$$Q_n^\dagger = X_n \mathcal{P}_n^\dagger - Y_n \mathcal{P}_n$$

$$|0\rangle = \mathcal{N} \mathrm{e}^{dB^\dagger B^\dagger} |\rangle$$

- where

$$\begin{aligned}
\mathcal{P}_n^\dagger &\overset{n=0}{=} 1 \\
&\overset{n=1}{=} B^\dagger \\
&\overset{n=2}{=} (B^{\dagger 2} + c) \\
&\overset{n=3}{=} (B^{\dagger 3} + 3cB^\dagger) \\
&\overset{n=4}{=} (B^{\dagger 4} + 6cB^{\dagger 2} + 3c^2) \\
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&\overset{n=6}{=} (B^{\dagger 6} + 15cB^{\dagger 4} + 45c^2 B^{\dagger 2} + 15c^3) \\
&= \ldots
\end{aligned}$$

QRPA FOR EVERY EXCITED STATE

- let's define the phonon operators as

$$\boxed{Q_n^\dagger = X_n \mathcal{P}_n^\dagger - Y_n \mathcal{P}_n}$$

$$|0\rangle = \mathcal{N} \mathrm{e}^{dB^\dagger B^\dagger} |\rangle$$

- the same procedure leads to

$$\begin{aligned} d &= \frac{1}{2}\sqrt[n]{\frac{Y_n}{X_n}} = -\frac{(2\varepsilon + \lambda_1) - E}{4\lambda_2} \\ c &= \frac{(2d)}{(2d)^2 - 1} = -X_1 Y_1 \end{aligned}$$

- interesting relation

$$\frac{Y_1^n}{X_1^n} = (2d)^n = \frac{Y_n}{X_n}$$

QRPA FOR EVERY EXCITED STATE

- analytic expressions for RPA matrix elements is available

$$\mathcal{A}_n/\mathcal{U}_n = n\left[(2\varepsilon+\lambda_1)+2\lambda_2\frac{(2d)-(2d)^{2n-1}}{1-(2d)^{2n}}\right]$$

$$\mathcal{B}_n/\mathcal{U}_n = 2\lambda_2 n\frac{(2d)^{n-1}-(2d)^{n+1}}{1-(2d)^{2n}}$$

$$\mathcal{U}_n = n!\frac{1-(2d)^{2n}}{\left(1-(2d)^2\right)^n}$$

- and finaly the energy spectrum

$$E_n = n\sqrt{(2\varepsilon+\lambda_1)^2-4\lambda_2^2} = nE$$

SIMULTANEOUS DESCRIPTION FOR MORE EXCITED STATES

- (dim of RPA matrix)/2 = # of states described

- for simultaneous description of more states we need more forward and backward amplitudes
  - e.g.

$$Q_{13}^{\dagger} = X_3 B^{\dagger 3} + X_1 B^{\dagger} - Y_3 B^3 + Y_1 B$$

  - or in general

$$\boxed{Q_{n_{\max}}^{\dagger} = \sum_{i=1}^{n_{\max}} \left(X_i B^{\dagger i} - Y_i B^i \right)} \qquad |0\rangle = \mathcal{N} \mathrm{e}^{dB^{\dagger} B^{\dagger}} |\rangle$$

- it reproduces the correct spectrum of first $n_{\max}$ excited states

$$E_i = i\sqrt{(2\varepsilon + \lambda_1)^2 - 4\lambda_2^2} \quad \text{for} \quad i = 1, \ldots, n_{\max}$$

CONCLUSIONS

CONCLUSIONS

- Our goal is to describe better the nuclear structure related to double beta decay.
- Namely, we want to formulate the simultaneous QRPA description of mother, daughter and transition nuclei states

- We have studied the simplistic model and formulated nonlinear QRPA for that.
- In fact we have formulated new QRPA approach to exact solution of harmonic oscillator.

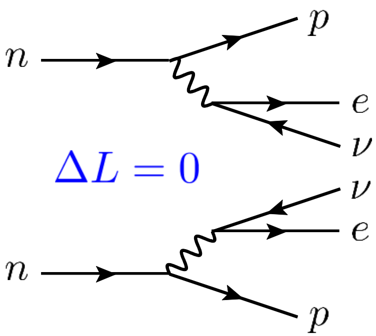
  (might be interesting result per se)

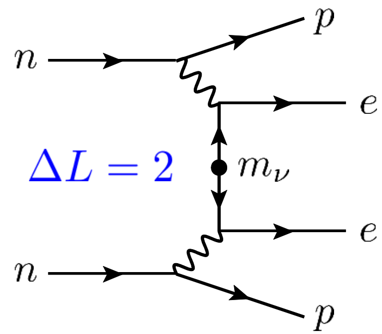
- We plan to apply the nonlinear phonon definitions onto more complicated systems.
  - H_{BM}, H_F, *realistic models*

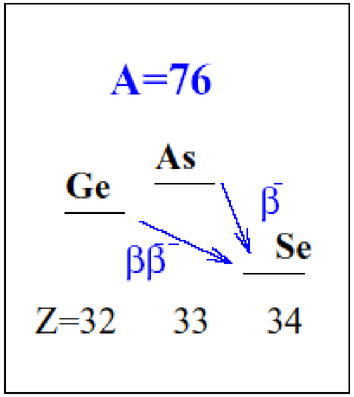
- The presented results might be a basis for perturbation calculations within models with perturbative anharmonic interactions.

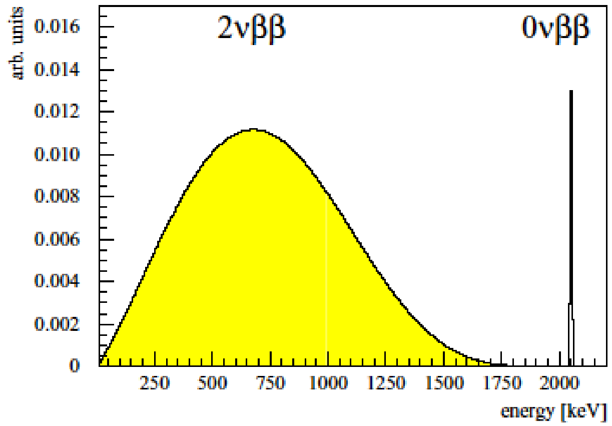
DOUBLE BETA DECAY

$$(A,Z) \longrightarrow (A,Z+2) + e^- + e^- (+\bar{\nu}_e + \bar{\nu}_e)$$







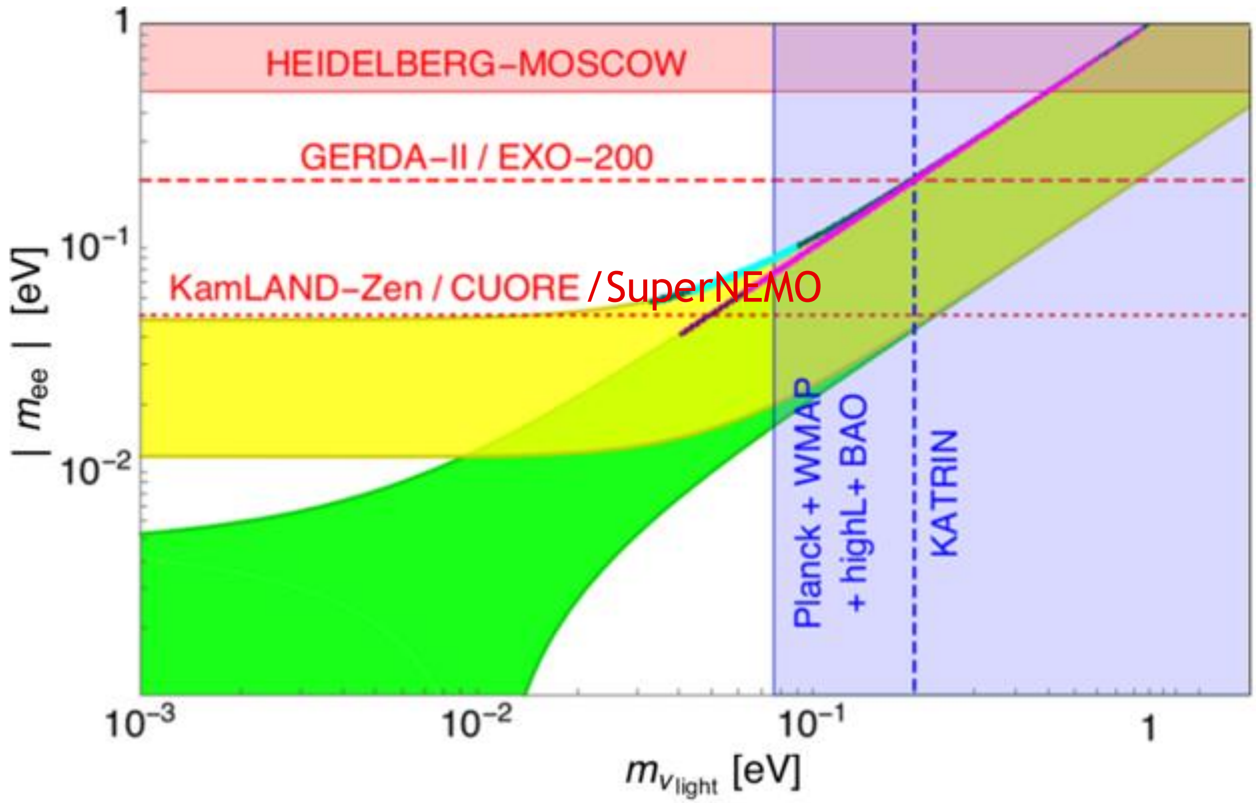


- ^{48}Ca, ^{76}Ge, ^{82}Se, ^{96}Zr, ^{100}Mo, ^{116}Cd, ^{128}Te, ^{130}Te, ^{150}Nd, ^{238}U
- 2nbb: $T_{1/2} \approx (10^{18} - 10^{24})\,\mathrm{y}$
- 0nbb: $T_{1/2} > 10^{26}\,\mathrm{y}$

NEUTRINOLESS DOUBLE BETA DECAY

$$\left(T_{1/2}^{0\nu}\right)^{-1} = \left|\frac{m_{\beta\beta}}{m_e}\right|^2 g_A^4 \left|M_\nu^{0\nu}\right|^2 G^{0\nu}$$

$$|m_{\beta\beta}| = |c_{12}^2 c_{13}^2 e^{i\alpha_1} m_1 + s_{12}^2 c_{13}^2 e^{i\alpha_2} m_2 + s_{13}^2 m_3|$$



NEUTRINOLESS DOUBLE BETA DECAY

- The predicted values of half-lifes have big uncertainties mainly from nuclear matrix elements.