A Logic of Questions Based on Łukasiewicz Fuzzy Logic

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The aim of this paper is to enrich Łukasiewicz fuzzy logic (see Hájek, 1993) with a new operator, known from inquisitive semantics (Ciardelli & Roelofsen, 2011) as *inquisitive disjunction*. This operator allows to form new type of sentences that represent questions. The resulting system, which we will call *The Inuqisitive Extension of Łukasiewicz Fuzzy Logic*, will be a logic of questions based on Łukasiewicz Fuzzy Logic of declarative sentences. The results are taken from (Punčochář, 201X).

I will start with a brief introduction of an abstract semantic framework for substurctural logics. It is a modification and extension of the semantics proposed in (Došen, 1989). The semantic structures of this framework will be called information models. An informational model is a structure of the type $\mathcal{M} = \langle S, +, \cdot, 0, 1, C, V \rangle$ that satisfies the following conditions: $\langle S, + \rangle$ is a join-semilattice, determining an ordering: $a \leq b$ iff $a + b = b$; 0 is the least element, i.e. $0 + a = a$; moreover, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$, and $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$; $1 \cdot a = a$ and $0 \cdot a = 0$; C is a binary (compatibility) relation such that: there is no a such that $0Ca$, if aCb then bCa, and $(a + b)Cc$ iff aCc or bCc; finally, V is a valuation defined as a function assigning an ideal (a nonempty downset closed under $+$) to every atomic formula.

L will denote a language standardly used in substructural logics. $L^?$ is the inquisitive extension of L, i.e. L enriched with one binary connective ? (inquisitive disjunction). For example, the formula $p?q$ represents the question *whether p or q*.

Given any information model $\mathcal{M} = \langle S, +, \cdot, 0, 1, C, V \rangle$, we will define a relation between the elements of S and formulas of $L^?$ by the following semantic clauses:

- $a \vDash p$ iff $p \in V(a)$.
- $a \vDash \bot$ iff $a = 0$.
- $a \vDash t$ iff $a \leq 1$.
- $a \vDash \neg\varphi$ iff for any b, if bCa then $b \nvDash \varphi$.
- $a \vDash \varphi \to \psi$ iff for any b, if $b \vDash \varphi$, then $a \cdot b \vDash \psi$.
- $a \vDash \varphi \wedge \psi$ iff $a \vDash \varphi$ and $a \vDash \psi$.
- $a \vDash \varphi \otimes \psi$ iff for some b, c: $b \vDash \varphi$, $c \vDash \psi$, and $a \leq b \cdot c$.
- $a \vDash \varphi \vee \psi$ iff for some b, c: $b \vDash \varphi$, $c \vDash \psi$, and $a \leq b + c$.
- $a \vDash \varphi?\psi$ iff $a \vDash \varphi$ or $a \vDash \psi$.

A formula φ of the language $L^?$ is valid in $\mathcal{M}$ iff $1 \vDash \varphi$ in $\mathcal{M}$. The set of L-formulas valid in all information models is a non-distributive modification of the logic known as Full Lambek enriched with a paraconsistent negation. A suitable corresponding axiomatic system for this logic (that will be presented during the talk) will be denoted as FL. I will present also an axiomatization of the set of all $L^?$-formulas valid in class of all information models. The axiomatic system will be denoted as $InqFL$ (an inquisitive extension of FL).

Let us denote the set of L-formulas that are valid in a class of informational models $\mathcal{C}$ as $Log(\mathcal{C})$. A set of L-formulas λ is called a logic of declarative sentences if there is a class of informational models $\mathcal{C}$ such that $\lambda = Log(\mathcal{C})$.

Let us denote the set of $L^?$-formulas that are valid in a class of informational models $\mathcal{C}$ as $Log^?(\mathcal{C})$ and the class of models of some given set of L-formulas Δ as $Mod(\Delta)$.

Let λ be a logic of declarative sentences. The *inquisitive extension* of λ, denoted as $\lambda^?$, is the set of all $L^?$-formulas that are valid in every model of λ. In symbols, $\lambda^? = Log^?(Mod(\lambda))$.

Theorem 1. *If FL plus a set of axioms A axiomatizes λ, then $InqFL$ plus A axiomatizes $\lambda^?$.*

A product of two information models will be defined in a natural way and the following result will be shown.

Theorem 2. *Let $\mathcal{C}$ be a class of informational models. If $Log(\mathcal{C}) = \lambda$ and $\mathcal{C}$ is closed under products, then $Log^?(\mathcal{C}) = \lambda^?$.*

In the next step, I will define a class of information models that will determine the inquisitive extension of Łukasiewicz fuzzy logic.

Fuzzy models are structures of the form $\mathcal{M}^n_E = \langle S, +, \cdot, 0_n, 1_n, C, V\rangle$, where $n \geq 1$ is a natural number, $E = \langle e_1, \ldots, e_n\rangle$ is an n-tuple of functions from atomic formulas to the closed interval $[0, 1]$, and it holds:

- $S = \{\langle a_1, \ldots, a_n\rangle; a_1, \ldots a_n \in [0, 1]\}$,
- $\langle a_1, \ldots, a_n\rangle + \langle b_1, \ldots, b_n\rangle = \langle max\{a_1, b_1\}, \ldots, max\{a_n, b_n\}\rangle$,
- $\langle a_1, \ldots, a_n\rangle \cdot \langle b_1, \ldots, b_n\rangle = \langle a_1 * b_1, \ldots, a_n * b_n\rangle$, where $a * b = max\{0, a + b - 1\}$.
- $1_n = \langle 1, \ldots, 1\rangle$, where 1 is n-times.
- $0_n = \langle 0, \ldots, 0\rangle$, where 0 is n-times.
- $\langle a_1, \ldots, a_n\rangle C\langle b_1, \ldots, b_n\rangle$ iff for some i ($1 \leq i \leq n$), $1 - b_i < a_i$.
- $\langle a_1, \ldots, a_n\rangle \in V(p)$ iff for all i ($1 \leq i \leq n$), $a_i \leq e_i(p)$.

Lemma 1. *Every fuzzy model is an informational model.*

Lemma 2. *The class of fuzzy models is closed under products.*

Let Ł represent the set of L-formulas valid in Łukasiewicz fuzzy logic.

Theorem 3. *For any L-formula α, $\alpha \in$ Ł iff α is valid in every fuzzy model.*

Theorem 4. *For any $L^?$-formula φ, $\varphi \in$ Ł$^?$ iff φ is valid in every fuzzy model.*

If time allows I will discuss also the possibility to extend other fuzzy logics with the inquisitive disjunction.

References

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