# The Semantics of Empirical Unverifiability

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**Abstract.** Pavel Cmorej has argued that the existence of unverifiable and unfalsifiable empirical propositions follows from certain plausible assumptions concerning the notions of possibility and verification. Cmorej proves, it the context of a bi-modal alethic-epistemic axiom system AM4, that (1) '*p* and it is not verified that *p*' is unverifiable; (2) '*p* or it is falsified that *p*' is unfalsifiable; (3) every unverifiable *p* is logically equivalent to '*p* and it is not verifiable that *p*'; (4) every unverifiable *p* entails that *p* is unverifiable. This article elaborates on Cmorej's results in three ways. Firstly, we formulate a version of neighbourhood semantics for AM4 and prove completeness. This allows us to replace Cmorej's axiomatic derivations with simple model-theoretic arguments. Secondly, we link Cmorej's results to two well-known paradoxes, namely Moore's Paradox and the Knowability Paradox. Thirdly, we generalise Cmorej's results, show them to be independent of each other and argue that results (3) and (4) are independent of any assumptions concerning the notion of verification.

**Keywords.** Completeness – epistemic logic – knowability – verifiabiliy.

# 1 Introduction

Cmorej (1988, 1990) argues that the existence of unverifiable and unfalsifiable empirical propositions is a consequence of certain plausible assumptions concerning the notions of possibility and verification.[1] His argument is proof-theoretic and employs

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[1](Cmorej, 1990) is a translation of the Slovak original (Cmorej, 1988). I'll refer to the internationally accessible (Cmorej, 1990) for the rest of the article.

an alethic-epistemic axiom system. Cmorej's main result is that schemas

(1) $\sim MV(\alpha \wedge \sim V\alpha)$

(2) $\sim MF(\alpha \vee F\alpha)$,

are provable in the axiom system in question, where $M$ stands for 'it is possible that', $V$ stands for 'it is verified that' and $F$ stands for 'it is falsified that' ($F\alpha$ is defined as $V\sim\alpha$). If $\alpha$ is a hitherto unverified empirical proposition, then $\alpha \wedge \sim V\alpha$ is empirical as well. Yet, according to (1), it is unverifiable. Similarly, if $\alpha$ is empirical and not falsified, then $\alpha \vee F\alpha$ is empirical and, according to (2), not falsifiable.

Cmorej then goes on to establish two further results concerning unverifiable propositions. Firstly, each unverifiable proposition $\alpha$ is necessarily equivalent to $\alpha \wedge \sim V\alpha$. In other words,

(3) $\sim MV\alpha \supset L(\alpha \equiv (\alpha \wedge \sim V\alpha))$

is provable (where $L$ stands for 'it is necessary that'). Secondly, each unverifiable proposition $\alpha$ entails a proposition saying that $\alpha$ is unverifiable, i.e.

(4) $\sim MV\alpha \supset L(\alpha \supset \sim MV\alpha)$

is provable. (Similar results are established for falsifiability, but these are easily derivable form the results stated above by applying the definition of $F$.)

This article elaborates on Cmorej's results and sets them into a wider philosophical context. Firstly, Cmorej's arguments are simplified by replacing the complex axiomatic proofs of the results concerning (1) – (4) by simple model-theoretic arguments. Secondly, Cmorej's result concerning (1) is linked to two well-known paradoxes, namely Moore's Paradox (Green & Williams, 2007; Moore, 1942) and the Knowability Paradox (Fitch, 1963; Salerno, 2009). Thirdly, the results are generalised and shown to be independent. In particular, we set up a weak bi-modal logic that validates (1) and (2) without validating (3), (4) and most Cmorej's assumptions concerning $V$ and $M$. We also formulate a bi-modal logic that validates (3) and (4) without validating (1) or (2). The precise nature of the latter logic suggests that the results concerning (3) and (4) are independent of any assumptions concerning the notion of verification.

The article is organised as follows. Section 2 introduces Cmorej's axiom system AM4 and establishes completeness with respect to a specific class of modal neighbourhood frames (Chellas, 1980; Segerberg, 1971). This allows us to formulate simple model-theoretic arguments establishing (1) – (4). Section 3 relates Cmorej's result concerning (1) to Moore's Paradox and the Knowability Paradox. Section 4 shows that the results concerning (1) and (2) are independent from the results concerning (3) and (4), and that the latter two are independent of any assumptions concerning the notion of verification. The final section 5 sums up the main points of the article.

# 2 Semantic Arguments

This section introduces the axiom system AM4 (2.1) discusses models (2.2), proves completeness (2.3) and provides simple model-theoretic arguments establishing (1) – (4) (2.4).

## 2.1 AM4

Let us fix a denumerable set $Var$ of propositional variables. Every propositional variable $p, q, \ldots$ is a *formula*. If $\alpha$ and $\beta$ are formulas, then so are $\sim\alpha$, $\alpha \wedge \beta$, $L\alpha$ and $V\alpha$. Other Boolean connectives are defined in the usual fashion. $L\alpha$ is read as 'it is necessary that $\alpha$' (or '$\alpha$ is necessary') and $V\alpha$ as 'it is verified that $\alpha$' (or '$\alpha$ is verified'). $M\alpha$ is defined as $\sim L\sim\alpha$ and is read as 'it is possible that $\alpha$' (or '$\alpha$ is possible'). $F\alpha$ is defined as $V\sim\alpha$ and is read as 'it is falsified that $\alpha$' (or '$\alpha$ is falsified'). A formula is *tautologous* if it is a substitution instance of a tautology of classical propositional logic.

**Definition 2.1 (AM4, Cmorej (1990)).** The axiom system AM4 is given by the following axiom schemas and rules of inference. Every tautologous formula is an axiom. Other axioms are all formulas of the form:

(A1) $L\alpha \supset \alpha$

(A2) $L(\alpha \supset \beta) \supset (L\alpha \supset L\beta)$

(A3) $\sim L\alpha \supset L\sim L\alpha$

(B1) $V\alpha \supset \alpha$

(B2) $V(\alpha \wedge \beta) \supset (V\alpha \wedge V\beta)$

(B3) $(V\alpha \wedge V\beta) \supset V(\alpha \wedge \beta)$

(B4) $V\alpha \supset VV\alpha$

(C) $L(\alpha \supset \beta) \supset (V\alpha \supset V\beta)$

There are two rules of inference, namely Modus Ponens and $L$-Necessitation ('If $\vdash \alpha$, then $\vdash L\alpha$'). Proofs and derivations are defined as usual. ■

The choice of 'alethic' $L$-axioms and rules and 'methodological' $V$-axioms makes it clear that $L$ is a *normal* modality governed by axioms of the system S5 (see Hughes & Cresswell, 1996), while $V$ is a *regular* modality governed at least by the axioms of the system RT4 (see Chellas, 1980). We shall see later on that, in fact, $V$ is a non-normal modality as the rule of $V$-Necessitation is not a derivable rule. In other words, verification is not closed under admissible zero-premise inference rules. However, as the 'interaction axiom' (C) suggests, verification is closed under admissible one-premise rules. In fact, a consequence of the inclusion of (B3) among axioms entails that verification is closed under admissible multi-premise rules as well.

**Lemma 2.2.** *$L\alpha \supset LL\alpha$ is derivable in AM4.*

*Proof.* Folklore (see Hughes & Cresswell, 1996, 58). □

**Lemma 2.3.** *If $\alpha \equiv \beta$ is provable in AM4, then so is $V\alpha \equiv V\beta$.*

*Proof.* We make use of some obviously admissible S5-rules. If $\vdash \alpha \equiv \beta$, then $\vdash L(\alpha \equiv \beta)$, then $\vdash L(\alpha \supset \beta) \wedge L(\beta \supset \alpha)$, then $\vdash (V\alpha \supset V\beta) \wedge (V\beta \supset V\alpha)$. □

## 2.2 Models

The models of our choice are neighbourhood models, where neighbourhoods (to be defined shortly) are closed under intersection. The assumption of closure under supersets, standard when regular systems are dealt with, is simulated by a non-standard truth-condition for $V\alpha$. $L$ is treated as a universal modality.

**Definition 2.4 (Frames).** A *frame* is a couple

$$\mathcal{F} = \langle \mathcal{W}, \mathcal{N} \rangle$$

where $\mathcal{W}$ is a non-empty set ('states', '(possible) worlds') and $\mathcal{N}$ is a function from $\mathcal{W}$ to subsets of the power-set of $\mathcal{W}$ ('neighbourhood function'). Hence, $\mathcal{N}(w)$ is a set of sets of worlds ('neighbourhoods of $w$'). It is assumed that

- (c) If $X, Y \in \mathcal{N}(w)$, then $X \cap Y \in \mathcal{N}(w)$;
- (t) If $X \in \mathcal{N}(w)$, then $w \in X$;
- (iv) If $X \in \mathcal{N}(w)$, then $\{v \mid X \in \mathcal{N}(v)\} \in \mathcal{N}(w)$. ■

Sets $X \in \mathcal{N}(w)$ can be thought of as propositions 'directly' verified at $w$. The assumption (c) guarantees that (B3) is valid in every frame (to be defined shortly); (t) ensures (B1) and (iv) ensures (B4), (see Chellas, 1980).

**Definition 2.5 (Models and Truth-Sets).** A *model* based on $\mathcal{F}$ is a couple

$$\mathcal{M} = \langle \mathcal{F}, \mathcal{V} \rangle$$

where $\mathcal{V}$ is a function from $Var$ to subsets of $\mathcal{W}$ ('valuation'). The *truth-set* $|\alpha|_{\mathcal{M}}$ of a formula $\alpha$ in model $\mathcal{M}$ is defined recursively as follows:

- $|p|_{\mathcal{M}} = \mathcal{V}(p)$;
- $|{\sim}\alpha|_{\mathcal{M}} = \mathcal{W} \setminus |\alpha|_{\mathcal{M}}$;
- $|\alpha \wedge \beta|_{\mathcal{M}} = |\alpha|_{\mathcal{M}} \cap |\beta|_{\mathcal{M}}$;
- $|V\alpha|_{\mathcal{M}} = \{w \mid X \subseteq |\alpha|_{\mathcal{M}} \text{ for some } X \in \mathcal{N}(w)\}$;
- $|L\alpha|_{\mathcal{M}} = \mathcal{W}$ if $|\alpha|_{\mathcal{M}} = \mathcal{W}$; $|L\alpha|_{\mathcal{M}} = \emptyset$ otherwise. ■

$w \in |\alpha|_{\mathcal{M}}$ is read as '$\alpha$ is true in $w$ (in the context of $\mathcal{M}$)'. This will be written also as $\mathcal{M}, w \models \alpha$. Informally, $V\alpha$ is true in $w$ iff there is a proposition directly verified at $w$ that 'entails' $\alpha$. $L\alpha$ is true at any world iff $\alpha$ is true in every world. $\mathcal{M}$ will not be mentioned when the identity of the model in question is clear from the context or immaterial.

**Definition 2.6 (Consequence).** $\alpha$ is a $\mathcal{M}$-*consequence* of a set of formulas $\Gamma$ iff

$$\bigcap_{\beta \in \Gamma} |\beta|_{\mathcal{M}} \subseteq |\alpha|_{\mathcal{M}}$$

('$\Gamma$ $\mathcal{M}$-entails $\alpha$'). $\alpha$ is $\mathcal{M}$-valid iff it is a $\mathcal{M}$-consequence of the empty set. $\alpha$ is a $\mathcal{F}$-*consequence* of $\Gamma$ iff it is a $\mathcal{M}$-consequence of $\Gamma$ for every $\mathcal{M}$ based on $\mathcal{F}$. If $\mathcal{C}$ is a class of frames (models), then $\alpha$ is a $\mathcal{C}$-consequence of $\Gamma$ iff it is a $\mathcal{F}$-consequence ($\mathcal{M}$-consequence) of $\Gamma$ for every $\mathcal{F}$ ($\mathcal{M}$) in $\mathcal{C}$. Similarly for $\mathcal{C}$-validity. ■

$\Gamma$ $\mathcal{M}$-entails $\alpha$ iff there is no world in $\mathcal{M}$ where all the 'assumptions' in $\Gamma$ are true, but $\alpha$ is false. $\alpha$ is $\mathcal{M}$-valid iff it is true 'throughout the model $\mathcal{M}$'.

**Example 2.7.** Let us consider an example. Let the set of worlds be $\{v, u\}$ and assume that the truth-set of $p$ is $\{v\}$, while the truth-set of $q$ is $\{v, u\}$. In addition, let $\mathcal{N}(v) = \{\{v\}\}$ and $\mathcal{N}(u) = \emptyset$. It is easy to check that this model satisfies the conditions (c), (t) and (iv). $q$ is valid in the model and, hence, $Lq$ holds in both worlds. So does does ${\sim}L(p \wedge q)$. $Vq$ holds in $v$, because $\{v\} \in \mathcal{N}(v)$ and $\{v\} \subseteq \{v, u\}$, the truth-set of $p$. However, $Vq$ does not hold in $u$. Note that even $V{\sim}q$ is false in $u$. The truth-set of ${\sim}q$ is $\emptyset$, but, obviously, $\emptyset \notin \emptyset$. In fact, ${\sim}V\alpha$ holds in $u$ for every formula $\alpha$. In conjunction with our completeness proof of Section 2.3, this example shows that Cmorej's $V$ is not a normal modality (as such, it would have to satisfy $V$-necessitation). ■

Neighbourhood semantics has a wide range of applications, including models of coalitions within games (Pauly, 2002). Neighbourhood models have recently been applied to an epistemic language with both normal an non-normal modalities within the project of *evidence logics* (see van Benthem, Fernández-Duque, & Pacuit, 2014; van Benthem & Pacuit, 2011). In view of our completeness result established below, Cmorej may be credited with an early contribution to evidence logic.

## 2.3 Completeness

The goal of the present subsection is to show that $\alpha$ is derivable from a set of assumptions $\Gamma$ in AM4 iff $\Gamma$ $\mathcal{F}$-entails $\alpha$ in every $\mathcal{F}$. One half of the claim is established easily.

**Proposition 2.8 (Soundness).** *If $\alpha$ is derivable from a set of assumptions $\Gamma$ in AM4, then $\Gamma$ $\mathcal{F}$-entails $\alpha$ in every $\mathcal{F}$.*

*Proof.* It is sufficient to show that every axiom is valid in every frame and that the rules of inference preserve validity. All cases are straightforward. Nevertheless, let us prove the validity of (B3) and (B4). First, (B3). Consider any $\mathcal{M}, w$. If $\mathcal{M}, w \models V\alpha \wedge V\beta$, then there is $X \in \mathcal{N}(w)$ such that $X \subseteq |\alpha|$ and there is $Y \in \mathcal{N}(w)$ such that $Y \subseteq |\beta|$. But then $X \cap Y \in \mathcal{N}(w)$ by (c). Obviously, $X \cap Y \subseteq |\alpha \wedge \beta|$. Hence, $\mathcal{M}, w \models V(\alpha \wedge \beta)$. Next, (B4). If $M, w \models V\alpha$, then there is $X \in \mathcal{N}(w)$ such that $X \subseteq |\alpha|$. By (iv), $\{v \mid X \in \mathcal{N}(v)\} \in \mathcal{N}(w)$. It is plain that $\{v \mid X \in \mathcal{N}(v)\} \subseteq |V\alpha|$. In other words, there is $Y \in \mathcal{N}(w)$ such that $Y \subseteq |V\alpha|$. Consequently, $\mathcal{M}, w \models VV\alpha$. □

To establish the other half of the main claim, we employ the standard canonical model technique (see Chellas, 1980). A specific feature of our situation is the presence of the universal modality $L$. To deal with this extra machinery, we combine the standard completeness argument for regular systems with a simple strategy that is used within completeness proofs for normal systems with the universal modality (Blackburn, de Rijke, & Venema, 2001, ch. 7.1). But first, let us re-capitulate some standard terminology.[2]

**Definition 2.9 (AM4-sets).** A set $\Gamma$ of formulas is *maximal AM4-consistent* ('an AM4-set') iff

- $\Gamma$ is *consistent*, i.e. there is no $\{\alpha_1, \ldots, \alpha_n, \beta\} \subseteq \Gamma$ such that $\alpha_1 \wedge \ldots \wedge \alpha_n \supset {\sim}\beta$ is provable in AM4; and
- $\Gamma$ is *maximal*, i.e. if $\alpha \notin \Gamma$, then $\Gamma \cup \{\alpha\}$ is not consistent. ■

**Lemma 2.10.** *Some well-known properties of maximal consistent sets:*

- *If $\Gamma$ is an AM4-set, $\Delta \subseteq \Gamma$ and $\alpha$ is derivable form $\Delta$ in AM4, then $\alpha \in \Gamma$;*

[2]More details on maximal consistent sets and modal completeness proofs are provided by Blackburn et al. (2001, ch. 4), Chellas (1980, ch. 2.6–2.7, 4.5, 5.3) and Hughes and Cresswell (1996, ch. 6), who discuss normal systems. Chellas (1980, ch. 9) discusses completeness proofs for some non-normal systems.

- *If $\Delta$ is consistent then there is an AM4-set $\Gamma$ such that $\Delta \subseteq \Gamma$ (Lindenbaum's Lemma);*

*Proof.* Standard (see Chellas, 1980, ch. 2.6). $\square$

Note that the above Lemma entails that if $\Gamma$ is an AM4-set, then $\sim\alpha \in \Gamma$ iff $\alpha \notin \Gamma$ and $\alpha \wedge \beta \in \Gamma$ iff $\alpha, \beta \in \Gamma$.

**Definition 2.11 (Pre-models).** A *pre-model* is a tuple

$$\mathcal{M}_0 = \langle \mathcal{W}_0, \mathcal{R}, \mathcal{N}_0, \mathcal{V}_0 \rangle,$$

where

- $\mathcal{W}_0$ is the set of all AM4-sets, and $|\alpha|_0 = \{\Gamma \in \mathcal{W}_0 \mid \alpha \in \Gamma\}$;
- $\mathcal{R}\Gamma\Delta$ iff $\{\alpha \mid L\alpha \in \Gamma\} \subseteq \Delta$, and $\mathcal{R}(\Gamma) = \{\Delta \mid \mathcal{R}\Gamma\Delta\}$;
- $\mathcal{N}_0(\Gamma) = \{|\alpha|_0 \mid V\alpha \in \Gamma\}$;
- $\mathcal{V}_0(p) = |p|_0$. $\blacksquare$

**Lemma 2.12.** *For all $\Gamma \in \mathcal{W}_0$, $\mathcal{N}_0(\Gamma)$ is closed under (binary) intersections.*

*Proof.* Assume that $X, Y \in \mathcal{N}_0(\Gamma)$. By the definition of $\mathcal{N}_0$, $X = |\alpha|_0$ and $Y = |\beta|_0$ for some $V\alpha, V\beta \in \Gamma$. By Lemma 2.10, $V(\alpha \wedge \beta) \in \Gamma$. Hence, $|\alpha \wedge \beta|_0 \in \mathcal{N}_0$. In other words, $|\alpha|_0 \cap |\beta|_0 \in \mathcal{N}_0(\Gamma)$. $\square$

**Lemma 2.13.** *If $\Gamma \in \mathcal{W}_0$ and $L\alpha \notin \Gamma$, then there is $\Delta \in \mathcal{W}_0$ such that*

- *$\mathcal{R}\Gamma\Delta$ and*
- *$\sim\alpha \in \Delta$.*

*Proof.* Standard (see Hughes & Cresswell, 1996, 115–117). $\square$

It is clear that, in pre-models, we can have some $\Gamma, \Delta, \alpha$ such that $L\alpha \in \Gamma$, but $\alpha \notin \Delta$ (if not $\mathcal{R}\Gamma\Delta$). Hence, in the context of pre-models, $L$ is not a universal modality. To fix this, we use a standard 'trick'.

**Definition 2.14 (Canonical $\Lambda$-model).** Let $\Lambda \in \mathcal{W}_0$. A canonical $\Lambda$-model is a tuple

$$\mathcal{M}_\Lambda = \langle \mathcal{W}_\Lambda, \mathcal{N}_\Lambda, \mathcal{V}_\Lambda \rangle$$

where

- $\mathcal{W}_\Lambda = \mathcal{R}(\Lambda)$ and $|\alpha|_\Lambda = |\alpha|_0 \cap \mathcal{W}_\Lambda$;

- $\mathcal{N}_\Lambda(\Gamma) = \{X \subseteq \mathcal{W}_\Lambda \mid X = X_0 \cap \mathcal{W}_\Lambda \text{ for some } X_0 \in \mathcal{N}_0(\Gamma)\}$;
- $V_\Lambda(p) = |p|_\Lambda$. ■

$\Lambda$ is seen as the 'centre' of the model, which universe is the set of AM4-sets reachable from the centre via $\mathcal{R}$. Crucially, every $\Lambda$-neighbourhood of any $\Gamma \in \mathcal{W}_0$ is a 'pre-neighbourhood' of $\Gamma$ with every AM4-set *not* reachable from the centre 'bitten off'.

Now the goal is to show that, for every $\Lambda$, $\mathcal{M}_\Lambda$ is indeed a model.

**Lemma 2.15.** $|\alpha|_\Lambda \subseteq |\beta|_\Lambda$ *iff* $L(\alpha \supset \beta) \in \Lambda$.

*Proof.* We omit the simple argument establishing the right-to-left direction. To prove the converse, assume that $L(\alpha \supset \beta) \notin \Lambda$. By Lemma 2.13, there is $\Gamma \in \mathcal{R}(\Lambda)$ such that $\alpha \in \Gamma$ and $\beta \notin \Gamma$. Hence, $\Gamma \in |\alpha|_0 \cap \mathcal{W}_\Lambda$, but $\Gamma \notin |\beta|_0 \cap \mathcal{W}_\Lambda$. In other words, $|\alpha|_\Lambda \nsubseteq |\beta|_\Lambda$. □

**Lemma 2.16.** *If* $\Gamma \in \mathcal{W}_\Lambda$ *and* $L\alpha \in \Lambda$, *then* $L\alpha \in \Gamma$.

*Proof.* Follows from Lemmas 2.2 and 2.10. □

**Lemma 2.17 (Frame Lemma).** *For all* $\Lambda \in \mathcal{W}_0$ *and* $\Gamma \in \mathcal{W}_\Lambda$*:*

- *(c) If* $X, Y \in \mathcal{N}_\Lambda(\Gamma)$, *then* $X \cap Y \in \mathcal{N}_\Lambda(\Gamma)$*;*
- *(t) If* $X \in \mathcal{N}_\Lambda(\Gamma)$, *then* $\Gamma \in X$*;*
- *(iv) If* $X \in \mathcal{N}_\Lambda(\Gamma)$, *then* $\{\Delta \mid X \in \mathcal{N}_\Lambda(\Delta)\} \in \mathcal{N}_\Lambda(\Gamma)$.

*Proof.* (c) Assume that $X, Y \in \mathcal{N}_\Lambda(\Gamma)$. Then $X = |\alpha|_\Lambda$ and $Y = |\beta|_\Lambda$ for some $V\alpha, V\beta \in \Gamma$. By Lemma 2.12, $|\alpha|_0 \cap |\beta|_0 \in \mathcal{N}_0(\Gamma)$. Hence, $X \cap Y = |\alpha|_0 \cap |\beta|_0 \cap \mathcal{W}_\Lambda \in \mathcal{N}_\Lambda(\Gamma)$.

(t) Assume that $X \in \mathcal{N}_\Lambda(\Gamma)$. Then $X = |\alpha|_0 \cap \mathcal{W}_0$ for some $V\alpha \in \Gamma$. By Lemma 2.10 and axiom (B1), $\alpha \in \Gamma$, i.e. $\Gamma \in |\alpha|_0$. Consequently, $\Gamma \in X$.

(iv) Assume that $X \in \mathcal{N}_\Lambda(\Gamma)$. Then $X = |\alpha|_0 \cap \mathcal{W}_\Lambda$ for some $V\alpha \in \Gamma$. By Lemma 2.10 and axiom (B4), $VV\alpha \in \Gamma$. Now assume that $\{\Delta \mid X \in \mathcal{N}_\Lambda(\Delta)\} \notin \mathcal{N}_\Lambda(\Gamma)$. This means that $\{\Delta \mid X \in \mathcal{N}_\Lambda(\Delta)\} \neq |\beta|_0 \cap \mathcal{W}_\Lambda$ for no $V\beta \in \Gamma$. In particular, then, this holds for $VV\alpha$. In other words,

$$\{\Delta \mid X \in \mathcal{N}_\Lambda(\Delta)\} \neq |V\alpha|_0 \cap \mathcal{W}_\Lambda$$

Now there are two cases to check.

1. There is $\Delta \in \mathcal{W}_\Lambda$ such that $\Delta \in |V\alpha|_0 \cap \mathcal{W}_\Lambda$ but $|V\alpha|_0 \cap \mathcal{W}_\Lambda \notin \mathcal{N}_\Lambda(\Delta)$. The latter means that $|\alpha|_\Lambda \neq |\beta|_\Lambda$ for no $V\beta \in \Delta$. But $V\alpha \in \Delta$, so the assumption entails that $|\alpha|_\Lambda \neq |\alpha|_\Lambda$. Contradiction.

2. There is $\Delta \in \mathcal{W}_\Lambda$ such that $|V\alpha|_0 \cap \mathcal{W}_\Lambda \in \mathcal{N}_\Lambda(\Delta)$ but $\Delta \notin |V\alpha|_0 \cap \mathcal{W}_\Lambda$. In other words, $|\alpha|_\Lambda = |\beta|_\Lambda$ for some $V\beta \in \Delta$, but $V\alpha \notin \Delta$. The former entails, by Lemma 2.15, that $L(\alpha \supset \beta) \wedge L(\beta \supset \alpha) \in \Lambda$. By Lemma 2.16, $L(\beta \supset \alpha) \in \Delta$. But then, by Lemma 2.10 and axiom (C), $V\beta \supset V\alpha \in \Delta$. Consequently, $V\alpha \in \Delta$. Contradiction.

□

**Lemma 2.18 (Model Lemma).** *For all $\Lambda \in \mathcal{W}_0$ and $\Gamma \in \mathcal{W}_\Lambda$, $\alpha \in \Gamma$ iff $\mathcal{M}_\Lambda, \Gamma \models \alpha$.*

*Proof.* We need to check that $\alpha \in \Gamma$ iff the truth-condition for $\alpha$ is satisfied with respect to $\Gamma$. The proof is by induction on the complexity of $\alpha$. The base case $\alpha = p$ holds by definition. The cases of $\sim$ and $\wedge$ are easy (and standard) and we omit them. Only the 'modal' cases are checked explicitly.

We check that $L\alpha \in \Gamma$ iff $\mathcal{M}_\Lambda, \Delta \models \alpha$ for all $\Delta \in \mathcal{W}_\Lambda$. The right-hand side is equivalent to the claim that $\alpha \in \Delta$ for all $\Delta \in \mathcal{W}_\Lambda$ by the induction hypothesis. Now the left-to-right implication is an obvious consequence the definition of $\mathcal{W}_\Lambda$. The right-to-left implication follows from Lemma 2.13 and the definition of $\mathcal{W}_\Lambda$.

Next, we check that $V\alpha \in \Gamma$ iff there is a $X \in \mathcal{N}_\Lambda(\Gamma)$ such that $X \subseteq |\alpha|_\Lambda$. If $V\alpha \in \Gamma$, then $|\alpha|_0 \in \mathcal{N}_0(\Gamma)$ and, hence, $|\alpha|_0 \cap \mathcal{W}_\Lambda \in \mathcal{N}_\Lambda(\Gamma)$. Conversely, if there is $X \in \mathcal{N}_\Lambda(\Gamma)$ such that $X \subseteq |\alpha|_\Lambda$, then $X = |\beta|_\Lambda$ for some $V\beta \in \Gamma$. $V\alpha \in \Gamma$ follows by Lemmas 2.15 and 2.16. □

The Frame and Model Lemmas ensure that every canonical $\Lambda$-model is a model and that membership in $\Gamma$ is equivalent to truth in $\Gamma$. Completeness follows immediately.

**Theorem 2.19 (Strong Completeness).** *Let $\Theta$ be any set of formulas. If $\Theta$ $\mathcal{F}$-entails $\alpha$ for every $\mathcal{F}$, then $\alpha$ is derivable from $\Theta$ in AM4.*

*Proof.* Assume that $\alpha$ is not derivable from $\Theta$. Then the set $\Theta \cup \{\sim\alpha\}$ is consistent. By Lindenbaum's Lemma, there is an AM4-set $\Lambda \supseteq \Theta \cup \{\sim\alpha\}$. Construct the $\Lambda$-canonical model $\mathcal{M}_\Lambda$. By Lemmas 2.17 and 2.18, there is a model $\mathcal{M}$ (namely $\mathcal{M}_\Lambda$) and a world $w$ (namely $\Lambda$) such that $\mathcal{M}, w \models \beta$ for every $\beta \in \Theta$, but $\mathcal{M}, w \not\models \alpha$. Hence, $\Theta$ does not $\mathcal{F}$-entail $\alpha$ for all $\mathcal{F}$. □

## 2.4 Cmorej's Results, Semantically

A direct consequence of the Completeness Theorem is that Cmorej's results may be established by using simple model-theoretic arguments.

Assume that (1) is not provable. Then, by the Completeness Theorem, there is a model $\mathcal{M}$ and a world $w$ such that $MV(\alpha \wedge \sim V\alpha)$ is true in $w$. But then, by the truth-condition for $L$, $V(\alpha \wedge \sim V\alpha)$ is true in some $u$ in the model $\mathcal{M}$. Soundness and (B2) imply that $V\alpha \wedge V\sim V\alpha$ holds in $u$ and (B1) leads to the contradiction that $V\alpha \wedge \sim V\alpha$ holds in $u$.

The provability of (2) is a direct consequence of the provability of (1). If the schema (1) is valid then so is $\sim MV(\sim\alpha \wedge \sim V\sim\alpha)$ and, by Lemma 2.3, $\sim MV\sim(\alpha \vee V\sim\alpha)$ is valid as well.

Now assume that (3) is false in some $\mathcal{M}, w$. Hence, $L\sim V\alpha \wedge M\sim(\alpha \equiv (\alpha \wedge \sim V\alpha))$ is true in $w$. This means that there is some $u$ in $\mathcal{M}$ such that $\sim V\alpha \wedge \sim(\alpha \equiv (\alpha \wedge \sim V\alpha))$ holds in $u$. But this is impossible, since the latter formula is a substitution instance of a contradiction of classical propositional logic.

Finally, assume that (4) is false in $\mathcal{M}, w$. Then $L\sim V\alpha \wedge M(\alpha \wedge MV\alpha)$ in $w$. By Lemma 2.2 and Soundness, $L\sim V\alpha \wedge \alpha \wedge MV\alpha$ in $u$. Contradiction. The nature of the latter two arguments suggests that the results concerning (3) and (4) are independent of *any* assumptions concerning $V$. We will return to this point in Section 4.

# 3 Unverifiability, Absurdity, and Unknowability

This section links Cmorej's results to two well-known philosophical problems, Moore's Paradox and the Knowability Paradox. Our sole aim is to point out some similarities between Cmorej's findings and the two paradoxes without going into philosophical detail.

Cmorej's main result is that

(5) $p \wedge \sim Vp$,

as well as all its substitution instances, is provably unverifiable. (5) is similar in form to so-called (omissive) *moorean sentences*, i.e. sentences of the form

(6) $p$ and I do not believe that $p$,

with 'I believe that' replaced by 'It is verified that'. Moorean sentences and the air of absurdity surrounding them are at the heart of a famous problem, known as *Moore's Paradox*. Green and Williams (2007, 3) explain that

> G. E. Moore observed that to say, 'I went to the pictures last Tuesday but I don't believe that I did' would be 'absurd' (1942, 543). Over half a century later, such sayings continue to perplex philosophers and other students of language,

> logic, and cognition. On the one hand, such sayings seem distinct from semantically odd Liar-type sayings such as 'What I'm now saying is not true'. Unlike Liar-type sentences, what Moore said might be true: One can readily imagine a situation in which Moore went to the pictures last Tuesday but does not believe that he did so. On the other hand, it does seem absurd to *assert* a proposition while, with no apparent change of mind, or aside to a different audience, going on to deny that one believes it. It seems no less absurd to *judge true* the following proposition: *p and I do not believe that p*. (Original emphasis.)

(5) may itself be labelled as 'absurd to utter' or 'absurd to judge true'. Assume that I assert that $p$ and that $p$ is not verified at the same time. It seems, then, that my assertion implies that it lacks appropriate grounds: If the assertion is true, then one of the statements being asserted is unverified. But on what grounds is it asserted, then?

Cmorej's result concerning (1) can be construed as providing an explanation of the air of absurdity surrounding (5): (5) is unverifiable and, therefore, un-$X$-able for every $X$ that requires verification.[3] This explanation is similar in spirit to Hintikka's (1962, 52–54) solution to Moore's Paradox, who argues that it is impossible for the speaker to believe (6).

Nevertheless, belief may be thought to be far too distant in nature from verification to ground any comparisons of Cmorej's (5) to the moorean (6). Verification, it might be argued, is closer to (empirical) *knowledge*. Hence, it may seem more plausible to construe (5) along the lines of

(7) $p$ and it is not known that $p$

On this account, Cmorej's result implies that propositions of the form (7) are *unknowable*. This observation is, of course, at the heart of another famous problem, the *Knowability Paradox* due to Frederic Fitch and Alonzo Church (Fitch, 1963; Salerno, 2009). Its gist is that the plausible assumption that every truth is *knowable* entails the ridiculous conclusion that every truth is *known*. For assume that every truth is knowable. Then, given the fact that (7) is unknowable, (7) is false. In other words 'If $p$, then it is known that $p$' is true. But $p$ is arbitrary, so the claim holds for every $p$, i.e. every truth is known.

# 4 Independence Results

This section is devoted to showing that the results concerning (1) and (2) are independent of the results concerning (3) and (4), and that the latter two are independent

[3] In the sense that if some $p$ is $X$-ed then $p$ is verified.

of any assumptions concerning the notion of verification. Consequently, the results concerning (1) and (2) are generalised, i.e. shown to hold for weaker notions of verification, and the results concerning (3) and (4) are shown to hold for every unary operator in place of $V$ whatsoever.

The results are established as follows. Firstly, in section 4.1 we formulate AM1, a bi-modal logic for $L$ and $V$ that is rather weaker than AM4, but validates (1) and (2) without validating (3) or (4). Secondly, in section 4.2 we formulate another bi-modal logic AM0 with some very weak assumptions concerning $L$ and no assumptions concerning $V$ at all, and show that the logic validates (3) and (4) without validating (1) or (2). Section 4.3 provides some additional remarks. We note that both AM1 and AM0 will be formulated semantically, i.e. as sets of formulas valid in a class of frames. Axiom systems will be mentioned, but completeness will not be proved. The reason is that both completeness arguments are simple exercises extending the standard completeness proofs for 'classical' logics (see Chellas, 1980).

## 4.1 (1) and (2) without (3) or (4)

AM1 will be defined as a set of formulas valid in a special class of bi-neighbourhood frames. Hence, we shall use neighbourhood models where both operators $L$ and $V$ are given truth-conditions in terms of neighbourhood functions. As a result, $L$ in AM1 is a non-normal modality.

**Definition 4.1 (AM1-Frames and Models).** An AM1-frame is a triple

$$\mathcal{F} = \langle \mathcal{W}, \mathcal{N}_L, \mathcal{N}_V \rangle$$

where $\mathcal{W}$ is a non-empty set (interpreted as before) and both $\mathcal{N}_L, \mathcal{N}_V$ are functions from $\mathcal{W}$ to subsets of the power-set of $\mathcal{W}$. It is assumed that (for all $w$)

- (l) $\mathcal{W} \in \mathcal{N}_L(w)$;
- (m) For all $Z$ and all $X \in \mathcal{N}_V(w)$, if $X \subseteq Z$, then for all $Y \in \mathcal{N}_V(w)$, there is $u \in Y$ and some $U \in \mathcal{N}_V(u)$ such that $U \subseteq Z$, for some $U$.

An AM1-model is an AM1-frame with a valuation, i.e. $\mathcal{M} = \langle \mathcal{F}, \mathcal{V} \rangle$. Truth-sets are defined as before, with the exception of

- $|L\alpha|_{\mathcal{M}} = \{w \mid |\alpha|_{\mathcal{M}} \in \mathcal{N}_L(w)\}$.

($V\alpha$ is dealt with as before, but in terms of $\mathcal{N}_V$.) Validity is defined as usual. AM1 is the set of formulas valid in every AM1-frame. ■

$\mathcal{N}_L(w)$, the set of $L$-neighbourhoods of $w$, is seen as the set of propositions necessary at $w$. It is assumed only that the 'maximal proposition' $\mathcal{W}$ is always necessary (l). The condition (m) might seem confusing, but its role is made clear by the proof of the following fact.

**Fact 4.2.** *If $\alpha \in$ AM1, then $L\alpha \in$ AM1. Moreover, every formula of the form*

$$V{\sim}V\alpha \supset {\sim}V\alpha$$

*belongs to AM1.*

*Proof.* Assume that $\alpha \in$ AM1 and take any $\mathcal{M}, w$. It follows that $|\alpha|_{\mathcal{M}} = \mathcal{W}$. Consequently, $|\alpha|_{\mathcal{M}} \in \mathcal{N}_L(w)$ and, hence, $L\alpha$ is true in $w$.

By propositional logic, $V{\sim}V\alpha \supset {\sim}V\alpha$ is equivalent to $V\alpha \supset {\sim}V{\sim}V\alpha$. Now assume that $\mathcal{M}, w \models V\alpha$. We have to show that $\mathcal{M}, w \models {\sim}V{\sim}V\alpha$. Assume that this is not the case (indirect assumption). The first assumption entails that there is $X \in \mathcal{N}_V(w)$ such that $X \subseteq |\alpha|$. The indirect assumption entails that there is $Y \in \mathcal{N}_V(w)$ such that $Y \subseteq |{\sim}V\alpha|$. In other words, for all $u \in Y$ and all $U \in \mathcal{N}_V(u)$, $U \nsubseteq |\alpha|$. But this is precisely the negation of our condition (m). □

It is easy to show that the only non-tautologous axiom schema of AM4 that belongs to AM1 is (B2). This is done by constructing countermodels for all other axiom schemas. We give one example and leave the rest to the reader as an exercise.

**Example 4.3.** Let $\mathcal{W} = \{v, u\}$ and $|p| = \{u\}$. Moreover, let $\{\{v, u\}, \{u\}\}$ ($\emptyset$) by the value of $\mathcal{N}_L(x)$ ($\mathcal{N}_V(x)$) for every $x \in \mathcal{W}$. It is easily checked that both (l) and (m) are satisfied. Moreover, $Lp$ holds in $v$. However, $p$ is false in $v$. The axiom schema (A1) fails as $p$ is necessary but not true in some world of some model. ■

To facilitate comparison with AM4, we state (without proof) the following axiomatization result.

**Proposition 4.4.** *AM1 is soundly and completely axiomatized by the following axiom system. Every tautologous formula is an axiom and, moreover, every formula of the form*

*(B1')* $V{\sim}V\alpha \supset {\sim}V\alpha$

*(B2)* $V(\alpha \wedge \beta) \supset (V\alpha \wedge V\beta)$

*is an axiom as well. The rules of inference are Modus Ponens, L-Necessitation and*

*(RE) If* $\vdash \alpha \equiv \beta$*, then* $\vdash X\alpha \equiv X\beta$*, where* $X$ *is* $L$ *or* $V$*.*

Note that (B1') is a weak version of the axiom (B1), which is stating that every verified proposition is true. (B1') requires only that every verified proposition *of the form* $\sim V\alpha$ be true. The main observation is that this suffices to validate (1) and (2), while there are AM1-countermodels to both (3) and (4).

**Proposition 4.5.** (1) *and* (2) *are valid in AM1, but* (3) *and* (4) *are not.*

*Proof.* (1). Fact 4.2 and propositional logic entail that

$$(V\alpha \wedge V{\sim}V\alpha) \supset {\sim}(V\alpha \wedge V{\sim}V\alpha)$$

belongs to AM1. But (B2) is valid and, hence,

$$V(\alpha \wedge {\sim}V\alpha) \supset {\sim}V(\alpha \wedge {\sim}V\alpha)$$

is in AM1, which, by propositional logic, means that ${\sim}V(\alpha \wedge {\sim}V\alpha)$ belongs to AM1. By Fact 4.2 again, $L{\sim}V(\alpha \wedge {\sim}V\alpha)$ belongs to AM1.

(2). From the validity of (1) by propositional logic and repeated applications of (semantic counterparts of) the rule (RE).

(3). Our countermodel is as follows. $\mathcal{W} = \{v, u\}$ and $\mathcal{N}_V(x) = \{\{v\}\}$ for all $x \in \mathcal{W}$; $\mathcal{N}_L(v) = \{\emptyset, \mathcal{W}\}$ and $\mathcal{N}_L(u) = \{\mathcal{W}\}$; $|p| = \{v\}$. It is readily seen that this is indeed an AM1-model (the key to (m) is that $\mathcal{N}_V(x)$ is the same singleton for all $x \in \mathcal{W}$). Obviously, $|Vp| = \mathcal{W}$, $|{\sim}Vp| = \emptyset$ and $|{\sim}(p \wedge Vp)| = \{u\}$. Consequently, $L{\sim}Vp$ holds in $v$ (as $\emptyset \in \mathcal{N}_L(v)$), but $L{\sim}(p \wedge Vp)$ does not hold in $v$ (as $\{u\} \notin \mathcal{N}_L(v)$). But, as is easily checked, $L{\sim}Vp \wedge {\sim}L{\sim}(p \wedge Vp)$ entails the negation of (3).

(4). The countermodel is just like the countermodel to (3) except for $|p| = \{u, v\}$. It is easily checked that, as before, $|{\sim}Vp| = \emptyset$ and, moreover, $|{\sim}L{\sim}Vp| = \{u\}$. Hence, $|{\sim}(p \wedge {\sim}L{\sim}Vp)| = \{v\}$. But this means that, as before, $L{\sim}Vp$ holds in $v$. However, as $\{v\} \notin \mathcal{N}_L(v)$, $L{\sim}(p \wedge MVp)$ is false in $v$. Consequently, (4) is false in $v$. □

Proposition 4.5 generalises Cmorej's results concerning (1) and (2). It shows that the original results can be obtained by building on assumptions concerning the notions of verification and necessity that are far weaker that the ones originally used by Cmorej. The second upshot is that the results concerning (1) and (2) are independent of those concerning (3) and (4). In other words, one may construe 'verified' and 'necessary' in such a manner that $\alpha \wedge {\sim}V\alpha$ turns out to be 'unverifiable' (and $\alpha \vee F\alpha$ to be 'unsatisfiable'), but not every 'unverifiable' $\alpha$ is logically equivalent to $\alpha \wedge {\sim}V\alpha$ and not every 'unverifiable' $\alpha$ entails a proposition that says that $\alpha$ is 'unverifiable'.

## 4.2 (3) and (4) without (1) or (2)

The logic AM0 is defined similarly as AM1.

**Definition 4.6 (AM0-Frames and Models).** An AM0-frame is a couple $\mathcal{F} = \langle \mathcal{W}, \mathcal{N} \rangle$ where all the components are as before, but only one condition is enforced:

- (iv) If $X \in \mathcal{N}(w)$, then $\{v \mid X \in \mathcal{N}(v)\} \in \mathcal{N}(w)$.

An AM0-model $\mathcal{M} = \langle \mathcal{F}, \mathcal{V} \rangle$, as before. The truth-sets for Boolean formulas are defined as usual. Moreover:

- $|V\alpha|_{\mathcal{M}}$ is arbitrary;
- $|L\alpha|_{\mathcal{M}} = \{w \mid X \subseteq |\alpha|_{\mathcal{M}}$ for some $X \in \mathcal{N}(w)\}$.

AM0 is defined as the set of formulas valid in every AM0-frame. ■

In AM0, $L$ takes the place of $V$ and is given a truth-condition in terms of a neighbourhood function. It is the same truth-condition that was given to $V$ in the semantics for AM4, but fewer restrictions ale placed on $\mathcal{N}$. The absence of any specific truth-condition for formulas of the form $V\alpha$ reflects the absence of any assumptions concerning the notion of verification. A formal consequence of this absence is that formulas of the form $V\alpha$ behave like propositional variables. Of course, substitution of equivalents then fails. $\alpha$ is necessary in $w$ iff it 'follows from' some proposition in $\mathcal{N}_L(w)$, the set of 'core necessities' of $w$.

**Fact 4.7.** *The following schemas belong to AM0:*

- $MM\alpha \supset M\alpha$
- $M(\alpha \wedge \beta) \supset M\alpha$

*Moreover, if $\alpha \supset \beta$ belongs to AM0, then so does $L\alpha \supset L\beta$.*

*Proof.* The first validity is a consequence of (iv). Note that $MM\alpha \supset M\alpha$ belongs to AM0 if $L\alpha \supset LL\alpha$ does. It is routine to check that (iv) ensures that the latter in fact belongs to AM0. The second validity follows from the truth-conditions for $L\alpha$ and $L(\alpha \wedge \beta)$. The final claim is a standard consequence of the truth-condition for $L\alpha$ (see Chellas, 1980). □

We skip the examples of AM0-models and the arguments that most AM4-axioms are not valid in AM0. To facilitate comparison with AM4, however, we state (without proof) the following axiomatization result.

**Proposition 4.8.** *AM0 is soundly and completely axiomatized by the following axiom system. Every tautologous formula is an axiom and, moreover, every formula of the form*

*(A2')* $L(\alpha \wedge \beta) \supset (L\alpha \wedge L\beta)$

*(A4)* $L\alpha \supset LL\alpha$

*is an axiom as well. The rules of inference are Modus Ponens and*

*(REL)* *If* $\vdash \alpha \equiv \beta$, *then* $\vdash L\alpha \equiv L\beta$.

The main observation is that AM0 validates (3) and (4), but not so for (1) and (2).

**Proposition 4.9.** (3) *and* (4) *are valid in AM0, but* (1) *and* (2) *are not.*

*Proof.* (3) is quite easy. Note (again) that

$$\sim V\alpha \supset (\alpha \equiv (\alpha \wedge \sim V\alpha))$$

is a tautologous formula. The rest follows by Fact 4.7.

(4). $MMV\alpha \supset MV\alpha$ and $M(\alpha \wedge MV\alpha) \supset MMV\alpha$ are valid by Fact 4.7. It follows by propositional logic that

$$M(p \wedge MVp) \supset MV\alpha$$

is valid in AM0. The rest follows by propositional logic and the definition of $M$.

(1) and (2) are very easy. Formulas of the form $V\alpha$ have arbitrary truth-sets. Hence, we can easily construct a model over $\mathcal{W} = \{v, u\}$ such that $|\sim V(p \wedge \sim Vp)| = \{v\}$ and $|\sim V\sim(p \vee V\sim p)| = \{u\}$, but $\mathcal{N}(v) = \{\{v, u\}\}$, for example. But then both $L\sim V(p \wedge \sim Vp)$ and $L\sim V\sim(p \vee V\sim p)$ are false in $v$. □

Proposition 4.9 shows that Cmorej's results concerning (3) and (4) are obtainable rather easily. In fact, they follow from two very weak assumptions concerning necessity and are independent of any specific interpretation of the operator '$V$'.

## 4.3 Additional Remarks

The results of the above two sections suggest that AM4 is not the weakest possible logic of necessity and verification for which Cmorej's results are derivable. Let us consider AM2, the combination of AM0 and AM1. We could discuss its semantics in terms of $\mathcal{N}_L$ and $\mathcal{N}_V$, but we only mention the corresponding axiom system. As usual, every tautologous formula is an axiom and Modus Ponens is a rule of inference. The additional axiom schemata are (B1'), (B2), (A2') and (A4). Additional inference rules are (RE) and $L$-necessitation. It is clear that AM2 is weaker that AM4, but all of (1) – (4) are valid in AM4. Hence, Cmorej's original system is not the weakest one for which his main results hold.

Let us note that the converses of (3) and (4) are derivable in AM0.3, a system that results from AM0 by adding (A1) and (B1). (Again, providing a semantics for this system is easy.) Let us see why. Firstly, if both (A1) and (B1) are valid, then so is

$$V\alpha \supset (\alpha \wedge MV\alpha)$$

But then, by Fact 4.7 (which obviously holds for AM0.3 as well),

$$MV\alpha \supset M(\alpha \wedge MV\alpha)$$

is valid. The validity of the converse of (4) follows by propositional logic and the definition of $M$. Secondly, let us assume, that $MV\alpha$ holds in some world $w$ for some $\alpha$. Then $M(\alpha \wedge V\alpha)$ holds in $w$ by (B1). By propositional reasoning and (REL), $M(\alpha \wedge \sim(\alpha \wedge \sim V\alpha))$. Consequently,

$$M\left((\alpha \wedge \sim(\alpha \wedge \sim V\alpha)) \vee ((\alpha \wedge \sim V\alpha) \wedge \sim\alpha)\right)$$

in $w$. But the latter means that $\sim L(\alpha \equiv (\alpha \wedge \sim V\alpha))$ in $w$.

Hence, a system in which all of (1) – (4) plus the converses of (3) and (4) hold is the combination of AM0.3 with AM2, which we can call AM3. (In an axiomatization of AM3, (B1') can be omitted in favour of (B1).) Again, it is rather clear that AM3 is weaker that AM4. This could be shown rigorously by model-theoretic arguments, but we shall not engage in this exercise here.

# 5 Conclusion

The present article has elaborated on Cmorej's (1990) interesting results concerning unverifiable and unfalsifiable empirical propositions in three ways. Firstly, we have provided simple model-theoretic arguments establishing the main results with respect to the logic AM4. This was made possible by our soundness and completeness results for AM4 using a version of neighbourhood semantics. Secondly, we have pointed out some striking similarities of Cmorej's findings to aspects of two well-known philosophical problems, Moore's Paradox and the Knowability Paradox. Thirdly, we have generalised Cmorej's results and discussed logics weaker that AM4 in which some combinations of the results hold. It has been argued that, in fact, AM4 is not the weakest logic in which all of Cmorej's original results hold. Perhaps AM4 is to be preferred to such weaker logics on some other grounds, but we leave this issue open.

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