# Substructural Epistemic Logics

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**Abstract.** The article introduces substructural epistemic logics of belief supported by evidence. The logics combine normal modal epistemic logics (implicit belief) with distributive substructural logics (available evidence). Pieces of evidence are represented by points in substructural models and availability of evidence is modelled by a function on the point set. The main technical result is a general completeness theorem. Axiomatizations are provided by means of two-sorted Hilbert-style calculi. It is also shown that the framework presents a natural solution to the problem of logical omniscience.

**Keywords:** epistemic logic; evidence; knowledge representation; logical omniscience; substructural logics

## 1 Introduction

Epistemic logics based on normal modal logics (Hintikka, 1962; van Benthem, 2011) are well-suited for representing 'implicit' epistemic attitudes, i.e. what conclusions can be drawn from agents' 'active' knowledge, beliefs, etc. This makes them a useful tool in many areas of computer science, for example in formalizing 'external' reasoning about databases and multi-agent systems (Fagin, Halpern, Moses, & Vardi, 1995; Meyer, 2003; Meyer & van der Hoek, 1995; Moses, 2008). In some contexts, however, the 'internal' perspective of the agent is relevant as well. For instance, if a conclusion of an agent's active beliefs is to play a role in

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reasoning about the agent's actions and decisions, it is important that it be related to the *evidence available to the agent*. Importantly, the warrant provided by the evidence, or recognized as such by the agent, might not be subject to the closure properties imposed by normal modal logics. This is related to the *logical omniscience problem*: the internal perspective of the agent might not obey the logic that is being used to derive consequences *about* the agent. Hence, there is a need for epistemic logics that combine normal modal logics with weaker logics representing the internal evidential perspective of the agent. These combined logics should be based on modular frameworks, where the 'internal' and 'external' components (and well as modes of their interaction) can be adjusted to suit the application at hand. This article presents such a framework.

### *1.1 Outline*

Our framework is a combination of normal modal logics with distributive substructural logics (Anderson & Belnap, 1975; Anderson, Belnap, & Dunn, 1992; Mares, 2004; Paoli, 2002; Read, 1988; Restall, 2000; Schröder-Heister & Došen, 1993). The combination yields a family of *substructural epistemic logics* that model implicit belief as well as the warrant provided by available evidence. The combination is rather natural, given the numerous epistemic *interpretations* (Restall, 1995; Routley & Meyer, 1973; Sequoiah-Grayson, 2009a, 2009c; Slaney & Meyer, 1997; Urquhart, 1972) and *applications* (Aucher, 2013, 2014; Belnap, 1977a, 1977b; Cozic, 2006; Dunn, 2010; Hjortland & Roy, 2015; Sedlár, 2013b; Sequoiah-Grayson, 2009b, 2013) of substructural logics.

We proceed as follows. First, relational models for distributive substructural logics (Restall, 2000) are equipped with additional machinery that represents the epistemic attitudes of our interest. States in our models are seen as 'bodies of evidence'. Implicit belief is modelled as usual, by means of a binary relation on states, while availability to the agent of a specific body of evidence is represented by a unary function. Second, possible worlds are represented by states that satisfy specific conditions. We add to the usual completeness and consistency conditions, discussed e.g. by Beall and Restall (2006), Mares (2004) and Restall (2000), three new requirements concerning the ternary accessibility relation. As a result, worlds are states where negation as well as implication and intensional conjunction (fusion) turn out to behave extensionally. Validity is defined in terms of worlds and the binary accessibility relation is required to connect worlds with worlds only. Consequently, the substructural epistemic logics studied here are extensions of normal modal logics that add a non-normal operator construed in terms of warrant by available evidence. As such, the logics also present a natural solution to the problem of logical omniscience.

Our framework is modular on three levels, i) with respect to the normal modal component governing the implicit modality $\Box$, ii) the substructural component governing warrant provided by evidence, and iii) the modes of interaction and iteration of $\Box$ and the evidence operator $A$. The main technical result of the article is a general completeness theorem. It is noteworthy that the complete axiomatizations are given in terms of two-sorted Hilbert-style calculi, where proofs are ordered couples of finite sequences of formulas. Several observations pertaining to substructural epistemic correspondence theory are pointed out as well. The article establishes the 'basic theory' of substructural epistemic logics and is seen as laying the groundwork for further developments and applications.

### *1.2 Related work*

Epistemically motivated combinations of normal modal and substructural logics have a long history, dating back to at least the 1980s (Lakemeyer, 1987; Lakemeyer & Levesque, 1988; Levesque, 1984).[1] The main advantages of this approach include i) the fact that it *incorporates* normal modal epistemic logics (and, hence, is able to represent 'implicit' epistemic attitudes as well as the 'explicit' ones), ii) a *semantic* approach to resource-bounded attitudes (as opposed to the syntactic approach of awareness logics, discussed by Fagin and Halpern (1988), and related systems), iii) *compositional* semantics (as opposed to the approaches based on arbitrary impossible worlds, see (Rantala, 1982), for example) and iv) a *hyperintensional* treatment of resource-bounded attitudes (i.e. logically equivalent statements are not intersubstitutable in epistemic contexts, unlike the approaches based on non-normal modal logics (Chellas, 1980)).

Our framework is designed to retain these features while presenting a more general and homogeneous picture. This sets it apart form two recent related contributions based on substructural logics. In a series of papers (Bílková, Majer, & Peliš, 2015; Bílková, Majer, Peliš, & Restall, 2010; Majer & Peliš, 2009, 2010) Bílková, Majer and Peliš have set out a sophisticated epistemic framework based on substructural semantics. However, their logics are alternatives to, not extensions of, normal modal epistemic logics. Sedlár (2013a, 2015) introduces a general substructural epistemic framework that extends normal modal epistemic logics. However, the framework has some problematic features, which are avoided by the present approach. For instance, the membership problem concerning the set of axioms of some axiomatizations provided by Sedlár (2015) is undecidable. We return to a more detailed comparison of our framework with these two approaches

[1] See also the first-order extensions introduced by Lakemeyer (1994, 1996), the detailed survey in the monograph (Lakemeyer & Levesque, 2000) and applications to belief revision discussed in Lakemeyer and Lang (1996). Fagin, Halpern, and Vardi (1995) provide a notable related approach.

in section 5.3.

It should be noted that there are established epistemic logics that represent evidence explicitly. *Justification logics* treat evidence syntactically in the style of awareness logics (Artemov, 2001, 2008; Baltag, Renne, & Smets, 2014; Fitting, 2005). They started out as explicit provability logics and therefore offer a rather fine-grained representation of evidential warrant. *Evidence logics* are based on the neighbourhood semantics for monotonic non-normal modal logics and represent pieces of evidence by sets of possible worlds (van Benthem, Fernández-Duque, & Pacuit, 2014; van Benthem & Pacuit, 2011). As a consequence, evidential warrant as represented in the framework is closed under classical one-premiss inference rules. Our framework is a middle road between these other two. Our representation of evidential warrant is more fine-grained than the representation offered by evidence logics, but it renders evidence subject to a greater number of general principles than the syntactic framework of justification logics.

We note that the 'functional' approach to epistemic attitudes has been explored recently by Holliday (2014). In fact, our work can be seen as a generalization of this interesting contribution.

### *1.3 Overview of the article*

The rest of the article is set out as follows. Section 2 explains our motivations – the kind of scenarios we aim to formalize and the reasons we think the substructural approach is needed to do that. Section 3 reviews the necessary basics of substructural logics and briefly outlines an epistemic interpretation of relational substructural models. Section 4 introduces 'worlds', a special kind of points in substructural models and discusses some of their properties. Epistemic substructural models are introduced in Section 5, and Section 6 establishes the main completeness result. Section 7 concludes the article and outlines some possible paths for future research.

## 2 Motivations

This section explains our motivations for introducing the substructural epistemic framework. We aim at outlining the general picture, not at elaborating the philosophical details.

**Example 1.** Suppose Alice is working in her study when she is suddenly distracted by rain beating on her windowpane. As usual, she has her radio on, and news has just come on. Interestingly enough, the forecast for today calls for 'sunny and pleasant' weather.

Everything around Alice, including her study, its surroundings, and the radio announcement, constitutes the evidence available to her at that moment – her *evidential situation*. Her evidential situation can cause Alice to have specific beliefs and prevent her from having others. What will be more important for us, however, is the fact that the evidential situation can be *used as a warrant* for specific beliefs. For example, it can be used to warrant the belief that it is raining outside and that the radio is on.

This particular evidential situation has two interesting features. First, it is *incomplete*. For example, it cannot be used as a warrant for the belief that Groningen's city center is surrounded by a canal, but it cannot be used to warrant that Groningen's city center is *not* surrounded by a canal, either. There is nothing in Alice's evidential situation that can be produced as a *good reason* for claiming either one of these assertions. Hence, we have $p$ such that neither $p$ nor $\neg p$ is warranted by the given evidential situation. Second, the situation is *inconsistent*. It can be used to warrant two contradictory beliefs, namely that it is raining outside Alice's study and that it is not. Taken individually, seeing the rain and hearing the forecast are both good reasons for the respective beliefs. Hence, we have $q$ such that both $q$ and $\neg q$ are warranted. These features are rather common. Real-life agents often find themselves in incomplete and inconsistent evidential situations and it is therefore interesting to inquire into frameworks that are able to deal with such scenarios satisfactorily.

In modal epistemic logic, evidential situations are modelled as sets of states in a Kripke model (let us call these 'set-situations'). Warrant is then seen as truth in every state in the given set. This takes care of incompleteness as there obviously are set-situations that contain a $p$-state and also a $\neg p$-state. Hence, there are set-situations that do not warrant $p$ nor $\neg p$.

However, set-situations have some counterintuitive features. Firstly, they seem to *warrant too much*. Being sets of states in Kripke models, set-situations warrant every modal consequence of a set of warranted formulas. Consequently, every valid formula is warranted and for every pair of logically equivalent formulas, one is warranted if and only if the other is. Nevertheless, we could easily find a complex propositional tautology and plausibly argue that there is nothing in Alice's evidential situation of Example 1 that would warrant a belief in the tautology. Similarly, we could produce a pair of equivalent statements, one of which is warranted by Alice's evidence while the other is not, simply because the evidential situation does not warrant the equivalence of the two statements.

Secondly, set-situations do not seem to be able to handle inconsistent evidential situations very well. Simply put, there is only one inconsistent set-situation, namely the empty set. In addition, the empty set warrants every formula in the language. However, we have already argued that, although inconsistent, Alice's situation in Example 1 does not warrant any of the claims about the canal in Groningen.

Thirdly, models using set-situations are usually equipped by a rather limited notion of *information update*. In epistemic product models discussed by Baltag and Moss (2004); Baltag, Moss, and Solecki (1998), update is always 'classically monotonic' – Boolean formulas remain warranted after new information is provided. Of course, knowledge representation is more interested in scenarios where updates are *non-monotonic* even in the Boolean case. Modal epistemic logic models non-monotonic update by using plausibility models, see van Benthem (2011), where warrant is represented as truth in the most plausible states, given the agent's current plausibility ordering. However, this modelling retains all of the unwelcome traits of set-situations.

Hence, there is a good case for looking around for better models of evidential situations. Substructural logics are an obvious place where to start as they (by definition) abandon principles that directly contribute to the problems of set-situation-models mentioned above.

This is not to mean that a satisfactory representation of evidential situations should abandon set-situations completely. They are useful in representing the relationship between agents' evidential states and (the consequences of) their actual beliefs.

**Example 2.** This is a variation on Example 1. Suppose that Alice is taking a break from working and is listening to the radio. As before, the weather forecast for today is 'sunny and pleasant'. Being rather absorbed in listening to the news, Alice does not notice the rain outside. As a result of the information provided by the news reader, she believes that it is not raining. Furthermore, she mishears a report about an accident that occurred on a canal, thinking it took place on the canal surrounding Groningen's city center. In reality, however, the accident took place on a canal in a different city. As a result of her mishearing though, Alice now believes that Groningen's city center is surrounded by a canal.

This example points out that the agents' actual beliefs are not always determined by the available evidence. Moreover, agents may even be uncertain or wrong as to the exact shape of their evidential situation. Hence, it is useful to model agents' (implicit) beliefs *independently* of their evidential situations. We will use set-situations (rather standardly) to represent implicit beliefs.

Putting things together, we arrive at something like the picture in Figure 1. The circled state on the left represents the actual world, where $p \wedge q$ is the case. The circled state on the right is the only epistemically accessible world that makes Alice's beliefs true. The usual epistemic accessibility relation is represented by the solid arrow. The non-circled state above the actual world represents Alice's evidential situation in the actual world and the dashed arrow represents the 'availability

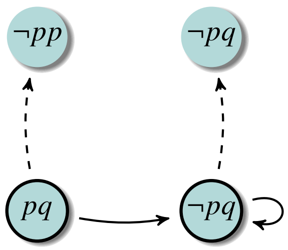


Figure 1: A visual representation of Example 2.

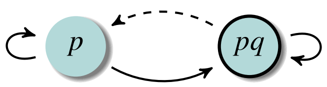


Figure 2: A visual representation of Example 3.

function'. As in the example, the evidence available to Alice in the actual world warrants both $\neg p$ (it includes the radio announcement) and $p$ (it includes the visible rain), but it does not warrant $q$ nor $\neg q$. However, Alice does not 'know' that her evidential situation warrants $p$. In fact, she takes it that the evidence available to her warrants $\neg p$ and $q$.

**Example 3.** Assume that Beth is taking a course in first-order logic. She has just acquainted herself with a sound and complete axiomatization. Let $p$ represent a description of the axiomatization. Assume that $q$ is some rather complicated first-order theorem. Due to soundness, $q$ is valid and, hence, true in every possible world. But assume that Beth has never actually proved $q$, nor has she checked it for validity. Hence, her evidential situation does not support $q$. More generally, her evidence does not support the fact that $q$ follows from Beth's beliefs, i.e. that it is not possible for $q$ to be false, for all Beth believes.

This example may be represented by the picture in Figure 2. The state on the right is the actual world, where both $p$ and $q$ hold and are both implicitly believed by Beth. The state on the left represents Beth's evidential situation that warrants $p$ but not $q$. Importantly, neither does the evidential situation carry the information that $q$ follows from Beth's beliefs. This may be represented by epistemic accessibility in an obvious manner: according to the evidential situation, '$q$ is true' and '$q$ is not true' are both possible, for all Beth believes (according to the evidential situation). This can be represented by letting both states be epistemically accessible from the evidential situation.

Note that, in addition to facts and (implicit) beliefs, evidential situations may provide information about the *available evidence* as well.

**Example 4.** Assume that Carol is conducting an experiment. Carol's evidential situation may be seen as comprising of her background knowledge, the lab, the experiment and its results, together with Carol's interpretation of the results. Assume that the experiment does not support a conclusion $p$, but Carol's interpretation of the experimental results is overly optimistic – she assumes that the experiment does support $p$. As a result, Carol believes that $p$.

The situation in Example 4 may be characterised by saying that Carol's evidential situation provides wrong information about itself, i.e. about the evidence available to Carol. According to her evidential situation, she is in a different evidential situation, one that does support $p$. This may be represented by the picture in Figure 3. The left circled state represents the actual world where $p$ is false. Carol believes that $p$ and, hence, the only epistemically accessible world is represented by the right circled state. Carol's evidential situation in the actual world does not support $p$. However, the evidential situation supports the information that Carol's evidential situation supports $p$. In other words, the evidence available according to the actual evidential situation is a body of information supporting $p$. This is represented by the dashed arrow leading from the above left state, representing the actual evidential situation. In addition, the picture assumes that Carol believes that her evidential situation is one which supports $p$.

Structures that correspond to such examples are easily obtained by extending relational models for distributive substructural logics. It is the latter that we turn to now.

## 3 Substructural Logics

This section is a brief overview of substructural logics. We introduce the 'basic' substructural logic that will be the primary object of our elaboration (3.1) and discuss some of its extensions (3.2). An epistemic interpretation of substructural models is briefly discussed as well (3.3). The concepts and results introduced in 3.1

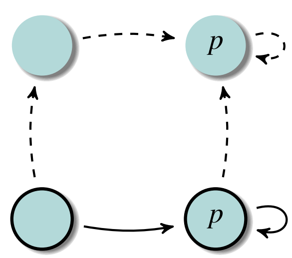


Figure 3: A visual representation of Example 4.

and 3.2 are standard (our presentation builds on Bílková et al. (2015) and Restall (2000)). The interpretation provided in 3.3 is 'folklore' (see Beall et al. (2012) on the ternary relation and Berto (2015); Dunn (1993); Mares (2004) on the compatibility relation).

### *3.1 The basic logic*

We follow Bílková et al. (2015) in taking as our basic logic the commutative distributive non-associative full Lambek calculus with a simple negation **DFNLe**, defined over weakly commutative substructural frames with a symmetric compatibility relation. Our reasons for this choice of the basic logic follow. Firstly, the logic is sufficiently simple as to its corresponding frame conditions. Secondly, it has only one implication connective and only one negation. Thirdly, the underlying frames have been given an epistemic interpretation before, namely by Sequoiah-Grayson (2013) (although the interpretation used here is slightly different).

**Definition 5 (Language $\mathcal{L}_0$).** The basic substructural language $\mathcal{L}_0$ is the set of formulas defined thus:

(1) $\varphi ::= p \mid \top \mid \bot \mid t \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \otimes \varphi \mid \varphi \to \varphi$

where $p \in At$, a countable set of propositional letters (fixed throughout the article).

'$\top$' represents trivial truth, '$\bot$' trivial falsehood and '$t$' is taken to stand for 'non-trivial truth' or 'logic' (this will become more clear once we discuss models). '$\otimes$' is the intensional conjunction (or 'fusion'). Restall (2000) goes into more detail.

**Definition 6 (Weakly Commutative Simple Frames).** A weakly commutative simple frame is a tuple

$$F = \langle P, \leq, L, R, C \rangle$$

such that

- $\langle P, \leq \rangle$ is a poset with a non-empty domain $P$
- $L \subseteq P$ is $\leq$-closed ($x \in L$ and $x \leq y$ only if $y \in L$) and

  (2) $x \leq y \iff (\exists z \in L).Rzxy$

- $R \subseteq P^3$ such that

  (3) $Rxyz$ and $x' \leq x$ and $y' \leq y$ and $z \leq z' \Longrightarrow Rx'y'z'$

  and (weak commutativity)

  (4) $Rxyz \Longrightarrow Ryxz$

- $C \subseteq P^2$ such that

  (5) $Cxy$ and $x' \leq x$ and $y' \leq y \Longrightarrow Cx'y'$

  and

  (6) $Cxy \Longrightarrow Cyx$

**Definition 7 (Basic Models).** A basic model is a tuple

$$M = \langle F, V \rangle$$

where $F$ is a weakly commutative simple frame and $V : At \mapsto 2^P$ such that $V(p)$ is $\leq$-closed (a valuation on $F$). The model $M = \langle F, V \rangle$ is said to be 'on $F$'. The valuation generates the following satisfaction relation between $\varphi \in \mathcal{L}_0$ and $x \in P$:

- $x \Vdash p$ iff $x \in V(p)$
- $x \Vdash \top$ iff $x = x$ (i.e. always)
- $x \Vdash \bot$ iff $x \neq x$ (i.e. never)
- $x \Vdash t$ iff $x \in L$
- $x \Vdash \neg\varphi$ iff for all $y$, $Cxy$ implies $y \nVdash \varphi$
- $x \Vdash \varphi \wedge \psi$ iff $x \Vdash \varphi$ and $x \Vdash \psi$
- $x \Vdash \varphi \vee \psi$ iff $x \Vdash \varphi$ or $x \Vdash \psi$
- $x \Vdash \varphi \otimes \psi$ iff there are $y, z$ such that $Ryzx$ and $y \Vdash \varphi$ and $z \Vdash \psi$
- $x \Vdash \varphi \to \psi$ iff for all $y, z$, if $Rxyz$ and $y \Vdash \varphi$, then $z \Vdash \psi$

Moreover:

- $\varphi$ is $L$-valid in $M$ ($M \Vdash \varphi$) iff $x \Vdash \varphi$ for all $x \in L$
- $\varphi$ entails $\psi$ in $M$ ($\varphi \Vdash_M \psi$) iff for all $x$, $x \Vdash \varphi$ only if $x \Vdash \psi$
- $\varphi$ is $L$-valid in $F$ ($F \Vdash \varphi$) iff $\langle F, V \rangle \Vdash \varphi$ for all $V$ on $F$
- $\varphi$ is $L$-valid in a class of frames $\mathcal{F}$ ($\mathcal{F} \Vdash \varphi$) iff $F \Vdash \varphi$ for all $F \in \mathcal{F}$

Every language considered in this article is an extension of $\mathcal{L}_0$ and it is assumed that the truth conditions for the $\mathcal{L}_0$-fragment of every such language are as in the above definition.

**Definition 8 (Monotonicity).** A language $\mathcal{L}$ is monotone over a class of substructural frames $\mathcal{F}$ iff for every $M$ on $F \in \mathcal{F}$, $x \Vdash \varphi$ and $x \leq y$ only if $y \Vdash \varphi$, for all $\varphi \in \mathcal{L}$.

If $\mathcal{L}$ is monotone over $\mathcal{F}$ and $F \in \mathcal{F}$, then we shall say that $\mathcal{L}$ is monotone over $F$.

**Lemma 9 (Deduction).** *Let $\mathcal{L}$ be monotone over $F$ and let $M$ be on $F$. Then $\varphi \to \psi$ is L-valid in $M$ iff $\varphi \Vdash_M \psi$.*

*Proof.* First, assume that $M \Vdash \varphi \to \psi$, i.e. $x \Vdash \varphi \to \psi$ for all $x \in L$. Let us have $y \in P$ such that $y \Vdash \varphi$. It is plain that $y \leq y$. By (2) of Definition 6, there is $z \in L$ such that $Rzyy$. But $z \Vdash \varphi \to \psi$ and, hence, $y \Vdash \psi$. Second, assume that $M \nVdash \varphi \to \psi$, i.e. $x \nVdash \varphi \to \psi$ for some $x \in L$. In other words, there are $y, z$ such that $Rxyz$ and $y \Vdash \varphi$ and $z \nVdash \psi$. By (2), $y \leq z$. By the monotonicity assumption, $z \Vdash \varphi$. In other words, $\varphi \nVdash_M \psi$. □

**Lemma 10 ($\mathcal{L}_0$-monotonicity).** *$\mathcal{L}_0$ is monotone over the class of weakly commutative simple frames. (In fact, it is monotone over all substructural frames).*

*Proof.* This is a standard result, see (Restall, 2000, p. 244). As an example, we prove the case for $\varphi = \psi \to \chi$. Assume that $x \leq y$. The condition (3) of Definition 6 implies that if $Ryzz'$, then $Rxzz'$. Hence, $y \nVdash \psi \to \chi$ implies $x \nVdash \psi \to \chi$. □

**Definition 11 (Axiomatization of DFNLe, Bílková et al. (2015)).** The basic Hilbert-style system $\mathcal{H}_0$ comprises of the following axiom schemata and rules. Axiom schemata:

- $\varphi \to \varphi$
- $\varphi \wedge \psi \to \varphi$ and $\varphi \wedge \psi \to \psi$
- $\varphi \to \varphi \vee \psi$ and $\psi \to \varphi \vee \psi$
- $\varphi \to \top$ and $\bot \to \varphi$
- $\varphi \wedge (\psi \vee \chi) \to (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$

Rules ('//' indicates a two-way rule):

- $\varphi, \varphi \to \psi \,/\, \psi$
- $\varphi \to \psi, \psi \to \chi \,/\, \varphi \to \chi$
- $\chi \to \varphi, \chi \to \psi \,/\, \chi \to (\varphi \wedge \psi)$
- $\varphi \to \chi, \psi \to \chi \,/\, (\varphi \vee \psi) \to \chi$
- $\varphi \to (\psi \to \chi) \,//\, (\psi \otimes \varphi) \to \chi$
- $\varphi \to (\psi \to \chi) \,//\, \psi \to (\varphi \to \chi)$
- $t \to \varphi \,//\, \varphi$
- $\varphi \to \neg\psi \,//\, \psi \to \neg\varphi$

| Name | Schema | Property |
|---|---|---|
| Associativity | $(\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$ | $R(xy)zu \to Rx(yz)u$ |
| Contraction | $(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$ | $Rxyz \to R(xy)yz$ |
| Weak Contraction | $((\varphi \to \psi) \land \varphi) \to \psi$ | $Rxxx$ |
| Mingle | $\varphi \to (\varphi \to \varphi)$ | $Rxxy \to x \leq y$ |
| Weakening | $\varphi \to (\psi \to \varphi)$ | $Rxyz \to x \leq z$ |
| Excluded Middle | $\top \to (\varphi \lor \neg\varphi)$ | $Cxy \to y \leq x$ |
| Explosion | $\varphi \land \neg\varphi \to \bot$ | $Cxx$ |
| Double Negation Elimination | $\neg\neg\varphi \to \varphi$ | $\exists y.(Cxy \land \forall z.(Cyz \to z \leq x))$ |

Figure 4: Substructural frame conditions and the corresponding schemata

A basic proof of $\varphi$ is a finite sequence of formulas $\overrightarrow{\chi_n}\varphi = \langle \chi_1, \ldots, \chi_n, \varphi \rangle$ such that for all $\psi$ in $\overrightarrow{\chi_n}\varphi$, either

- $\psi$ is an instance of an axiom schema, or
- there is a rule $\psi_1, \ldots, \psi_n \,/\, \psi$ and $\psi_1, \ldots, \psi_n$ precede $\psi$ in $\overrightarrow{\chi_n}\varphi$

**Theorem 12.** *There is a basic proof of $\varphi \in \mathcal{L}_0$ iff $\varphi$ is L-valid in the class of weakly commutative simple frames.*

*Proof.* Standard. See Restall (2000), for example. □

**Corollary 13.** *There is a basic proof of $\varphi \to \psi \in \mathcal{L}_0$ iff $\varphi \Vdash_M \psi$ for all $M$.*

### 3.2 Extensions of the basic logic

Stronger substructural logics are obtained either by focusing on a narrower class of frames or by extending $\mathcal{H}_0$ by extra axiom schemata. As in modal logic, specific axiom schemata correspond to specific frame conditions. Some of the correspondences are laid out in Figure 4, where $R(xy)zu$ stands for $\exists x'.(Rxyx' \land Rx'zu)$ and $Rx(yz)u$ stands for $\exists x'.(Ryzx' \land Rxx'u)$.

For example, adding Associativity and Weak Contraction, together with Double Negation Elimination gives the relevant logic **R**. A further addition of Weakening, Explosion and Excluded Middle gives classical logic.

**Theorem 14.** *Let $X$ be a selection of the axiom schemata shown in Figure 4 and $X'$ the list of corresponding frame conditions. Let $\mathcal{H}_X$ be $\mathcal{H}_0$ with members of $X$ as extra axioms and $\mathcal{F}_{X'}$ the class of frames satisfying the conditions in $X'$. Then $\varphi \in \mathcal{L}_0$ is provable in $\mathcal{H}_X$ iff $\varphi$ is L-valid in $\mathcal{F}_{X'}$.*

*Proof.* Standard. See Restall (2000), for example. □

### 3.3 An epistemic interpretation of substructural models

We construe points $x \in P$ in the usual fashion, as 'bodies of information', 'bodies of evidence' or simply *evidential situations*. The satisfaction relation is construed in terms of warrant or *support*: $x \Vdash \varphi$ is read '(the evidential situation) $x$ supports the information expressed (carried) by $\varphi$', or '$x$ supports $\varphi$' for short. We are being deliberately vague here: support can be seen as 'containment' of (the information carried by) $\varphi$ in $x$, or just the relation that obtains between $x$ and $\varphi$ when $\varphi$ is a reasonable conclusion, given $x$. The partial order $\leq$ is seen as 'support-containment' (or 'support-extension'): $x \leq y$ means that $y$ supports (warrants) at least as much information as $x$.

The relation $C$ is construed as *compatibility* between bodies of evidence, see Dunn (1993) and (Berto, 2015), but also (Mares, 2004, ch. 5) and Restall (1999). Hence, $\neg\varphi$ is supported by $x$ iff every $y$ that satisfies $\varphi$ is incompatible with $x$. Our basic logic assumes that compatibility is symmetric: if $x$ is compatible with $y$, then $y$ is compatible with $x$ (6). The condition (5) is clearly plausible: if $Cxy$ and $x', y'$ do not support more information than $x, y$, respectively, then $x'$ should clearly be compatible with $y'$. Evaluating $\neg\varphi$ by means of $C$ makes it possible that some evidence supports both $\varphi$ and $\neg\varphi$ (only if not $Cxx$), or does not support $\varphi$ nor $\neg\varphi$ (if $x \nVdash \varphi$ and $y \Vdash \varphi$ for some $y$ such that $Cxy$).

$Rxyz$ means that at least everything warranted by the *combination* of $x$ and $y$ is warranted by $z$, see Beall et al. (2012). Hence, $\varphi \to \psi$ is supported by $x$ iff the combination of $x$ with any $y$ that supports $\varphi$ is support-extended only by evidence that warrants $\psi$. In other words, adding any evidence that supports $\varphi$ yields evidence that supports $\psi$. Fusion $\varphi \otimes \psi$ is a form of dynamic conjunction: $x$ supports $\varphi \otimes \psi$ iff $x$ extends evidence that is obtained by combining some $y, z$ that support $\varphi, \psi$, respectively. It is plain that both implication and fusion have a rather dynamic flavour. It is also possible to read $Rxyz$ alternatively as 'Everything that is obtainable from $x$ and $y$ by using at least Modus Ponens is warranted by $z$'. This reading qualifies the 'combination' referred to in the folklore interpretation by requiring a minimal action that has to be done while performing the combination, namely applying Modus Pones wherever it is possible. This makes (3) trivial. Condition (4) means that if $Rxyz$, then $x$ and $y$ are 'equal' sources of premisses to be used in the combination.[2]

Last, but not least, the set $L$ is construed as the set of 'logical' points (Mares, 2004; Restall, 2000), where all the valid implications of the logic hold true. The

[2]It should be pointed out that the 'combination' referred to in the folklore interpretation of $R$ is not (necessarily) a specific point in $P$ (unlike in Urquhart's semilattice semantics (Urquhart, 1972) and Fine's operational semantics (Fine, 1974)). This is similar to the situation in modal epistemic logic, where the 'active beliefs' that hold in all the accessible states are not directly represented in the model either.

condition (2) is rather plausible given the second interpretation of $R$ and the fact that $\varphi \to \varphi$ should indeed come out as valid in any reasonable logic.

## 4 Worlds

The main aim of this article involves combining normal modal logics with substructural logics. Hence, we need a way to represent *classical possible worlds*. Our approach is to represent worlds by specific bodies of evidence, i.e. specific points in the substructural model. We extend the approach towards defining worlds known from the literature on relevant logic, see (Beall & Restall, 2006, ch. 5.1), (Mares, 2004, ch. 5) and (Restall, 2000, ch. 16). Our worlds are designed so that not only '$\neg$' but also '$\to$' and '$\otimes$' turn out to be extensional.

**Definition 15.** Let $F$ be a substructural frame. A point $w \in P$ is a *world in F* iff

1. $Cww$
2. $Cwx$ implies $x \leq w$
3. $Rwww$
4. $Rwxy$ implies $x \leq w \leq y$
5. $Rxyw$ implies $x \leq w$ and $y \leq w$

**Lemma 16 (Extensionality).** *Let $w$ be a world in $F$ and let $\mathcal{L}$ be monotone over $F$. Then for every $M$ on $F$ and every $\varphi, \psi \in \mathcal{L}$*

1. $w \Vdash \neg\varphi$ *iff* $w \nVdash \varphi$
2. $w \Vdash \varphi \to \psi$ *iff* $w \nVdash \varphi$ *or* $w \Vdash \psi$
3. $w \Vdash \varphi \otimes \psi$ *iff* $w \Vdash \varphi$ *and* $w \Vdash \psi$

*Proof.* All the 'items' referred to in the proof are items of Definition 15.

(1) Left-to-right is entailed by reflexivity of $C$. If $w \nVdash \neg\varphi$, then $Cwx$ and $x \Vdash \varphi$ for some $x$. Item (2) implies that $x \leq w$ and $w \Vdash \varphi$ follows by the monotonicity assumption.

(2) Assume that $w \Vdash \varphi \to \psi$ and $w \Vdash \varphi$. Then $Rwww$ entails $w \Vdash \psi$. Right-to-left: Assume that $w \nVdash \varphi \to \psi$, i.e. here are $x, y$ such that $Rwxy$, $x \Vdash \varphi$ and $y \nVdash \psi$. Item (4) implies that $x \leq w \leq y$. But then monotonicity implies that $w \Vdash \varphi$ and $w \nVdash \psi$.

(3) Right-to-left is entailed by item (3). Left-to-right follows from item (5) and monotonicity. □

Again, 'worlds' are pieces of evidence that represent classical possible worlds. If $w$ is a world in $F$, then we shall often read $w \Vdash \varphi$ as '$\varphi$ holds (is true) in $w$' instead of '$\varphi$ holds according to $w$' or '$\varphi$ is supported by $w$'.

The first two 'world-conditions' are known from the literature. They are usually interpreted in terms of consistency (1) and completeness (2). The condition (3) can be seen as a requirement of closure under Modus Ponens. Conditions (4) and (5) are designed specifically to yield extensionality of '$\rightarrow$' and '$\otimes$'. Providing a deeper philosophical interpretation of the latter two conditions is an open problem.

**Example 17.** We show that items (4) and (5) are independent. In this example, '$R(xy)z$' is short for '$Rxyz$ and $Ryxz$'. Firstly, let us construct a weakly commutative simple frame as follows. Let $P = \{x, y, w\}$, $C$ is the universal relation on $P$ and

- $\leq$ is the smallest partial order on $P$ such that $y \leq w$
- $R(yu)u$ for all $u \in P$
- $Ruuw$ for all $u \in P$
- $L = \{y, w\}$

It is tedious but not hard to check that this is in fact a weakly commutative simple frame. Note that $w$ satisfies conditions (1) – (4) of Definition 15. Yet, $Rxxw$ without $x \leq w$. Hence, condition (5) fails.

Secondly, let us have the same $P$ and $C$, but

- $\leq$ is the smallest partial order on $P$ such that $x \leq w$ and $x \leq y$
- $R(wx)y$, $R(xy)y$, $R(xw)w$ and $Rwww$
- $Rxxu$ for all $u \in P$
- $L = \{x, y, w\}$

Again, this is a weakly commutative simple frame and $w$ satisfies (1) – (3) and (5) of Definition 15. Yet, $Rwxy$ without $w \leq y$. Hence, condition (4) fails.

**Lemma 18.** *For every $F$, the set of worlds in $F$ is a subset of $L$.*

*Proof.* It is clear that $w \leq w$ for all worlds $w$ in $F$. But then there is a $x \in L$ such that $Rxww$. By the definition of worlds, item (5) (or item (4) and weak commutativity), $x \leq w$. But $L$ is $\leq$-closed and, hence, $w \in L$. □

**Proposition 19.** *Consider any weakly commutative simple frame $F$ with point-set $P$ and the set of logical points $L$. If $L$ is a subset of the set of worlds in $F$, then every point in $P$ is a world in $F$.*

*Proof.* Take any $x \in P$. Obviously $x \leq x$. Then, by (2) of Definition 6, there is some $y \in L$ such that $Ryxx$. But then, by our assumption, $y$ is a world in $F$. Hence, by item (4) of Definition 15, $x \leq y \leq x$. Hence, by antisymmetry of $\leq$, $x$ is a world in $F$. □

Let us call a frame *flat* iff the ordering relation $\leq$ is the identity relation on $P$. It is clear that $w$ is a world in a flat frame iff

1. $Cwx$ implies $w = x$, and
2. $Rwxy$ or $Rxwy$ or $Rxyw$ implies $w = x = y$

Let us define $Th^{\mathcal{L}}_{M}(x) = \{\varphi \in \mathcal{L} \mid M, x \Vdash \varphi\}$. We shall write this as '$Th(x)$' if $\mathcal{L}$ and $M$ are clear from the context.

**Proposition 20.** *Let $F$ be a weakly commutative simple frame and let $w, v$ be worlds in $F$. Let $\mathcal{L}$ be monotone over $F$. Fix a model $M$ on $F$. Then $w \leq v$ implies $Th(w) = Th(v)$.*

*Proof.* First, assume that $w \leq v$ and $\varphi \in Th(w)$. Then $\varphi \in Th(v)$ follows from monotonicity of $\mathcal{L}$ over $F$. Second, assume that $\varphi \in Th(v)$. If $\varphi \notin Th(w)$, then $\neg\varphi \in Th(w)$ by Lemma 16. Consequently, $\neg\varphi \in Th(v)$ by monotonicity and $\varphi \notin Th(v)$ by Lemma 16 again, a contradiction. □

## 5 Epistemic Models

This section introduces *substructural epistemic logics*, defined over extended substructural models. Standard substructural models are extended by a binary epistemic accessibility relation $S$ and an available-evidence function $|\cdot|$, whilst validity is defined as truth in every world of the underlying substructural frame.

We begin by introducing the core definitions and discussing interesting validities and invalidities (5.1). Then we look at some basic results in substructural epistemic correspondence theory (5.2). The present approach is a generalization of the framework introduced in Sedlár (2015), and is also related to Bílková et al. (2015). Section 5.3 explains the main differences between the present framework and (Bílková et al., 2015; Sedlár, 2015) in more detail.

### *5.1 Basic definitions and properties*

In general, substructural epistemic logics are classes of formulas of an epistemically interpreted language valid in specific classes of epistemically interpreted extensions of substructural frames.

**Definition 21 (Epistemic Language).** Formulas of the epistemic language $\mathcal{L}_B$ are built up as follows:

(7) $\varphi ::= \varphi_0 \mid \Box\varphi \mid A\varphi$

($\varphi_0$ is any formula of $\mathcal{L}_0$) Moreover, we define $B\varphi := \Box\varphi \wedge A\varphi$.

Henceforth, $\varphi, \psi, \ldots$ range over $\mathcal{L}_B$, unless stated otherwise. $\Box\varphi$ is read 'The agent (implicitly) believes that $\varphi$' and $A\varphi$ as 'The body of evidence available to the agent supports $\varphi$'. Hence, $B\varphi$ represents implicit belief backed up by evidence, reading 'The agent implicitly believes that $\varphi$ and $\varphi$ is supported by the evidence available to the agent'. We note that this way of defining $B$ is related to some well-known definitions of 'explicit' epistemic operators, see (Fagin & Halpern, 1988), for example.

**Definition 22 (Epistemic Frame).** Let $F$ be a weakly commutative simple frame. An *epistemic frame on $F$* is a tuple

$$\mathfrak{F} = \langle F, W, S, |\cdot| \rangle$$

such that

- $W \subseteq P$ is a set of worlds in $F$ (possibly empty and not necessarily the set of all worlds in $F$)
- $S$ is a binary relation on $P$ such that

  (8) $Sxy$ and $x' \leq x$ and $y \leq y' \Longrightarrow Sx'y'$

  and

  (9) $Sxy$ and $Wx \Longrightarrow Wy$

- $|\cdot|$ is a monotone unary operation on $P$, i.e.

  (10) $x \leq y \Longrightarrow |x| \leq |y|$

**Definition 23 (Epistemic Model).** An *epistemic model* (on $\mathfrak{F}$) is a couple

$$\mathfrak{M} = \langle \mathfrak{F}, V \rangle$$

such that $\mathfrak{F}$ is an epistemic frame (on some $F$) and $V$ is a valuation on $F$. The valuation generates a satisfaction relation obeying Definition 7 and, moreover:

- $x \Vdash \Box\varphi$ iff for all $y$, $Sxy$ implies $y \Vdash \varphi$

- $x \Vdash A\varphi$ iff $|x| \Vdash \varphi$

Moreover:

- $\varphi$ is valid in $\mathfrak{M}$ ($\mathfrak{M} \Vdash \varphi$) iff $x \Vdash \varphi$ for all *worlds* $x \in W$.
- $\varphi$ entails $\psi$ in $\mathfrak{M}$ ($\varphi \Vdash_{\mathfrak{M}} \psi$) iff for all $x$: if $x \Vdash \varphi$, then $x \Vdash \psi$ (i.e. entailment in an epistemic $\mathfrak{M}$ coincides with entailment in the underlying substructural $M$).
- $\varphi$ is valid in $\mathfrak{F}$ on $F$ ($\mathfrak{F} \Vdash \varphi$) iff $\langle \mathfrak{F}, V \rangle \Vdash \varphi$ for all $V$ on $F$.
- $\varphi$ is valid in a class of epistemic frames $\mathcal{E}$ ($\mathcal{E} \Vdash \varphi$) iff $\mathfrak{F} \Vdash \varphi$ for all $\mathfrak{F} \in \mathcal{E}$ and $\varphi$ is a consequence of $\Sigma = \{\psi_i \mid i \in I\}$ over $\mathcal{E}$ ($\Sigma \Vdash_{\mathcal{E}} \varphi$) iff for every $\mathfrak{F} \in \mathcal{E}$: if $x \Vdash \psi_i$ for all $\psi_i \in \Sigma$ in some *world* $x \in W$ and model $\langle \mathfrak{F}, V \rangle$, then $x \Vdash \varphi$ in $\langle \mathfrak{F}, V \rangle$. Consequences in frames are defined as consequences over singleton sets of frames.
- A rule $\varphi_1, \ldots, \varphi_n \,/\, \psi$ is valid in $\mathfrak{M}$ ($\mathfrak{F}$, $\mathcal{E}$) iff the following is the case: if every $\varphi_i (1 \leqslant i \leqslant n)$ is valid in $\mathfrak{M}$ ($\mathfrak{F}$, $\mathcal{E}$), then so is $\psi$.

$L$-validity of formulas and rules in epistemic models, frames and classes of frames is defined in terms of $L$-validity in the underlying basic models, weakly commutative simple frames and classes thereof (notation: $\mathfrak{M} \Vdash^L \varphi$ etc.).

We have argued that a satisfactory representation of scenarios such as those in Examples 1 – 4 should involve classical possible worlds, evidential situations that can fail to satisfy some classical logical laws, a doxastic accessibility relation representing implicit belief and an availability function specifying the evidential situation of the agent. Now we have everything we need to set up such a representation. Section 3.3 pointed out that there is a folklore interpretation of relational substructural models according to which points are seen as bodies of information or, equivalently, evidential situations. Section 4 has shown that points of a special sort behave like classical possible worlds. We have now added a doxastic accessibility relation and an availability function.

But first, let us discuss the third crucial feature of the present framework, namely the re-definition of validity as truth in all *classical possible worlds*, i.e. all $x \in W$. In conjunction with Lemma 16, the definition entails that the set of formulas valid in any epistemic model contains the set of propositional tautologies. This reflects the assumption that validity is a notion pertaining to the notion of *truth*, not the more general notion of *support*.

Let us now return to accessibility and availability. $S$ is a doxastic accessibility relation for a contextually fixed agent: $Sxy$ means that everything the agent believes according to $x$ is supported by $y$. The condition (8) is then rather plausible. Assume that everything the agent believes according to $x$ is supported by

$y$. If $x' \leq x$, then $x'$ does not support more information about the agent's beliefs than $x$. Hence, everything the agent believes according to $x'$ is supported by $y$ and by every $y'$ such that $y \leq y'$. We do not focus here on the most general version of the framework and we assume that if $x$ represents a possible world, then $Sxy$ only if $y$ represents a possible world as well (9). Hence, if $x$ represents a possible world, then $Sxy$ means that everything the agent believes in $x$ is true in world $y$. The rationale behind this assumption is twofold. First, it makes the necessitation rule valid. In conjunction with our definition of validity, this entails that the substructural epistemic logics we study in this article are extensions of normal modal epistemic logics. This reflects the notion that normal modal logics are a suitable representation of 'external' reasoning about agents. Second, formulas '$\Box\varphi$' are seen as statements concerning implicit belief, i.e. what conclusions can be drawn form an agent's explicit beliefs 'externally'. Hence, implicit belief should be closed under valid implication.

The function $|\cdot|$ assigns to every body of information $x$ the body of information (evidence) $|x|$ available to the agent *according to* $x$. The monotonicity condition (10) secures monotonicity of $\mathcal{L}_B$ but can be motivated independently as well: if $y$ supports at least as much information as $x$, then it supports at least as much information *concerning the available evidence* as $x$. Note that if $x$ represents a possible world, then $|x|$ can also be construed as the body of evidence available to the agent *in* world $x$, or simply the evidential situation of the agent in $x$. Importantly, $|x|$ can be a non-worldly situation even if $x$ is a world. This feature is crucial as it makes it possible that the set of formulas supported by evidence available in some world $x$ is not closed under classical consequence (unlike the set of implicitly believed formulas). It is important to note that, although we adopt (9), $S$ is not required to satisfy $S(x) \subseteq W$ for all $x \in P$. This is to account for the possibility of an agent not having evidence that warrants for every classical consequence of her beliefs *that it is a consequence* of her beliefs, as in Example 3.

A language $\mathcal{L}$ is monotone over a class $\mathcal{E}$ of epistemic frames iff it is monotone over $\mathcal{F}_{\mathcal{E}} = \{F \mid \langle F, W, S, |\cdot| \rangle \in \mathcal{E}\}$.

**Lemma 24 ($\mathcal{L}_B$-monotonicity).** *$\mathcal{L}_B$ is monotone over every $\mathfrak{F}$.*

*Proof.* The result is a simple extension of Lemma 10. Assume that $x \leq y$, $y \nVdash \Box\varphi$ and that the result holds for $\varphi$. Then there is a $z$ such that $Syz$ and $z \nVdash \psi$. By (8), $Sxz$ and, hence, $x \nVdash \Box\varphi$.

Next assume that $x \Vdash A\varphi$, i.e. $|x| \Vdash \varphi$, and that the result holds for $\varphi$. By (10), $|x| \leq |y|$ and, hence, $|y| \Vdash \varphi$. Consequently, $y \Vdash A\varphi$. □

**Lemma 25.** *If $\varphi$ is L-valid in $\mathfrak{M}$, then it is valid in $\mathfrak{M}$.*

*Proof.* Follows from Lemma 18. □

Let $\bigwedge \varphi_{(n)}$ be shorthand for $\bigwedge_{1\leqslant i\leqslant n} \varphi_i$ and similarly for $\bigvee \varphi_{(n)}$.

**Proposition 26.** *The following schemata and rules are valid in every epistemic frame:*

1. *Propositional tautologies (in $\mathcal{L}_B$) and Modus Ponens*
2. $(\varphi \otimes \psi) \leftrightarrow (\varphi \wedge \psi)$
3. $t \leftrightarrow \top$
4. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
5. $\varphi \,/\, \Box\varphi$
6. $\top \to A\top$ *and* $A\bot \to \bot$
7. $(\bigwedge A\varphi_{(n)}) \leftrightarrow A\,(\bigwedge \varphi_{(n)})$
8. $(\bigvee A\varphi_{(n)}) \leftrightarrow A\,(\bigvee \varphi_{(n)})$
9. *If* $\bigwedge \varphi_{(n)} \to \bigvee \psi_{(m)}$ *is L-valid in* $\mathfrak{M}$*, then*
    - $\mathfrak{M} \Vdash \bigwedge \varphi_{(n)} \to \bigvee \psi_{(m)}$ *and*
    - $\mathfrak{M} \Vdash^L \bigwedge A\varphi_{(n)} \to \bigvee A\psi_{(m)}$ *and, consequently*
    - $\mathfrak{M} \Vdash \bigwedge A\varphi_{(n)} \to \bigvee A\psi_{(m)}$

*Proof.* Items (1) and (2) follow directly from Lemma 16 and Lemma 24. Item (3) follows from Lemma 18. Item (4) follows from $Rwww$ (for all $w \in W$) and (9) of Definition 22, while item (5) follows from (9) of Definition 22 alone. Item (6) follows from the truth conditions of $\top, \bot$. Items (7) and (8) follow from Definition 7. Item (9) follows from items (7) and (8) and Lemmas 9, 24, 25. □

**Proposition 27.** *If* $\bigwedge \varphi_{(n)} \to \psi$ *is L-valid in* $\mathfrak{M}$*, then so is* $\bigwedge \Box\varphi_{(n)} \to \Box\psi$*. Moreover, the necessitation rule* $\varphi/\Box\varphi$ *does not preserve L-validity.*

*Proof.* If $\bigwedge \varphi_{(n)} \to \psi$ is $L$-valid in $\mathfrak{M}$, then, by Lemmas 9 and 24, if there is $x \Vdash \varphi_i$ for all $1 \leq i \leq n$, then $x \Vdash \psi$. Hence, if there is some $y \Vdash \Box\varphi_i$ for every $i$, then $y \Vdash \Box\psi$. To establish the second claim, it is sufficient to take a model where $p$ is true in $x$ iff $x \in L$ and there is some $y$ such that $Sxy$ and $y \notin L$. □

**Corollary 28.** *The following schemata and rules are valid in every* $\mathfrak{M}$*:*

1. $(\bigwedge B\varphi_{(n)}) \leftrightarrow B\,(\bigwedge \varphi_{(n)})$
2. $(\bigvee B\varphi_{(n)}) \to B\,(\bigvee \varphi_{(n)})$
3. *If* $\mathfrak{M} \Vdash^L \bigwedge \varphi_{(n)} \to \psi$*, then* $\mathfrak{M} \Vdash \bigwedge B\varphi_{(n)} \to B\psi$
4. *If* $\mathfrak{M} \Vdash^L \varphi \leftrightarrow \psi$*, then* $\mathfrak{M} \Vdash B\varphi \leftrightarrow B\psi$

An important detail concerning item (9) of Proposition 26 and items (3) and (4) of Corollary 28 is that $\bigwedge \varphi_{(n)} \to \bigvee \psi_{(m)}$, $\bigwedge \varphi_{(n)} \to \psi$ and $\varphi \leftrightarrow \psi$, respectively, are assumed to be *L-valid* in $\mathfrak{M}$ (i.e. true in all logical states $x \in L$), not *valid* in $\mathfrak{M}$ (i.e. true in all worlds $x \in W$). The corresponding claims with 'valid' instead of '$L$-valid' *do not* hold, as witnessed by items (10) – (12) of the following proposition. In fact, it is the failure of the latter claims that allows our framework to avoid the classical omniscience problem.

**Proposition 29.** *The following schemata and rules are not valid in every* $\mathfrak{M}$ *(for every* $n \geq 0$*):*

1. $\Box\varphi \to A\varphi$
2. $A\varphi \to \Box\varphi$
3. $A\varphi \to \varphi$
4. $A\varphi \to AA\varphi$
5. $\neg A\varphi \to A\neg A\varphi$
6. $A\varphi \to \Box A\varphi$
7. $\neg A\varphi \to \Box\neg A\varphi$
8. $\Box\varphi \to A\Box\varphi$
9. $\neg\Box\varphi \to A\neg\Box\varphi$
10. $B\left(\bigvee \varphi_{(n)}\right) \to \left(\bigvee B\varphi_{(n)}\right)$
11. $\bigwedge \varphi_{(n)} \to \psi \ / \ \bigwedge B\varphi_{(n)} \to B\psi$
12. $\varphi \leftrightarrow \psi \ / \ A\varphi \leftrightarrow A\psi$
13. $A(\varphi \otimes \psi) \leftrightarrow (A\varphi \otimes A\psi)$

*Proof.* A point $x$ in $F$ is *local* if and only if for all $y, z$: (a) $x \leq y$ iff $x = y$ iff $Cxy$, (b) $Rxyz$ iff $x = y = z$. It is clear that every local point (in $F$) is a world (in $F$). Consider the model in Figure 5. Locality is indicated by a thick circle and $W$ is the set of local points. The solid arrow represents $S$ and the dashed arrow from $x$ to $y$ indicates that $y = |x|$. It is assumed that $x_i \leq y$ iff $x_i = y$ for $i \in \{1,2\}$, $Rx_1x_1x_1$, $Rx_1x_2x_2$ and $Rx_2x_1x_2$, while $Cx_ix_j$ for $i, j \in \{1,2\}$. Moreover, $L = \{w_1, w_2, x_1\}$. Hence, the model is indeed an epistemic model. Now $w_1$ invalidates items (1) – (11), while $w_2$ invalidates (12) and (13).

(1) - (3) are trivial. (4) $|w_1| \Vdash q$, but $||w_1|| \nVdash q$. (5) $|w_1| \nVdash p$, but $C|w_1|x_2$ and $x_2 \Vdash Ap$. (6) $|w_1| \Vdash q$, but $|w_2| \nVdash q$. (7) $|w_1| \nVdash p$, but $|w_2| \Vdash p$. (8) $w_1 \Vdash \Box p$, but $|w_1| \nVdash \Box p$. (9) $w_1 \nVdash \Box q$, but $|w_1| \nVdash \neg\Box q$ as $C|w_1|x_2$ and $x_2 \Vdash \Box q$. (10) $w_1 \Vdash A(p \vee q) \wedge \Box(p \vee q)$, but $w_1 \nVdash Ap \vee \Box q$. (11) $w_1 \Vdash B(p \vee q)$ and $p \vee q \to p \vee \neg p$ is valid. However, $|w_1| \nVdash p$ and $|w_1| \nVdash \neg p$ as $x_2 \Vdash p$. (12)

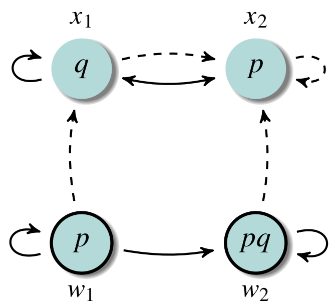


Figure 5: Counterexamples for Proposition 29.

$\varphi \otimes \psi \leftrightarrow \varphi \wedge \psi$ and $|w_2| \Vdash p \otimes q$, but $|w_2| \nVdash p \wedge q$. The same world invalidates the left-to-right direction of (13). The right-to-left direction is easily invalidated by any frame that contains a world $w$ such that $Rx_1x_2|w|$ for some $x_1$, $x_2$ such that $x_i \nleq |w|$. □

Substructural epistemic logics are extensions of normal modal epistemic logics (Proposition 26, (1) – (5)) that avoid the 'classical' logical omniscience problem (Proposition 29, (11)). Informally, 'classical reasoning' may fail in the scope of the evidence-operator $A$. Support by evidence and evidence-based belief $(A, B)$ are nevertheless closed under the entailments of the underlying substructural logic (Proposition 26, (9); Corollary 28). Moreover, it is assumed that trivial truths are supported by every piece of evidence and trivial falsehoods by no piece of evidence at all (Proposition 26, (6)).

Closure under conjunction introduction and elimination, disjunction introduction and the 'primeness' of pieces of evidence (Proposition 26, (7) and (8)) may be considered unwelcome in specific contexts of application. Two comments on this. Firstly, note that the conjunction-closure properties hold only for the extensional '$\wedge$', not the intensional '$\otimes$' (Proposition 29, (13)). Hence, conjunction-closure does not hold for every conjunction connective in the language. One might even argue, as Sequoiah-Grayson (2013) does, that the 'proper' conjunction to be used in epistemic contexts is '$\otimes$', not '$\wedge$'.

Secondly, the unintuitive nature of $A(\varphi \vee \psi) \rightarrow (A\varphi \vee A\psi)$ might be explained away in a similar fashion. We might argue that, speaking epistemically, a disjunction is supported by some evidence iff it is incompatible with the evidence to assume that both disjuncts are false. To make this precise, let us define a 'compatibility disjunction' as follows:

(11) $\varphi \sqcup \psi := \neg(\neg\varphi \wedge \neg\psi)$

It is easy to show that in frames with symmetric $C$, the following principle of closure under compatibility disjunction introduction is valid:

(12) $(A\varphi \vee A\psi) \rightarrow A(\varphi \sqcup \psi)$

However, the converse implication is not valid if the negation used does not satisfy double negation elimination, i.e. is weaker than de Morgan negation.

Note that a crucial feature of substructural epistemic logics is the failure of the substitution-of-equivalents rule (Proposition 29, (12)). However, a weaker version of the rule holds.

**Lemma 30 (Cautious Substitution).** *Let $\varphi_1, \varphi_2 \in \mathcal{L}_B$ and*

- $\mathfrak{M} \Vdash \varphi_1 \leftrightarrow \varphi_2$ *for some* $\mathfrak{M}$
- $\psi$ *contains an occurrence* $[\varphi_1]$ *of* $\varphi_1$ *that is not in the scope of 'A'*
- $\psi'$ *results form* $\psi$ *by replacing* $[\varphi_1]$ *by an occurrence of* $\varphi_2$

*Then* $\mathfrak{M} \Vdash \psi \leftrightarrow \psi'$.

*Proof.* By induction on the complexity of $\psi$. □

The Lemma implies that classical reasoning fails only in the scope of $A$.

### 5.2 Definability

**Definition 31.** A schema $\chi$ *defines* a class of epistemic frames $\mathcal{E}$ iff

$$\mathfrak{F} \in \mathcal{E} \iff \mathfrak{F} \Vdash \varphi_\chi$$

for every instance $\varphi_\chi$ of $\chi$ (i.e. $\mathfrak{F} \in \mathcal{E}$ iff every instance of $\chi$ is valid in $\mathfrak{F}$). $\chi$ *L-defines* $\mathcal{E}$ iff

$$\mathfrak{F} \in \mathcal{E} \iff \mathfrak{F} \Vdash^L \varphi_\chi$$

for every $\varphi_\chi$. A class of frames is definable ($L$-definable) iff there is a schema that defines ($L$-defines) it. We shall say that $\mathcal{E}_1$ is $\mathcal{E}_2$-definable ($\mathcal{E}_2$-$L$-definable) iff $\mathcal{E}_1 \cap \mathcal{E}_2$ is definable ($L$-definable).

**Theorem 32.** *In Figure 6, the class of epistemic frames satisfying the property on the right is defined by the ('corresponding') schema on the left.*

*Proof.* As usual, left-to-right is proven by transposition: on the assumption that the frame condition in question is not satisfied, we construct a specific model that invalidates the corresponding schema. Proofs of the most obvious cases are left out.

| Schema | Property |
|---|---|
| $\Box\varphi \to A\varphi$ | $Wx \wedge \lvert x\rvert \leq y \to Sxy$ |
| $A\varphi \to \Box\varphi$ | $Wx \wedge Sxy \to \lvert x\rvert \leq y$ |
| $A\varphi \to \varphi$ | $Wx \to \lvert x\rvert \leq x$ |
| $A\varphi \to AA\varphi$ | $Wx \to \lvert x\rvert \leq \lvert x\rvert^2$ |
| $A\varphi \to \neg A\neg\varphi$ | $Wx \to C\lvert x\rvert\lvert x\rvert$ |
| $\neg A\varphi \to A\neg A\varphi$ | $Wx \wedge C\lvert x\rvert y \to \lvert y\rvert \leq \lvert x\rvert$ |
| $A\varphi \to \Box A\varphi$ | $Wx \wedge Sxy \to \lvert x\rvert \leq \lvert y\rvert$ |
| $\neg A\varphi \to \Box\neg A\varphi$ | $Wx \wedge Sxy \to \lvert y\rvert \leq \lvert x\rvert$ |
| $\Box\varphi \to A\Box\varphi$ | $Wx \wedge S\lvert x\rvert y \to \exists z.(Sxz \wedge z \leq y)$ |
| $\neg\Box\varphi \to A\neg\Box\varphi$ | $Wx \wedge Sxy \wedge C\lvert x\rvert z \to \exists u.(Szu \wedge u \leq y)$ |
| $\Box\varphi \to \varphi$ | $Wx \to Sxx$ |
| $\Box\varphi \to \Box\Box\varphi$ | $Wx \wedge Sxy \wedge Syz \to Sxz$ |
| $\neg\Box\varphi \to \Box\neg\Box\varphi$ | $Wx \wedge Sxy \wedge Sxz \to Syz$ |

Figure 6: Some definable classes of epistemic frames and the defining schemata.

- ($\Box\varphi \to A\varphi$) Right-to-left is entailed by the fact that the frame condition in question implies $Wx \to Sx|x|$. Next, assume that there are $x, y$ such that $Wx, |x| \leq y$ and $\neg Sxy$. Let us define a model as follows: $z \Vdash p$ iff $Sxz$, for all $z$. It is clear that $x \Vdash \Box p$, but $x \nVdash Ap$.
- ($\neg A\varphi \to A\neg A\varphi$) Right-to-left: Assume that $Wx$ and $|x| \nVdash \varphi$. If $C(|x|) = \emptyset$, then we are done. Assume that $C(|x|) \neq \emptyset$ and pick an arbitrary $y \in C(|x|)$. The frame condition entails that $|y| \leq |x|$, hence $|y| \nVdash \varphi$ and $y \nVdash A\varphi$. But $y$ was arbitrary and, hence, $|x| \Vdash \neg A\varphi$. Left-to right: Assume that $Wx$ and $C|x|y$, but $|y| \nleq |x|$. Let $z \Vdash p$ iff $z \nleq |x|$, for all $z$. Hence, $|x| \nVdash p$ and $|y| \Vdash p$. Consequently, $x \nVdash Ap$, but $x \nVdash A\neg Ap$.
- ($\neg A\varphi \to \Box\neg A\varphi$) Right-to-left: Assume that $|x| \nVdash \varphi$. If $S(x) = \emptyset$ then we are done. Assume that $S(x) \neq \emptyset$ and pick an arbitrary $y \in S(x)$. The frame condition tells us that $|y| \leq |x|$. Consequently, $|y| \nVdash \varphi$, i.e. $y \nVdash A\varphi$. But $y$ is arbitrary and, hence, $x \Vdash \Box\neg A\varphi$. Left-to-right: Assume that $Wx, Sxy$ and $|y| \nleq |x|$. Let $z \Vdash p$ iff $z \nleq |x|$, for all $z$. Consequently, $|x| \nVdash p$ and $|y| \Vdash p$. Hence, $x \Vdash \neg Ap$ and $y \Vdash Ap$. In other words, $x \Vdash \neg\Box\neg Ap$.
- ($\neg\Box\varphi \to A\neg\Box\varphi$) Right-to-left: Assume that $Wx$ and $Sxy$ and $y \nVdash \varphi$. If $C(|x|)$ is empty, then we are done. Assume that $C(|x|) \neq \emptyset$ and pick an arbitrary $z \in C(|x|)$. The frame condition tells us that there is $u$ such that $Szu$ and $u \leq y$. Consequently, $u \nVdash \varphi$. Hence, $z \nVdash \Box\varphi$. But $z$ is arbitrary and, hence, $|x| \Vdash \neg\Box\varphi$. Left-to-right: Assume that there are $x, y, z$ such that $Wx, Sxy, C|x|z$ and for all $u$, if $Szu$, then $u \nleq y$. Let $v \Vdash p$ iff there is $u$ such that $Szu$ and $u \leq v$, for all $v$. Obviously, $z \Vdash \Box p$ and, hence, $|x| \nVdash \neg\Box p$.

However, the assumption that $y \Vdash p$ is in contradiction with the assumption that the frame condition in question fails. Consequently, $x \nVdash \Box p$.

□

**Definition 33 (Directed Disjoint Union).** Let $\mathfrak{F}$ and $\mathfrak{F}'$ be two disjoint epistemic frames, i.e. $P \cap P' = \emptyset$. The directed disjoint union of $\mathfrak{F}$ and $\mathfrak{F}'$ is the tuple

(13) $\mathfrak{F} \biguplus \mathfrak{F}' = \langle P^{\cup}, \leq^{\cup}, L^{\cup}, C^{\cup}, R^{\cup}, W^{\cup}, S^{\cup}, |\cdot|^{\cup} \rangle$

where $W^{\cup} = W$ and $X^{\cup} = X \cup X'$ for $X \in \{P, \leq, L, C, R, S, |\cdot|\}$.

Hence, the directed disjoint union of $\mathfrak{F}$ and $\mathfrak{F}'$ is a disjoint union such that the set of worlds in the disjoint union is identified with the set of worlds of the 'first' component $\mathfrak{F}$.

**Lemma 34.** *$\mathfrak{F} \uplus \mathfrak{F}'$ is an epistemic frame, for every disjoint $\mathfrak{F}, \mathfrak{F}'$.*

*Proof.* This is straightforward. For example, assume that $x \leq^{\cup} y$. Since $P$ and $P'$ are disjoint, either $x \leq y$ or $x \leq' y$. But in both cases there is $z \in L^{\cup}$ such that $R^{\cup}zxy$. Now assume that $R^{\cup}xyz$ and $x \in L^{\cup}$. Again, since $P$ and $P'$ are disjoint, either $x \in L$ and $Rxyz$ or $x \in L'$ and $R'xyz$. Both cases entail that $y \leq^{\cup} y$. □

**Lemma 35.** *$\mathfrak{F} \Vdash \varphi$ iff $\mathfrak{F} \uplus \mathfrak{F}' \Vdash \varphi$ for all $\mathfrak{F}'$ disjoint from $\mathfrak{F}$.*

**Definition 36 (Global Frame Conditions).** A frame condition $\xi$ is *global* iff it does not contain occurrences of the predicate '$W$'. If $\xi$ is a frame condition, then its global version is the condition $\xi'$ obtained by replacing every occurrence of '$Wx_i$' by '$x_i = x_i$', for every $x_i$ that occurs in $\xi$.

**Theorem 37.** *Let $\xi$ be a first-order global frame condition such that neither the class of $\xi$-frames nor its complement is empty. Then the class of $\xi$-frames is not definable.*

*Proof.* Let $\mathfrak{F}$ be a $\xi$-frame and $\mathfrak{F}'$ not a $\xi$-frame. Let $\mathfrak{F}''$ be an isomorphic copy of $\mathfrak{F}'$ disjoint from $\mathfrak{F}$. It is immediate that $\mathfrak{F}''$ is not a $\xi$-frame either. Assume that $\varphi$ defines $\xi$-frames. Hence, $\mathfrak{F} \Vdash \varphi$. By Lemma 35, $\mathfrak{F} \uplus \mathfrak{F}'' \Vdash \varphi$. But it is clear that $\mathfrak{F} \uplus \mathfrak{F}''$ is not a $\xi$-frame, a contradiction. □

**Corollary 38.** *Global versions of the frame conditions mentioned in Theorem 32 are not definable.*

A schema will be called *negative* (*positive*) iff it does (not) contain an 'explicit' occurrence of negation. For example, $A\varphi \to \Box\varphi$ is positive, but $A\neg\varphi \to \Box\neg\varphi$ is negative.

**Theorem 39.** *Let '$\xi$' range over frame properties in Figure 6 and let $\varphi^{\xi}$ denote the schema corresponding to $\xi$. Let $\mathcal{C}$ be the class of epistemic frames where*

- $Cxx$
- $Cxy \to y \leq x$

*for every $x, y \in P$. Then the following holds:*

1. *If $\varphi^{\xi}$ is positive, then it L-defines the global version of $\xi$.*
2. *If $\varphi^{\xi}$ is negative, then it does not L-define the global version of $\xi$.*
3. *If $\varphi^{\xi}$ is negative, the it $\mathcal{C}$-L-defines the global version of $\xi$.*

Another angle at (3) of this theorem is the fact that the global version of $\xi$ is $L$-definable in a language with Boolean negation.

### *5.3 Notes on two related approaches*

We are now in a position to assess the main differences between the present approach and two related contributions, namely Bílková et al. (2015) and Sedlár (2015).

In effect, Bílková et al. (2015) provide an epistemic interpretation of substrucutral modal logics. The approach builds on substructural models extended with the relation $S$, but our (9) is not assumed. The relation is construed in terms of reliable sources of information: $Sxy$ is taken to mean that $x$ is a reliable source of information for state $y$. Two conditions (not adopted here) are assumed: $Sxy$ implies $x \leq y$ and $Cxy$. The language $\mathcal{L}_0$ is extended by a unary operator '$K$' where $K\varphi$ is read as 'The agent knows that $\varphi$'. Knowledge is construed in terms of support by a reliable source. Formally, $K$ is a 'backward-looking' existential modality: $K\varphi$ holds in $x$ iff there is a $y$ such that $Syx$ and $\varphi$ holds in $y$. Validity is defined in the usual substructural way as truth in every logical state in $L$. The semantics does not recognize worlds and, as a result, there is no counterpart of the 'external environment' in addition to information states. Hence, the epistemic logics discussed by Bílková et al. (2015) are not extensions of normal modal epistemic logics. Importantly, the logics are not able to formalize the interplay between implicit and evidence-based attitudes. Moreover, their main focus is on the rather strong operator of knowledge (consistent, factive etc.)

Sedlár (2015) provides a framework that overcomes these limitations. Modal substructural frames are extended by 'worlds' and an epistemic indistinguishability relation $E$ on worlds is introduced. A function $\sigma$ from worlds to the set of states of the original substructural frame is added. A substructural language is extended by the familiar operators '$\Box$' and '$A$', where the corresponding formulas are read as in the present framework. Validity is defined as truth in all worlds and a family of substructural epistemic logics is introduced. Sound and complete axiomatizations are provided.

The framework of Sedlár (2015) has three unwelcome features. Firstly, the epistemic models are, in effect, unions of modal Kripke models with relational substructural models with a function from the modal part to the substructural part. This works, but it is not very elegant. Importantly, meanings of the connectives of the language are not the same 'in worlds' and 'in states'. In other words, the truth conditions adopted in Sedlár (2015) are relative to the 'kind' of state *by definition*.[3] Our approach introduces homogeneous models: worlds are defined as a *special kind* of substructural information states.

Secondly, (Sedlár, 2015) assumes that $A\varphi \leftrightarrow AA\varphi$ and discusses a more general case only very briefly. The present homogeneous framework provides a generalisation.

Thirdly, the axiomatizations provided by Sedlár (2015) are unnecessarily involved. In particular, specific epistemic axioms are 'generated' by the natural-deduction system for the underlying modal substructural logic. An important drawback of this approach is that if the underlying modal substructural logic is undecidable, then membership in *the set of axioms* of the corresponding substructural epistemic logic is undecidable as well. The next section demonstrates that the present framework avoids this problem and provides straightforward axiomatizations of a wider class of logics.

## 6 Axiomatization, Soundness and Completeness

This section establishes the main result of the article, a general (strong) completeness theorem for a number of substructural epistemic logics. We define our basic axiom system in Section 6.1. Section 6.2 establishes completeness for the basic substructural epistemic logic, using a variant of the canonical model technique. Section 6.3 discusses the extensions of our result to stronger substructural epistemic logics.

### *6.1 Axioms, proofs and soundness*

In the substructural epistemic semantics, the notion of validity in epistemic models differs from the notion of validity in the underlying substructural models, namely

[3]For example, conditionals $\varphi \to \psi$ are *defined* to behave as material implications in worlds and as substructural implications in states. Such relative truth conditions go back to at least Kripke (1965) and a criticism of this approach can be found in Mares (2004), for example. Variations of this defect are present in some logical frameworks with 'impossible worlds' (Cresswell, 1970, 1972, 1973; Jago, 2014; Rantala, 1982; Rescher & Brandom, 1980), many of which are epistemically motivated as well.

$L$-validity. Validity is defined in terms of worlds in $W$, but $L$-validity is defined in terms of the logical states in $L \supseteq W$. Hence, not every validity-preserving rule is also $L$-validity-preserving (e. g. Proposition 27), nor vice versa (e.g. Proposition 29, item (12)). Therefore, proof systems for substructural epistemic logics need to provide control over the application of these two kinds of rules. To achieve this, we formulate 'two-sorted' Hilbert-style calculi, where proofs are defined as ordered couples of sequences of formulas. Informally, the 'left side' corresponds to proofs of $L$-validity ('the underlying substructural logic plus the $L$-valid schemata and rules involving epistemic operators') whilst the 'right side' corresponds to proofs of validity ('the substructural epistemic logic in question').

**Definition 40.** The Hilbert-style calculus for epistemic frames $\mathcal{H}$ is a two-sorted system defined as follows.

**Axioms.** The set of *l-axioms* is the smallest set containing $\mathcal{L}_B$-instances of axiom schemata of $\mathcal{H}_0$ given in Definition 11:

- $\varphi \to \varphi$
- $\varphi \wedge \psi \to \varphi$ and $\varphi \wedge \psi \to \psi$
- $\varphi \to \varphi \vee \psi$ and $\psi \to \varphi \vee \psi$
- $\varphi \to \top$ and $\bot \to \varphi$
- $\varphi \wedge (\psi \vee \chi) \to (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$

together with two new axioms:

- $\top \to A\top$ and $A\bot \to \bot$

The set of *r-axioms* consists of

- propositional tautologies in $\mathcal{L}_B$
- $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
- $\varphi \wedge \psi \leftrightarrow \varphi \otimes \psi$
- $t \leftrightarrow \top$

**Rules.** The set of *l-rules* is the smallest set of $\mathcal{L}_B$-instances of the $\mathcal{H}$-rules given in Definition 11:

- $\varphi, \varphi \to \psi \,/\, \psi$
- $\varphi \to \psi, \psi \to \chi \,/\, \varphi \to \chi$
- $\chi \to \varphi, \chi \to \psi \,/\, \chi \to (\varphi \wedge \psi)$
- $\varphi \to \chi, \psi \to \chi \,/\, (\varphi \vee \psi) \to \chi$

- $\varphi \to (\psi \to \chi) \,//\, (\psi \otimes \varphi) \to \chi$
- $\varphi \to (\psi \to \chi) \,//\, \psi \to (\varphi \to \chi)$
- $t \to \varphi \,//\, \varphi$
- $\varphi \to \neg\psi \,//\, \psi \to \neg\varphi$

together with the '$A$-rule':

- $\bigwedge \varphi_{(n)} \to \bigvee \psi_{(m)} \,/\, \bigwedge A\varphi_{(n)} \to \bigvee A\psi_{(m)}$, for $n, m \geq 1$

and the '$\Box$-rule':

- $\bigwedge \varphi_{(n)} \to \psi \,/\, \bigwedge \Box\varphi_{(n)} \to \Box\psi$, for $n \geq 1$

The set of $r$-rules contains Modus Ponens and Necessitation ($\varphi \,/\, \Box\varphi$).

**Proofs.** $\mathcal{H}$-proofs are ordered couples of (possibly empty) sequences of $\mathcal{L}_B$-formulas defined inductively as follows:

1. If $\overrightarrow{\chi_n} \mid \overrightarrow{\chi_m}$ is a proof and $\varphi$ is a $l$-axiom, then $\overrightarrow{\chi_n}\varphi|\overrightarrow{\chi_m}$ is a proof ($n, m \geq 0$)
2. If $\overrightarrow{\chi_n} \mid \overrightarrow{\chi_m}$ is a proof and $\varphi$ is a $r$-axiom, then $\overrightarrow{\chi_n}|\overrightarrow{\chi_m}\varphi$ is a proof ($n, m \geq 0$)
3. If $\overrightarrow{\chi_n} \mid \overrightarrow{\chi_m}$ is a proof such that $\overrightarrow{\chi_n}$ contains $\varphi_1, \ldots, \varphi_n$ and $\varphi_1, \ldots, \varphi_n \,/\, \psi$ is a $l$-rule, then $\overrightarrow{\chi_n}\psi|\overrightarrow{\chi_m}$ is a proof
4. If $\overrightarrow{\chi_n} \mid \overrightarrow{\chi_m}$ is a proof such that $\overrightarrow{\chi_m}$ contains $\varphi_1, \ldots, \varphi_n$ and $\varphi_1, \ldots, \varphi_n \,/\, \psi$ is a $r$-rule, then $\overrightarrow{\chi_n}|\overrightarrow{\chi_m}\psi$ is a proof
5. If $\overrightarrow{\chi_n}\psi|\overrightarrow{\chi_m}$ is a proof, then $\overrightarrow{\chi_n}\psi|\overrightarrow{\chi_m}\psi$ is a proof ('the jump rule')

$\varphi$ is provable in $\mathcal{H}_0$ ($\mathcal{H}_0 \vdash \varphi$ or simply $\vdash \varphi$) iff there is a proof $\overrightarrow{\chi_n}|\overrightarrow{\chi_m}\varphi$. $\varphi$ is *derivable* from $\Sigma \subseteq \mathcal{L}_B$ in $\mathcal{H}_0$ ($\Sigma \vdash_{\mathcal{H}_0} \varphi$ or simply $\Sigma \vdash \varphi$) iff there is a $\{\psi_1, \ldots, \psi_n\} \subseteq \Sigma$ such that $\vdash \bigwedge \psi_{(n)} \to \varphi$. $\vdash_l \varphi$ iff there is a proof $\overrightarrow{\chi_n}\varphi|\overrightarrow{\chi_m}$.

The $A$-rule reflects the fact that if $\bigwedge \varphi_{(n)} \to \bigvee \psi_{(m)}$ is $L$-valid, then so is $\bigwedge A\varphi_{(n)} \to \bigvee A\psi_{(m)}$ (see Proposition 26, item (9)). The $\Box$-rule reflects the fact that if $\bigwedge \varphi_{(n)} \to \psi$ is $L$-valid, then so is $\bigwedge \Box\varphi_{(n)} \to \Box\psi$ (see Proposition 27). The jump rule represents the fact that every $L$-valid formula is valid (Lemma 25). It is plain that membership in the union of the sets of $l$-axioms and $r$-axioms, respectively, is decidable.

**Example 41.** An example of (a shortened version of) a $\mathcal{H}_0$-proof is shown in Figure 7. The 4th 'line' is obtained by using the jump rule. Note that the rule used in deriving the 5th line is 'classically admissible', i.e. validity-preserving, but not $L$-validity preserving. Therefore, the rule could not have been used on the left side of the proof (where 'we reason about $L$-validity'). A similar remark applies to the use of necessitation in step 6.

| | |
|---|---|
| 1. $(p \wedge \neg Ap) \to \neg Ap$ | 4. $\Box Ap \to \Box\neg(p \wedge \neg Ap)$ |
| 2. $Ap \to \neg(p \wedge \neg Ap)$ | 5. $\Box Ap \to \Box(p \to Ap)$ |
| 3. $\Box Ap \to \Box\neg(p \wedge \neg Ap)$ | 6. $\Box(\Box Ap \to \Box(p \to Ap))$ |

Figure 7: A shortened proof of $\Box(\Box Ap \to \Box(p \to Ap))$

**Lemma 42.** *Every instance of each l-axiom except for the new axioms $\top \to A\top$ and $A\bot \to \bot$ is a $\mathcal{L}_B$-tautology. In addition, every l-rule apart from*

- $\varphi \to (\psi \to \chi) \,//\, (\psi \otimes \varphi) \to \chi$
- $t \to \varphi \,/\, \varphi$
- *the A-rule and the $\Box$-rule*

*is a propositionally admissible rule.*

**Theorem 43 (Soundness).** *If $\Sigma \vdash_{\mathcal{H}_0} \varphi$, then $\Sigma \Vdash_{\mathcal{E}} \varphi$, where $\mathcal{E}$ is the class of all epistemic frames.*

*Proof.* It is sufficient to prove that $\vdash \psi$ implies $\mathfrak{F} \Vdash \psi$ for all $\psi, \mathfrak{F}$. This is done as usual, by induction on the complexity of proofs. First, we have to show that every $r$-axiom is valid in every frame and every $r$-rule preserves validity. This follows from Proposition 26. Second, we have to show that every formula provable by using the jump rule is valid. To show this, it is sufficient to show that every $l$-axiom is valid and every $l$-rule preserves validity. The first claim and most of the cases of the second claim follow from Proposition 26 and Lemma 42. It remains to establish validity-preservation for the rules explicitly mentioned in Lemma 42. Validity-preservation of the first two rules follows from Proposition 26 ('$\wedge$ and $\otimes$', '$t$ and $\top$') and Lemma 30. Validity-preservation of the $A$-rule follows from Proposition 26 and the straightforward extension of Corollary 13 to the case of whole $\mathcal{L}_B$. Finally, the $\Box$-rule is easily shown to be valid by usual modal reasoning. □

### *6.2 Theories and completeness*

**Definition 44 (Theories).** A $r$-theory is a maximally $\vdash$-consistent theory, i. e. a set $\Gamma$ of $\mathcal{L}_B$-formulas such that

- There is no finite $\Gamma' \subseteq \Gamma$ such that $\vdash \neg \bigwedge \Gamma'$
- If $\varphi \notin \Gamma$, then there is a finite $\Gamma' \subseteq \Gamma$ such that $\vdash \bigwedge \Gamma' \to \neg\varphi$

An $l$-theory is a non-trivial prime $\vdash_l$-theory, i.e. a set $\Gamma$ of $\mathcal{L}_B$-formulas such that

- If $\varphi, \psi \in \Gamma$, then $\varphi \wedge \psi \in \Gamma$

- If $\varphi \in \Gamma$ and $\vdash_l \varphi \to \psi$, then $\psi \in \Gamma$
- $\varphi \vee \psi \in \Gamma$ only if $\varphi \in \Gamma$ or $\psi \in \Gamma$
- $\Gamma \neq \emptyset$ and $\Gamma \neq \mathcal{L}_B$

We note that $r$-theories have the standard properties of maximally consistent sets: $\varphi \in \Gamma$ iff $\neg\varphi \notin \Gamma$; if $\vdash \varphi$, then $\varphi \in \Gamma$; if $\vdash \varphi \to \psi$, then $\varphi \in \Gamma$ only if $\psi \in \Gamma$.

**Lemma 45.** *Every $r$-theory is a $l$-theory.*

*Proof.* Let $\Gamma$ be a $r$-theory. 1. Let $\varphi, \psi \in \Gamma$. Then $\varphi \wedge \psi \in \Gamma$ follows from the fact that $\vdash \varphi \to (\psi \to (\varphi \wedge \psi))$. 2. Let $\vdash_l \varphi \to \psi$ and $\varphi \in \Gamma$. But $\vdash_l \varphi \to \psi$ implies $\vdash \varphi \to \psi$ (by the jump rule) and, hence, $\psi \in \Gamma$. 3. If $\varphi, \psi \notin \Gamma$, then $\neg\varphi, \neg\psi \in \Gamma$ and, consequently, $\varphi \vee \psi \notin \Gamma$. 4. $\vdash \varphi$ implies $\varphi \in \Gamma$ and $\vdash \neg\varphi$ implies $\varphi \notin \Gamma$. Hence, $\Gamma \neq \emptyset$ and $\Gamma \neq \mathcal{L}_B$. □

In what follows, $\Gamma_A = \{\varphi \mid A\varphi \in \Gamma\}$ and $\Gamma_\Box = \{\varphi \mid \Box\varphi \in \Gamma\}$.

**Lemma 46.** *If $\Gamma$ is a $l$-theory, then so is $\Gamma_A$.*

*Proof.* 1. Assume that $\varphi, \psi \in \Gamma_A$, i.e. $A\varphi \wedge A\psi \in \Gamma$. But $\vdash_l A\varphi \wedge A\psi \to A(\varphi \wedge \psi)$ (from $\varphi \wedge \psi \to \varphi \wedge \psi$ by the $A$-rule) and, hence, $\varphi \wedge \psi \in \Gamma_A$. 2. If $\vdash_l \varphi \to \psi$, then $\vdash_l A\varphi \to A\psi$. But then $\varphi \in \Gamma_A$ only if $\psi \in \Gamma_A$. 3. If $\varphi \vee \psi \in \Gamma_A$, then $A(\varphi \vee \psi) \in \Gamma$, then $A\varphi \vee A\psi \in \Gamma$ (since $\vdash_l A(\varphi \vee \psi) \to A\varphi \vee A\psi$ by the $A$-rule). Consequently, $\varphi \in \Gamma_A$ or $\psi \in \Gamma_A$. 4. $\vdash_l \top \to A\top$ implies $\vdash_l \varphi \to A\top$ for every $\varphi$. But $\Gamma \neq \emptyset$ and, hence, $\top \in \Gamma_A$. $\vdash A\bot \to \bot$ implies $\vdash_l A\bot \to \varphi$ for every $\varphi$. Now assume that $\Gamma_A = \mathcal{L}_B$. Hence, $\bot \in \Gamma_A$, i.e. $A\bot \in \Gamma$. But then $\Gamma = \mathcal{L}_B$, which is impossible. □

**Definition 47 (Canonical Model and Frame).** The canonical frame is a tuple

$$\mathfrak{F}' = \langle P', \leq', L', R', C', W', S', |\cdot|' \rangle$$

such that $P'$ is the set of all $l$-theories, $\leq'$ is set inclusion, $W'$ is the set of (all) $r$-theories and $|\cdot|'$ is a unary operation on $P'$ such that $|\Gamma|' = \Gamma_A$. Moreover:

- $\Gamma \in L'$ iff $t \in \Gamma$
- $C'\Gamma\Delta$ iff $\neg\varphi \in \Gamma$ only if $\varphi \notin \Delta$
- $R'\Gamma\Delta\Theta$ iff for all $\varphi, \psi$: if $\varphi \to \psi \in \Gamma$ and $\varphi \in \Delta$, then $\psi \in \Theta$
- $S'\Gamma\Delta$ iff $\begin{cases} \Gamma_\Box \subseteq \Delta & \text{if } \Gamma \notin W' \\ \Gamma_\Box \subseteq \Delta \text{ and } \Delta \in W' & \text{otherwise} \end{cases}$

We shall drop the prime if no confusion is likely to arise.

The canonical model is

$$\mathfrak{M}' = \langle \mathfrak{F}', V' \rangle$$

such that

- $V'(p) = \{\Gamma \mid p \in \Gamma\}$

Moreover, we define the canonical satisfaction relation $\Vdash'$ thus:

$$\Gamma \Vdash' \varphi \iff \varphi \in \Gamma$$

**Lemma 48.** *Let us define*

- $R_{\otimes}\Gamma\Delta\Theta$ *iff for all* $\varphi \in \Gamma, \psi \in \Delta$*:* $\varphi \otimes \psi \in \Theta$

*Then* $R\Gamma\Delta\Theta$ *implies* $R_{\otimes}\Gamma\Delta\Theta$*.*

*Proof.* Assume that $R\Gamma\Delta\Theta$, $\varphi \in \Gamma$ and $\psi \in \Delta$. We have to show that $\varphi \otimes \psi \in \Theta$. But $\vdash_l \varphi \to (\psi \to (\varphi \otimes \psi))$ and, hence, $\psi \to (\varphi \otimes \psi) \in \Gamma$. By the definition of $R$, $\varphi \otimes \psi \in \Theta$. □

**Lemma 49.** *Every* $\Gamma \in W$ *is a world in the canonical* $\mathfrak{F}$*.*

*Proof.* We have to show that every $r$-theory $\Gamma$ complies with the conditions of Definition 15.

- $C\Gamma\Gamma$ follows from the fact that $\varphi \in \Gamma$ iff $\neg\varphi \notin \Gamma$.
- $C\Gamma\Delta \Rightarrow \Delta \subseteq \Gamma$. If $\varphi \in \Delta$, then $\neg\varphi \notin \Gamma$ by the definition of $C$. But then $\varphi \in \Gamma$.
- $R\Gamma\Gamma\Gamma$ is obvious, since $r$-theories are closed under Modus Ponens.
- $R\Gamma\Delta_1\Delta_2 \Rightarrow \Delta_1 \subseteq \Gamma \subseteq \Delta_2$. First, assume that $\varphi \in \Delta_1$. If $\varphi \notin \Gamma$, then $\neg\varphi \in \Gamma$, then $\varphi \to \bot \in \Gamma$, then $\bot \in \Delta_2$, a contradiction. Now assume that $\varphi \in \Gamma$. Then $\top \to \varphi \in \Gamma$ and, hence, $\varphi \in \Delta_2$.
- $R\Delta_1\Delta_2\Gamma \Rightarrow \Delta_1, \Delta_2 \subseteq \Gamma$. By Lemma 48, the antecedent implies that $R_{\otimes}\Delta_1\Delta_2\Gamma$. Assume that $\varphi_i \in \Delta_i$. But then $\varphi_1 \otimes \varphi_2 \in \Gamma$ and, since $\vdash (\varphi_1 \otimes \varphi_2) \leftrightarrow (\varphi_1 \wedge \varphi_2)$, $\varphi_i \in \Gamma$.

□

**Lemma 50 (Canonical Frame Lemma).** *The canonical frame is an epistemic frame.*

*Proof.* The proof that $R, C, L$ and $S$ satisfy (3), (5), (2) of Definition 6 and (8) of Definition 22 is standard, see (Restall, 2000, ch. 11.3), and similarly for (6) and (4) of Definition 6. $W \subseteq P$ follows from Lemma 45. $|\Gamma|$ is well-defined by Lemma 46 and its monotonicity is obvious. $S$ complies with (9) by definition. The rest follows from Lemma 49. □

**Lemma 51 (Extended Witness Lemma).** *Let $\Gamma, \Delta, \Theta$ range over l-theories.*

- *If $\Box\varphi \notin \Gamma$, then there is $\Delta$ such that $S\Gamma\Delta$ and $\varphi \notin \Delta$*
- *If $\neg\varphi \notin \Gamma$, then there is $\Delta$ such that $C\Gamma\Delta$ and $\varphi \in \Delta$*
- *If $\varphi \to \psi \notin \Gamma$, then there are $\Delta, \Theta$ such that $R\Gamma\Delta\Theta$, $\varphi \in \Delta$ and $\psi \notin \Theta$*
- *If $\varphi \otimes \psi \in \Gamma$, then there are $\Delta, \Theta$ such that $R\Delta\Theta\Gamma$, $\varphi \in \Delta$ and $\psi \in \Theta$*

*Moreover, if $\Omega$ is a r-theory, then*

- *If $\Box\varphi \notin \Omega$, then there is a r-theory $\Omega'$ such that $S\Omega\Omega'$ and $\varphi \notin \Omega'$.*

*Proof.* The first four cases are standard, see (Restall, 2000, p. 255). It is perhaps good to note that the $\Box$-rule is needed to prove the first case.

The final case is proven by a technique well-known from modal logic. If $\varphi \notin \Omega$, then $\{\neg\varphi\} \cup \Omega_{\Box}$ is $\vdash$-consistent. Hence, this union can be extended to a maximally $\vdash$-consistent $r$-theory $\Omega'$ by the Lindenbaum Lemma. □

**Lemma 52 (Canonical Model Lemma).** *The canonical model is an epistemic model.*

*Proof.* Most of this is obvious and the rest follows from the Extended Witness Lemma 51. □

**Theorem 53 (Strong Completeness).** *If $\Sigma \Vdash_{\mathcal{E}} \varphi$, then $\Sigma \vdash_{\mathcal{H}_0} \varphi$, where $\mathcal{E}$ is the class of all epistemic frames.*

### *6.3 Completeness of Extensions*

Let $X$ contain (names for) some (zero or more) ingredients of the following kind:

1. The substructural frame conditions discussed in Sect. 3.2
2. The substructural epistemic frame conditions mentioned in Theorem 32
3. Global versions of the 'positive' frame conditions mentioned in Theorem 32

Let $\mathcal{E}_X$ be the class of epistemic frames satisfying all of the ingredients (referred to) in $X$. Then $Log_X$, the substructural epistemic logic of $\mathcal{E}_X$, is soundly and completely axiomatized by $\mathcal{H}_X$, where

- The set of $l$-axioms is extended by the schemata defining the extra substructural frame conditions (1) and the schemata defining the global versions of the positive 'epistemic' frame conditions (3).
- The set of $r$-axioms is extended by the schemata defining the extra epistemic frame conditions (2).

This claim is easily established by routine extension of the completeness proof for $\mathcal{E}_\emptyset$. The proof makes substantial use of Theorems 32 and 39.

## 7 Conclusion

We have introduced a family of substructural epistemic logics that formalize an agent's 'internal' evidential situation in addition to her implicit beliefs. The use of substructural logics is motivated by the fact that the evidential situations are often inconsistent and yet non-trivial, and, in general, not closed under classical consequence. Our main technical contribution is a general completeness theorem. In addition, we have pointed out some observations concerning substructural epistemic correspondence theory. Of course, many issues remain open and many paths are as yet unexplored. The rest of the article outlines at least some of these issues and paths. The outline should be read as an overview of possible topics of future research.

### *7.1 The Transference-of-Decidability Conjecture*

We have not studied the decidability of our logics at all. However, the following is a plausible conjecture.

**Conjecture 54 (Transference of Decidability).** *Let $\mathcal{H}$ be an extension of $\mathcal{H}_0$ such that*

1. *the set of $l$-axioms and rules without $\top \to A\top$, $A\bot \to \bot$, the $A$-rule and the $\Box$-rule is an axiomatization of a decidable substructural logic, and*
2. *the set of $r$-axioms and rules without the equivalence axioms $t \leftrightarrow \top$ and $(\varphi \otimes \psi) \leftrightarrow (\varphi \wedge \psi)$ is an axiomatization of a decidable modal logic.*

*Then membership in $\{\langle \Sigma, \varphi \rangle \mid \Sigma \vdash_{\mathcal{H}} \varphi\}$ is decidable.*

The conjecture states that if we combine a decidable substructural logic with a decidable modal logic and then add $A$, the result is a decidable substructural epistemic logic. The work on the conjecture is beyond the scope of this article and is a natural topic for future research.

### 7.2 Groups and group attitudes

We have presented a mono-agent framework mainly as a matter of simplification. Multi-agent versions for $n$ agents are defined in an obvious way. A more interesting extension is to introduce group attitudes such as universal, common and distributed belief, and their 'evidential' versions. Evidential versions of universal and common belief are relatively straightforward. Define a relation $H \subseteq P^2$ as follows:

- $Hxy$ iff $\exists i.(y = |x|_i)$

where $|x|_i$ is the evidence available to agent $i$. An obvious way to interpret 'universal evidential support' $E\varphi$ and 'common evidential support' $C\varphi$ would be

- $x \Vdash E\varphi$ iff $Hxy$ implies $y \Vdash \varphi$
- $x \Vdash C\varphi$ iff $H^*xy$ implies $y \Vdash \varphi$ (where $H^*$ is the reflexive transitive closure of $H$)

An encouraging observation is that the language with $E$ and $C$ is monotone. However, sound and complete axiomatizations of such extended logics are yet to be provided.

'Distributed evidential support' is more elusive. A natural approach is to define the distributed evidential support according to $x$ for a group of agents $\{1, \ldots, n\}$ in terms of the greatest lower bound of the set $\{|x|_1, \ldots, |x|_n\}$. However, there is no guarantee that a greatest lower bound exists for every such set.

### 7.3 Dynamic extensions

We have seen that a certain amount of dynamics is already 'built-in' when it comes to our semantics. In particular, substructural $\varphi \to \psi$ is warranted by $x$ iff 'adding' any $\varphi$-supporting $y$ yields support for $\psi$. Hence, we might read $A(\varphi \to \psi)$ dynamically as 'After adding any $\varphi$-supporting evidence, $A\psi$ holds'. In fact, we might extend the language with the 'adding' modality $[+\varphi]$ such that

- $\mathfrak{M}, x \Vdash [+\varphi]\psi$ iff $\exists yz.R|x|yz \wedge \mathfrak{M}, y \Vdash \varphi$ only if $\mathfrak{M}', x \Vdash \psi$

where $\mathfrak{M}'$ is as $\mathfrak{M}$ with the exception that $|x|' = z$ such that there is a $y$ with $R|x|yz$ and $\mathfrak{M}, y \Vdash \varphi$. However, a fuller investigation of dynamic extensions and interpretations of the present framework is a topic for future research.

### 7.4 Relations to similar approaches

We might construe a multi-agent version of the present framework as representing distinct evidence-yielding 'resources'. On this interpretation, $A_\sigma\varphi$ is read 'Resource $\sigma$ yields evidence that supports $\varphi$', for every $\sigma \in Res$. An interesting extension is to consider operators on resources. One particular choice of operators brings our approach very close to justification logics.

Let us define a resource algebra $\mathfrak{Res} = \langle Res, \circ, \bullet, \dagger \rangle$, where $\circ$ and $\bullet$ are binary operations on $Res$ and $\dagger$ is unary. An epistemic frame over $\mathfrak{Res}$ is $\mathfrak{F}$ such that (for all $x \in P$)

- $R|x|_\sigma |x|_{\sigma'} |x|_{\sigma \circ \sigma'}$
- $|x|_\sigma \leq |x|_{\sigma \bullet \sigma'}$ and$|x|_{\sigma'} \leq |x|_{\sigma \bullet \sigma'}$
- $|x|_\sigma \leq ||x|_{\dagger\sigma}|_\sigma$

These conditions yield $L$-validity of the following principles:

- $A_\sigma(\varphi \to \psi) \wedge A_{\sigma'}\varphi \to A_{\sigma \circ \sigma'}\psi$
- $A_\sigma\varphi \vee A_{\sigma'}\varphi \to A_{\sigma \bullet \sigma'}\varphi$
- $A_\sigma\varphi \to A_{\dagger\sigma}A_\sigma\varphi$

These are, of course, counterparts of the justification axioms governing the operations of 'application', 'sum' and 'proof-checker' (Artemov, 2001, 2008).[4] However, our substructural version of justification logic is much less fine-grained than the original justification framework. For example, we have $A_\sigma\varphi \to A_\sigma\psi$ $L$-valid for every $L$-valid $\varphi \to \psi$. It is nevertheless interesting to inquire into the formal properties of substructural justification logics and their applications.


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[4]In an unpublished talk, Sebastian Sequoiah-Grayson suggested that there is a link between the operation of 'application' on justifications and the substructural truth condition for $\varphi \to \psi$. The present proposal is an elaboration of this suggestion.

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