Prague Gathering of Logicians
&
Beauty of Logic
2018

Volume of Abstracts

Organized by

Institute of Computer Science & Institute of Philosophy
The Czech Academy of Sciences

25-27 January 2018
Villa Lanna
Prague

Programme: Prague Gathering of Logicians & Beauty of Logic 2018

Villa Lanna, 25–27 January, 2018

Thursday, 25 January

9:30 – 10:00	*registration and morning coffee*	
9:50	*opening*	
10:00 – 11:00	Pavel Hrubeš	Random formulas and the interpolation technique in proof complexity
11:00 – 11:15	*break*	
11:15 – 12:15	Carles Noguera	A graded model theory
14:00 – 15:00	Francesc Esteva	On first order and modal Łukasiewicz and product logics
15:00 – 15:30	*coffee break*	
15:30 – 16:30	George Metcalfe	Finite model properties for the one-variable fragment of first-order Gödel logic
16:30 – 16:45	*break*	
16:45 – 17:10	Vít Punčochář	A logic of questions based on Łukasiewicz fuzzy logic
17:10 – 17:35	Sándor Jenei	Strong standard completeness of **IUL** plus $\mathbf{t} \Leftrightarrow \mathbf{f}$ via a structure theorem for finitely generated group-like FL_e-algebras à la Hahn
17:35 – 18:00	Vilém Novák	On partial fuzzy type theory
18:00 – 18:25	Luca San Mauro	Trial and error mathematics: dialectical, p-dialectical, and q-dialectical systems

Friday, 26 January

9:30 – 10:00	*morning coffee*	
10:00 – 11:00	Sy David Friedman	The current state of the foundations of set theory
11:00 – 11:15	*break*	
11:15 – 12:15	Joel David Hamkins	The universal algorithm and the universal finite set

Friday, 26 January (cont'd)

14:00 – 15:00	PETER SCHROEDER-HEISTER	Ekman's Paradox and the quest for an intensional theory of proofs
15:00 – 15:30	*coffee break*	
15:30 – 16:30	ALBERT VISSER	Intuitionistic provability logic
16:30 – 16:45	*break*	
16:45 – 17:10	ANNA HORSKÁ	What is the height of Gentzen reduction trees?
17:10 – 17:35	BERTALAN BODOR	Homogeneous structures and a new poset
19:00	*banquet dinner*	

Saturday, 27 January

9:30 – 10:00	*morning coffee*	
10:00 – 11:00	MANUEL BODIRSKY	Complexity classifications for fragments of existential second-order logic
11:00 – 11:15	*break*	
11:15 – 12:15	MICHAEL PINSKER	Equations in algebras induced by beautiful first-order structures
14:00 – 15:00	TOMÁŠ KROUPA	Hájek's probability logic and its two-sorted algebraic semantics
15:00 – 15:30	*coffee break*	
15:30 – 16:30	VINCENZO MARRA	Universal expectations, generic measure spaces, and Hájek's probability logic
16:30 – 16:45	*break*	
16:45 – 17:10	ANTONÍN DVOŘÁK	First-order fuzzy modal logics with variable domains
17:10 – 17:35	LIBOR BĚHOUNEK	Hájek-style modalities in fuzzy intensional semantics
17:35 – 18:00	IGOR SEDLÁR	Reasoning about weighted graphs in many-valued modal logic
18:00 – 18:25	PETR CINTULA	On Hájek's (half-)forgotten treasures
18:30	*closing*	

Random formulas and the interpolation technique in proof complexity

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Abstract

I will discuss the interpolation technique as a means to obtain lower bounds in proof complexity. The method had been previously applied to a host of proof systems; I will discuss its applicability to randomly chosen unsatisfiable formulas.

A graded model theory

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Abstract

Hájek's monograph "Metamathematics of Fuzzy Logic" was the starting point of a long stream of research on mathematical fuzzy logic. Among many other aspects, Hájek's proposed program included the study of first-order logics to deal with reasoning with graded predicates. This gave rise to a number of papers by himself and others on a graded model theory, which generalizes classical model theory by considering structures where predicates are interpreted on a many-valued scale. In this talk we will discuss the motivations of such theory and survey its main results and latest contributions.

On First order and Modal
Łukasiewicz and Product Logics

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January 12, 2018

1 Abstract

In the book "Metamathematics of Fuzzy Logic" Petr Hájek introduced BL logic and its first order version BL∀ as the many-valued logic underlying fuzzy logic in narrow sense. In this talk we begin with a brief summary of basic results on BL logic and the so-called t-norm based logics (axiomatic extensions of BL) and their first order versions. Then we will give basic results on semantics of these first order logics with special attention to Łukasiewicz and product logics and their completeness results w.r.t. witnessed and quasi-witnessed models (See Hájek's book, Hájek-Cintula paper and Cerami's thesis).

In order to introduce Modal many-valued logics, the main topic of the talk, we summarize previous results on first order Łukasiewicz and Product logics when we restrict the semantics to standard semantics, semantics over chains on the real unit interval. Then we will define and study Modal Łukasiewicz and Product logics and summarize results obtained in this setting and a new one. Taking into account the relation between Modal and Description many-valued logics, we summarize decidability results for VAL, SAT and SAT_{pos} for Modal Łukasiewicz and Product Logics. We end up with an open problem in Modal (and also first order) Product logics.

Finite Model Properties for the One-Variable Fragment of First-Order Gödel Logic

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Abstract

In his pioneering monograph "Metamathematics of Fuzzy Logic" Petr Hájek proved that the crisp S5 Gödel modal logic — equivalently, the one-variable fragment of first-order Gödel logic — does not have the finite model property, and posed the question as to whether this logic is decidable. In this talk, I will give a positive answer to this question, making use of a correspondence between monadic Heyting algebras — algebraic semantics for the one-variable fragment of first-order intuitionistic logic — and Heyting algebras with a relatively complete subalgebra, that have been studied by, among others, G. Bezhanishvili, L. Esakia, and A. Monteiro. Notably, although the crisp S5 Gödel modal logic does not have the finite model property with respect to its standard semantics, it does have this property with respect to its algebraic semantics.

A Logic of Questions Based on Łukasiewicz Fuzzy Logic

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The aim of this paper is to enrich Łukasiewicz fuzzy logic (see Hájek, 1993) with a new operator, known from inquisitive semantics (Ciardelli & Roelofsen, 2011) as *inquisitive disjunction*. This operator allows to form new type of sentences that represent questions. The resulting system, which we will call *The Inuqisitive Extension of Łukasiewicz Fuzzy Logic*, will be a logic of questions based on Łukasiewicz Fuzzy Logic of declarative sentences. The results are taken from (Punčochář, 201X).

I will start with a brief introduction of an abstract semantic framework for substurctural logics. It is a modification and extension of the semantics proposed in (Došen, 1989). The semantic structures of this framework will be called information models. An informational model is a structure of the type $\mathcal{M} = \langle S, +, \cdot, 0, 1, C, V\rangle$ that satisfies the following conditions: $\langle S, +\rangle$ is a join-semilattice, determining an ordering: $a \leq b$ iff $a + b = b$; 0 is the least element, i.e. $0 + a = a$; moreover, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$, and $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$; $1 \cdot a = a$ and $0 \cdot a = 0$; C is a binary (compatibility) relation such that: there is no a such that $0Ca$, if aCb then bCa, and $(a + b)Cc$ iff aCc or bCc; finally, V is a valuation defined as a function assigning an ideal (a nonempty downset closed under $+$) to every atomic formula.

L will denote a language standardly used in substructural logics. $L^?$ is the inquisitive extension of L, i.e. L enriched with one binary connective ? (inquisitive disjunction). For example, the formula $p?q$ represents the question *whether p or q*.

Given any information model $\mathcal{M} = \langle S, +, \cdot, 0, 1, C, V\rangle$, we will define a relation between the elements of S and formulas of $L^?$ by the following semantic clauses:

- $a \vDash p$ iff $p \in V(a)$.
- $a \vDash \bot$ iff $a = 0$.
- $a \vDash t$ iff $a \leq 1$.
- $a \vDash \neg\varphi$ iff for any b, if bCa then $b \nvDash \varphi$.
- $a \vDash \varphi \rightarrow \psi$ iff for any b, if $b \vDash \varphi$, then $a \cdot b \vDash \psi$.
- $a \vDash \varphi \wedge \psi$ iff $a \vDash \varphi$ and $a \vDash \psi$.
- $a \vDash \varphi \otimes \psi$ iff for some b, c: $b \vDash \varphi$, $c \vDash \psi$, and $a \leq b \cdot c$.
- $a \vDash \varphi \vee \psi$ iff for some b, c: $b \vDash \varphi$, $c \vDash \psi$, and $a \leq b + c$.
- $a \vDash \varphi?\psi$ iff $a \vDash \varphi$ or $a \vDash \psi$.

A formula φ of the language $L^?$ is valid in $\mathcal{M}$ iff $1 \vDash \varphi$ in $\mathcal{M}$. The set of L-formulas valid in all information models is a non-distributive modification of the logic known as Full Lambek enriched with a paraconsistent negation. A suitable corresponding axiomatic system for this logic (that will be presented during the talk) will be denoted as FL. I will present also an axiomatization of the set of all $L^?$-formulas valid in class of all information models. The axiomatic system will be denoted as $InqFL$ (an inquisitive extension of FL).

Let us denote the set of L-formulas that are valid in a class of informational models $\mathcal{C}$ as $Log(\mathcal{C})$. A set of L-formulas λ is called a logic of declarative sentences if there is a class of informational models $\mathcal{C}$ such that $\lambda = Log(\mathcal{C})$.

Let us denote the set of $L^?$-formulas that are valid in a class of informational models $\mathcal{C}$ as $Log^?(\mathcal{C})$ and the class of models of some given set of L-formulas Δ as $Mod(\Delta)$.

Let λ be a logic of declarative sentences. The *inquisitive extension* of λ, denoted as $\lambda^?$, is the set of all $L^?$-formulas that are valid in every model of λ. In symbols, $\lambda^? = Log^?(Mod(\lambda))$.

Theorem 1. *If FL plus a set of axioms A axiomatizes λ, then $InqFL$ plus A axiomatizes $\lambda^?$.*

A product of two information models will be defined in a natural way and the following result will be shown.

Theorem 2. *Let $\mathcal{C}$ be a class of informational models. If $Log(\mathcal{C}) = \lambda$ and $\mathcal{C}$ is closed under products, then $Log^?(\mathcal{C}) = \lambda^?$.*

In the next step, I will define a class of information models that will determine the inquisitive extension of Łukasiewicz fuzzy logic.

Fuzzy models are structures of the form $\mathcal{M}_E^n = \langle S, +, \cdot, 0_n, 1_n, C, V\rangle$, where $n \geq 1$ is a natural number, $E = \langle e_1, \ldots, e_n\rangle$ is an n-tuple of functions from atomic formulas to the closed interval $[0, 1]$, and it holds:

- $S = \{\langle a_1, \ldots, a_n\rangle; a_1, \ldots a_n \in [0, 1]\}$,
- $\langle a_1, \ldots, a_n\rangle + \langle b_1, \ldots, b_n\rangle = \langle max\{a_1, b_1\}, \ldots, max\{a_n, b_n\}\rangle$,
- $\langle a_1, \ldots, a_n\rangle \cdot \langle b_1, \ldots, b_n\rangle = \langle a_1 * b_1, \ldots, a_n * b_n\rangle$, where $a * b = max\{0, a + b - 1\}$.
- $1_n = \langle 1, \ldots, 1\rangle$, where 1 is n-times.
- $0_n = \langle 0, \ldots, 0\rangle$, where 0 is n-times.
- $\langle a_1, \ldots, a_n\rangle C \langle b_1, \ldots, b_n\rangle$ iff for some i ($1 \leq i \leq n$), $1 - b_i < a_i$.
- $\langle a_1, \ldots, a_n\rangle \in V(p)$ iff for all i ($1 \leq i \leq n$), $a_i \leq e_i(p)$.

Lemma 1. *Every fuzzy model is an informational model.*

Lemma 2. *The class of fuzzy models is closed under products.*

Let *Ł* represent the set of L-formulas valid in Łukasiewicz fuzzy logic.

Theorem 3. *For any L-formula α, $\alpha \in$ Ł iff α is valid in every fuzzy model.*

Theorem 4. *For any $L^?$-formula φ, $\varphi \in$ Ł$^?$ iff φ is valid in every fuzzy model.*

If time allows I will discuss also the possibility to extend other fuzzy logics with the inquisitive disjunction.

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Strong Standard Completeness of **IUL** plus $\mathbf{t} \Leftrightarrow \mathbf{f}$ via a Structure Theorem for Finitely Generated Group-like FL_e-algebras à la Hahn

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Hahn's structure theorem [2] states that totally ordered Abelian groups can be embedded in the *lexicographic product* of *real groups*. Residuated lattices are semigroups only, and are algebraic counterparts of substructural logics [1]. Involutive commutative residuated chains (aka. involutive FL_e-chains) form an algebraic counterpart of the logic **IUL** [5]. The focus of our investigation is a subclass of them, called commutative *group-like* residuated chains, that is, totally ordered, involutive commutative residuated lattices such that the unit of the monoidal operation coincides with the constant that defines the involution. These algebras are algebraic counterparts of **IUL** plus $\mathbf{t} \Leftrightarrow \mathbf{f}$.

Group-like commutative residuated chains can be characterized as generalizations of totally ordered Abelian groups, hence their name, see Theorem 2. Thirdly, in quest for establishing a structural description for all commutative group-like residuated chains à la Hahn, so-called partial-lexicographic product constructions will be introduced. Roughly, only a cancellative subalgebra of a commutative group-like residuated chain is used as a first component of a lexicographic product, and the rest of the algebra is left unchanged. This results in group-like FL_e-algebras, see Theorem 1. The main theorem is about the structure of group-like FL_e-chains with a finite number of idempotents. Each such algebra is embeddable into a finite partial-lexicographic product of totally ordered Abelian groups, see Theorem 4. This result extends the famous structural description of totally ordered Abelian groups by Hahn, to, e.g, finitely generated group-like FL_e-chains. A corollary is the strong standard completeness of the logic **IUL** plus $\mathbf{t} \Leftrightarrow \mathbf{f}$.

Definition 1. *(Partial-lexicographic products)* Let $\mathbf{X} = (X, \wedge_X, \vee_X, *, \to_*, t_X, f_X)$ be a group-like FL_e-algebra and $\mathbf{Y} = (Y, \wedge_Y, \vee_Y, \star, \to_\star, t_Y, f_Y)$ be an involutive FL_e-algebra, with residual complement $'^*$ and $'^\star$, respectively. Add a top element $\top$ to Y, and extend $\star$ by $\top \star y = y \star \top = \top$ for $y \in Y \cup \{\top\}$, then add a bottom element $\perp$ to $Y \cup \{\top\}$, and extend $\star$ by $\perp \star y = y \star \perp = \perp$ for $y \in Y \cup \{\perp, \top\}$. Let $\mathbf{X}_1 = (X_1, \wedge_X, \vee_X, *, \to_*, t_X, f_X)$ be any cancellative subalgebra of $\mathbf{X}$. We define $\mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1}, \mathbf{Y}^{\perp\top})} = \left(X_{\Gamma(X_1, Y^{\perp\top})}, \leq, \circledast, \to_\circledast, (t_X, t_Y), (f_X, f_Y)\right)$, where $X_{\Gamma(X_1, Y^{\perp\top})} = (X_1 \times (Y \cup \{\perp, \top\})) \cup ((X \setminus X_1) \times \{\perp\})$, $\leq$ is the restriction of the lexicographic order of $\leq_X$ and $\leq_{Y \cup \{\perp, \top\}}$ to $X_{\Gamma(X_1, Y^{\perp\top})}$, $\circledast$ is defined coordinatewise, and the operation $\to_\circledast$ is given by $(x_1, y_1) \to_\circledast (x_2, y_2) = ((x_1, y_1) \circledast (x_2, y_2)')'$ where

$$(x, y)' = \begin{cases} (x'^*, y'^\star) & \text{if } x \in X_1 \\ (x'^*, \perp) & \text{if } x \notin X_1 \end{cases} .$$

Call $\mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1}, \mathbf{Y}^{\perp\top})}$ the *(type-I) partial-lexicographic product* of X, X_1, and Y, respectively.

Let $\mathbf{X} = (X, \leq_X, *, \to_*, t_X, f_X)$ be a group-like FL_e-chain, $\mathbf{Y} = (Y, \leq_Y, \star, \to_\star, t_Y, f_Y)$ be an involutive FL_e-algebra, with residual complement $'^*$ and $'^\star$, respectively. Add a top element $\top$ to

*This work was supported by the GINOP 2.3.2-15-2016-00022 grant.

Y, and extend $\star$ by $\top\star y = y\star\top = \top$ for $y \in Y\cup\{\top\}$. Further, let $\mathbf{X_1} = (X_1, \wedge, \vee, *, \to_*, t_X, f_X)$ be a cancellative, discrete, prime (that is, $(X \setminus X_1) * (X \setminus X_1) \subseteq X \setminus X_1$) subalgebra of $\mathbf{X}$. We define $\mathbf{X_{\Gamma(X_1,Y^\top)}} = \left(X_{\Gamma(X_1,Y^\top)}, \leq, \circledast, \to_\circledast, (t_X, t_Y), (f_X, f_Y)\right)$, where $X_{\Gamma(X_1,Y^\top)} = (X_1 \times (Y \cup \{\top\})) \cup ((X \setminus X_1) \times \{\top\})$, $\leq$ is the restriction of the lexicographic order of $\leq_X$ and $\leq_{Y\cup\{\top\}}$ to $X_{\Gamma(X_1,Y)}$, $\circledast$ is defined coordinatewise, and the operation $\to_\circledast$ is given by $(x_1, y_1) \to_\circledast (x_2, y_2) = ((x_1, y_1) \circledast (x_2, y_2)')'$ where

$$(x, y)' = \begin{cases} ((x'^*), \top) & \text{if } x \notin X_1 \text{ and } y = \top \\ (x'^*, y'^\star) & \text{if } x \in X_1 \text{ and } y \in Y \\ ((x'^*)_\downarrow, \top) & \text{if } x \in X_1 \text{ and } y = \top \end{cases} .$$

[1] Call $\mathbf{X_{\Gamma(X_1,Y^\top)}}$ the *(type-II) partial-lexicographic product* of X, X_1, and Y, respectively.

Theorem 1. *$\mathbf{X_{\Gamma(X_1,Y^{\perp\top})}}$ and $\mathbf{X_{\Gamma(X_1,Y^\top)}}$ are involutive FL$_e$-algebras. If $\mathbf{Y}$ is group-like then also $\mathbf{X_{\Gamma(X_1,Y^{\perp\top})}}$ and $\mathbf{X_{\Gamma(X_1,Y^\top)}}$ are group-like.*

Theorem 2. *For a group-like FL$_e$-algebra $(X, \wedge, \vee, \circledast, \to_\circledast, t, f)$ the following statements are equivalent: $(X, \wedge, \vee, \circledast, t)$ is a lattice-ordered Abelian group if and only if $\circledast$ is cancellative if and only if $x \to_\circledast x = t$ for all $x \in X$ if and only if the only idempotent element in the positive cone of X is t.*

Theorem 3. *Any order-dense group-like FL$_e$-chain which has only a finite number of idempotents can be built by iterating finitely many times the partial-lexicographic product constructions using only totally ordered groups, as building blocks. More formally, let $\mathbf{X}$ be an order-dense group-like FL$_e$-chain which has $n \in \mathbf{N}$ idempotents in its positive cone. Denote $I = \{\perp\top, \top\}$. For $i \in \{1, 2, \ldots, n\}$ there exist totally ordered Abelian groups $\mathbf{G}_i$, $\mathbf{H}_1 \leq \mathbf{G}_1$, $\mathbf{H}_i \leq \mathbf{\Gamma}(\mathbf{H}_{i-1}, \mathbf{G}_i)$ $(i \in \{2, \ldots, n-1\})$, and a binary sequence $\iota \in I^{\{2,\ldots,n\}}$ such that $\mathbf{X} \simeq \mathbf{X}_n$, where $\mathbf{X}_1 := \mathbf{G}_1$ and $\mathbf{X}_i := \mathbf{X}_{i-1\,\mathbf{\Gamma}(\mathbf{H}_{i-1}, \mathbf{G}_i{}^{\iota_i})}$ $(i \in \{2, \ldots, n\})$.*

We say that a group-like FL$_e$-chain is represented as a *finite* partial-lexicographic product of linearly ordered Abelian groups $\mathbf{G}_1 \ldots, \mathbf{G}_n$, if it arises via finitely many iterations of the type I and type II constructions using linearly ordered Abelian groups $\mathbf{G}_1 \ldots, \mathbf{G}_n$ in the way it is described in Theorem 3

Theorem 4. *Any group-like FL$_e$-chain, which has only a finite number of idempotents, can be embedded into the finite partial-lexicographic product of totally ordered Abelian groups.*

Lemma 1. *Any finitely generated group-like FL$_e$-chain has only a finite number of idempotents.*

Theorem 5. *The logic* **IUL** *extended by the axiom* $\mathbf{t} \Leftrightarrow \mathbf{f}$ *is strongly standard complete.*

References

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[1] $x_\downarrow = \begin{cases} u & \text{if there exists } u < x \text{ such that there is no element in } X \text{ between } u \text{ and } x, \\ x & \text{if for any } u < x \text{ there exists } v \in X \text{ such that } u < v < x \text{ holds.} \end{cases}$

On Partial Fuzzy Type Theory

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This paper is a study of fuzzy type theory (FTT) with partial functions. We introduce a special value "$*$" to all the types which represents "undefined". In the interpretation of FTT, this value lays outside of the corresponding domains. The value $*_o$ of type o is defined as the formula $\iota_{o(oo)} \cdot \lambda x_o \bot$ which means application of the description operator to the empty set. Similarly, the $*_\epsilon$ is defined as $\iota_{\epsilon(o\epsilon)} \cdot \lambda x_\epsilon *_o$, i.e., the description operator is applied to a fuzzy set on M_ϵ whose membership function is nowhere defined. For higher types we define

$$*_{\beta\alpha} \equiv \lambda x_\alpha *_\beta$$

which means that "undefined" is a nowhere defined function from the set M_α of type α to a set M_β of type β.

In the development of FTT with partial functions, we must be careful because the value "undefined" is a well formed formula. The outcome is that $T \vdash A_o \equiv *_o$ means that the formula A_o is in the theory T equal to "undefined". This cannot be true because otherwise A_o would have to be also undefined. Consequently, a formula A_o is defined if $T \vdash \neg(A_o \equiv *_o)$. We thus introduce two special predicates "?" (the given formula is undefined) and "!" (the given formula is defined) which can be extended to all types.

Important outcome of our approach is that the λ-conversion is preserved which makes our system of FTT very powerful. Among many results, we show that $T \vdash *_o$ implies that T is contradictory. We prove that any consistent theory of FTT with partial functions has a model. We can also include the theory presented in the papers [4, 5] as a special theory of partial FTT. The proposed extension of FTT works of all (so far considered) kinds of algebras of truth values.

References

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Acknowledgement This paper was supported by the grant 16-19170S of GAČR, Czech Republic.

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Trial and error mathematics: dialectical, *p*-dialectical, and *q*-dialectical systems

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Joint work with Amidei, Andrews, Pianigiani, and Sorbi.

Abstract. Formal systems represent mathematical theories in a rather static way, in which axioms of the represented theory have to be defined from the beginning, and no further modification is permitted. As is clear, this representation is not comprehensive of all aspects of real mathematical theories: for instance, when defining a new theory, a mathematician might choose axioms through some trial and error process, instead of fixing them, once for all, at the initial stage. One way of characterizating such cases is provided by the so-called experimental logics, firstly studied by Jeroslow in the 1970's [2], and also explored by Hajek [1]. Our approach is based on a different – yet related – notion, introduced by Magari [3] in the same period: dialectical systems.
The basic ingredients of a *dialectical system* are a number c, encoding a contradiction; a deduction operator H, that tells us how to derive consequences from a finite set of statements D; and a proposing function f, that proposes statements to be accepted or rejected as provisional theses of the system. We prove several results concerning dialectical systems and their expressiveness. Furthermore, we investigate two additional class of systems that enrich Magari's original proposal with a natural mechanism of revision: p-dialectical systems and q-dialectical systems.
We prove the following: dialectical, p-dialectical, and p-dialectical sets (i.e., the sets of statements that are eventually accepted by, respectively, dialectical, p-dialectical, and q-dialectical systems) are always Δ^0_2 sets; the three systems display the same computational power, in that dialectical, p-dialectical, and q-dialectical sets have the same Turing degrees (namely, the computably enumerable Turing degrees), and the same enumeration degrees (namely, the Π^0_1 enumeration degrees); yet, p-dialectical and q-dialectical sets form a class which is much larger than that of dialectical sets, since the first two inhabit each level of Ershov hierarchy, while dialectical sets are always ω-computably enumerable.
Finally we show that, if we restrict to the case of *systems with connectives*, i.e. systems in which the deduction operator H has to satisfy the rules of classical logic, then we obtain the following: if S is a system that does not derive the contradiction from the empty set of premises, then S is the *completion* of a given theory. We make use of this fact to study our systems as machines to build, in the limit, completions of first-order theories. We show that dialectical and q-dialectical completions coincide, while Peano Arithmetic has a p-dialectical completion which is neither a dialectical completion, nor a q-dialectical completion.

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The Current State of the Foundations of Set Theory

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Abstract

Set-theorists have for many years had a pretty good system of axioms for mathematics, the ZFC axioms. Nearly all of the theorems of mathematics can be translated into set theory and then shown to follow from the ZFC axioms. But Goedel's incompleteness theorem tells us that no system of axioms, not even ZFC, is really complete: there always are statements that can be neither proved nor disproved in any formal system. The most famous example for ZFC is Cantor's continuum hypothesis (CH), stating that any two uncountable sets of real numbers have the same cardinality.

Goedel conjectured that one might resolve this incompleteness problem by adding axioms of large infinity to ZFC, now called large cardinal axioms, in order to resolve many of the natural problems of set theory like CH. Goedel was only partly right: Many natural questions concerning nicely definable sets of reals are resolved by large cardinal axioms as well as virtually any question about the consistency (freedom from contradiction) of statements of set theory. But many questions, including CH, remain untouched by large cardinal axioms.

Is the incompleteness of ZFC relevant for mathematics? In other words, are there questions that are important for areas of mathematics other than logic which are undecidable in ZFC? There is evidence for a positive answer: the Whitehead problem (Abelian group theory), the Kaplansky Conjecture (Banach algebras), the existence of outer automorphisms of the Calkin algebra (C* algebras), the Borel Conjecture (measure theory) are all undecidable in ZFC. But some will regard these examples as disguised versions of questions in abstract set theory, lying outside of "core mathematics". Whether the mathematicians of the future will need axioms beyond ZFC to resolve questions at the heart of mathematics remains a fascinating open question.

However there is no doubt that set-theorists themselves must go beyond ZFC if they wish to resolve questions at the heart of set theory. This problem has been approached in two distinct ways, through "intrinsic" or "extrinsic" evidence for new axioms of set theory. The former makes use of principles concerning sets that result from our intuitive understanding of the concept; only recently has it been discovered that such principles can lead to new axioms which go far beyond ZFC. The latter has until now been based on the choice of axioms which best facilitate the mathematical development of the subject. A new proposal is to expand this to the choice of axioms which best resolve questions outside set theory, such as those mentioned above, which are known to be undecidable in ZFC.

The universal algorithm and the universal finite set

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Abstract

The universal algorithm is a Turing machine program e that can in principle enumerate any finite sequence of numbers, if run in the right model of PA, and furthermore, can always enumerate any desired extension of that sequence in a suitable end-extension of that model. The universal finite set is a Σ_2 definition in set theory that can in principle define any finite set, in the right model of set theory, and can always define any desired finite extension of that set in a suitable top-extension of that model. I shall give an account of both results and describe applications to the model theory of arithmetic and set theory.

Ekman's Paradox and the Quest for an Intensional Theory of Proofs

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Abstract

I plead to give intensional considerations in proof theory, in particular the question of identity of proofs, a stronger stance than it is often given. A crucial aspect is what can be counted as a proper step of proof reduction, as this is what constitutes the identity of proofs. I discuss in particular Ekman's paradox, which is an interesting example of a derivation in intuitionistic propositional logic, which is non-normalizable given a (prima facie) plausible extension of the standard notion of reduction. In spite of its simplicity, Ekman's paradox points at fundamental issues of proof theory and proof-theoretic semantics. (Joint work with Luca Tranchini)

Intuitionistic Provability Logic

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Abstract

One of the most striking features of Solovay's arithmetical completeness theorem for Löb's Logic is its great stability. As a first approximation: It holds for any classical Σ^0_1-sound theory that extends Elementary Arithmetic. (We stipulate that theories are equipped with a designated elementary axiom set.)

The situation is dramatically different when we consider constructive theories like Heyting Arithmetic. Different theories satisfy very different sets of principles. Solovay style completeness theorems are only known for very few constructive theories.

My talk will offer a scenic tour across the landscape of possible principles. I will discuss what is known about closed fragments. (We will see a surprising appearance of Gödel-Dummett Logic here.) Moreover, I will discuss the characterization of the Σ^0_1-provability logic of Heyting Arithmetic due to Mojtahedi and Ardeshir. I will also sketch the recent work of Jetze Zoethout that gives an arithmetical completeness theorem for a reasonably natural theory. Jetzes work also provides an alternative route to the Σ^0_1-provability logic of Heyting Arithmetic.

Tremendous beauty is hidden in this subject, but we have to work hard to make it visible.

What is the height of Gentzens reduction trees?

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Abstract

Gentzen's consistency proof of 1935 uses cut elimination in an infinitary calculus to obtain the consistency of Peano arithmetic. The cut elimination theorem is called the Hilfssatz and it is interesting because of two reasons:
(1) The cut elimination strategy applied there eliminates always an uppermost cut regardless of the complexity. This is in contrast to the commonly used cut elimination strategy, called Tait's strategy, that eliminates one of the most complex cuts.
(2) The proof of the Hilfssatz makes implicit use of transfinite induction up to the height of cut free infinitary derivations that have been already constructed.
If Gentzen had applied Tait's strategy in the Hilfssatz, he would have obtained transfinite induction up to ϵ_0. Based on the analysis of Gentzen's cut elimination strategy, we want to explain that Gentzen's original proof might require transfinite induction up to some ordinal that is bigger than ϵ_0. Another interesting question is whether and how both cut elimination strategies differ in finite calculi.

Homogeneous structures and a new poset

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Abstract

In 1991 Thomas conjectured that every homogeneous countable structure over a finite relational language has finitely many reducts. This has been solved for several individual structures, but we still don't know much in general. One of the intermediate steps would be to show that every structure which is a reduct of such a structure has finitely many *minimal* reducts. By Manuel Bodirsky, Michael Pinsker, and Todor Tsankov we know that the analogous statement is true for existentially positive reducts under some Ramsey assumptions. However we know that Thomas' conjecture is not true for existentially positive reducts in general. In order to get around this problem, in this talk I will introduce a new poset of structures giving rise to a possible generalization of Thomas' conjecture.

Equations in algebras induced by beautiful first-order structures

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Abstract

Every first-order structure gives rise to a general algebra via its *polymorphism clone*, which consists of all homomorphisms of finite powers of the structure into itself. This algebra can be examined from the viewpoint of universal algebra, and its equational structure sheds some light on the original first-order structure. We summarize recent results on algebras which are induced in this way by countable *omega*-categorical structures, and how their equational structure relates to the equational structure of finite algebras.

Hájek's Probability Logic and Its Two-sorted Algebraic Semantics

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Abstract

Petr Hájek together with his co-workers developed a two-tier modal calculus for reasoning about probabilistic uncertainty. The lower syntactical layer is interpreted as a logic for inferring statements about events. Łukasiewicz logic, which is employed on the upper level, makes it possible to express probabilities of formulas directly by truth-degrees of the formulas "probably ϕ". We will mention the original Kripke-style semantics and then we introduce a two-sorted algebraic semantics, which will be further developed in the follow-up talk by V. Marra. Various completeness results (standard, finite) with respect to different models of Hájek's logic will be discussed.

Universal expectations, generic measure spaces, and Hájek's Probability Logic

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Abstract

Beginning where T. Kroupa's talk left us, we have (i) Hájek's Probability Logic, a tool to reason about expected values of bounded real random variables, and (ii) a two-sorted algebraisation of this logic. I attempt to sketch how (i) and (ii) fit into a broader current research programme aimed at algebraising probability and measure. One of the reasons why I pursue this programme (together with T. Kroupa, further collaborators, and students) is because I would like to know the answers to questions such as the following two. (a) Given a Boolean algebra, is there a universal (=most general) way of assigning a probability degree to its elements? (b) Does there exist a probability space that is generic for the class of all probability spaces, in the same sense that the two-element Boolean algebra is generic for the class of all Boolean algebras? The questions are hand-wavy in this abstract, but will be made precise in the talk. Then the answers are: (a) Yes; (b) Yes. The further implied questions — what do universal expectations and generic probability spaces look like? — can be answered, too: (a) requires considerable involvement with affine representations in the style of Choquet, and is best left to the talk; as to (b), the Cantor space equipped with a specific rational-valued probability measure turns out to be generic. In terms of Hájek's seminal ideas on Probability Logic, the universal expectations of (a) are just the Lindenbaum-Tarski algebras of his logic (relative to a theory in the sort of events); and a generic probability space as in (b) is a single model with respect to which his logic is complete.

First-Order Fuzzy Modal Logics with Variable Domains

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In modal discourse of various kinds (alethic, temporal, doxastic, deontic, etc.) it is often appropriate to regard some individuals as existing only in some (or even none) of the possible worlds: for instance, the individual called Socrates existed in some of the past world-times, but not in the world-times before his birth or after his death; the golden mountain, being possible, does exist in some possible world(s), though not in the actual world; and the largest natural number, being contradictory, is commonly modeled as not existing in any world. In other words, in predicate modal logic it is quite natural to consider different domains of individuals for different possible worlds. However, in normal predicate modal logics it is easy to prove for any term t that $\vdash \Box(\exists x)(x = t)$, i.e., that the referent of t necessarily exists in every possible world, contrary to our starting point. A natural solution is to employ quantification principles of some member of the family of *free logics,* or logics 'free of existential assumptions' [Nol07, Pos07, Gar01]. Free logics permit to block some classically valid inferences that would lead to the formula above.

In recent decades, modal logics have been studied not just in classical bivalent settings, but also in gradual settings over suitable fuzzy logics; an important early contribution to fuzzy modal logic was made by Hájek in [Háj98, Sect. 8.3]. To handle non-existing individuals analogously not just in bivalent, but also in gradual contexts, free variants of predicate fuzzy logics are needed. The first variant of free fuzzy logic has recently been proposed by the present authors [BD18]. Here we focus on its application in predicate fuzzy modal logic with variable domains of individuals.

As argued in [BD18], a reasonable choice for free fuzzy logic is the fuzzification of *positive* free logic, in which empty-termed formulas can be true (to a degree) or truth-valueless. To accommodate truth value gaps, a simple system L^* of partial fuzzy logic proposed in [BN15, BD16] is employed (further options are left for future work). In L^* (definable over any $\triangle$-core fuzzy logic L), truth-value gaps are represented by an 'error code' $*$ for an undefined truth degree. Propositional connectives and quantifiers in formulas are marked by the manner in which the error code truth-functionally propagates from subformulas (e.g., Bochvar-style for a fatal error, Sobociński-style for an ignorable error, and Kleene-style for an overridable error); auxiliary defined connectives provide the means to explicitly express the definability preconditions of valid inference rules.

The free fuzzy logic of [BD18] employs the *dual-domain* semantics, in which each first-order model is equipped with a crisp non-empty *outer* domain D_0, and an *inner* domain D_1, which is a crisp or fuzzy subset of D_0. All terms have (possibly dummy) referents in D_0, while D_1 collects existent individuals. Predicate and function symbols are interpreted over the outer domain, with truth-value gaps allowed. The usual ('inner') quantifiers range over D_1. The 'outer' quantifiers (over D_0), useful for formalizing such propositions as "some things do not exist", behave as in the usual (non-free) fuzzy logic; the inner quantifiers can be defined by restricting the outer ones (by Kleene connectives) to the inner domain. The resulting apparatus enables making the existence preconditions of inference rules explicit.

In our talk we will present a semantics for predicate fuzzy modal logic with variable domains, based on the free fuzzy logic of [BD18]. Similarly to the latter, each possible world $w \in W$ in a fuzzy Kripke frame comes equipped with an inner domain D_1^w, comprising individuals that exist in w. Furthermore, there is a common outer domain $D_0 \supseteq \bigcup_{w \in W} D_1^w$, which enables formulating statements about objects that exist in only some (or even none) of the possible worlds. The language and evaluation of formulas are defined in a straightforward manner, combining the Tarski conditions of non-modal free fuzzy logic and modal fuzzy logic with constant domains; the modification works for a broad class of underlying fuzzy modal logics. We will give the initial observations on the proposed semantics and discuss the (in)validity of important quantified modal formulas in resulting fuzzy modal logics. Finally we will outline its further development.

Acknowledgment: The work is supported by projects 16–19170S of GA ČR and LQ1602 of MŠMT ČR (NPU II).

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Hájek-style modalities in fuzzy intensional semantics

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Kripke-style fuzzy modal logic with a fuzzy accessibility relation notoriously invalidates the modal axiom K, due to the failure of the rule of contraction in most (t-norm based) fuzzy logics. While the failure of K may be desirable in some fuzzy modal logics (for example epistemic, where K entails logical omniscience), in others it is commonly seen as problematic.

In [3, §8.3], Hájek sketched a hierarchy of fuzzy Kripke-style modal operators $\Box_n, \Diamond_n$ for each $n \in \mathbb{N}$, defined w.r.t. n-times iterated self-intersection of the accessibility relation, i.e., with the following Tarski conditions in a fuzzy Kripke frame (W, R, L):

$$\|\Box_n\varphi\|_w = \bigwedge_{w' \in W} \big(R^n ww' \rightarrow_L \|\varphi\|_{w'}\big), \qquad \|\Diamond_n\varphi\|_w = \bigvee_{w' \in W} \big(R^n ww' \,\&_L \|\varphi\|_{w'}\big),$$

where L is a BL-algebra, $R\colon W^2 \to L$, and $R^n ww' = Rww' \,\&\, \dots \,\&\, Rww'$ (n times). Rather than to failure, this definition leads to a contraction-sensitive variant of K, namely $\Box_n(\varphi \to \psi) \to (\Box_m\varphi \to \Box_{n+m}\psi)$. Some other modal axioms that fail in simple Kripke-style fuzzy modal logics receive a multiplicity-sensitive variants, too.

In the talk we will elaborate Hájek's sketched idea in the systematic framework of fuzzy intensional semantics, developed by the present authors (full paper in progress). The formal semantics consists in a suitable translation of modal formulae into Russell-style higher-order fuzzy logic (a variant of which was introduced in [1]). As a case study demonstrating the applicability of the formal framework, we will slightly extend Hájek's results on $\Box_n, \Diamond_n$ in several directions and discuss the significance of his approach to fuzzy modalities. Furthermore, we will discuss the envisaged applications of the apparatus in further areas of intensional fuzzy logic (incl. probabilistic, epistemic, counterfactual [2], or non-monotonic reasoning in both classical and fuzzy settings).

Acknowledgment: The first author was supported by projects 16–19170S of GA ČR and LQ1602 of MŠMT ČR (NPU II).

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Reasoning about weighted graphs in many-valued modal logic

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Summary. Classical Kripke frames are (directed) graphs, so it is not surprising that classical modal logic has been suggested as a natural formalism for reasoning about graphs [1, 5, 6]. In this talk we propose *many-valued* modal logics as a natural formalism for reasoning about *weighted* graphs. We introduce a family of many-valued modal logics suitable for formalizing reasoning about weighted graphs and prove completeness for some of these logics.

Weighted graphs and graph logics. Let $\boldsymbol{A}$ be a complete FL_{ew} algebra (with bounds $0, 1$) and let Lab be a countable set of labels. A *labeled* $\boldsymbol{A}$*-weighted directed semi-simple graph* is a pair $\mathfrak{G} = \langle V, E, f \rangle$ where V is a non-empty set (of vertexes), E is a function from $V \times V$ to $\boldsymbol{A}$ (the $\boldsymbol{A}$-weighted edge function) and $f : V \times \mathsf{Lab} \to \{0, 1\}$ (the labeling function).

Intuitively, $\boldsymbol{A}$ is seen as an algebra of *distances* and $E(v, v')$ is the distance between v and v'. The element 1 represents the smallest possible distance ("zero distance") and 0 represents the largest possible distance ("infinite distance"). While $\wedge$ and $\vee$ represent the infimum (largest distance) and supremum (smallest distance), $\odot$ is a fusion (merge) operation on distances used when calculating the result of "adding distances".

Let $\mathcal{L} = \left\{\wedge, \vee, \odot, \to, \bar{1}, \bar{0}, \Diamond, \Box\right\}$ and let $\boldsymbol{Fm}_{\mathcal{L}}$ be the absolutely free $\mathcal{L}$-algebra generated by Lab. Elements of this algebra are called $\mathcal{L}$-formulas; φ, ψ etc. range over $\mathcal{L}$-formulas, α, β range over $\mathcal{L}$-formulas without occurrences of $\Box, \Diamond$, and Γ, Δ etc. range over sets of $\mathcal{L}$-formulas.

For every $v \in V$, the labeling function f induces a function $f_v : \boldsymbol{Fm}_{\mathcal{L}} \to \boldsymbol{A}$ satisfying

$$
\begin{aligned}
f_v(\bar{c}) &= c && \text{for } c \in \{0, 1\} \\
f_v(\varphi \circ \psi) &= f_v(\varphi) \circ^{\boldsymbol{A}} f_v(\psi) && \text{for } \circ \in \{\wedge, \vee, \odot, \to\} \\
f_v(\Box\varphi) &= \bigwedge_{w \in V} \left\{E(v, w) \to^{\boldsymbol{A}} f_w(\varphi)\right\} \\
f_v(\Diamond\varphi) &= \bigvee_{w \in V} \left\{E(v, w) \odot^{\boldsymbol{A}} f_w(\varphi)\right\}
\end{aligned}
$$

We will sometimes write $v(\varphi)$ instead of $f_v(\varphi)$.

Given $\boldsymbol{A}$, a formula φ is a *global* $\boldsymbol{A}$*-consequence* of a set of formulas Γ (notation $\Gamma \vdash^g_{\boldsymbol{A}} \varphi$) iff, for every $\boldsymbol{A}$-weighted labeled graph $\mathfrak{G}$, if $v[\Gamma] = \{1\}$ for every $v \in \mathfrak{G}$, then $v(\varphi) = 1$ for every $v \in \mathfrak{G}$. A formula φ is a *local* $\boldsymbol{A}$*-consequence* of Γ (notation $\Gamma \vdash^l_{\boldsymbol{A}} \varphi$) iff, for every $\boldsymbol{A}$-weighted labeled graph $\mathfrak{G}$ and every $v \in \mathfrak{G}$, if $v[\Gamma] = \{1\}$, then $v(\varphi) = 1$. Importantly, neither of these consequence relations is *structural*, i.e. closed under arbitrary substitutions. For example, $p \vee \neg p$ follows from the empty set, but $\Box p \vee \neg \Box p$ does not.

Expressiveness. The set $\{E(v,u) \odot u(p)\,;\, u \in V\}$ contains $E(v,w)$ for all w such that $w(p) = 1$ and 0 if there is u such that $u(p) = 0$. This means that $v(\Diamond p)$ is the smallest distance from v to a vertex labeled with p (0 if there is no such vertex). We may call this the *minimal cost of (reaching) p in v*. As a special case, $v(\Diamond \bar{1})$ is the distance from v to the closest vertex.

It is clear that $1 \leq v(\Diamond p) \to v(\Diamond q)$ iff $v(\Diamond p) \leq v(\Diamond q)$. So, $v(\Diamond p \to \Diamond q) = 1$ means that the smallest distance (from v) to a vertex labeled by p is at least as big as the smallest distance to a vertex labeled by q; in other words, "q is at least as close as p".

It is remarkable that, in some sense, the Diamond operator is now the main one: while $\Diamond p$ will be evaluated as a supremum of values of the algebra, the value of $\Box p$ can only be evaluated to (infima of) negated values of $\boldsymbol{A}$, and so, in many cases, while $\Diamond p$ can indeed take any value, $\Box p$ will be limited to the negated elements of $\boldsymbol{A}$. This fact is not extended to arbitrary formulas (that is to say, $\Box\varphi$ is non longer limited to the negated values of the algebra), but nevertheless let us show some results on the partial interdefinability of $\Box$ from $\Diamond$ that support the previous idea.

Proposition. *The following formulas are valid in all $\boldsymbol{A}$-weighted graphs:*

- $\Box^n \alpha \leftrightarrow \neg\Diamond^n\neg\alpha$,
- $\Box(\varphi \to \Box^n\alpha) \leftrightarrow \neg\Diamond(\varphi \odot \Diamond^n\neg\alpha)$

Axiomatization. Axiomatizations of the local and global consequence relations over $\boldsymbol{A}$-weighted directed graphs are straightforward in cases where the axiomatization of the $\boldsymbol{A}$-valued modal logic is known. This amounts simply to defining a two-layered axiomatic system in the line of [4]. Formally, the logic of all directed $\boldsymbol{A}$-weighted labeled graphs is complete with respect to the axiomatic system $W_{\boldsymbol{A}}$ defined by:

- An axiomatic system for the modal logic of $\boldsymbol{A}$-valued Kripke models
- Axioms of Classical Logic for formulas without modalities

This provides us with axiomatic systems for the logic of weighted graphs over the standard Gödel algebra (using [3]), and over arbitrary finite residuated lattices (by means of the axiomatization presented in [2]).

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On Hájek's (half-)forgotten treasures

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Petr Hájek was the founder and main developer of Mathematical Fuzzy Logic (MFL). In his 1998 book [1] or in early MFL papers he opened numerous lines of research. In the following years he and his colleagues deeply studied many of these lines. It this talk we survey some of his ideas which perhaps have not received as much attention as they deserve, but have been recently revived and are being currently developed by his disciples and friends.

We will focus on logical models of reasoning with vagueness and uncertainty, in particular on various forms of propositional modal logics (with two-layer formalism, S5-like modalities, indexed modalities, etc.), fragments of first-order systems (monadic and description logics), theories in full first-order logics (lattice-valued set theory, Cantor-Łukasiewicz set theory, weak fuzzy arithmetics) and finally expansions of the usual first-order fuzzy logics with generalized quantifiers.

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