Topology

Coming together to make Grothendieck's generalized topology

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Grothendieck made a wonderful discovery of a generalization of topological spaces, namely toposes. It encompasses examples he needed from algebraic geometry, but also many that are easier to understand, such as the generalized spaces of sets or of groups. They bring a fresh perspective on many apparently non-negotiable features of mathematical reality. For example, they suggest that the difference between sets and proper classes is not so much one of size, as of topology. They also suggest that continuity is, ultimately, a logical phenomenon: a map is continuous if it can be defined within the constraints of a so-called "geometric" logic.

Making mathematical sense of toposes brings together algebra, categories and logic very intimately. However, those subjects get modified in ways that will feel... a little weird. Lindenbaum algebras become categories; partial, many sorted algebraic theories become too important to neglect; logic needs some infinitary connectives; and, above all else, constructive reasoning frequently becomes essential, rejecting excluded middle and choice.

The aim of this tutorial is to give some feel for where and why these modifications start to take place, where it becomes insufficient just to elucidate the classical details of the old approaches.

Sheaves: continuous set-valued maps

This first session looks at sheaves over topological spaces, as they are at the heart of topos theory. There are two equivalent definition - as presheaves with pasting or as local homeomorphisms -, and we shall focus more particularly on the latter.

It is in local homeomorphisms that we can see most clearly the idea of continuous set-valued map - base point maps to fibre, or stalk. This is a first clue that there might somehow be a "space of sets", even though it is not a topological space in the conventional sense. (It is an important example of topos, known as the "object classifier".)

The local homeomorphisms also bring out a clear distinction between the "geometric" constructions that work fibrewise, and other constructions that use germs and are infected by the behaviour on neighbouring fibres. The geometric constructions for sheaves boil down to finite limits and arbitrary colimits, and these are analogous to the finite intersections and arbitrary unions of opens in topology.

Theories and models

The second session looks at the categorical approach to logic, for many-sorted, first-order theories. We follow the approach described in Johnstone's "Sketches of an Elephant", vol. 2, section D1.

A signature will specify sorts, functions and predicates, and from these can be built terms and formulae.

A subtle feature that might feel a little weird is that we do not assume a countable, pre-given stock of free variables. Instead, finite sets of free variables are declared ad-hoc as "contexts" whenever they are needed. An immediate benefit of this is that, by allowing an empty context, it deals correctly with empty carriers.

The semantics of sorts, terms and formulae is as sets, functions and subsets, but an important next step is that this can be generalized to semantics in any category, as objects, morphisms and subobjects, provided it has the structure needed to interpret the logical connectives being used.

Given the signature, a theory will then be presented as a set of sequents. This is unnecessary if the logic already allows implication and universal quantification as connectives - we might as well present the theory with formulae. But we are interested in weaker logics such as - a prime example - geometric logic.

In the special case of the propositional fragment (no sorts) we see how geometric theories can be used to describe topological spaces, so that the models of the theory are the points of the space.

This leads to the following idea. Ordinary "point-set" topology describes a space by first saying what its points are, as a set, and then adding extra structure (the lattice of opens) to describe the topology. The "point-free" approach describes the points and topology all in one indecomposable step as a geometric theory. The points are the models, the opens are the formulae; the finite intersections and arbitrary unions of opens come from the connectives in the logic. In the next two sessions we shall see how to generalize this to predicate theories, and make a mathematical object - the classifying topos - that represents the theory in a presentation-independent way.

Classifying categories

The third session examines how to make a category out of a theory. You might expect to use a category of models, but in fact this doesn't work in general. A fundamental problem is that is that geometric theories are incomplete in general, so there may be too few models in any given category such as *Set*. Instead we look for a presentation-independent way to describe "models in arbitrary categories". The answer in the end is the classifying topos of a geometric theory, but this is an extreme example of a more general notion of "classifying category". Understanding these will help to put the classifying toposes in context.

One example that is already familiar is for propositional theories. Then the classifying category is just a poset, and is the Lindenbaum algebra. For a geometric theory this will be a frame, as in locale theory. (See Johnstone's "Stone Spaces".)

Other simple examples come from algebraic theories such as that of groups. The corresponding classifying category is the Lawvere theory, and this can be related easily to other presentation-independent forms such as abstract clones and monads. It is the finite-product-category freely generated by a generic group. Its universal property is that models of the algebraic theory in any finite-product-category are equivalent to finite product preserving functors from the Lawvere theory.

The same principle can be be applied with more complicated logics. It shows how to characterize the classifying topos by a universal property, although the concrete construction gets too complicated to be related to clones or monads.

For finitary logics, it is useful to know classifying categories can be constructed by universal algebra, freely generated by the generic model understood as generators and relations. This comes out of an initial model theorem for cartesian (essentially algebraic) theories, like algebraic ones but using finite limits instead of finite products. (See Palmgren and Vickers "Partial Horn logic and cartesian categories".)

For geometric theories, which are infinitary, we can still specify the classifying topos by a universal characterization, but the methods of universal algebra cannot be used to construct it. However, it turns out that it is still possible to use ad hoc methods, of presheaves and sheaves.

An interesting phenomenon, currently being explored, is that there is a finitary logic of "arithmetic universes", strong enough to include the real line, whose classifying categories are good approximations to the classifying toposes.

Toposes and geometric reasoning

In the fourth session I shall describe how we can exploit the classifying toposes once we have them.

What the universal characterization tells us is that the classifying topos for a theory T is the "geometric mathematics freely generated by a generic model of T". To perform constructions in the classifying topos, we declare "Let M be a model of T", and then, within the scope of that declaration, work within the constructive constraints of geometric mathematics. In particular, this means we must avoid using choice or excluded middle.

In that context, if you construct a model of another geometric theory, then you have constructed a continuous map between the two spaces (a geometric morphism between the classifying toposes). Hence points can still be validly used for these point-free spaces, and continuity becomes a logical issue: reasoning must be geometric.

Going further, we can use geometric mathematics to construct a space out of a generic point. This turns out to construct a bundle, the fibres being the spaces constructed when the generic construction is applied to specific points.