An introduction to Abstract Algebraic Logic

Part 2

Tommaso Moraschini

Institute of Computer Science of the Czech Academy of Sciences

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Abstract algebraizability

Definition

Let $\mathbf{M} = \langle M, \cdot, 1\rangle$ be a monoid.

1. An **M**-set is a pair $\mathbb{A} = \langle A, *_{\mathbb{A}}\rangle$ where $*_{\mathbb{A}} \colon M \times A \to A$ s.t.
$$(\sigma \cdot \pi)(a) = \sigma *_{\mathbb{A}} (\pi *_{\mathbb{A}} a) \text{ and } 1 *_{\mathbb{A}} a = a.$$
2. A consequence relation $\vdash$ on an **M**-set $\mathbb{A}$ is **M**-structural when for every $\sigma \in M$,
$$\text{if } X \vdash x, \text{ then } \sigma *_{\mathbb{A}} X \vdash \sigma *_{\mathbb{A}} x.$$
3. Let $\mathbb{A}$ and $\mathbb{B}$ be two **M**-sets. An structural transformer from $\mathbb{A}$ to $\mathbb{B}$ is a residuated map $\boldsymbol{\tau} \colon \mathcal{P}(A) \to \mathcal{P}(B)$ such that for every $\sigma \in M$ and $X \subseteq A$,
$$\sigma *_{\mathbb{B}} \boldsymbol{\tau}(X) = \boldsymbol{\tau}(\sigma *_{\mathbb{A}} X).$$

Abstract algebraizability

Definition

Let $\mathbb{A}$ and $\mathbb{B}$ be **M**-sets. Two **M**-structural consequences relations $\vdash_{\mathbb{A}}$ and $\vdash_{\mathbb{B}}$ (resp. on A and B) are equivalent if there are structural transformers
$$\boldsymbol{\tau} \colon \mathcal{P}(\mathbb{A}) \longleftrightarrow \mathcal{P}(\mathbb{B}) \colon \boldsymbol{\rho}$$
such that
$$X \vdash_{\mathbb{A}} x \iff \boldsymbol{\tau}(X) \vdash_{\mathbb{B}} \boldsymbol{\tau}(x)$$
$$y \dashv\vdash_{\mathbb{B}} \boldsymbol{\tau\rho}(y)$$

Remark

Let **M** be the monoid of substitutions, $\vdash$ be a logic and K be a gen. quasi-variety. $\vdash$ is algebraizable with eq. alg. sem. K iff $\vdash$ and $\vDash_{\mathsf{K}}$ are equivalent as **M**-structural consequences.

Examples: order-algebraizability

- Let *Ineq* be set of inequalities $\varphi \preccurlyeq \psi$ in ω variables.

Definition

1. Let K be a class of ordered algebras and $\Theta \cup \{\varphi \preccurlyeq \psi\} \subseteq Ineq$,
$$\Theta \vDash_{\mathsf{K}}^{\preccurlyeq} \varphi \preccurlyeq \psi \iff \text{for every } \langle \boldsymbol{A}, \leq\rangle \in \mathsf{K} \text{ and hom } v \colon \boldsymbol{Fm} \to \boldsymbol{A}, \text{ if } v(\alpha) \leq v(\beta) \text{ for every } \alpha \preccurlyeq \beta \in \Theta, \text{ then } v(\varphi) \leq v(\psi).$$
The relation $\vDash_{\mathsf{K}}^{\preccurlyeq}$ is the inequational consequence relative to K.
2. A logic $\vdash$ is order algebraizable if it is equivalent to the consequence $\vDash_{\mathsf{K}}^{\preccurlyeq}$ of some class of ordered algebras K.

Remark

Every algebraizable logic is order algebraizable.

Examples: order-algebraizability

Example: BCI

- A CRPM is a structure $\langle A, \leq, *, 1, \to\rangle$ where $\langle A, \leq, *, 1\rangle$ is an ordered commutative monoid and
$$a \to b = \max\{c \in A : a * c \leq b\}.$$
- Consider the implication fragment BCI of linear logic:
$$\emptyset \vdash (x \to y) \to ((z \to x) \to (z \to y))$$
$$\emptyset \vdash (x \to (y \to z)) \to (y \to (x \to z))$$
$$\emptyset \vdash x \to x$$
$$x, x \to y \vdash y.$$
- BCI is order algebraizable w.r.t. $\langle \leq, \to\rangle$-subreducts of CRPMs.
- BCI is not algebraizable (hint: use semantic characterization of algebraizability).

Examples: Gentzen-algebraizability

Definition (very informal version)

A Gentzen system $\vdash_{\mathbf{G}}$ is algebraizable if it is equivalent to $\vDash_{\mathsf{K}}$ for some generalized quasi-variety K.

Example: substructural logic

- Let **FL** be the Full Lambek Calculus and FL be the variety of FL-algebras.
- **FL** is algebraizable w.r.t. FL via the translations
$$\boldsymbol{\tau} : \mathcal{P}(Seq) \longleftrightarrow \mathcal{P}(Eq) : \boldsymbol{\rho}$$
$$\boldsymbol{\tau}(\gamma_1, \ldots, \gamma_n \rhd \varphi) := \{(1 \cdot \gamma_1 \cdot \cdots \cdot \gamma_n) \leq \varphi\}$$
$$\boldsymbol{\tau}(\gamma_1, \ldots, \gamma_n \rhd \emptyset) := \{(1 \cdot \gamma_1 \cdot \cdots \cdot \gamma_n) \leq 0\}$$
$$\boldsymbol{\rho}(\varphi \approx \psi) := \{\varphi \rhd \psi, \psi \rhd \varphi\}.$$

Metalogical properties and bridge theorems

- Since algebraizale logics are essentially the same as relative equational consequences,they form a nice framework for the formulation of transfer theorems:

variants of interpolation	$\longleftrightarrow$	variants of amalgamation
variants of deduction theorem	$\longleftrightarrow$	variants of EDPC and CEP
variants of Beth definability	$\longleftrightarrow$	variants of ES
inconsistency lemma	$\longleftrightarrow$	variant of filtrality
having a disjunction	$\longleftrightarrow$	congruence distributivity

- We consider only very few of them...

Deduction theorems: local deduction theorem

Definition

A logic $\vdash$ has the local deduction detachment theorem (LDDT) if there is a family $\{\Phi_i(x, y) : i \in I\}$ of sets of formulas such that
$$\Gamma, \psi \vdash \varphi \iff \text{there is } i \in I \text{ such that } \Gamma \vdash \Phi_i(\psi, \varphi).$$

Example: Łucasiewicz logic

- In Łukasiewicz logic **Ł** we have
$$\Gamma, \psi \vdash_{\text{Ł}} \varphi \iff \text{there is } n \in \omega \text{ s.t. } \Gamma \vdash \underbrace{(\psi * \cdots * \psi)}_{n\text{-times}} \to \varphi.$$
- **Ł** has LDDT with $\{\Phi_n(x, y) : n \in \omega\}$ as follows:
$$\Phi_n(x, y) = \{\underbrace{(x * \cdots * x)}_{n\text{-times}} \to y\}.$$

Deduction theorems: local deduction theorem

Example: global modal logic

$$\Gamma, \psi \vdash^g_{\mathbf{K}} \varphi \iff \exists n \in \omega \text{ s.t. } \Gamma \vdash^g_{\mathbf{K}} (\varphi \wedge \Box\varphi \wedge \cdots \wedge \Box^n\varphi) \to \psi.$$

Then $\vdash^g_{\mathbf{K}}$ has LDDT with $\{\Phi_n(x,y) : n \in \omega\}$ as follows:

$$\Phi_n(x,y) = \{(x \wedge \Box x \wedge \cdots \wedge \Box^n x) \to y\}.$$

Definition

A generalized quasi-variety K has the relative congruence extension property (RCEP) if for every $\boldsymbol{A} \leq \boldsymbol{B} \in \mathsf{K}$ and $\theta \in \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$, there is $\phi \in \mathrm{Con}_{\mathsf{K}}\boldsymbol{B}$ such that $\theta = \phi \cap A^2$.

Theorem

Let $\vdash$ be a finitary algebraizable logic with equivalent algebraic semantics K. Then $\vdash$ has the LDDT if and only if K has the RCEP.

Deduction theorems: deduction theorem

Definition

A logic $\vdash$ has the deduction datachment theorem (DDT) if there is a finite set $\Phi(x,y)$ of formulas such that

$$\Gamma, \psi \vdash \varphi \iff \Gamma \vdash \Phi(\psi, \varphi).$$

Example

- Intuitionistic logic **IPC** has the deduction theorem witnessed by $\{x \to y\}$, i.e.

$$\Gamma, \psi \vdash_{\mathbf{IPC}} \varphi \iff \Gamma \vdash_{\mathbf{IPC}} \psi \to \varphi.$$

- The global consequence of **K4** have the deduction theorem witnessed by $\{(x \wedge \Box x) \to y\}$, i.e.

$$\Gamma, \psi \vdash^g_{\mathbf{K4}} \varphi \iff \Gamma \vdash^g_{\mathbf{K4}} (\psi \wedge \Box\psi) \to \varphi.$$

Deduction theorems: deduction theorem

Definition

A quasi-variety K has equationally definable principal relative congruences (EDPRC) if there is a finite set of equations $\Phi(x,y,z,v)$ such that for every algebra $\boldsymbol{A} \in \mathsf{K}$ and $a,b,c,d \in A$,

$$\langle a,b \rangle \in \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{K}}(c,d) \iff \boldsymbol{A} \vDash \Phi(a,b,c,d).$$

Theorem

Let $\vdash$ be a finitary algebraizable logic with equivalent algebraic semantics the quasi-variety K. Then $\vdash$ has the DDT if and only if K has EDPRC.

- Remark: This is useful to disprove the fact that a logic has the DDT, e.g. **Ł** and $\vdash^g_{\mathbf{K}}$ have not DDT.

Deduction theorems: contextual deduction theorem

Definition

A logic $\vdash$ has the contextual deduction detachment theorem (CDDT) when for every $n \in \omega$ there is a finite set of formulas $\Phi_n(x_1,\ldots,x_n,y_1,y_2)$ such that for all $\Gamma \cup \{\varphi,\psi\}$ in $x_1,\ldots,x_n$,

$$\Gamma, \psi \vdash \varphi \iff \Gamma \vdash \Phi_n(x_1,\ldots,x_n,\varphi,\psi).$$

Example: relevance logic

Relevance logic **R** has the CDDT as follows: for $\Gamma \cup \{\varphi,\psi\}$ in $x_1,\ldots,x_n$,

$$\Gamma, \psi \vdash_{\mathbf{R}} \varphi \iff \Gamma \vdash_{\mathbf{R}} ((x_1 \to x_1) \wedge \cdots \wedge (x_n \to x_n) \wedge \psi) \to \varphi.$$

R has not the DDT since relevant algebras lack the (R)CEP.

Deduction theorems: contextual deduction theorem

Definition

A quasi-variety K has equationally semidefinable principal relative congruences (ESPRC) if for every $n \in \omega$ there is a finite set of equations $\Phi_n(x_1, \dots, x_n, y_1, y_2, y_3, y_4)$ such that whenever $e_1, \dots, e_n$ generate an algebra $\boldsymbol{A} \in \mathsf{K}$ and $a, b, c, d \in A$,

$$\langle a, b \rangle \in \mathrm{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(c, d) \iff \boldsymbol{A} \vDash \Phi_n(\vec{e}, a, b, c, d).$$

Theorem

Let $\vdash$ be a finitary algebraizable logic with equivalent algebraic semantics the quasi-variety K. Then $\vdash$ has the CDDT if and only if K has ESPRC.

Deduction theorems: recap

Transfer results

$$\begin{aligned} \text{local deduction theorem} &\longleftrightarrow \text{RCEP} \\ \text{deduction theorem} &\longleftrightarrow \text{EDPRC} \\ \text{contextual deduction theorem} &\longleftrightarrow \text{ESPRC}. \end{aligned}$$

- In quasi-varieties we have

$$\text{EDPRC} \iff \text{ESPRC and RCEP}.$$

Observation

Let $\vdash$ be a finitary algebraizable logic whose equivalent algebraic semantics is a quasi-variety. Then $\vdash$ has DDT if and only if it has LDDT and CDDT.

Inconsistency lemma

Definition

A logic $\vdash$ has the classical inconsistency lemma (CIL) if for every $n \in \omega$ there is a finite set of formulas $\Phi_n(x_1, \dots, x_n)$ such that

$$\begin{aligned} \Gamma \cup \alpha_1, \dots, \alpha_n \text{ is inconsistent } &\iff \Gamma \vdash \Phi_n(\alpha_1, \dots, \alpha_n) \\ \Gamma \cup \Phi_n(\alpha_1, \dots, \alpha_n) \text{ is inconsistent } &\iff \Gamma \vdash \{\alpha_1, \dots, \alpha_n\}. \end{aligned}$$

Example

- **CPC** has the CIL witnessed by

$$\Phi_n(x_1, \dots, x_n) = \{\neg(x_1 \wedge \dots \wedge x_n)\}.$$

- Let **L** be a substructural logic algebraized by a subvariety of FL_{ew} satisfying $x \vee (x^k \to 0) \approx 1$ for some $k \geq 1$. **L** has CIL with

$$\Phi_n(x_1, \dots, x_n) = \{(x_1 * \dots * x_n)^k \to 0\}.$$

Inconsistency lemma

Definition

Let K be a quasi-variety.

1. Let $\boldsymbol{A} \leq \prod_{i \in I} \boldsymbol{A}_i$ be a K-representation and F a filter over I. Then define $\theta_F \subseteq A \times A$ as

$$\langle a, b \rangle \in \theta_F \iff \{i \in I : a(i) = b(i)\} \in F.$$

   This representation admits only filtral K-congruences if every congruence of $\boldsymbol{A}$ has the form θ_F for some filtrer F.
2. K is filtral if every K-representation admits only K-congruences.

Theorem

A quasi-variety is filtral if and only if it is relatively semi-simple and it has EDPRC.

Inconsistency lemma

Theorem

Let $\vdash$ be a finitary algebraizable logic with equivalent algebraic semantics the quasi-variety K. TFAE:

1. $\vdash$ has CIL.
2. K has EDPRC, is relatively semi-simple, and for every $\boldsymbol{A} \in \mathsf{K}$ the total congruence is compact in $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$.
3. K is filtral and the subalgebras of its non-trivial members are non-trivial.

In particular, in this case $\vdash$ has the DDT.