Categories

Beck's Monadicity Theorem: A categorical approach to universal algebra

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Many objects of interest in mathematics organise themselves into what is called a category. In fact, category-theoretic formulations of most global assertions in logic, algebra and topology seem inevitable.

In the series of talks we focus on deriving a characterisation of categories that have a distinct algebraic flavour: they are generalised varieties of algebras, or, in, modern parlance, monadic categories. A characterisation of monadic categories has been given by Jonathan Mock Beck in his PhD thesis in 1967. We will essentially follow Beck's exposition but we will also give new examples and applications.

As a prerequisite, the course needs only some very basic understanding of category theory: to know what a category, functor and natural transformation is will be ample. The tutorial will be divided into four sessions, more about the topics follow:

Adjunctions

One of the fundamental concepts of category theory is an adjunction. We present various guises of the subject and we focus on how adjunctions naturally generalise the concept of free objects known from universal algebra.

Limits and colimits

Limits and colimits of diagrams generalise various constructions such as the direct product of algebras, quotient algebras, etc. We mention the basic types of (co)limits, prove Maranda's Theorem that products and "subobjects" suffice for all limits and we prove Freyd's General Adjoint Functor Theorem.

Monads and their categories of algebras

Every adjunction yields a monad. We show various differerent facets of monads: they are certain "monoids", they are "formations of terms", etc. These facets will allow us to introduce easily how monads give rise to algebras for a monad. We also show that every monad yields at least two adjunctions: the Kleisli adjunction (="the category of term substitutions") and the Eilenberg-Moore adjuction (="the category of modules").

The monadicity theorems

Finally, we summarise the above into proving when a functor $U\colon \boldsymbol{A} \to \boldsymbol{X}$ exhibits the category $\boldsymbol{A}$ as "the category of modules" for a monad on $\boldsymbol{X}$. This is the gist of Beck's Monadicity Theorem and it essentially means that "$\boldsymbol{A}$ is a variety of algebras for an equational theory in $\boldsymbol{X}$".