Reasoning about weighted graphs in many-valued modal logic

Igor Sedlár and Amanda Vidal

Institute of Computer Science, Czech Academy of Sciences

Prague, Czech Republic

Summary. Classical Kripke frames are (directed) graphs, so it is not surprising that classical modal logic has been suggested as a natural formalism for reasoning about graphs [1, 5, 6]. In this talk we propose *many-valued* modal logics as a natural formalism for reasoning about *weighted* graphs. We introduce a family of many-valued modal logics suitable for formalizing reasoning about weighted graphs and prove completeness for some of these logics.

Weighted graphs and graph logics. Let $\boldsymbol{A}$ be a complete FL_{ew} algebra (with bounds $0, 1$) and let Lab be a countable set of labels. A *labeled $\boldsymbol{A}$-weighted directed semi-simple graph* is a pair $\mathfrak{G} = \langle V, E, f \rangle$ where V is a non-empty set (of vertexes), E is a function from $V \times V$ to $\boldsymbol{A}$ (the $\boldsymbol{A}$-weighted edge function) and $f : V \times \mathsf{Lab} \to \{0, 1\}$ (the labeling function).

Intuitively, $\boldsymbol{A}$ is seen as an algebra of *distances* and $E(v, v')$ is the distance between v and v'. The element 1 represents the smallest possible distance ("zero distance") and 0 represents the largest possible distance ("infinite distance"). While $\wedge$ and $\vee$ represent the infimum (largest distance) and supremum (smallest distance), $\odot$ is a fusion (merge) operation on distances used when calculating the result of "adding distances".

Let $\mathcal{L} = \left\{\wedge, \vee, \odot, \to, \bar{1}, \bar{0}, \Diamond, \Box\right\}$ and let $\boldsymbol{Fm}_{\mathcal{L}}$ be the absolutely free $\mathcal{L}$-algebra generated by Lab. Elements of this algebra are called $\mathcal{L}$-formulas; φ, ψ etc. range over $\mathcal{L}$-formulas, α, β range over $\mathcal{L}$-formulas without occurrences of $\Box, \Diamond$, and Γ, Δ etc. range over sets of $\mathcal{L}$-formulas.

For every $v \in V$, the labeling function f induces a function $f_v : \boldsymbol{Fm}_{\mathcal{L}} \to \boldsymbol{A}$ satisfying

$$\begin{aligned}
f_v(\bar{c}) &= c && \text{for } c \in \{0, 1\} \\
f_v(\varphi \circ \psi) &= f_v(\varphi) \circ^{\boldsymbol{A}} f_v(\psi) && \text{for } \circ \in \{\wedge, \vee, \odot, \to\} \\
f_v(\Box\varphi) &= \bigwedge_{w \in V} \left\{E(v, w) \to^{\boldsymbol{A}} f_w(\varphi)\right\} \\
f_v(\Diamond\varphi) &= \bigvee_{w \in V} \left\{E(v, w) \odot^{\boldsymbol{A}} f_w(\varphi)\right\}
\end{aligned}$$

We will sometimes write $v(\varphi)$ instead of $f_v(\varphi)$.

Given $\boldsymbol{A}$, a formula φ is a *global $\boldsymbol{A}$-consequence* of a set of formulas Γ (notation $\Gamma \vdash^{g}_{\boldsymbol{A}} \varphi$) iff, for every $\boldsymbol{A}$-weighted labeled graph $\mathfrak{G}$, if $v[\Gamma] = \{1\}$ for every $v \in \mathfrak{G}$, then $v(\varphi) = 1$ for every $v \in \mathfrak{G}$. A formula φ is a *local $\boldsymbol{A}$-consequence* of Γ (notation $\Gamma \vdash^{l}_{\boldsymbol{A}} \varphi$) iff, for every $\boldsymbol{A}$-weighted labeled graph $\mathfrak{G}$ and every $v \in \mathfrak{G}$, if $v[\Gamma] = \{1\}$, then $v(\varphi) = 1$. Importantly, neither of these consequence relations is *structural*, i.e. closed under arbitrary substitutions. For example, $p \vee \neg p$ follows from the empty set, but $\Box p \vee \neg\Box p$ does not.

Expressiveness. The set $\{E(v,u) \odot u(p)\,;\, u \in V\}$ contains $E(v,w)$ for all w such that $w(p) = 1$ and 0 if there is u such that $u(p) = 0$. This means that $v(\Diamond p)$ is the smallest distance from v to a vertex labeled with p (0 if there is no such vertex). We may call this the *minimal cost of (reaching) p in v*. As a special case, $v(\Diamond \bar{1})$ is the distance from v to the closest vertex.

It is clear that $1 \leq v(\Diamond p) \to v(\Diamond q)$ iff $v(\Diamond p) \leq v(\Diamond q)$. So, $v(\Diamond p \to \Diamond q) = 1$ means that the smallest distance (from v) to a vertex labeled by p is at least as big as the smallest distance to a vertex labeled by q; in other words, "q is at least as close as p".

It is remarkable that, in some sense, the Diamond operator is now the main one: while $\Diamond p$ will be evaluated as a supremum of values of the algebra, the value of $\Box p$ can only be evaluated to (infima of) negated values of $\boldsymbol{A}$, and so, in many cases, while $\Diamond p$ can indeed take any value, $\Box p$ will be limited to the negated elements of $\boldsymbol{A}$. This fact is not extended to arbitrary formulas (that is to say, $\Box\varphi$ is non longer limited to the negated values of the algebra), but nevertheless let us show some results on the partial interdefinability of $\Box$ from $\Diamond$ that support the previous idea.

Proposition. *The following formulas are valid in all* $\boldsymbol{A}$*-weighted graphs:*

- $\Box^n \alpha \leftrightarrow \neg\Diamond^n\neg\alpha$,
- $\Box(\varphi \to \Box^n\alpha) \leftrightarrow \neg\Diamond(\varphi \odot \Diamond^n\neg\alpha)$

Axiomatization. Axiomatizations of the local and global consequence relations over $\boldsymbol{A}$-weighted directed graphs are straightforward in cases where the axiomatization of the $\boldsymbol{A}$-valued modal logic is known. This amounts simply to defining a two-layered axiomatic system in the line of [4]. Formally, the logic of all directed $\boldsymbol{A}$-weighted labeled graphs is complete with respect to the axiomatic system $W_{\boldsymbol{A}}$ defined by:

- An axiomatic system for the modal logic of $\boldsymbol{A}$-valued Kripke models
- Axioms of Classical Logic for formulas without modalities

This provides us with axiomatic systems for the logic of weighted graphs over the standard Gödel algebra (using [3]), and over arbitrary finite residuated lattices (by means of the axiomatization presented in [2]).

References

[1] Mario R. F. Benevides and L. Menasché Schechter. Using modal logics to express and check global graph properties. *Logic Journal of the IGPL*, 17(5):559–587, 2009.

[2] Félix Bou, Francesc Esteva, Lluís Godo, and Ricardo Oscar Rodríguez. On the minimum many-valued modal logic over a finite residuated lattice. *Journal of Logic and Computation*, 21(5):739–790, 2011.

[3] Xavier Caicedo and Ricardo Oscar Rodríguez. Bi-modal Gödel logic over [0,1]-valued Kripke frames. *Journal of Logic and Computation*, 25(1):37–55, 2015.

[4] Petr Cintula and Carles Noguera. Modal logics of uncertainty with two-layer syntax: A general completeness theorem. In Kohlenbach U., Barceló P., and de Queiroz R., editors, *Logic, Language, Information, and Computation. WoLLIC 2014*, LNCS 8652, pages 124–136, Berlin, Heidelberg, 2014. Springer.

[5] James Gate and Iain A. Stewart. The expressibility of fragments of hybrid graph logic on finite digraphs. *Journal of Applied Logic*, 11(3):272 – 288, 2013.

[6] Yutaka Miyazaki. Graph theory and modal logic. Talk at BLAST 2013, Chapman University, 2013.