

ACL T: Algebra, Categories, Logic in Topology  
- Grothendieck's generalized topological spaces (toposes)

Steve Vickers  
CS Theory Group  
Birmingham

### 3. Classifying categories

Maths generated by a generic model

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Geometric theories may be incomplete

- not enough models in **Set**
- category of models in **Set** doesn't fully describe theory

Classifying category - e.g. Lawvere theory

= stuff freely generated by generic model

- there's a universal characterization of what this means

For finitary logics, can use universal algebra

- theory presents category (of appropriate kind) by generators and relations

For geometric logic, classifying topos is constructed by more ad hoc methods.

#### Outline of course

1. Sheaves: Continuous set-valued maps

2. Theories and models: Categorical approach to many-sorted first-order theories.

3. Classifying categories: Maths generated by a generic model

4. Toposes and geometric reasoning: How to "do generalized topology".

# Presentation independence

Theory is really theory presentation

- start from its ingredients (sorts, functions, predicates, axioms)
- can generate much more (terms, formulae, valid sequents)
- different presentations can generate essentially the same stuff

e.g. can rebrand -



up to isomorphism

- sort lists as sorts with projection functions
- predicates as sorts with inclusion functions
- functions as predicates (the graphs) with axioms for functionhood
- axioms as sequents derivable from other axioms

Two questions

1. How can we describe the abstract essence of a theory

- independent of how it was presented?

2. Can we define maps (model transformers) from one theory to another

- in a way that doesn't depend on presentation?

# Example

T<sub>1</sub> has - one sort  $\sigma$

- a unary predicate P

- model M = object M( $\sigma$ ) equipped with designated subobject M(P)

T<sub>2</sub> has - two sorts  $\sigma, \tau$

- a unary function  $i: \tau \rightarrow \sigma$

- an axiom to force i to be monic

Exercise: How does axiom work?

$$i(x) =_{\sigma} i(y) \vdash_{x,y:\tau} x =_{\tau} y$$

- model N = two objects N( $\sigma$ ), N( $\tau$ ) with monic N(i): N( $\tau$ )  $\rightarrow$  N( $\sigma$ )

Same models, despite totally different signatures and axioms

Same homomorphisms too - categories of models are equivalent

Two questions

1. How can we describe the abstract essence of a theory
  - independent of how it was presented?
2. Can we define maps (model transformers) from one theory to another
  - in a way that doesn't depend on presentation?

## Category of models?

No -

Relies on *completeness* - enough models

Geometric logic is incomplete

Two questions

1. How can we describe the abstract essence of a theory
  - independent of how it was presented?
2. Can we define maps (model transformers) from one theory to another
  - in a way that doesn't depend on presentation?

## Classifying category

*Everything* derivable from presentation

Derived formulae and sequents

Derived sorts

Derived functions

Derivations apply to models in categories other than Set

# Presentation independence: Propositional case

## Lindenbaum algebras

Lindenbaum algebra for theory  
= formulae modulo logical equivalence

Everything derivable from presentation

Derived formulae and sequents

look for models in arbitrary poset  $A$   
with sufficient structure

- to interpret connectives
- and validate logical rules

~~Derived sorts~~

~~Derived functions~~

Derivations apply to models in categories other than Set

posets

$\{0, 1\}$

Same process as before:

- Interpretation  $I$  interprets propositional symbols as elements of  $A$
- $I$  extends to arbitrary formulae using structure of  $A$
- For a model, the sequents  $\phi \vdash \psi$  must be satisfied:
- require  $I(\phi) \leq I(\psi)$

Example: classical logic uses Boolean algebras

Connectives - conjunction, disjunction, negation  
+ logical rules - including excluded middle  
=> algebra of Boolean algebras

Given a signature (propositional symbols):  
- formula = term in Boolean algebra formed from those symbols



## Example: geometric logic uses *frames*

Connectives - finite conjunction, arbitrary disjunction  
+ logical rules - not including excluded middle  
=> algebra of Boolean algebras

Frame = lattice with finite meets, arbitrary joins  
+ meet distributes over all joins

hence a complete lattice

Frame homomorphism preserves finite meets, all joins

Frames as objects are the same as complete Heyting algebras!

But frame homomorphisms only preserve the geometric structure  
- not the Heyting arrow or negation

# Universal property of Lindenbaum algebra $L(T)$

construction - formulae modulo equivalence

derived from signature  
- using logical connectives

derived from sequents  
- using logical rules

specification - has universal (generic) model

interpret each propositional symbol  $P$  as its equivalence class in  $L(T)$

sequents all satisfied in  $L(T)$  - by construction  
- so get a model in  $L(T)$   
- generic model

# Universal property of Lindenbaum algebra $L(T)$

let  $A$  be a poset of appropriate kind e.g. Boolean algebra for classical logic

let  $M$  be a model of  $T$  in  $A$  frame for geometric

Then there is a unique homomorphism  $M': L(T) \rightarrow A$  of appropriate kind  
that transforms generic model to (specific model)  $M$



## Proof sketch

For each prop. symbol  $P$ , must have  $M'([P]) = I_M(P)$

For homomorphism  $M'$ , must have  $M'([\phi]) = I_M(\phi)$

Well defined, because  $M$  a model

- hence if  $\phi, \psi$  equivalent then  $I_M(\phi) = I_M(\psi)$

$M'$  is a homomorphism, because equivalence is algebraic congruence.

Models of  $T$  in  $A$  are equivalent to homomorphisms  $L(T) \rightarrow A$

# Universal algebra

Universal property says:

$L(T)$  is algebra presented by generators and relations

generators  $G$  = propositional symbols from signature

relations  $R$  = axioms from theory

- in equational form

-  $\phi \vdash \psi$  becomes relation  $\phi \vee \psi = \psi$  (i.e.  $\phi \leq \psi$ )

e.g. for classical logic (Boolean algebras)

$$L(T) = BA\langle G|R \rangle$$

Boolean algebra presented by generators  $G$  subject to relations  $R$

# Universal algebra - for geometric logic (frames)

$$L(T) = \text{Fr}\langle G|T \rangle$$

- also written  $\Omega[T]$       **see why later**

Care needed to show it exists in geometric case

Formulae form a proper class

- because of unbounded infinitary disjunctions

"formulae modulo equivalence" is problematic

logical rules (idempotence, distributivity)

=> every formula equivalent to disjunction of finite conjunctions

=> get set of equivalence classes

=> can construct frame  $\text{Fr}\langle G|R \rangle$  with correct universal property for

Lindenbaum algebra

# Presentations = propositional geometric theories

Algebra

Logic

generators

G

signature  
- propositional symbols

relations

R

axioms

$$\bigwedge_i a_i \leq \bigvee_j \bigwedge_R b_{jR}$$

$$\bigwedge_i a_i \vdash \bigvee_j \bigwedge_R b_{jR}$$

presentation

$T = (G, R)$

theory (signature, axioms)

frame presented

$\Omega[T] =$   
 $\text{Fr}\langle G|R \rangle$

Lindenbaum algebra  
(formulae modulo equivalence)

connectives: finite conjunction, arbitrary disjunction

# Universal property of $\Omega[T] = \text{Fr}\langle G|R \rangle$

For any frame  $A$ , and for any -

Algebra

Function  $f: G \rightarrow A$   
respecting the relations  $R$

Logic

Model of  $T$  in  $A$

there is a unique frame homomorphism  $f': \text{Fr}\langle G|R \rangle \rightarrow A$   
that agrees with  $f$  on generators  $G$

Locales: write  $[T]$  for locale with  $\Omega[T] = \text{Fr}\langle G|R \rangle$   
For any locale  $X$ ,  
maps  $f: X \rightarrow [T]$  in bijection with  
models of  $T$  in  $\Omega X$  - models of  $T$  "at  $X$ "  
Points of  $[T] =$  models of  $T$

Easier to see when  $X = 1$ ,  
 $A = \Omega X = P(X)$   
 $= \{\text{truth values}\}$

# Sequents v. congruence

logic -- algebra

Congruence on algebraic terms

- represents logical equivalence of formulae

Logical equivalence

= both sequents can be derived  
- from axioms of theory  
+ rules of logic

$$\phi \dashv\vdash \psi$$

$$\phi \vdash \psi$$

$$\psi \vdash \phi$$

Sequents correspond to partial order in algebra

$$\phi \leq \psi$$

if (equivalently)

$$\phi = \phi \wedge \psi \quad \text{or} \quad \phi \vee \psi = \psi \quad \text{or} \quad \top = \phi \rightarrow \psi$$



# Classifying categories = categorical Lindenbaum algebras

How to do the same for first-order theories?

Again, must understand models in categories other than Set

Look for category with generic model

- and appropriate universal property

more important than knowing  
how to construct it

Structure needed in category (limits? colimits? images? etc.)

- depends on logic being used

For geometric logic use Grothendieck toposes

# Models in categories (other than Set)

Interpret:

sorts ... objects (the carriers)

sort lists ... products of carriers

function symbols, terms in context ... morphisms

predicate symbols, formulae in context ... subobjects (monics)

~~X~~ sort  $\sigma$   
 sort list  $\vec{\sigma}$   
 context  $\vec{x} : \vec{\sigma}$

~~X~~ function symbol  
 $f : \vec{\sigma} \rightarrow \tau$

~~X~~ predicate  
 $P \hookrightarrow \vec{\sigma}$

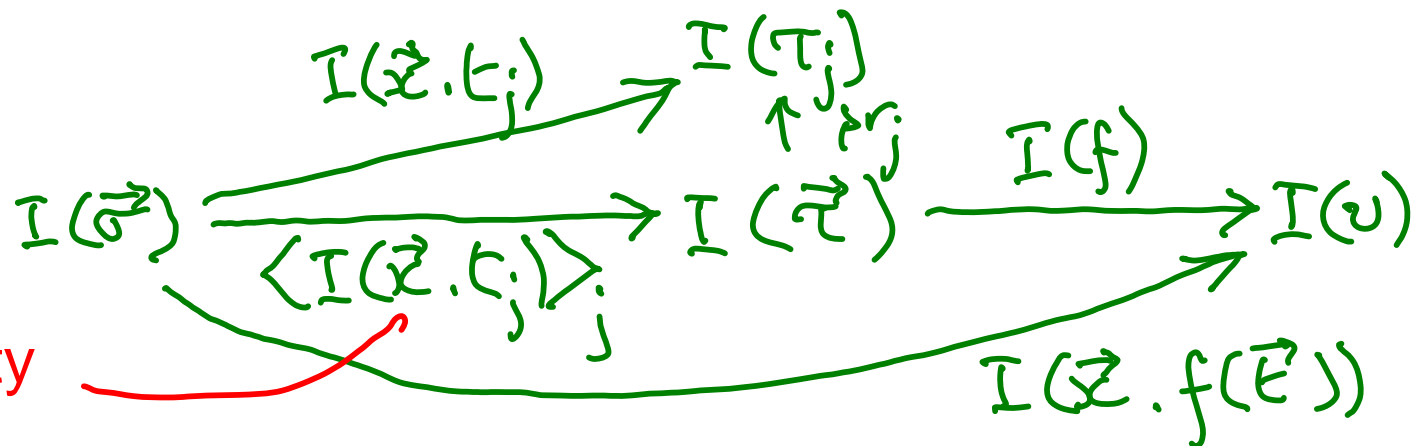
as set  $I(\sigma)$  - carrier  
 $I(\vec{\sigma})$  } product of carriers  
 $I(\vec{x})$  }  $I(\sigma_1) \times \dots \times I(\sigma_n)$

function  
 $I(f) : I(\vec{\sigma}) \rightarrow I(\tau)$

subset  
 $I(P) \subseteq I(\vec{\sigma})$

Use categorical structure to analyse what is needed to interpret logic

To interpret terms: need finite products

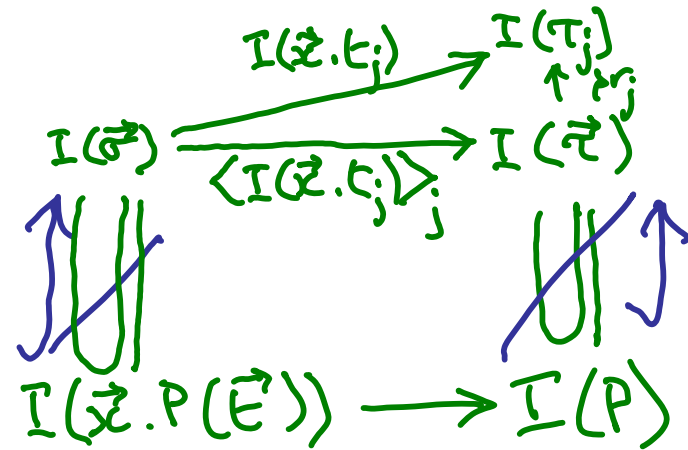


tupling uses  
 product property

# To interpret formulae

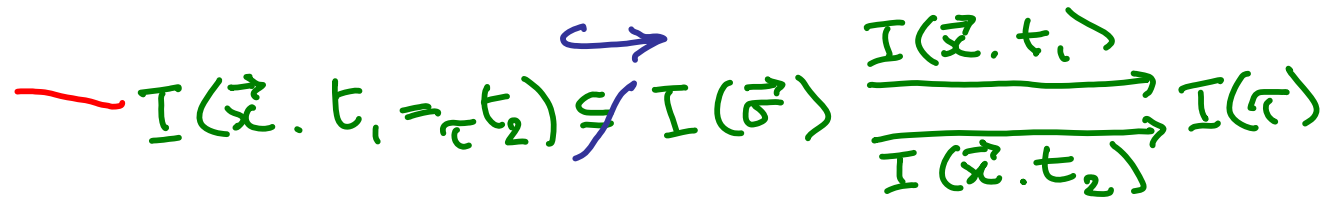
Finite limits are enough to get  
 - predicate symbol applied to

inverse image is pullback



- equations

equalizer



- conjunction

intersection is pullback

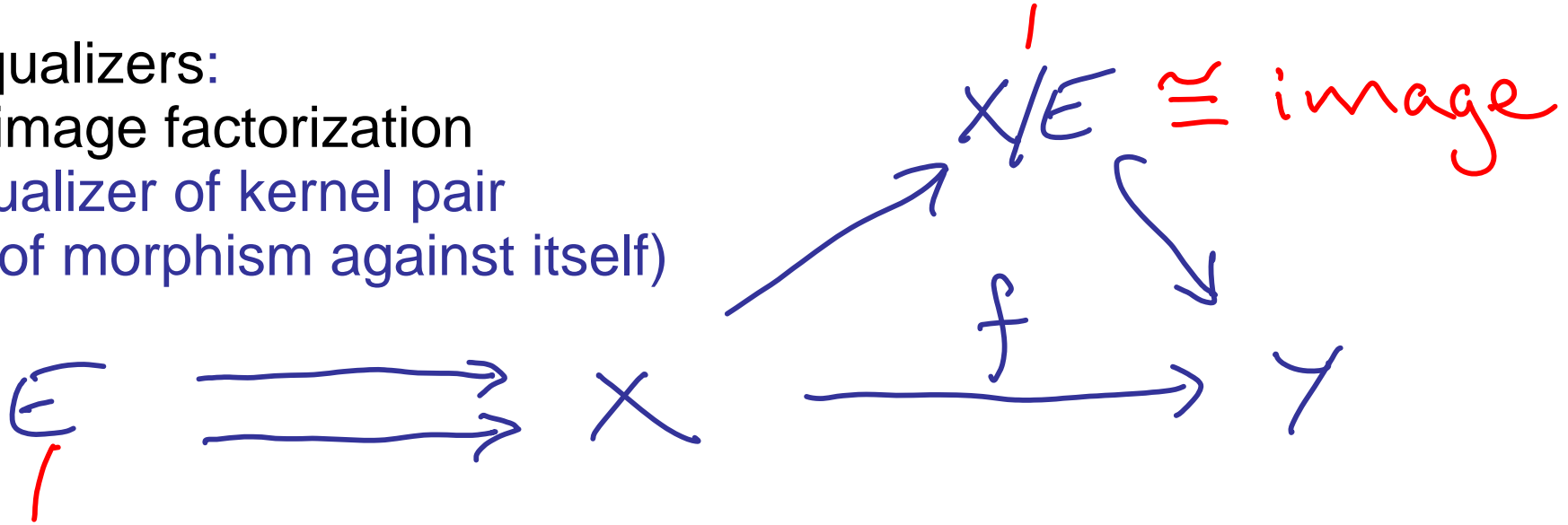


# In addition to finite limits

quotient = coequalizer

With coequalizers:

- can get image factorization
- as coequalizer of kernel pair (pullback of morphism against itself)

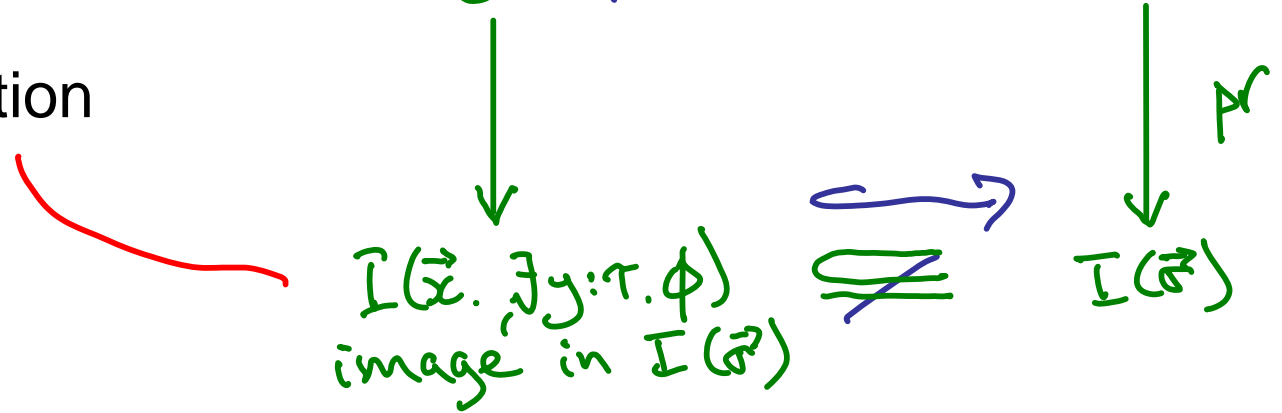


$E =$  equivalence relation for  $f$ .  $x E x'$  if  $f(x) = f(x')$

$E =$  kernel pair for  $f =$  pullback of  $f$  against itself

e.g.  $I(\vec{x}, y, \phi) \cong I(\vec{\sigma}, \tau) = I(\vec{\sigma}) \times I(\tau)$

Get existential quantification

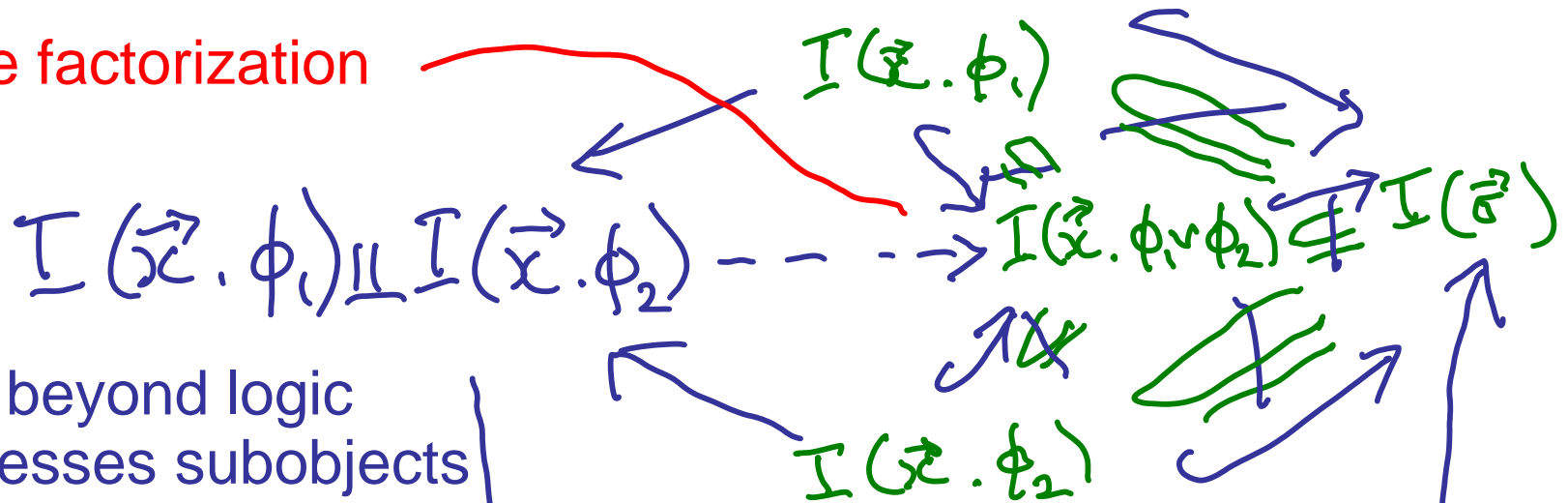


# In addition to finite limits

With coequalizers and coproducts (hence colimits):

- can get disjunction

image factorization



Coproducts go beyond logic

- logic only accesses subobjects of products
- but they're still useful

With infinite coproducts

- get infinite disjunctions

copairing

# Summary

With finite limits, can interpret

- n-ary functions and predicates, and their applications to terms
- finite conjunctions, equations

With arbitrary colimits as well,

- arbitrary disjunctions
- existential quantification
- hence geometric logic


Also need the limits and colimits to be well behaved

- or logic doesn't work correctly
- e.g. to get distributivity

Category of sheaves over  $X$  is good for geometric theories

Grothendieck discovered more, toposes.

# Grothendieck topos (Giraud's Theorem)

- each hom-set is small
  - has finite limits
  - has set-indexed coproducts
  - for which coproduct injections are monic and disjoint
  - and which are stable under pullback
  - has image factorization
  - and it is stable under pullback
  - equivalence relations are kernel pairs
  - has a "separating" set  $A$  of objects: for any  $f, g: X \rightarrow Y$ ,  
If for all  $Z$  in  $A$ , and for all  $h: Z \rightarrow X$ , we have  $fh = gh$ , then  $f = g$ .
- all set-indexed colimits
- 

# Appropriate functors between Grothendieck toposes

important (geometric) structure is finite limits, arbitrary colimits  
- cf. finite meets, arbitrary joins for propositional case (topology)

appropriate functors preserve that structure

Also interested in natural transformations between them

Note

Grothendieck toposes have other non-geometric structure

- e.g. exponentials, subobject classifier, powerobjects  
they are elementary toposes

The extra structure is not preserved by these functors  $F$



## Other connectives - e.g. negation

Natural deduction rules for introduction and elimination imply:

$$\frac{\phi \vdash \neg\psi}{\phi \wedge \psi \vdash \perp}$$

excluded middle is separate

$$\top \vdash \psi \vee \neg\psi$$

$\neg\psi$  is weakest  $\phi$  such that  $\phi \wedge \psi \vdash \perp$

Need more categorical structure to get this Heyting negation.

Category of sheaves has it

- but without excluded middle
- and it doesn't work fibrewise
- not geometric

# Geometric morphism

$$f: \mathcal{F} \rightarrow \mathcal{E}$$

= adjunction  $\mathcal{F} \begin{array}{c} \xleftarrow{f^*} \\ \perp \\ \xrightarrow{f_*} \end{array} \mathcal{E}$  such that

inverse image part  $f^*$  (left adjoint)  
preserves finite limits

$f^*$  automatically preserves all colimits

Conversely: If  $F: \mathcal{E} \rightarrow \mathcal{F}$  preserves finite limits, arbitrary colimits then it has a right adjoint  $G$ , and  $(F \dashv G)$  is geometric morphism

Geometric morphisms are equivalent to the appropriate functors between Grothendieck toposes

but backwards!

# Natural transformations between geometric morphisms

$$f, g: \mathcal{E} \rightarrow \mathcal{F}$$

Natural transformation  $\alpha$  from  $f$  to  $g$

= (definition) natural transformation  $\alpha^*$  from  $f^*$  to  $g^*$

- equivalent to natural transformation  $\alpha_*$  from  $g_*$  to  $f_*$

# Universal property of classifying category $L(T)$

There is a generic model of  $T$  in  $L(T)$

let  $A$  be a category of appropriate kind

let  $M$  be a model of  $T$  in  $A$

Then there is a unique (up to isomorphism) functor  $M': L(T) \rightarrow A$  that preserves the structure used by the logic and transforms generic model to (specific model)  $M$  - up to isomorphism

$L(T)$  defined up to categorical equivalence

Same as for propositional theories and Lindenbaum algebras, but using categories instead of posets.

# More precisely: take care of homomorphisms

objects: geometric morphisms  
morphisms: natural transformations

objects: models of T in E  
morphisms: homomorphisms

$$\begin{array}{ccc} \text{Top}(\Sigma, \mathcal{S}[\Pi]) & \longrightarrow & \text{Mod}_\Sigma(\Pi) \\ f & \longmapsto & f^*(M_G) \end{array}$$

is equivalence of categories for every Grothendieck topos E

1. If M a model of T in E, then it's isomorphic to  $f^*(M_G)$  for some f
2. If  $f, g: E \rightarrow \mathcal{S}[T]$ , and  $h: f^*(M_G) \rightarrow g^*(M_G)$  a homomorphism, then there is unique  $\alpha: f \rightarrow g$  such that  $h = \alpha^*(M_G)$

is equivalence of categories for every Grothendieck topos  $E$

1. If  $M$  a model of  $T$  in  $E$ , then it's isomorphic to  $f^*(M_G)$  for some  $f$

Second part says -

2. If  $f, g: E \rightarrow S[T]$ , and  $h: f^*(M_G) \rightarrow g^*(M_G)$  a homomorphism, then there is unique  $\alpha: f \rightarrow g$  such that  $h = \alpha^*(M_G)$

the carrier morphisms in  $h$  lift to carrier morphisms for all derived sorts, giving natural transformation  $\alpha$

Works for **positive** logics, such as geometric logic

- and fragments such as algebraic logic (operators and equational laws)

For more general logics, e.g. classical or intuitionistic,

- must restrict morphisms on both sides

- e.g. restrict to isomorphisms

see lecture 2:

"Homomorphisms preserve *some* formulae"

# Classifying category for finitary positive logics

Use universal algebra to present  $L(T)$

- as "category of appropriate kind"
- using generators and relations got from theory  $T$

Theory of "categories of appropriate kind" is not single-sorted algebraic

- two sorts, for objects and morphisms
- operations may be partial
- e.g. composition of morphisms, formation of pullbacks

It is usually cartesian (essentially algebraic)

- domains of definition for partial operations are defined by equations

Then presentation by generators and relations still works

For non-positive logics?

Can still work if careful with notion of homomorphism

## Example: Lawvere theories

Suppose  $T$  single sorted algebraic (e.g. theory of monoids)

- one sort  $\sigma$
- no predicate symbols, only functions (operators)
- equational axioms

$$\text{true} \vdash_{\vec{x}} t_1 = t_2$$

"category of appropriate kind" has finite products

- and we want functors that preserve finite products

classifying category has countably many objects

- for  $\sigma$  and its finite powers

morphisms for operators and all derived operations

- and tuple maps to powers of  $\sigma$

 abstract clone

$L(T)$  is the Lawvere theory for  $T$



# Universal property of classifying topos $S[T]$

$S[T]$  has generic model  $M_G$

Let  $E$  be a Grothendieck topos with model  $M$  of  $T$ .

Then there is a unique (up to isomorphism) geometric morphism

$$f: E \rightarrow S[T]$$

such that  $M \cong f^*(M_G)$

$S[T]$  is:

"geometric mathematics freely generated by the generic model  $M_G$ "

Whole of  $S[T]$  can be derived from ingredients of  $M_G$ , using finite limits and arbitrary colimits.

# Constructing $S[T]$

Most important message:

Classifying topos can be constructed, with required universal property

How to do it?

1. Manipulate  $T$  into form of a site:

- signature is a category  $C$
- interpretations are required to be "flat" functors from  $C$
- axioms of form

where  $u_i: X_i \rightarrow Y$  morphisms in  $C$

$$\text{true} \vdash_{y:Y} \bigvee_i \exists x: X_i. y = u_i(x)$$

2. Use presheaves over  $C$  to adjoin arbitrary colimits **and preserves finite limits already in  $C$**
3. Use sheaves (pasting condition) to make axioms hold

# $S[T]$ for propositional geometric $T$

Already have frame  $\Omega[T]$  as poset form of Lindenbaum algebra

Theorem  $S[T]$  is equivalent to topos of sheaves over  $\Omega[T]$

Lecture 1:

For a topological space, defn of sheaf as presheaf with pasting depends only on frame of opens.

It works equally well for an arbitrary frame.

Lecture 2:

$S[T]$  classifies theory of completely prime filters of  $\Omega[T]$

Theorem For two propositional geometric theories  $T, T'$ :

Geometric morphisms  $S[T] \rightarrow S[T']$

are equivalent to

frame homomorphisms  $\Omega[T] \leftarrow \Omega[T']$

# Further reading

Frames and their presentations

Johnstone - Stone Spaces

Vickers - Topology via Logic

Grothendieck toposes and classifying toposes

Mac Lane and Moerdijk

Johnstone - Elephant

Also Vickers "Locales and toposes as spaces" - reader's guide through the standard texts

Universal algebra for cartesian theories

Palmgren and Vickers