Chapter 1

Introduction

The theoretical nuclear physics in the low-energy regime aims to study the properties of atomic nuclei as bound systems of nucleons. Most microscopic models of nuclear structure are rooted in the non-relativistic quantum mechanics [1]. It is assumed that nucleons are "point-like" particles despite general empirical evidence of their inner quark-gluon structure. The quark and gluonic degrees of freedom, as well as non-nucleonic degrees of freedom (mesons, Δ), are considered at most for construction of the interactions among nucleons. Differently, in the high-energy regime and especially in the heavy-ion collisions, theoretical framework of quantum field theory, specifically Quantum Chromodynamics (QCD) is used. Here, the quark and gluon degrees of freedom are relevant and various statistical methods are employed to study the equation of state and other properties of nuclear matter. [cite?]

In this thesis, we restrict ourselves to the theoretical description of nuclear structure in the low-energy regime. Two main issues must be resolved, first the formulation of the force acting among nucleons and, second, the application of this force in quantum many-body methods. There are several approaches to the construction of the nucleon-nucleon (NN) potential.

The first class of the NN potentials is of phenomenological type. These effective forces rely on a set of parameters which are usually fitted directly to the bulk properties of certain set of doubly-magic nuclei [2]. Examples of the effective potentials are the Skyrme [3] and Gogny [4] forces. These interactions are traditionally used in self-consistent mean-field models. Within these approaches, the ground-state properties of the nuclear systems are described by using the Hartree-Fock (HF) method while the excitation spectra are calculated within the Random Phase Approximation (RPA), or its extensions, in the quasi-boson approximation [2].

The NN potentials of the second class are derived from the microscopic theory of the NN scattering and reproduce the experimental phase shifts. These potentials are

often called realistic or high-precision. They are either inspired by the idea of meson exchanges, such is the case of Nijmegen [5], Argonne V18 [6], and CD-Bonn [7], or are derived from the effective field theory which satisfies all symmetries of QCD with nucleons and pions as the degrees of freedom (ChPT potentials) [8]. The perturbative character of the ChPT allows to improve step by step the precision of the NN interactions by going into the higher orders of perturbation. Starting from the next-to-next-to leading order (N^2LO), the NNN interactions appear. A detailed discussion of the ChPT can be found in Refs. [9, 10]. Using the bare realistic NN potentials as the input for various nuclear models often brings practical computational challenges which are caused by the slow convergence or the presence of the strong repulsive core in these forces. Typically, the renormalization methods, such as V_{low-k} [11], SRG [12], UCOM [13], and G-Matrix [14] are introduced to address these difficulties. The G-Matrix can be also interpreted as an effective potential which describes the NN interaction inside the nuclear medium [15].

The second issue of nuclear physics is to solve the many-body problem with given interactions among nucleons. Theoretical nuclear physics adopts various physical models to describe the nuclear observables, such as the binding energy, charge radius and the excitation spectra.

The lightest nuclear systems with $A = 3$ and $A = 4$ can be described directly from the free-space NN interactions by solving the Fadeev [16] and Fadeev-Yakubovsky [17] equations, respectively.

Generally, nuclei up to $A \approx 50$ can be described by *ab initio* methods. These models describe the nuclear structure with minimum of approximations. However, their computational complexity rapidly increases with A. Examples of *ab initio* models are No Core Shell Model [18], Coupled Cluster Model [19], Fermionic Molecular Dynamics [20], Self-Consistent Green's Function Method [21], and Green's Function Monte Carlo Method [22].

The structure of nuclei starting from ^{16}O up to the heaviest elements is often described by the self-consistent mean-field models, such as the Hartree-Fock (HF) or Hartree-Fock-Bogoliubov (HFB) methods [23].

The HF calculations starting from the renormalized realistic potentials yield unrealistic single-particle spectra with too big gaps between the major shells as well as unrealistic nuclear radii. These phenomena have been shown in an HF calculation with the Argonne V18 + UCOM interaction [24] and independently in a calculation based on the CD-Bonn + V_{low-k} interaction [25]. In both studies, the realistic NN potential was corrected by the phenomenological density-dependent (DD) term which simulates the effect of the three-body NNN interactions. The DD term was found to be of crucial importance for generating the single-particle spectra in qual-

itative agreement with the empirical ones. The self-consistent mean field generated by the HF method with the realistic NN interactions corrected by the DD term was used as a starting point for the beyond mean-field calculations of several medium and heavy nuclei within the Equation of Motion Phonon Method (EMPM) [26, 27, 28]. The EMPM solves a set of equations of motion to generate a multi-phonon basis that can be used for diagonalizing the nuclear Hamiltonian. The many-body correlations in the nuclear eigenstates obtained in the EMPM calculations are crucial for the description of the nuclear ground-state properties [29] as well as the properties of the excited states [30, 31, 32, 33].

The crucial importance of the DD term in the aforementioned calculations calls for the direct implementation of the three-body NNN force which naturally occurs in the potential derived from the ChPT.

Throughout this work, the chiral NN + NNN potential up to the next-to-next-to leading order N^2LO_{sat} [34] is employed. The parameters of the N^2LO_{sat} potential were fitted not only to reproduce the experimental phase shifts of the NN scattering but also other properties of light nuclear systems [34]. Thus, it is more suitable interaction for calculations of medium-mass and heavy nuclear systems than the standard bare chiral interactions. However, it remains very desirable to include also short-range correlation effects into our Hamiltonian, although this task is beyond the scope of this thesis.

The theory of nuclear structure can be extended to study exotic nuclear systems, namely hypernuclei. A hypernucleus is a bound nuclear system, in which a hyperon with non-zero strangeness $(\Lambda, \Sigma, \Xi, \Omega)$ is present. With the exception of the Σ^0, the hyperons decay predominantly weakly which results in their rather long lifetime $\approx 10^{-10}$ s. The first hypernucleus has been observed in 1952 by J. Pniewski and M. Danysz [35]. To this day, approximately 30 species of the Λ hypernuclei have been discovered starting from the lightest hypernucleus ${}^{3}_{\Lambda}$H to the heaviest hypernuclear systems ${}^{208}_{\Lambda}$Pb and ${}^{208}_{\Lambda}$Bi. Experimental study of hypernuclei have been performed by many collaborations worldwide (CERN, BNL, KEK, FINUDA, JLab, JPARC, GSI, MAMI-C) [36, 37, 38].

Several Λ-nucleon (ΛN) potentials – both bare and effective – have been developed for implementation in hypernuclear models. Examples of these potentials are effective G-Matrix potentials derived from the Nijmegen model ESC08 [39], Jülich

meson exchange model [40], and the chiral ΛN interaction at the LO [41] and recently at the NLO [42].

In Ref. [43], the hypernuclear mean-field model based on realistic baryon-baryon forces has been introduced. The NN interactions were described by the chiral $\mathrm{NNLO_{opt}}$ with parameters fitted to minimize the effect of the three-body force [44]. However, the effect of the three-body force was non-negligible. Therefore, the potential $\mathrm{NNLO_{opt}}$ was corrected by the phenomenological density-dependent (DD) term which simulated the effect of the NNN forces. The ΛN force was described by the effective G-matrix Nijmegen interaction with parametrization ESC08c [45].

The main drawback of the G-matrix Nijmegen potential is the strong dependence on the Fermi momentum k_{F}. This parameter is evaluated self-consistently by the Thomas-Fermi Approximation from the overlap density calculated by the Average Density Approximation [37]. In addition, the symmetric spin-orbit (SLS) and anti-symmetric spin-orbit (ALS) parts of the potential are not treated explicitly but only within the Scheerbaum Approximation [45, 37]. This would not allow us to study the dependence of hypernuclear spectra on the spin part of the ΛN potential beyond the approximation of mean field.

In this thesis, the NΛ-NΛ channel of the chiral LO YN potential is employed as the ΛN force [41]. The results of hypernuclear calculations presented in this thesis document the effect of the NNN interactions on the behavior of the Λ hyperon in the nuclear medium.

The main focus of our research is the study of the spectra of medium-mass and heavy single-Λ hypernuclei. Our long-term goal is to provide an *ab initio* description of structure of these hypernuclei. The thesis provides first steps towards this goal. We implement the Hamiltonian based on realistic baryon chiral forces and develop two methods based on mean-field approximation - the Hartree-Fock (HF) method in the proton-neutron-Λ (p-n-Λ) formalism and the extension of the Tamm-Dancoff Approximation (TDA), nucleon-Λ TDA (NΛ TDA) method. In this work, we do not address several issues. Namely short-range correlations, $\Lambda - \Sigma$ mixing in the YN interaction, and the many-body correlations. The solution of these issues is beyond the goal of this thesis which is the implementation of the full three-body NNN interactions into the formalism of hypernuclear mean-field model and study the effect of the NNN forces on the description of hypernuclei.

This thesis is organized as follows: The derivation of the Hartree-Fock method in the proton-neutron-Λ formalism is shown in Chapter 2. In Chapter 3, coupling of the Λ to the nuclear core is described within the NΛ TDA method which is an extension of the Tamm-Dancoff Approximation. Numerical implementation of the three-body NNN force is described in Chapter 4. We discuss our results in Chapter 5. We summarize our conclusions in Chapter 6. We define the matrix elements of the used Hamiltonian in harmonic oscillator basis in Appendix A. In B, we discuss the J-scheme formalism and we reformulate equations of the HF and NΛ TDA methods in this formalism.

Chapter 2

Hypernuclear mean-field model with three-body interactions

We describe the single-Λ hypernucleus as a many-body system consisting of the nuclear core and one Λ hyperon. Properties of the hypernucleus are determined by the Hamiltonian

$$\widehat{H} = \widehat{T}^N + \widehat{T}^\Lambda + \widehat{V}^{NN} + \widehat{V}^{NNN} + \widehat{V}^{\Lambda N} + \widehat{V}^{\Lambda NN} - \widehat{T}_{CM}. \tag{2.1}$$

Here, $\widehat{T}^N$ and $\widehat{T}^\Lambda$ denote the sums over kinetic operators of nucleons and the Λ particle, respectively. The terms $\widehat{V}^{NN}$ and $\widehat{V}^{\Lambda N}$ stand for sums over the two-body NN and ΛN potentials. Sums over the three-body interactions are included in the terms $\widehat{V}^{NNN}$ and $\widehat{V}^{\Lambda NN}$. The term $\widehat{T}_{CM}$ denotes the center-of-mass kinetic operator

$$\widehat{T}_{CM} = \frac{1}{2[(A-1)M + M_\Lambda]} \left(\sum_{a=1}^{A} \widehat{\vec{P}}_a^{\,2} + 2 \sum_{a<b} \widehat{\vec{P}}_a \cdot \widehat{\vec{P}}_b \right), \tag{2.2}$$

where $M \approx 938$ MeV is the mass of a nucleon, $M_\Lambda \approx 1116$ MeV is the mass of the Λ hyperon, A is the baryon number, and $\widehat{\vec{P}}_a$ is the momentum operator of the a-th particle.

The hypernuclear mean field is constructed self-consistently by the Hartree-Fock (HF) method. In Section 2.1, we derive the HF method for a system of identical fermions with the two-body interactions in the formalism of the second quantization. In Section 2.2, we show the HF method for the single-Λ hypernuclei including the three-body NNN and the ΛNN interactions in the proton-neutron-Λ formalism.

2.1 Hartree-Fock method in the second quantization

In this section, we show the derivation of the Hartree-Fock equation in the formalism of second quantization, i.e. in terms of creation (annihilation) operators $\alpha^\dagger(\alpha)$.

Let us consider a system of A identical fermions interacting through the two-body potential $\widehat{V}(\vec{r}_1, \vec{r}_2)$. The Hamiltonian of this system in the second quantization is given by

$$\widehat{H} = \sum_{ij} t_{ij} \alpha_i^\dagger \alpha_j + \frac{1}{4} \sum_{ijkl} V_{ijkl} \alpha_i^\dagger \alpha_j^\dagger \alpha_l \alpha_k, \tag{2.3}$$

where $\alpha_i^\dagger|0\rangle = |i\rangle$ creates the single-particle state $|i\rangle$ and $\alpha_i|i\rangle = |0\rangle$ annihilates the single-particle state $|i\rangle$. The ket $|0\rangle$ denotes the particle vacuum. In the Eq. (2.3), the matrix elements of the kinetic operator read $t_{ij} = \langle i|\widehat{T}|j\rangle$ and the matrix elements of the potential operator are antisymmetrized

$$V_{ijkl} = \langle ij|\widehat{V}(\vec{r}_1, \vec{r}_2)|kl\rangle - \langle ij|\widehat{V}(\vec{r}_1, \vec{r}_2)|lk\rangle = \langle ij|\widehat{V}(\vec{r}_1, \vec{r}_2)|kl - lk\rangle. \tag{2.4}$$

The antisymmetrized ground-state wave function of the studied system is a Slater determinant

$$|\Psi_0\rangle = \prod_{i=1}^{A} \alpha_i^\dagger |0\rangle, \tag{2.5}$$

where the product in Eq. (2.5) runs over the lowest single-particle states.

Throughout our work, we express all physical states in the basis of spherical harmonic oscillator. The creation and annihilation operators $\alpha_i^\dagger, \alpha_i$ correspond to the single-particle states $|i\rangle$ as $|i\rangle = \alpha_i^\dagger|0\rangle$. The wave functions $|i\rangle$ can be expanded into another basis represented by the operators $\alpha_i'^\dagger, \alpha_i'$. The bases states $|i\rangle$ and $|i'\rangle$ and their corresponding creation and annihilation operators are connected through a unitary transformation U

$$|i'\rangle = \sum_j U_{ij}|j\rangle, \tag{2.6a}$$

$$\alpha_i'^\dagger = \sum_{ij} U_{ij} \alpha_j^\dagger, \tag{2.6b}$$

$$\alpha_i' = \sum_{ij} \alpha_j U_{ij}^* = \sum_{ij} U_{ji}^\dagger a_j. \tag{2.6c}$$

The inverse transformation satisfies

$$|i\rangle = \sum_j U^*_{ji}|j'\rangle, \tag{2.7a}$$

$$\alpha^\dagger_i = \sum_{ij} U^+_{ij}\alpha'^\dagger_j = \sum_{ij} \alpha'^\dagger_j U^*_{ji}, \tag{2.7b}$$

$$\alpha_i = \sum_{ij} U_{ji}\alpha'_j. \tag{2.7c}$$

The basis represented by the operators $\alpha'^\dagger_i, \alpha'_i$ is denoted as the self-consistent basis. The ground state in the self-consistent basis is defined as the Hartree-Fock state $|\mathrm{HF}\rangle$

$$|\mathrm{HF}\rangle = \prod_{i=1}^{A} \alpha'^\dagger_i|0\rangle. \tag{2.8}$$

It is convenient to define the density matrix of the HF state ρ^{HF}:

$$\rho^{\mathrm{HF}}_{ji} = \langle \mathrm{HF}|\alpha^\dagger_i\alpha_j|\mathrm{HF}\rangle. \tag{2.9}$$

Using the transformation equations (2.7b), (2.7c) leads to the following relation

$$\rho^{\mathrm{HF}}_{ji} = \sum_{kl} U^*_{ki}U^T_{jl}\langle \mathrm{HF}|\alpha'^\dagger_k\alpha'_l|\mathrm{HF}\rangle. \tag{2.10}$$

Let us prove the following identity

$$\begin{aligned}\langle \mathrm{HF}|\alpha'^\dagger_k\alpha'_l|\mathrm{HF}\rangle &= \langle \mathrm{HF}|\{\alpha'^\dagger_k, \alpha'_l\}|\mathrm{HF}\rangle - \langle \mathrm{HF}|\alpha'_l\alpha'^\dagger_k|\mathrm{HF}\rangle \\ &= \delta_{kl}\langle \mathrm{HF}|\mathrm{HF}\rangle - \langle \mathrm{HF}|\alpha'_l\alpha'^\dagger_k|\mathrm{HF}\rangle = \delta_{kl}, \quad \forall k : \varepsilon_k \leq \varepsilon_F,\end{aligned} \tag{2.11}$$

where ε_k is the energy of the k-th level and ε_F is the energy of the highest occupied level (the Fermi level). We denote the occupied state k as $k-\mathrm{occ.}$. For all occupied single-particle states k, the expression (2.10) gives

$$\rho^{\mathrm{HF}}_{ji} = \sum_{k-\mathrm{occ.}} U^*_{ki}U^T_{jk} = \sum_{k-\mathrm{occ.}} U^T_{jk}U^*_{ki} = (U^TU^*)_{ji} \tag{2.12}$$

The next step of this calculation is construction of the energy functional

$$\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle = \sum_{ij} t_{ij}\langle \mathrm{HF}|\alpha^\dagger_i\alpha_j|\mathrm{HF}\rangle + \frac{1}{4}\sum_{ijkl} V_{ijkl}\langle \mathrm{HF}|\alpha^\dagger_i\alpha^\dagger_j\alpha_l\alpha_k|\mathrm{HF}\rangle. \tag{2.13}$$

The second term in Eq. (2.13) with the transformation relations (2.7b), (2.7c) gives

$$\begin{aligned}\langle \mathrm{HF}|\alpha^\dagger_i\alpha^\dagger_j\alpha_l\alpha_k|\mathrm{HF}\rangle &= \sum_{opqr} U^*_{oi}U^*_{pj}U^T_{lr}U^T_{kq}\langle \mathrm{HF}|\alpha'^\dagger_o\alpha'^\dagger_p\alpha'_r\alpha'_q|\mathrm{HF}\rangle \\ &= \sum_{opqr-\mathrm{occ.}} U^*_{oi}U^*_{pj}U^T_{lr}U^T_{kq}(\delta_{oq}\delta_{pr} - \delta_{or}\delta_{pq}) \\ &= \sum_{op-\mathrm{occ.}} (U^*_{oi}U^T_{ko}U^*_{pj}U^T_{lp} - U^*_{oi}U^T_{lo}U^*_{pj}U^T_{kp}) \\ &= (U^TU^*)_{ki}(U^TU^*)_{lj} - (U^TU^*)_{li}(U^TU^*)_{kj}\end{aligned} \tag{2.14}$$

The proof of the identity

$$\langle \mathrm{HF}|\alpha_o'^{\dagger}\alpha_p'^{\dagger}\alpha_r'\alpha_q'|\mathrm{HF}\rangle = \delta_{oq}\delta_{pr} - \delta_{or}\delta_{pq}, \quad \forall o,p : \varepsilon_o, \varepsilon_p \leq \varepsilon_F, \tag{2.15}$$

is analogous to the one in Eq. (2.11). The energy functional (2.13) therefore reads

$$\begin{aligned}\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle &= \sum_{ij}\sum_{o-\mathrm{occ.}} t_{ij}U^T_{jo}U^*_{oi} \\ &+ \frac{1}{4}\sum_{ijkl}\sum_{op-\mathrm{occ.}} V_{ijkl}U^T_{ko}U^*_{oi}U^T_{lp}U^*_{pj} - \frac{1}{4}\sum_{ijkl}\sum_{op-\mathrm{occ.}} V_{ijkl}U^T_{lo}U^*_{oi}U^T_{kp}U^*_{pj} \\ &= \sum_{ij} t_{ij}(U^TU^*)_{ji} + \frac{1}{4}\sum_{ijkl} V_{ijkl}(U^TU^*)_{ki}\left(U^TU^*\right)_{lj} \\ &- \frac{1}{4}\sum_{ijkl} V_{ijkl}(U^TU^*)_{li}(U^TU^*)_{kj}. \end{aligned} \tag{2.16}$$

We minimize the functional (2.16) with respect to the variation of the transformation U

$$\delta\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle = \frac{\delta\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle}{\delta U}\delta U + \frac{\delta\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle}{\delta U^*}\delta U^* = 0. \tag{2.17}$$

In general, U is an unitary matrix, hence we get two equivalent conditions

$$\frac{\delta\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle}{\delta U} = 0, \tag{2.18a}$$

$$\frac{\delta\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle}{\delta U^*} = 0. \tag{2.18b}$$

Variation of the functional in Eq. (2.16) with respect to U^* gives

$$\begin{aligned}\frac{\delta\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle}{\delta U^*_{op}} &= \sum_j t_{pj}U^T_{jo} + \frac{1}{4}\sum_{jkl} V_{pjkl}(U^TU^*)_{lj}U_{ko} \\ &+ \frac{1}{4}\sum_{ikl} V_{ipkl}(U^TU^*)_{ki}U_{lo} \\ &- \frac{1}{4}\sum_{jkl} V_{pjkl}(U^TU^*)_{kj}U_{lo} \\ &- \frac{1}{4}\sum_{ikl} V_{ipkl}(U^TU^*)_{li}U_{ko}. \end{aligned} \tag{2.19}$$

Due to hermiticity and antisymmetry, the matrix elements V_{ijkl} satisfy the following identities

$$V_{ijkl} = -V_{jikl} = -V_{ijlk} = V_{jilk}, \tag{2.20a}$$

$$V_{ijkl} = V_{klij}. \tag{2.20b}$$

Applying (2.20a) and (2.20b) on (2.19) yields the equation

$$\frac{\delta\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle}{\delta U^*_{op}} = \sum_j \left\{ t_{pj} + \sum_{kl} V_{pkjl}(U^T U^*)_{lk} \right\} U^T_{jo} = 0. \tag{2.21}$$

The unitarity of the matrix U presents following restriction

$$(U^T U^*)_{op} - \mathbb{I}_{op} = 0. \tag{2.22}$$

The variational problem with the restriction (2.22) is expressed as

$$\begin{aligned} &\frac{\delta}{\delta U^*_{op}} \left\{ \langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle - \varepsilon \left[U^T U^* - \mathbb{I} \right] \right\} \\ &\qquad = \sum_j \left\{ t_{pj} + \sum_{kl} V_{pkjl}(U^T U^*)_{lk} \right\} U^T_{jo} - \sum_j \varepsilon_p \mathbb{I}_{pj} U^T_{jo} = 0. \end{aligned} \tag{2.23}$$

By substituting the density matrix identity (2.12) we obtain the equation

$$\sum_j \left\{ t_{pj} + \sum_{kl} V_{pkjl} \rho^{\mathrm{HF}}_{lk} \right\} U^T_{jo} = \sum_j \varepsilon_p \mathbb{I}_{pj} U^T_{jo}, \tag{2.24}$$

which represents an eigenvalue problem of the matrix h_{mn} defined as

$$h_{mn} = t_{mn} + \sum_{kl} V_{mknl} \rho^{\mathrm{HF}}_{lk} = \varepsilon_m \delta_{mn}, \tag{2.25}$$

where we rename indices $p \to m, j \to n$. Equation (2.25) is called the Hartree-Fock equation. Through diagonalization of h_{mn} we obtain a self-consistent basis $|i'\rangle$ which is connected to the HO basis $|i\rangle$ by the unitary transformation U_{ij}, see Eq. (2.7a).

The Hamiltonian (2.3) can be rewritten into the form in which the creation and annihilation operators are normal ordered. For this procedure we use the Wick's theorem [46] and obtain the new expression of the Hamiltonian

$$\widehat{H} = E_{\mathrm{HF}} + \widehat{H}^{(1)} + \widehat{H}^{(2)}. \tag{2.26}$$

Operators $E_{\mathrm{HF}}, \widehat{H}^{(1)}$, and $\widehat{H}^{(2)}$ are defined as follows

$$E_{\mathrm{HF}} = \sum_{ij} t_{ij} \rho^{\mathrm{HF}}_{ji} + \frac{1}{2} \sum_{kl} V_{ijkl} \rho^{\mathrm{HF}}_{ki} \rho^{\mathrm{HF}}_{jl}, \tag{2.27a}$$

$$\widehat{H}^{(1)} = \sum_{ij} \left(t_{ij} + \sum_{kl} V_{kilj} \rho^{\mathrm{HF}}_{lk} \right) \ :\alpha^\dagger_i \alpha_j:, \tag{2.27b}$$

$$\widehat{H}^{(2)} = \frac{1}{4} \sum_{ijkl} V_{ijkl} \ :\alpha^\dagger_i \alpha^\dagger_j \alpha_l \alpha_k:, \tag{2.27c}$$

where $:\alpha_i^\dagger\alpha_j:$ and $:\alpha_i^\dagger\alpha_j^\dagger\alpha_l\alpha_k:$ denote the normal ordering of the operators $\alpha_i^\dagger\alpha_j$ and $\alpha_i^\dagger\alpha_j^\dagger\alpha_l\alpha_k$, respectively. Normal ordered operators satisfy

$$\begin{aligned}\langle \mathrm{HF}|\, :\alpha_i^\dagger\alpha_j:\, |\mathrm{HF}\rangle &= \langle \mathrm{HF}|\, :\alpha_i'^\dagger\alpha_j':\, |\mathrm{HF}\rangle = 0,\\ \langle \mathrm{HF}|\, :\alpha_i^\dagger\alpha_j^\dagger\alpha_l\alpha_k:\, |\mathrm{HF}\rangle &= \langle \mathrm{HF}|\, :\alpha_i'^\dagger\alpha_j'^\dagger\alpha_l'\alpha_k':\, |\mathrm{HF}\rangle = 0.\end{aligned} \tag{2.28}$$

Therefore, the value of the element $\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle$ is equal to the energy of the ground state E_{HF} which is defined by the Eq. (2.27a). The terms $\widehat{H}^{(1)}$ and $\widehat{H}^{(2)}$ in Eq. (2.27b) and (2.27c) represent one-body and two-body parts of the Hamiltonian (2.26). The operators in Eqs.(2.27a), (2.27b), and (2.27c) are expressed in the self-consistent basis spanned by the single-particle states $|i'\rangle$ as

$$E_{\mathrm{HF}} = \sum_{i-\mathrm{occ.}} \varepsilon_i - \frac{1}{2}\sum_{i,k-\mathrm{occ.}} \overline{V}_{kiki}, \tag{2.29a}$$

$$\widehat{H}^{(1)} = \sum_i \varepsilon_i\, :\alpha_i'^\dagger\alpha_i':, \tag{2.29b}$$

$$\widehat{H}^{(2)} = \frac{1}{4}\sum_{ijkl} \overline{V}_{ijkl}\, :\alpha_i'^\dagger\alpha_j'^\dagger\alpha_l'\alpha_k': . \tag{2.29c}$$

Here, $\overline{V}_{ijkl}$ denote the matrix elements V_{ijkl} transformed into the self-consistent basis $|i'\rangle$ by the equation

$$\overline{V}_{ijkl} = \langle i'j'|\widehat{V}(\vec{r}_1,\vec{r}_2)|k'l'-l'k'\rangle = \sum_{opqr} V_{opqr}U^*_{oi}U^*_{pj}U^T_{lr}U^T_{kq}. \tag{2.30}$$

2.2 Hartree-Fock method in the proton-neutron-Λ formalism with three-body interactions

In this section, we generalize the Hartree-Fock method for a single-Λ hypernucleus, the system consisting of Z protons, N neutrons and one Λ particle. We use a notation ${}^A_\Lambda X = (Z,N,1)$, where $A = Z+N+1$. We consider the subsystem ${}^{(A-1)}X = (Z,N,0)$, where $(A-1) = Z+N$, to be the nuclear core.

Proton-neutron-Λ (p-n-Λ) formalism adopts creation and annihilation operators $a^\dagger, a$ for protons, $b^\dagger, b$ for neutrons and $c^\dagger, c$ for the Λ particle. Moreover, we suppose that kinetic and potential operators in hypernuclear Hamiltonian (2.1) are defined

as following sums of operators:

$$\widehat{T}^{N} = \widehat{T}^{\mathrm{p}} + \widehat{T}^{\mathrm{n}}, \tag{2.31a}$$
$$\widehat{V}^{NN} = \widehat{V}^{\mathrm{pp}} + \widehat{V}^{\mathrm{pn}} + \widehat{V}^{\mathrm{nn}}, \tag{2.31b}$$
$$\widehat{V}^{N\Lambda} = \widehat{V}^{\mathrm{p}\Lambda} + \widehat{V}^{\mathrm{n}\Lambda}, \tag{2.31c}$$
$$\widehat{V}^{NNN} = \widehat{V}^{\mathrm{ppp}} + \widehat{V}^{\mathrm{nnn}} + \widehat{V}^{\mathrm{ppn}} + \widehat{V}^{\mathrm{pnn}}, \tag{2.31d}$$
$$\widehat{V}^{NN\Lambda} = \widehat{V}^{\mathrm{pp}\Lambda} + \widehat{V}^{\mathrm{nn}\Lambda} + \widehat{V}^{\mathrm{pn}\Lambda} \tag{2.31e}$$

The Hamiltonian of the hypernuclear system (2.1) is in the second quantization in p-n-Λ formalism defined as:

$$\begin{aligned}
\widehat{H} = &\sum_{ij} t^{\mathrm{p}}_{ij} a^{\dagger}_i a_j + \sum_{ij} t^{\mathrm{n}}_{ij} b^{\dagger}_i b_j + \sum_{ij} t^{\Lambda}_{ij} c^{\dagger}_i c_j \\
&+ \frac{1}{4}\sum_{ijkl} V^{\mathrm{pp}}_{ijkl} a^{\dagger}_i a^{\dagger}_j a_l a_k + \frac{1}{4}\sum_{ijkl} V^{\mathrm{nn}}_{ijkl} b^{\dagger}_i b^{\dagger}_j b_l b_k + \sum_{ijkl} V^{\mathrm{pn}}_{ijkl} a^{\dagger}_i a^{\dagger}_j a_l a_k \\
&+ \sum_{ijkl} V^{\mathrm{p}\Lambda}_{ijkl} a^{\dagger}_i c^{\dagger}_j c_l a_k + \sum_{ijkl} V^{\mathrm{n}\Lambda}_{ijkl} b^{\dagger}_i c^{\dagger}_j c_l b_k \\
&+ \frac{1}{36}\sum_{ijklmn} V^{\mathrm{ppp}}_{ijklmn} a^{\dagger}_i a^{\dagger}_j a^{\dagger}_k a_n a_m a_l + \frac{1}{36}\sum_{ijklmn} V^{\mathrm{nnn}}_{ijklmn} b^{\dagger}_i b^{\dagger}_j b^{\dagger}_k b_n b_m b_l \\
&+ \frac{1}{4}\sum_{ijklmn} V^{\mathrm{ppn}}_{ijklmn} a^{\dagger}_i a^{\dagger}_j b^{\dagger}_k b_n a_m a_l + \frac{1}{4}\sum_{ijklmn} V^{\mathrm{pnn}}_{ijklmn} a^{\dagger}_i b^{\dagger}_j b^{\dagger}_k b_n b_m a_l \\
&+ \frac{1}{4}\sum_{ijklmn} V^{\mathrm{pp}\Lambda}_{ijklmn} a^{\dagger}_i a^{\dagger}_j c^{\dagger}_k c_n a_m a_l + \frac{1}{4}\sum_{ijklmn} V^{\mathrm{nn}\Lambda}_{ijklmn} b^{\dagger}_i b^{\dagger}_j c^{\dagger}_k c_n b_m b_l \\
&+ \sum_{ijklmn} V^{\mathrm{pn}\Lambda}_{ijklmn} a^{\dagger}_i b^{\dagger}_j c^{\dagger}_k c_n b_m a_l.
\end{aligned} \tag{2.32}$$

The matrix elements of the kinetic operator, as well as all interaction terms in the HO basis in Eq. (2.32) are expressed in Appendix A.

The ground state of this system is described by the wave function

$$|\Psi_0\rangle = |\Psi_0\rangle_{\mathrm{p}} \otimes |\Psi_0\rangle_{\mathrm{n}} \otimes |\Psi_0\rangle_{\Lambda}, \tag{2.33}$$

where $|\Psi_0\rangle_{\mathrm{p}}$ and $|\Psi_0\rangle_{\mathrm{n}}$ are Slater determinants of protons and neutrons and $|\Psi_0\rangle_{\Lambda}$ is the single-particle wave function of the Λ hyperon. Respective wave functions in

Eq. (2.33) are defined as follows

$$|\Psi_0\rangle_{\rm p} = \prod_{i=1}^{Z} a_i^\dagger |0\rangle, \tag{2.34a}$$

$$|\Psi_0\rangle_{\rm n} = \prod_{i=1}^{N} b_i^\dagger |0\rangle, \tag{2.34b}$$

$$|\Psi_0\rangle_{\Lambda} = c_1^\dagger |0\rangle. \tag{2.34c}$$

Indices i and j run over Z and N lowest occupied states in the proton and the neutron potential wells, respectively. Unitary transformations for each type of creation and annihilation operators are defined as

$$a_i'^\dagger = \sum_{ij} A_{ij} a_j^\dagger; \quad a_i' = \sum_{ij} a_j A_{ij}^*, \tag{2.35a}$$

$$b_i'^\dagger = \sum_{ij} B_{ij} b_j^\dagger; \quad b_i' = \sum_{ij} b_j B_{ij}^*, \tag{2.35b}$$

$$c_i'^\dagger = \sum_{ij} C_{ij} c_j^\dagger; \quad c_i' = \sum_{ij} c_j C_{ij}^*. \tag{2.35c}$$

Ground-state wave function in the self-consistent basis gives

$$|{\rm HF}\rangle = |{\rm HF}\rangle_{\rm p} \otimes |{\rm HF}\rangle_{\rm n} \otimes |{\rm HF}\rangle_{\Lambda}, \tag{2.36}$$

where

$$|{\rm HF}\rangle_{\rm p} = \prod_{i=1}^{Z} a_i'^\dagger |0\rangle, \tag{2.37a}$$

$$|{\rm HF}\rangle_{\rm n} = \prod_{i=1}^{N} b_i'^\dagger |0\rangle, \tag{2.37b}$$

$$|{\rm HF}\rangle_{\Lambda} = c_1'^\dagger |0\rangle. \tag{2.37c}$$

$$\tag{2.37d}$$

Respective density matrices read

$$\rho_{ij}^{\rm p} = {}_{\rm p}\langle {\rm HF}|a_i^\dagger a_j|{\rm HF}\rangle_{\rm p}, \tag{2.38a}$$

$$\rho_{ij}^{\rm n} = {}_{\rm n}\langle {\rm HF}|b_i^\dagger b_j|{\rm HF}\rangle_{\rm n}, \tag{2.38b}$$

$$\rho_{ij}^{\Lambda} = {}_{\Lambda}\langle {\rm HF}|c_i^\dagger c_j|{\rm HF}\rangle_{\Lambda}. \tag{2.38c}$$

We construct the energy functional $\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle$ as follows:

$$\begin{aligned}
\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle = &\sum_{ij} t^{\mathrm{p}}_{ij}\langle \mathrm{HF}|a^\dagger_i a_j|\mathrm{HF}\rangle + \sum_{ij} t^{\mathrm{n}}_{ij}\langle \mathrm{HF}|b^\dagger_i b_j|\mathrm{HF}\rangle + \sum_{ij} t^{\Lambda}_{ij}\langle \mathrm{HF}|c^\dagger_i c_j|\mathrm{HF}\rangle \\
&+ \frac{1}{4}\sum_{ijkl} V^{\mathrm{pp}}_{ijkl}\langle \mathrm{HF}|a^\dagger_i a^\dagger_j a_l a_k|\mathrm{HF}\rangle + \frac{1}{4}\sum_{ijkl} V^{\mathrm{nn}}_{ijkl}\langle \mathrm{HF}|b^\dagger_i b^\dagger_j b_l b_k|\mathrm{HF}\rangle \\
&+ \sum_{ijkl} V^{\mathrm{pn}}_{ijkl}\langle \mathrm{HF}|a^\dagger_i b^\dagger_j b_l a_k|\mathrm{HF}\rangle + \sum_{ijkl} V^{\mathrm{p}\Lambda}_{ijkl}\langle \mathrm{HF}|a^\dagger_i c^\dagger_j c_l a_k|\mathrm{HF}\rangle \\
&+ \sum_{ijkl} V^{\mathrm{n}\Lambda}_{ijkl}\langle \mathrm{HF}|b^\dagger_i c^\dagger_j c_l b_k|\mathrm{HF}\rangle \\
&+ \frac{1}{36}\sum_{ijklmn} V^{\mathrm{ppp}}_{ijklmn}\langle \mathrm{HF}|a^\dagger_i a^\dagger_j a^\dagger_k a_n a_m a_l|\mathrm{HF}\rangle \\
&+ \frac{1}{36}\sum_{ijklmn} V^{\mathrm{nnn}}_{ijklmn}\langle \mathrm{HF}|b^\dagger_i b^\dagger_j b^\dagger_k b_n b_m b_l|\mathrm{HF}\rangle \\
&+ \frac{1}{4}\sum_{ijklmn} V^{\mathrm{ppn}}_{ijklmn}\langle \mathrm{HF}|a^\dagger_i a^\dagger_j b^\dagger_k b_n a_m a_l|\mathrm{HF}\rangle \\
&+ \frac{1}{4}\sum_{ijklmn} V^{\mathrm{pnn}}_{ijklmn}\langle \mathrm{HF}|a^\dagger_i b^\dagger_j b^\dagger_k b_n b_m a_l|\mathrm{HF}\rangle \\
&+ \frac{1}{4}\sum_{ijklmn} V^{\mathrm{pp}\Lambda}_{ijklmn}\langle \mathrm{HF}|a^\dagger_i a^\dagger_j c^\dagger_k c_n a_m a_l|\mathrm{HF}\rangle \\
&+ \frac{1}{4}\sum_{ijklmn} V^{\mathrm{nn}\Lambda}_{ijklmn}\langle \mathrm{HF}|b^\dagger_i b^\dagger_j c^\dagger_k c_n b_m b_l|\mathrm{HF}\rangle \\
&+ \sum_{ijklmn} V^{\mathrm{pn}\Lambda}_{ijklmn}\langle \mathrm{HF}|a^\dagger_i b^\dagger_j c^\dagger_k c_n b_m a_l|\mathrm{HF}\rangle. \qquad (2.39)
\end{aligned}$$

We express the following identities for the three-body terms:

$$\begin{aligned}
\langle \mathrm{HF}|a^\dagger_i a^\dagger_j a^\dagger_k a_n a_m a_l|\mathrm{HF}\rangle = &(A^TA^*)_{li}(A^TA^*)_{mj}(A^TA^*)_{nk} - (A^TA^*)_{nk}(A^TA^*)_{lj}(A^TA^*)_{mi} \\
&- (A^TA^*)_{mk}(A^TA^*)_{nj}(A^TA^*)_{li} - (A^TA^*)_{mk}(A^TA^*)_{lj}(A^TA^*)_{li} \\
&+ (A^TA^*)_{nj}(A^TA^*)_{lk}(A^TA^*)_{mi} - (A^TA^*)_{lk}(A^TA^*)_{mj}(A^TA^*)_{ni}, \qquad (2.40)
\end{aligned}$$

$$\begin{aligned}
\langle \mathrm{HF}|b^\dagger_i b^\dagger_j b^\dagger_k b_n b_m b_l|\mathrm{HF}\rangle = &(B^TB^*)_{li}(B^TB^*)_{mj}(B^TB^*)_{nk} - (B^TB^*)_{nk}(B^TB^*)_{lj}(B^TB^*)_{mi} \\
&- (B^TB^*)_{mk}(B^TB^*)_{nj}(B^TB^*)_{li} - (B^TB^*)_{mk}(B^TB^*)_{lj}(B^TB^*)_{li} \\
&+ (B^TB^*)_{nj}(B^TB^*)_{lk}(B^TB^*)_{mi} - (B^TB^*)_{lk}(B^TB^*)_{mj}(B^TB^*)_{ni}. \qquad (2.41)
\end{aligned}$$

The terms which contain multiple types of particles satisfy

$$\langle \mathrm{HF}|a_i^\dagger a_j^\dagger b_k^\dagger b_n a_m a_l|\mathrm{HF}\rangle = {}_{\mathrm{p}}\langle \mathrm{HF}|a_i^\dagger a_j^\dagger a_m a_l|\mathrm{HF}\rangle_{\mathrm{p}}\,{}_{\mathrm{n}}\langle \mathrm{HF}|b_k^\dagger b_n|\mathrm{HF}\rangle_{\mathrm{n}}, \tag{2.42a}$$

$$\langle \mathrm{HF}|a_i^\dagger b_j^\dagger c_k^\dagger c_n b_m a_l|\mathrm{HF}\rangle = {}_{\mathrm{p}}\langle \mathrm{HF}|a_i^\dagger a_l|\mathrm{HF}\rangle_{\mathrm{p}}\,{}_{\mathrm{n}}\langle \mathrm{HF}|b_j^\dagger b_m|\mathrm{HF}\rangle_{\mathrm{n}}\,{}_{\Lambda}\langle \mathrm{HF}|c_k^\dagger c_n|\mathrm{HF}\rangle_{\Lambda}. \tag{2.42b}$$

By minimizing the energy functional (2.39) with respect to the unitary transformations A, B, and C we obtain three Hartree-Fock equations – one for each type of particles. The HF equation for protons:

$$\begin{aligned} t_{ij}^{\mathrm{p}} + \sum_{kl} V_{ikjl}^{\mathrm{pp}}\rho_{lk}^{\mathrm{p}} + \sum_{kl} V_{ikjl}^{\mathrm{pn}}\rho_{lk}^{\mathrm{n}} + \sum_{kl} V_{ikjl}^{\mathrm{p}\Lambda}\rho_{lk}^{\Lambda} + \frac{1}{2}\sum_{klmn} V_{ikljmn}^{\mathrm{ppp}}\rho_{mk}^{\mathrm{p}}\rho_{nl}^{\mathrm{p}} \\ + \frac{1}{2}\sum_{klmn} V_{ikljmn}^{\mathrm{pnn}}\rho_{mk}^{\mathrm{n}}\rho_{nl}^{\mathrm{n}} + \sum_{klmn} V_{ikljmn}^{\mathrm{ppn}}\rho_{mk}^{\mathrm{p}}\rho_{nl}^{\mathrm{n}} + \sum_{klmn} V_{ikljmn}^{\mathrm{pp}\Lambda}\rho_{mk}^{\mathrm{p}}\rho_{nl}^{\Lambda} \\ + \sum_{klmn} V_{ijklmn}^{\mathrm{pn}\Lambda}\rho_{mk}^{\mathrm{n}}\rho_{nl}^{\Lambda} = \varepsilon_i^{\mathrm{p}}\delta_{ij}. \end{aligned} \tag{2.43}$$

The HF equation for neutrons:

$$\begin{aligned} t_{ij}^{\mathrm{n}} + \sum_{kl} V_{ikjl}^{\mathrm{nn}}\rho_{lk}^{\mathrm{n}} + \sum_{kl} V_{kilj}^{\mathrm{pn}}\rho_{lk}^{\mathrm{p}} + \sum_{kl} V_{ikjl}^{\mathrm{n}\Lambda}\rho_{lk}^{\Lambda} + \frac{1}{2}\sum_{klmn} V_{ikljmn}^{\mathrm{nnn}}\rho_{mk}^{\mathrm{n}}\rho_{nl}^{\mathrm{n}} \\ + \frac{1}{2}\sum_{klmn} V_{klimnj}^{\mathrm{ppn}}\rho_{mk}^{\mathrm{p}}\rho_{nl}^{\mathrm{p}} + \sum_{klmn} V_{klimnj}^{\mathrm{pnn}}\rho_{mk}^{\mathrm{p}}\rho_{nl}^{\mathrm{n}} + \sum_{klmn} V_{ikljmn}^{\mathrm{nn}\Lambda}\rho_{mk}^{\mathrm{n}}\rho_{nl}^{\Lambda} \\ + \sum_{klmn} V_{klimnj}^{\mathrm{pn}\Lambda}\rho_{mk}^{\mathrm{p}}\rho_{nl}^{\Lambda} = \varepsilon_i^{\mathrm{n}}\delta_{ij}. \end{aligned} \tag{2.44}$$

The HF equation for the Λ hyperon:

$$\begin{aligned} t_{ij}^{\Lambda} + \sum_{kl} V_{kilj}^{\mathrm{p}\Lambda}\rho_{lk}^{\mathrm{p}} + \sum_{kl} V_{kilj}^{\mathrm{n}\Lambda}\rho_{lk}^{\mathrm{n}} + \frac{1}{2}\sum_{klmn} V_{klimnj}^{\mathrm{pp}\Lambda}\rho_{mk}^{\mathrm{p}}\rho_{nl}^{\mathrm{p}} + \frac{1}{2}\sum_{klmn} V_{klimnj}^{\mathrm{nn}\Lambda}\rho_{mk}^{\mathrm{n}}\rho_{nl}^{\mathrm{n}} \\ + \sum_{klmn} V_{klimnj}^{\mathrm{pn}\Lambda}\rho_{mk}^{\mathrm{p}}\rho_{nl}^{\mathrm{n}} = \varepsilon_i^{\Lambda}\delta_{ij}. \end{aligned} \tag{2.45}$$

After solving the Hartree-Fock equations (2.43), (2.44), and (2.45) we obtain three self-consistent bases represented by the operators $(a'^\dagger, a')$, $(b'^\dagger, b')$, $(c'^\dagger, c')$. Using the Wick's theorem [46] on the hypernuclear Hamiltonian (2.32) gets us separable Hamiltonian of the form

$$\widehat{H} = E_{\mathrm{HF}} + \widehat{H}^{(1)} + \widehat{H}^{(2)} + \widehat{H}^{(3)}, \tag{2.46}$$

where E_{HF} is the ground-state (HF) energy, $\widehat{H}^{(1)}$ is a one-body operator, $\widehat{H}^{(2)}$ is a two-body operator, and $\widehat{H}^{(3)}$ is a three-body operator. Here, the HF energy is

defined as

$$
\begin{aligned}
E_{\mathrm{HF}} = & \sum_{i-\mathrm{occ.}} \varepsilon_i^{\mathrm{p}} + \sum_{i-\mathrm{occ.}} \varepsilon_i^{\mathrm{n}} + \sum_{i-\mathrm{occ.}} \varepsilon_i^{\Lambda} - \frac{1}{2} \sum_{ij-\mathrm{occ.}} \overline{V}_{ijij}^{\mathrm{pp}} - \frac{1}{2} \sum_{ij-\mathrm{occ.}} \overline{V}_{ijij}^{\mathrm{nn}} - \sum_{ij-\mathrm{occ.}} \overline{V}_{ijij}^{\mathrm{pn}} \\
& - \sum_{ij-\mathrm{occ.}} \overline{V}_{ijij}^{\mathrm{p}\Lambda} - \sum_{ij-\mathrm{occ.}} \overline{V}_{ijij}^{\mathrm{n}\Lambda} - \frac{1}{3} \sum_{ijk-\mathrm{occ.}} \overline{V}_{ijkijk}^{\mathrm{ppp}} - \frac{1}{3} \sum_{ijk-\mathrm{occ.}} \overline{V}_{ijkijk}^{\mathrm{nnn}} - \sum_{ijk-\mathrm{occ.}} \overline{V}_{ijkijk}^{\mathrm{ppn}} \\
& - \sum_{ijk-\mathrm{occ.}} \overline{V}_{ijkijk}^{\mathrm{pnn}} - \sum_{ijk-\mathrm{occ.}} \overline{V}_{ijkijk}^{\mathrm{pp}\Lambda} - \sum_{ijk-\mathrm{occ.}} \overline{V}_{ijkijk}^{\mathrm{nn}\Lambda} - 2 \sum_{ijk-\mathrm{occ.}} \overline{V}_{ijkijk}^{\mathrm{pn}\Lambda}. \qquad (2.47)
\end{aligned}
$$

The one-body operator $\widehat{H}^{(1)}$ reads

$$
\widehat{H}^{(1)} = \sum_i \varepsilon_i^{\mathrm{p}} :a_i'^{\dagger} a_i': + \sum_i \varepsilon_i^{\mathrm{n}} :b_i'^{\dagger} b_i': + \sum_i \varepsilon_i^{\Lambda} :c_i'^{\dagger} c_i':, \qquad (2.48)
$$

the two-body operator $\widehat{H}^{(2)}$ is defined as follows

$$
\begin{aligned}
\widehat{H}^{(2)} = & \frac{1}{4} \sum_{ijkl} \overline{V}_{ijkl}^{\mathrm{pp}} :a_i'^{\dagger} a_j'^{\dagger} a_l' a_k': + \frac{1}{4} \sum_{ijkl} \overline{V}_{ijkl}^{\mathrm{nn}} :b_i'^{\dagger} b_j'^{\dagger} b_l' b_k': + \sum_{ijkl} \overline{V}_{ijkl}^{\mathrm{pn}} :a_i'^{\dagger} b_j'^{\dagger} b_l' a_k': \\
& + \sum_{ijkl} \overline{V}_{ijkl}^{\mathrm{p}\Lambda} :a_i'^{\dagger} c_j'^{\dagger} c_l' a_k': + \sum_{ijkl} \overline{V}_{ijkl}^{\mathrm{n}\Lambda} :b_i'^{\dagger} c_j'^{\dagger} c_l' b_k': + \frac{1}{4} \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{ijmklm}^{\mathrm{ppp}} :a_i'^{\dagger} a_j'^{\dagger} a_l' a_k': \\
& + \frac{1}{4} \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{ijmklm}^{\mathrm{nnn}} :b_i'^{\dagger} b_j'^{\dagger} b_l' b_k': + \frac{1}{4} \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{ijmklm}^{\mathrm{ppn}} :a_i'^{\dagger} a_j'^{\dagger} a_l' a_k': \\
& + \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{mijmkl}^{\mathrm{ppn}} :a_i'^{\dagger} b_j'^{\dagger} b_l' a_k': + \frac{1}{4} \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{mijmkl}^{\mathrm{pnn}} :b_i'^{\dagger} b_j'^{\dagger} b_l' b_k': \\
& + \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{ijmklm}^{\mathrm{pnn}} :a_i'^{\dagger} b_j'^{\dagger} b_l' a_k': + \frac{1}{4} \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{ijmklm}^{\mathrm{pp}\Lambda} :a_i'^{\dagger} a_j'^{\dagger} a_l' a_k': \\
& + \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{mijmkl}^{\mathrm{pp}\Lambda} :a_i'^{\dagger} c_j'^{\dagger} c_l' a_k': + \frac{1}{4} \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{ijmklm}^{\mathrm{nn}\Lambda} :b_i'^{\dagger} b_j'^{\dagger} b_l' b_k': \\
& + \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{mijmkl}^{\mathrm{nn}\Lambda} :b_i'^{\dagger} c_j'^{\dagger} c_l' b_k': + \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{ijmklm}^{\mathrm{pn}\Lambda} :a_i'^{\dagger} b_j'^{\dagger} b_l' a_k': \\
& + \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{imjkml}^{\mathrm{pn}\Lambda} :a_i'^{\dagger} c_j'^{\dagger} c_l' a_k': + \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{mijmkl}^{\mathrm{pn}\Lambda} :b_i'^{\dagger} c_j'^{\dagger} c_l' b_k':, \qquad (2.49)
\end{aligned}
$$

the three-body part is defined as

$$
\begin{aligned}
\widehat{H}^{(3)} = & \frac{1}{36} \sum_{ijklmn} \overline{V}_{ijklmn}^{\mathrm{ppp}} :a_i'^{\dagger} a_j'^{\dagger} a_k'^{\dagger} a_n' a_m' a_l': + \frac{1}{36} \sum_{ijklmn} \overline{V}_{ijklmn}^{\mathrm{nnn}} :b_i'^{\dagger} b_j'^{\dagger} b_k'^{\dagger} b_n' b_m' b_l': \\
& + \frac{1}{4} \sum_{ijklmn} \overline{V}_{ijklmn}^{\mathrm{ppn}} :a_i'^{\dagger} a_j'^{\dagger} b_k'^{\dagger} b_n' a_m' a_l': + \frac{1}{4} \sum_{ijklmn} \overline{V}_{ijklmn}^{\mathrm{pnn}} :a_i'^{\dagger} b_j'^{\dagger} b_k'^{\dagger} b_n' b_m' a_l': \\
& + \frac{1}{4} \sum_{ijklmn} \overline{V}_{ijklmn}^{\mathrm{pp}\Lambda} :a_i'^{\dagger} a_j'^{\dagger} c_k'^{\dagger} c_n' a_m' a_l': + \frac{1}{36} \sum_{ijklmn} \overline{V}_{ijklmn}^{\mathrm{nn}\Lambda} :b_i'^{\dagger} b_j'^{\dagger} c_k'^{\dagger} c_n' b_m' b_l': \\
& + \sum_{ijklmn} \overline{V}_{ijklmn}^{\mathrm{pn}\Lambda} :a_i'^{\dagger} b_j'^{\dagger} c_k'^{\dagger} c_n' b_m' a_l': . \qquad (2.50)
\end{aligned}
$$

The interaction matrix elements in the Eqs. (2.47), (2.49), (2.50) are represented in the self-consistent basis. I.e. they are transformed from the interaction elements in the HO basis by the relations equivalent to the Eq. (2.30).

Chapter 3

Coupling of Λ with nuclear core

In this chapter, we introduce the coupling of the Λ particle with the nuclear core within the theoretical framework of the extended Tamm-Dancoff Approximation (TDA). The extension is denoted as the nucleon-Λ Tamm-Dancoff Approximation (NΛ TDA).

The NΛ TDA method describes the spectra of hypernuclei consisting of one Λ particle bound to the even-odd nuclear core. The hypernuclear wave functions are constructed by the annihilation of one nucleon from even-even nuclear system and then coupling of the even-odd core with the Λ particle through the ΛN interaction.

This chapter is organized as follows: In Section 3.1, we introduce the standard TDA method. In Section 3.2, we introduce the NΛ TDA method as a generalization of the TDA. In Section 3.3, we discuss possible extensions of our theoretical approach to general coupling of the Λ with multi-particle-hole excitations of the nuclear core.

3.1 Tamm-Dancoff approximation

The Hartree-Fock wave function (2.36) describes the hypernuclear ground state within the mean-field approximation. The general excitations of this state can be represented as the sum of one-, two-, and many- particle-hole excitations of the HF ground state. Here, we restrict ourselves on the basic approach, i.e. one-particle-hole excitations. This approach is called the Tamm-Dancoff Approximation (TDA).

The starting point of the TDA is the self-consistent basis obtained in the HF calculations. The operator which creates the particle-hole excitation is defined as follows:

$$Q^{\dagger}_{\mu} = \sum_{ph} \left(C^{\mu,\mathrm{p}}_{ph} a^{\prime\dagger}_{p} a^{\prime}_{\bar{h}} + C^{\mu,\mathrm{n}}_{ph} b^{\prime\dagger}_{p} b^{\prime}_{\bar{h}} \right), \tag{3.1}$$

where $a'^{\dagger}_{p}, a'_{\overline{h}}$ ($b'^{\dagger}_{p}, b'_{\overline{h}}$) denote the proton (neutron) creation and annihilation operators with respect to the particle (p) and hole (h) single-particle states in the self-consistent basis. In this thesis, we refer to the operator $Q^{\dagger}_{\mu}$ as the phonon operator. We use a following phase convention

$$|\overline{h}\rangle = |\alpha_h\ j_h\ \overline{m}_h\rangle = (-1)^{j_h+m_h}|\alpha_h\ j_h\ -m_h\rangle. \tag{3.2}$$

The need for this convention arises from the fact that the hole single-particle states $|\overline{h}\rangle$ transform under rotations as a jm spherical tensor [47]. The coefficients $C^{\mu,\mathrm{p}}_{ph}$ ($C^{\mu,\mathrm{n}}_{ph}$) in Eq. (3.1) represent the linear combinations of the proton (neutron) particle-hole (ph) excitations.

The derivation of the TDA method starts from the hypernuclear Hamiltonian in the separable form (2.46) with all matrix elements expressed in the self-consistent basis. The HF density in the self-consistent basis is defined as

$$\overline{\rho}^{\mathrm{p}}_{nm} = \langle \mathrm{HF}|a'^{\dagger}_{m}a'_{n}|\mathrm{HF}\rangle = \sum_{m-\mathrm{occ.}} \delta_{nm}, \tag{3.3a}$$

$$\overline{\rho}^{\mathrm{n}}_{nm} = \langle \mathrm{HF}|b'^{\dagger}_{m}b'_{n}|\mathrm{HF}\rangle = \sum_{m-\mathrm{occ.}} \delta_{nm}, \tag{3.3b}$$

$$\overline{\rho}^{\Lambda}_{nm} = \langle \mathrm{HF}|c'^{\dagger}_{m}c'_{n}|\mathrm{HF}\rangle = \sum_{m-\mathrm{occ.}} \delta_{nm}. \tag{3.3c}$$

We introduce the normal ordered interaction elements in the two-body part of the Hamiltonian (2.49)

$$\overline{V}^{\mathrm{pp,gen}}_{ijkl} = \overline{V}^{\mathrm{pp}}_{ijkl} + \sum_{mn}\left[\overline{V}^{\mathrm{ppp}}_{ijmkln}\overline{\rho}^{\mathrm{p}}_{nm} + \overline{V}^{\mathrm{ppn}}_{ijmkln}\overline{\rho}^{\mathrm{n}}_{nm} + \overline{V}^{\mathrm{pp}\Lambda}_{ijmkln}\overline{\rho}^{\Lambda}_{nm}\right], \tag{3.4a}$$

$$\overline{V}^{\mathrm{nn,gen}}_{ijkl} = \overline{V}^{\mathrm{nn}}_{ijkl} + \sum_{mn}\left[\overline{V}^{\mathrm{nnn}}_{ijmkln}\overline{\rho}^{\mathrm{n}}_{nm} + \overline{V}^{\mathrm{nnp}}_{ijmkln}\overline{\rho}^{\mathrm{p}}_{nm} + \overline{V}^{\mathrm{nn}\Lambda}_{ijmkln}\overline{\rho}^{\Lambda}_{nm}\right], \tag{3.4b}$$

$$\overline{V}^{\mathrm{pn,gen}}_{ijkl} = \overline{V}^{\mathrm{pn}}_{ijkl} + \sum_{mn}\left[\overline{V}^{\mathrm{ppn}}_{mijnkl}\overline{\rho}^{\mathrm{p}}_{nm} + \overline{V}^{\mathrm{pnn}}_{ijmkln}\overline{\rho}^{\mathrm{n}}_{nm} + \overline{V}^{\mathrm{pn}\Lambda}_{ijmkln}\overline{\rho}^{\Lambda}_{nm}\right], \tag{3.4c}$$

$$\overline{V}^{\mathrm{p}\Lambda,\mathrm{gen}}_{ijkl} = \overline{V}^{\mathrm{p}\Lambda}_{ijkl} + \sum_{mn}\left[\overline{V}^{\mathrm{pp}\Lambda}_{mijnkl}\overline{\rho}^{\mathrm{p}}_{nm} + \overline{V}^{\mathrm{pn}\Lambda}_{imjknl}\overline{\rho}^{\mathrm{n}}_{nm}\right], \tag{3.4d}$$

$$\overline{V}^{\mathrm{n}\Lambda,\mathrm{gen}}_{ijkl} = \overline{V}^{\mathrm{n}\Lambda}_{ijkl} + \sum_{mn}\left[\overline{V}^{\mathrm{nn}\Lambda}_{mijnkl}\overline{\rho}^{\mathrm{n}}_{nm} + \overline{V}^{\mathrm{pn}\Lambda}_{mijnkl}\overline{\rho}^{\mathrm{p}}_{nm}\right]. \tag{3.4e}$$

The matrix elements $\overline{V}^{\mathrm{pp,gen}}_{ijkl}, \overline{V}^{\mathrm{nn,gen}}_{ijkl}, \overline{V}^{\mathrm{pn,gen}}_{ijkl}, \overline{V}^{\mathrm{p}\Lambda,\mathrm{gen}}_{ijkl}, \overline{V}^{\mathrm{n}\Lambda,\mathrm{gen}}_{ijkl}$ represent the two-body interaction corrected by the presence of the three-body force in the A-body nuclear system. The terms in Eq. (2.50) correspond to the residual three-body interaction. It is the remaining part of the three-body force which was not summed into the parts of the Hamiltonian $E_{\mathrm{HF}}, \widehat{H}^{(1)}$, and $\widehat{H}^{(2)}$.

The two-body part $\widehat{H}^{(2)}$ of the Hamiltonian (2.46) is expressed with the normal-ordered two-body matrix elements as

$$\begin{aligned}\widehat{H}^{(2)} = \frac{1}{4}\sum_{ijkl}\overline{V}^{\mathrm{pp,gen}}_{ijkl} : a_i'^{\dagger}a_j'^{\dagger}a_l'a_k' : + \frac{1}{4}\sum_{ijkl}\overline{V}^{\mathrm{nn,gen}}_{ijkl} : b_i'^{\dagger}b_j'^{\dagger}b_l'b_k' : \\ + \sum_{ijkl}\overline{V}^{\mathrm{pn,gen}}_{ijkl} : a_i'^{\dagger}b_j'^{\dagger}b_l'a_k' : + \sum_{ijkl}\overline{V}^{\mathrm{p\Lambda,gen}}_{ijkl} : a_i'^{\dagger}c_j'^{\dagger}c_k'a_l' : \\ + \sum_{ijkl}\overline{V}^{\mathrm{n\Lambda,gen}}_{ijkl} : b_i'^{\dagger}c_j'^{\dagger}c_l'b_k' : \end{aligned} \tag{3.5}$$

The TDA method is based on the following equation of motion

$$\langle \mathrm{HF}|Q_{\nu'}[\widehat{H}, Q_\nu^\dagger]|\mathrm{HF}\rangle = (E_\nu - E_{\mathrm{HF}})\delta_{\nu\nu'}. \tag{3.6}$$

The commutator $[\widehat{H}, Q_\nu^\dagger]$ in Eq. (3.6) can be evaluated term by term from the Hamiltonian (2.46). The HF energy E_{HF} is and thus

$$[E_{\mathrm{HF}}, Q_\nu^\dagger]|\mathrm{HF}\rangle = 0. \tag{3.7}$$

Furthermore, we show that

$$\begin{aligned}[\widehat{H}^{(1)}, Q_\nu^\dagger]|\mathrm{HF}\rangle = \sum_{ph}(\varepsilon_p^{\mathrm{p}} - \varepsilon_h^{\mathrm{p}})C_{ph}^{\nu,\mathrm{p}}a_p'^{\dagger}a_{\overline{h}}'|\mathrm{HF}\rangle \\ + \sum_{ph}(\varepsilon_p^{\mathrm{n}} - \varepsilon_h^{\mathrm{n}})C_{ph}^{\nu,\mathrm{n}}b_p'^{\dagger}b_{\overline{h}}'|\mathrm{HF}\rangle. \end{aligned} \tag{3.8}$$

The two-body part of the Hamiltonian commutes with the phonon operator as follows:

$$\begin{aligned}[\widehat{H}^{(2)}, Q_\nu^\dagger]|\mathrm{HF}\rangle = \Bigg\{ - \sum_{ph}\sum_{p_1h_1}\overline{V}^{\mathrm{pp,gen}}_{p_1\overline{h}ph_1}C_{ph}^{\nu,\mathrm{p}}a_{p_1}'^{\dagger}a_{h_1}' - \sum_{ph}\sum_{p_1h_1}\overline{V}^{\mathrm{nn,gen}}_{p_1\overline{h}ph_1}C_{ph}^{\nu,\mathrm{n}}b_{p_1}'^{\dagger}b_{h_1}' \\ + \sum_{ph}\sum_{p_1h_1}\overline{V}^{\mathrm{pn,gen}}_{\overline{h}p_1ph_1}C_{ph}^{\nu,\mathrm{p}}b_{p_1}'^{\dagger}b_{h_1}' + \sum_{ph}\sum_{p_1h_1}\overline{V}^{\mathrm{pn,gen}}_{\overline{h}p_1ph_1}C_{ph}^{\nu,\mathrm{n}}a_{p_1}'^{\dagger}a_{h_1}' \Bigg\}|\mathrm{HF}\rangle. \end{aligned} \tag{3.9}$$

The term $[\widehat{H}^{(3)}, Q_\nu^\dagger]|\mathrm{HF}\rangle$ does not contribute to the TDA equation,

$$\langle \mathrm{HF}|Q_{\nu'}[\widehat{H}^{(3)}, Q_\nu^\dagger]|\mathrm{HF}\rangle = 0. \tag{3.10}$$

Substituting (3.8) and (3.9) into the TDA equation (3.6) leads to the eigenvalue problem

$$\sum_{ph}\begin{pmatrix} (\varepsilon_p^{\mathrm{p}} - \varepsilon_h^{\mathrm{p}})\delta_{pp'}\delta_{hh'} + \overline{V}^{\mathrm{pn,gen}}_{p'\overline{hh}'p} & \overline{V}^{\mathrm{pn,gen}}_{p'\overline{hh}'p} \\ \overline{V}^{\mathrm{pn,gen}}_{\overline{h}p'p\overline{h}'} & (\varepsilon_p^{\mathrm{p}} - \varepsilon_h^{\mathrm{p}})\delta_{pp'}\delta_{hh'} + \overline{V}^{\mathrm{pn,gen}}_{p'\overline{hh}'p} \end{pmatrix}\begin{pmatrix} C_{ph}^{\nu,\mathrm{p}} \\ C_{ph}^{\nu,\mathrm{n}} \end{pmatrix} = (E_\nu - E_{\mathrm{HF}})\begin{pmatrix} C_{p'h'}^{\nu,\mathrm{p}} \\ C_{p'h'}^{\nu,\mathrm{n}} \end{pmatrix}. \tag{3.11}$$

It should be noted that the operators $\overline{V}^{\mathrm{p}\Lambda,\mathrm{gen}}_{ijkl}$ and $\overline{V}^{\mathrm{n}\Lambda,\mathrm{gen}}_{ijkl}$ get subtracted in the commutators. Therefore, the ΛN interaction does not contribute to the TDA equation (3.11).

3.2 NΛ TDA method

The principle of the NΛ TDA method is the annihilation of one nucleon from the even-even nuclear system and coupling of this even-odd (or odd-even) core with the Λ hyperon. In analogy to the Eq. (3.1), the NΛ TDA phonon operators are defined as

$$R^{\dagger}_{\mu,\mathrm{p}\Lambda} = \sum_{ph} r^{\mu,\mathrm{p}\Lambda}_{ph} c'^{\dagger}_{p} a'_{\overline{h}}, \tag{3.12a}$$

$$R^{\dagger}_{\mu,\mathrm{n}\Lambda} = \sum_{ph} r^{\mu,\mathrm{n}\Lambda}_{ph} c'^{\dagger}_{p} b'_{\overline{h}}. \tag{3.12b}$$

The equations of motion of the NΛ TDA method are again formulated in analogy to the TDA:

$$\langle \mathrm{HF}|R_{\nu',\mathrm{p}\Lambda}[\widehat{H}, R^{\dagger}_{\nu,\mathrm{p}\Lambda}]|\mathrm{HF}\rangle = (E^{\mathrm{p}\Lambda}_{\nu} - E_{\mathrm{HF}})\delta_{\nu\nu'}, \tag{3.13a}$$

$$\langle \mathrm{HF}|R_{\nu',\mathrm{n}\Lambda}[\widehat{H}, R^{\dagger}_{\nu,\mathrm{n}\Lambda}]|\mathrm{HF}\rangle = (E^{\mathrm{n}\Lambda}_{\nu} - E_{\mathrm{HF}})\delta_{\nu\nu'}. \tag{3.13b}$$

By inserting the Hamiltonian $\widehat{H}$ as a sum of terms $E_{\mathrm{HF}}, \widehat{H}^{(1)}, \widehat{H}^{(2)}$, and $\widehat{H}^{(3)}$, we evaluate the commutation relations $[\widehat{H}, R^{\dagger}_{\nu,\mathrm{p}\Lambda}]$ and $[\widehat{H}, R^{\dagger}_{\nu,\mathrm{n}\Lambda}]$ term by term,

$$[E_{\mathrm{HF}}, R^{\dagger}_{\nu,\mathrm{p}\Lambda}]|\mathrm{HF}\rangle = 0, \tag{3.14a}$$

$$[E_{\mathrm{HF}}, R^{\dagger}_{\nu,\mathrm{n}\Lambda}]|\mathrm{HF}\rangle = 0, \tag{3.14b}$$

$$[\widehat{H}^{(1)}, R^{\dagger}_{\nu,\mathrm{p}\Lambda}]|\mathrm{HF}\rangle = \sum_{ph}(\varepsilon^{\Lambda}_{p} - \varepsilon^{\mathrm{p}}_{h}) r^{\nu,\mathrm{p}\Lambda}_{ph} c'^{\dagger}_{p} a'_{\overline{h}}|\mathrm{HF}\rangle, \tag{3.15a}$$

$$[\widehat{H}^{(1)}, R^{\dagger}_{\nu,\mathrm{n}\Lambda}]|\mathrm{HF}\rangle = \sum_{ph}(\varepsilon^{\Lambda}_{p} - \varepsilon^{\mathrm{n}}_{h}) r^{\nu,\mathrm{n}\Lambda}_{ph} c'^{\dagger}_{p} b'_{\overline{h}}|\mathrm{HF}\rangle, \tag{3.15b}$$

$$[\widehat{H}^{(2)}, R^{\dagger}_{\nu,\mathrm{p}\Lambda}]|\mathrm{HF}\rangle = -\sum_{p_1h_1}\sum_{ph} \overline{V}^{\mathrm{p}\Lambda,\mathrm{gen}}_{\overline{h}p_1h_1p} c'^{\dagger}_{p_1} a'_{h_1}|\mathrm{HF}\rangle, \tag{3.16a}$$

$$[\widehat{H}^{(2)}, R^{\dagger}_{\nu,\mathrm{n}\Lambda}]|\mathrm{HF}\rangle = -\sum_{p_1h_1}\sum_{ph} \overline{V}^{\mathrm{n}\Lambda,\mathrm{gen}}_{\overline{h}p_1h_1p} c'^{\dagger}_{p_1} b'_{h_1}|\mathrm{HF}\rangle. \tag{3.16b}$$

Again, it applies that

$$\langle \mathrm{HF}|R_{\nu',\mathrm{p}\Lambda}[\widehat{H}^{(3)}, R^{\dagger}_{\nu,\mathrm{p}\Lambda}]|\mathrm{HF}\rangle = \langle \mathrm{HF}|R_{\nu',\mathrm{n}\Lambda}[\widehat{H}^{(3)}, R^{\dagger}_{\nu,\mathrm{n}\Lambda}]|\mathrm{HF}\rangle = 0. \tag{3.17a}$$

The substitution of the Eqs. (3.15a) and (3.16a) into Eq. (3.13a) (and Eqs. (3.15b) and (3.16b) into Eq. (3.13b)) leads to

$$\langle \mathrm{HF}|R_{\nu',\mathrm{p}\Lambda}[\widehat{H}, R^{\dagger}_{\nu,\mathrm{p}\Lambda}]|\mathrm{HF}\rangle = \sum_{ph}\sum_{p'h'} r^{\nu',\mathrm{p}\Lambda}_{p'h'} r^{\nu,\mathrm{p}\Lambda}_{ph} \left((\varepsilon^{\Lambda}_{p} - \varepsilon^{\mathrm{p}}_{h})\delta_{pp'}\delta_{hh'} - \overline{V}^{\mathrm{p}\Lambda,\mathrm{gen}}_{\overline{h}p'\overline{h}'p} \right), \tag{3.18a}$$

$$\langle \mathrm{HF}|R_{\nu',\mathrm{n}\Lambda}[\widehat{H}, R^{\dagger}_{\nu,\mathrm{n}\Lambda}]|\mathrm{HF}\rangle = \sum_{ph}\sum_{p'h'} r^{\nu',\mathrm{n}\Lambda}_{p'h'} r^{\nu,\mathrm{n}\Lambda}_{ph} \left((\varepsilon^{\Lambda}_{p} - \varepsilon^{\mathrm{n}}_{h})\delta_{pp'}\delta_{hh'} - \overline{V}^{\mathrm{n}\Lambda,\mathrm{gen}}_{\overline{h}p'\overline{h}'p} \right). \tag{3.18b}$$

Eqs. (3.18a) and (3.18b) correspond to the following eigenvalue problems:

$$\sum_{ph} \left((\varepsilon^{\Lambda}_{p} - \varepsilon^{\mathrm{p}}_{h})\delta_{pp'}\delta_{hh'} - \overline{V}^{\mathrm{p}\Lambda,\mathrm{gen}}_{\overline{h}p'\overline{h}'p} \right) r^{\nu,\mathrm{p}\Lambda}_{ph} = (E^{\mathrm{p}\Lambda}_{\nu} - E_{\mathrm{HF}}) r^{\nu,\mathrm{p}\Lambda}_{p'h'}, \tag{3.19a}$$

$$\sum_{ph} \left((\varepsilon^{\Lambda}_{p} - \varepsilon^{\mathrm{n}}_{h})\delta_{pp'}\delta_{hh'} - \overline{V}^{\mathrm{n}\Lambda,\mathrm{gen}}_{\overline{h}p'\overline{h}'p} \right) r^{\nu,\mathrm{n}\Lambda}_{ph} = (E^{\mathrm{n}\Lambda}_{\nu} - E_{\mathrm{HF}}) r^{\nu,\mathrm{n}\Lambda}_{p'h'}. \tag{3.19b}$$

In practice, $\overline{V}^{\mathrm{p(n)}\Lambda,\mathrm{gen}}_{ijkl} \equiv \overline{V}^{\mathrm{p(n)}\Lambda}_{ijkl}$. We have developed a mathematical formalism of the NΛ TDA method which includes the three-body ΛNN interactions but we have not implemented the ΛNN interactions themselves in the numerical calculations. Here, we study the indirect effect of the NNN force which is accounted for in the HF calculations of the nuclear core (i.e. construction of the self-consistent basis).

3.3 General coupling of Λ with multi-particle-hole excitations of nuclear core

In Chapter 2, we derived the Hartree-Fock method in the proton-neutron-Λ formalism which is suitable for description of hypernuclear systems which consist of even-even nuclear core and one bound Λ hyperon.

In Section 3.1, we discussed the excitations of the even-even nuclear core within the TDA method as a general superposition of one-particle one-hole excitations of the HF ground state.

In Section 3.2, we generalized the TDA method and derived the NΛ TDA method which can be used for the calculations of hypernuclei with even-odd (or odd-even) nuclear core and one bound Λ hyperon.

The methods which are used for calculations of hypernuclear spectra (HF and NΛ TDA) can be understood as a starting point for two different generalizations of

the Equation of Motion Phonon Method (EMPM) [26, 27, 28]. Within the EMPM we split the Hilbert space into the direct sum of the n Hilbert subspaces:

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots \oplus \mathcal{H}_n. \tag{3.20}$$

In the first generalization we consider the coupling of the Λ particle with exictations of the nuclear core and thus construct the Hilbert subspaces in the following way:

$$\mathcal{H}_0 = \left\{ c'^{\dagger}_p |\mathrm{HF}\rangle_\mathrm{p} \otimes |\mathrm{HF}\rangle_\mathrm{n} \right\}, \tag{3.21a}$$
$$\mathcal{H}_1 = \left\{ Q^{\dagger}_{\mu_1} c'^{\dagger}_p |\mathrm{HF}\rangle_\mathrm{p} \otimes |\mathrm{HF}\rangle_\mathrm{n} \right\}, \tag{3.21b}$$
$$\mathcal{H}_2 = \left\{ Q^{\dagger}_{\mu_1} Q^{\dagger}_{\mu_2} c'^{\dagger}_p |\mathrm{HF}\rangle_\mathrm{p} \otimes |\mathrm{HF}\rangle_\mathrm{n} \right\}, \tag{3.21c}$$
$$\vdots$$
$$\mathcal{H}_n = \left\{ Q^{\dagger}_{\mu_1} Q^{\dagger}_{\mu_2} \ldots Q^{\dagger}_{\mu_n} c'^{\dagger}_p |\mathrm{HF}\rangle_\mathrm{p} \otimes |\mathrm{HF}\rangle_\mathrm{n} \right\}. \tag{3.21d}$$

The diagonalization of the hypernuclear Hamiltonian (2.32) in the Hilbert space defined in such way is suitable for description of hypernuclear systems with even-even core and one Λ particle.

In the second generalization we consider the excitations of the hypernuclear system created through the pΛ TDA (and nΛ TDA) method. The Hilbert subspaces are then generated as follows:

$$\mathcal{H}_0 = \left\{ R^{\dagger}_{\nu,p(n)\Lambda} |\mathrm{HF}\rangle_\mathrm{p} \otimes |\mathrm{HF}\rangle_\mathrm{n} \right\}, \tag{3.22a}$$
$$\mathcal{H}_1 = \left\{ Q^{\dagger}_{\mu_1} R^{\dagger}_{\nu,p(n)\Lambda} |\mathrm{HF}\rangle_\mathrm{p} \otimes |\mathrm{HF}\rangle_\mathrm{n} \right\}, \tag{3.22b}$$
$$\mathcal{H}_2 = \left\{ Q^{\dagger}_{\mu_1} Q^{\dagger}_{\mu_2} R^{\dagger}_{\nu,p(n)\Lambda} |\mathrm{HF}\rangle_\mathrm{p} \otimes |\mathrm{HF}\rangle_\mathrm{n} \right\}, \tag{3.22c}$$
$$\vdots$$
$$\mathcal{H}_n = \left\{ Q^{\dagger}_{\mu_1} Q^{\dagger}_{\mu_2} \ldots Q^{\dagger}_{\mu_n} R^{\dagger}_{\nu,p(n)\Lambda} |\mathrm{HF}\rangle_\mathrm{p} \otimes |\mathrm{HF}\rangle_\mathrm{n} \right\}. \tag{3.22d}$$

Similarly, diagonalization of the Hamiltonian (2.32) in this Hilbert space is suitable for description of hypernuclei with even-odd (or odd-even) nuclear cores and one Λ particle.

Chapter 4

Numerical implementation of the three-body force

The Hartree-Fock equations (2.43), (2.44), and (2.45) are solved numerically by the code which is an extension of the code that was used for the study of multipole response in neutron-rich nuclei [25].

All interaction elements of the NN, ΛN, and the NNN interactions are represented and stored in the J-scheme formalism (see Appendix B). I.e. we work with the J-coupled two-body elements $V^{J,\mathrm{pp}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)}$, $V^{J,\mathrm{pn}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)}$, $V^{J,\mathrm{nn}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)}$, $V^{J,\mathrm{p}\Lambda}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)}$, $V^{J,\mathrm{n}\Lambda}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)}$, and with the JT-coupled three-body elements $V^{J'_{12},J_{12},J,T'_{12},T_{12},T}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l),(n_m l_m j_m),(n_n l_n j_n)}$.

We need to place restrictions on the indices i, j, k, l, m, n which enumerate the $(n_\bullet l_\bullet j_\bullet)$ configurations within the NNN matrix elements to reduce the demands on the computer memory. We can use the antisymmetry and store only the matrix elements for the indices

$$i \geq j \geq k, \tag{4.1a}$$

$$l \geq m \geq n. \tag{4.1b}$$

Furthermore, we can use the fact the the NNN matrix elements are hermitian. Thus we can introduce another restriction which can be defined as $(ijk) \geq (lmn)$, e.g.

$$(i \cdot 10^6 + j \cdot 10^3 + k) \geq (l \cdot 10^6 + m \cdot 10^3 + n). \tag{4.2}$$

In Eq. (4.2), we suppose that we work within a single-particle basis with dim $< 10^3$.

The matrix elements of the NNN interaction that are not stored due to the restrictions (4.1a), (4.1b), and (4.2) need to be reconstructed on the fly.

In the code, we implement the following equations for protons (neutrons), re-

spectively,

$$
\begin{aligned}
&t^{\mathrm{p(n)}}_{(n_i l_i j_i),(n_j l_j j_j)}\delta_{l_i l_j}\delta_{j_i j_j}\delta_{m_i m_j} \\
&+\sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{pp(nn)}}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\mathrm{p(n)}}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_l l_k}\delta_{j_l j_k}\delta_{m_i m_j}\frac{(2J+1)}{(2j_i+1)} \\
&+\sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{pn(np)}}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\mathrm{n(p)}}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_l l_k}\delta_{j_j j_k}\delta_{m_i m_j}\frac{(2J+1)}{(2j_i+1)} \\
&+\sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{p\Lambda(n\Lambda)}}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\Lambda(\Lambda)}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_l l_k}\delta_{j_l j_k}\delta_{m_i m_j}\frac{(2J+1)}{(2j_i+1)}\frac{1}{(2j_k+1)} \\
&+\sum_{n_k l_k j_k m_k}\sum_{n_l l_l j_j m_l}\sum_{n_m l_m j_m m_m}\sum_{n_n l_n j_n m_n} \\
&\Big\{\frac{1}{2}V^{\mathrm{ppp(nnn)}}_{n_i l_i j_i m_i,n_k l_k j_k m_k,n_l l_l j_l m_l,n_j l_j j_j m_j,n_m l_m j_m m_m,n_n l_n j_n m_n}\rho^{\mathrm{p(n)}}_{n_m l_m j_m m_m,n_k l_k j_k m_k}\rho^{\mathrm{p(n)}}_{n_n l_n j_n m_n,n_l l_l j_l m_l} \\
&+\frac{1}{2}V^{\mathrm{pnn(ppn)}}_{n_i l_i j_i m_i,n_k l_k j_k m_k,n_l l_l j_l m_l,n_j l_j j_j m_j,n_m l_m j_m m_m,n_n l_n j_n m_n}\rho^{\mathrm{n(p)}}_{n_m l_m j_m m_m,n_k l_k j_k m_k}\rho^{\mathrm{n(n)}}_{n_n l_n j_n m_n,n_l l_l j_l m_l} \\
&+V^{\mathrm{ppn(pnn)}}_{n_i l_i j_i m_i,n_k l_k j_k m_k,n_l l_l j_l m_l,n_j l_j j_j m_j,n_m l_m j_m m_m,n_n l_n j_n m_n}\rho^{\mathrm{p(n)}}_{n_m l_m j_m m_m,n_k l_k j_k m_k}\rho^{\mathrm{n(n)}}_{n_n l_n j_n m_n,n_l l_l j_l m_l}\Big\} \\
&\qquad\qquad = \varepsilon^{\mathrm{p(n)}}_i \delta_{ij}. \qquad (4.3)
\end{aligned}
$$

The elements of the NNN interactions in Eq. (4.3) are decoupled into the M-scheme from the JT-coupled elements on the fly by using the transformation equations (B.9),(B.10), (B.11), and (B.12). In addition, the following HF equation for the Λ hyperon is implemented,

$$
\begin{aligned}
&t^{\Lambda}_{(n_i l_i j_i),(n_j l_j j_j)}\delta_{l_i l_j}\delta_{j_i j_j}\delta_{m_i m_j} \\
&+\sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{p\Lambda}}_{(n_k l_k j_k),(n_i l_i j_i),(n_l l_l j_l),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_k l_l}\delta_{j_k j_l}\delta_{m_i m_j}\frac{(2J+1)}{(2j_i+1)} \\
&+\sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{n\Lambda}}_{(n_k l_k j_k),(n_i l_i j_i),(n_l l_l j_l),(n_j l_j j_j)} \rho^{\mathrm{n}}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_k l_l}\delta_{j_k j_l}\delta_{m_i m_j}\frac{(2J+1)}{(2j_i+1)} \\
&\qquad\qquad = \varepsilon^{\Lambda}_i \delta_{ij}. \qquad (4.4)
\end{aligned}
$$

The HF code can run either in the static or in the dynamic mode. The code in the static mode at first solves the equations for protons and neutrons (4.3) without the proton-Λ and the neutron-Λ interactions. Afterwards, it solves the equation (4.4) for the Λ hyperon. In the static mode, the properties of protons and neutrons in the nuclear core are not affected by the presence of the Λ hyperon. We use the static mode for calculations of the bare nuclear core. The code in the dynamic mode

solves the equations (4.3) and (4.4) self-consistently for the whole hypernucleus. We use the dynamic mode for calculations of the Λ single-particle spectra.

The HF method is implemented in the spherical HO basis. This basis is infinite. In practical calculations, the basis is truncated by the maximal major shell number $N_{\max}$. I.e. the single-particle configuration space is defined as a set of the single-particle states $\{|i\rangle : 2n_i + l_i = N_i \leq N_{\max}\}$. The number n_i stands for the principal quantum number and l_i is the orbital angular momentum. The two-body interaction operators are represented as the matrix elements of the products of the two single-particle states $|i\rangle|j\rangle = |ij\rangle$, $|k\rangle|l\rangle = |kl\rangle$ (see equations (A.7a), (A.7b)). The two-particle basis is truncated consistently with the single-particle basis by the number $N_{\max}^{(12)}$. I.e. the two-particle configuration space is defined as a set of the states $\{|ij\rangle : 2n_i + l_i + 2n_j + l_j = N_i + N_j \leq 2N_{\max} = N_{\max}^{(12)}\}$. Similar logic applies to the three-body operators which are represented as matrix elements of products of the three single-particle states $|i\rangle|j\rangle|k\rangle = |ijk\rangle$, $|l\rangle|m\rangle|n\rangle = |lmn\rangle$. The three-particle basis is truncated by the number $N_{\max}^{(123)}$. The configuration space is defined as a set of the states $\{|ijk\rangle : 2n_i+l_i+2n_j+l_j+2n_k+l_k = N_i+N_j+N_k \leq 3N_{\max} = N_{\max}^{(123)}\}$. The conditions $N_{\max}^{(12)} = 2N_{\max}$ and $N_{\max}^{(123)} = 3N_{\max}$ lead to computational problems for any configuration space bigger than $N_{\max} = 5$. In such space, it is very complicated to generate, store, and operate with all three-body interaction matrix elements. For this reason we constrain the configuration space by the following condition $N_{\max} = N_{\max}^{(12)} = N_{\max}^{(123)}$.

The width of the potential well of the spherical harmonic oscillator is given by the parameter $\hbar\omega$. In this work, the parameter $\hbar\omega$ is fixed to 16 MeV.

In the NΛ TDA method, the three-body NNN interaction does not enter the equations explicitly as in the case of the HF method in the p-n-Λ formalism. The residual three-body interaction defined in the Eq. (2.50) does not contribute to the NΛ TDA Eqs. (3.19a) and (3.19b), respectively. The NNN force contributes only to the generalized two-body interaction terms defined in Eqs. (3.4a) - (3.4e). These matrix elements are called generalized interaction elements.

Chapter 5

Results

In this thesis, we study the effect of the three-body NNN interaction on the description of nuclei ^{16}O, ^{40}Ca, ^{48}Ca, and the single-Λ hypernuclei $^{17}_{\Lambda}$O, $^{41}_{\Lambda}$Ca, $^{49}_{\Lambda}$Ca, $^{16}_{\Lambda}$O, $^{40}_{\Lambda}$Ca, $^{48}_{\Lambda}$Ca. We calculate the radial density distributions, charge radii, nucleon single-particle spectra, and binding energies of ^{16}O, ^{40}Ca, and ^{48}Ca in the mean-field approximation by solving the Hartree-Fock (HF) Eq. (4.3) in the static mode. The single-Λ hypernuclei $^{17}_{\Lambda}$O, $^{41}_{\Lambda}$Ca, and $^{49}_{\Lambda}$Ca, which consist of one Λ hyperon bound to the even-even nuclear core, are described by the HF method in the proton-neutron-Λ (p-n-Λ) formalism. We calculate their Λ single-particle spectra by solving the HF Eqs. (4.3) and (4.4) in the dynamical mode. The single-Λ hypernuclei $^{16}_{\Lambda}$O, $^{40}_{\Lambda}$Ca, and $^{48}_{\Lambda}$Ca consist of one Λ particle bound to the even-odd nuclear core. We generate their energy spectra by first solving the HF Eqs. (4.3) and (4.4) in the static mode and then solving the NΛ TDA Eq. (3.19b).

In our calulcations, we do not assume short-range correlations in our Hamiltonian, $\Lambda - \Sigma$ mixing in the YN interaction [48], and many-body correlations. These issues are beyond the scope of this thesis.

We employ the chiral N^{2}LO$_{\text{sat}}$ NN and NNN interaction [34] which represents the force acting among nucleons. The force acting between the Λ hyperon and the nucleons is described by the NΛ–NΛ channel of the chiral LO YN interaction [41].

The configuration space was fixed by $N_{\max}$. We use the constraint $N_{\max} = N_{\max}^{(12)} = N_{\max}^{(123)}$. The parameter $\hbar\omega$ was set to 16 MeV for all calculations.

5.1 Calculations of ^{16}O, ^{40}Ca, and ^{48}Ca

In this section, we study the nuclei ^{16}O, ^{40}Ca, and ^{48}Ca within the mean-field approximation. First we check the convergence of the radial density distributions, charge radii, and the nucleon single-particle energies. Then we study the effect of

the NNN force on these observables.

The radial density distribution $\rho(r)$ is defined as follows:

$$\rho(r) = \rho_{\mathrm{p}}(r) + \rho_{\mathrm{n}}(r), \tag{5.1}$$

where $\rho_{\mathrm{p}}(r)$ and $\rho_{\mathrm{n}}(r)$ are proton and neutron density distributions, respectively. We calculate the proton radial density distribution $\rho_{\mathrm{p}}(r)$ as:

$$\rho_{\mathrm{p}}(r) = \frac{1}{4\pi} \sum_{ij} \sum_{k-\mathrm{occ.}} R_{n_i l_i}(r,b) R_{n_j l_j}(r,b) A_{kj} A_{ki} (2j_k + 1). \tag{5.2}$$

The neutron radial density distribution $\rho_{\mathrm{n}}(r)$ is of the form:

$$\rho_{\mathrm{n}}(r) = \frac{1}{4\pi} \sum_{ij} \sum_{k-\mathrm{occ.}} R_{n_i l_i}(r,b) R_{n_j l_j}(r,b) B_{kj} B_{ki} (2j_k + 1). \tag{5.3}$$

The functions $R_{n_i l_i}(r,b)$ and $R_{n_j l_j}(r,b)$ in Eqs. (5.2) and (5.3) are the radial parts of the HO wave functions and are discussed in detail in Appendix A. The matrices A and B represent the unitary transformations between the HO and the self-consistent bases and are defined in Chapter 2.

In Fig. 5.1, the radial density distributions of ^{16}O, ^{40}Ca, and ^{48}Ca for $N_{\max} = 6, 8, 10$, and 12 are plotted. These density distributions were calculated with NN and NNN interactions. In ^{16}O, the density distributions exhibit rapid convergence. In ^{40}Ca, the curves show slower convergence and differ for short distances under ~ 1 fm. In ^{48}Ca, the convergence is not reached. The configuration space is too small for this nucleus. We expect to reach the convergence in larger spaces.

In Figs. 5.2, 5.3, and 5.4, the radial density distributions of ^{16}O, ^{40}Ca, and ^{48}Ca, calculated with and without the NNN force, along with the ones calculated with the relativistic mean-field (RMF) NL-SH model [49], are shown. The RMF NL-SH is a phenomenological model which reproduces empirical density distributions [50]. The density distributions calculated within our model are the converged ones obtained for $N_{\max} = 12$. The NNN force has a repulsive effect which flattens and expands the density distributions. Moreover, the calculations which implement the NNN interaction yield results in qualitatively much better agreement with the empirical data.

Similar effect of the NNN force is observed for the charge radii. The mean-square charge radius $\langle r^2_{\mathrm{ch}} \rangle$ of a nucleus ${}^{A}_{Z}\mathrm{X}_{N}$ is defined as follows [51]:

$$\langle r^2_{\mathrm{ch}} \rangle = \left(1 - \frac{1}{A}\right) \langle r^2_{\mathrm{p}} \rangle + R^2_{\mathrm{p}} + \frac{N}{Z} R^2_{\mathrm{n}} + \frac{3\hbar^2}{4m_p^2 c^2}, \tag{5.4}$$

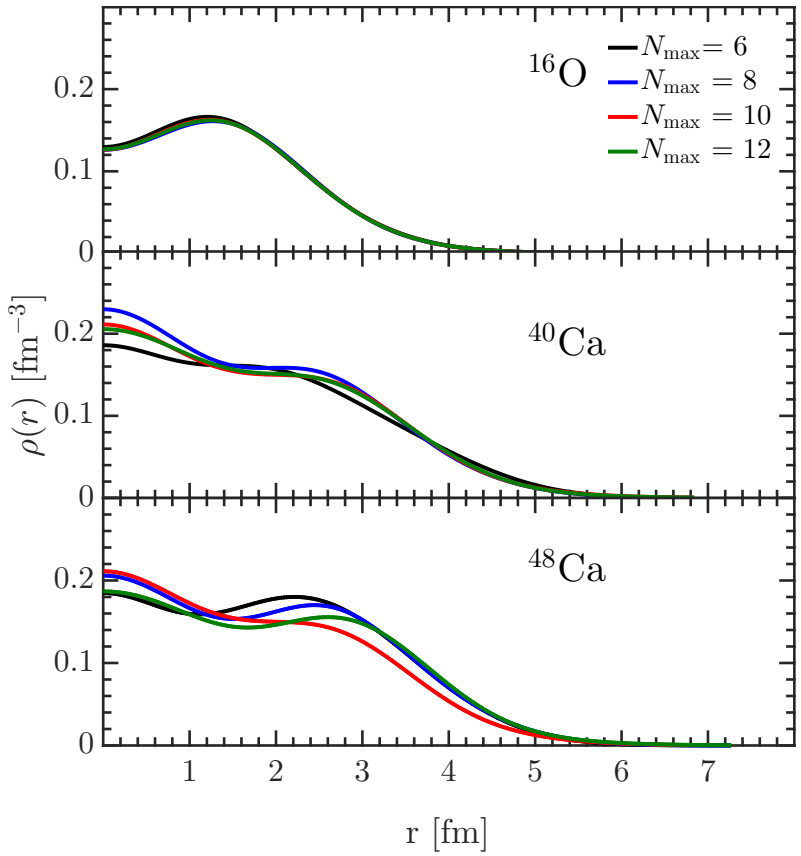

Fig. 5.1: The radial density distributions $\rho(r)$ of the nuclei ^{16}O, ^{40}Ca, and ^{48}Ca calculated with the two-body plus three-body (NN + NNN) interaction for $N_{\max} = 6, 8, 10, 12$.

where $R_p = 0.8775(51)$ fm, $R_n^2 = 0.1149(27)$ fm^2, and $\frac{3\hbar^2}{4m_p^2c^2} \sim 0.033$ fm^2. The term $\langle r_p^2 \rangle$ in Eq. (5.4) is the point proton mean-square radius

$$\langle r_p^2 \rangle = \frac{\int \mathrm{d}r\, r^4 \rho_{\mathrm{p}}(r)}{\int \mathrm{d}r\, r^2 \rho_{\mathrm{p}}(r)}. \tag{5.5}$$

The results are shown in Table 5.1. The quantity $\langle r_{\mathrm{ch}} \rangle$ is calculated as $\langle r_{\mathrm{ch}} \rangle = \sqrt{\langle r_{\mathrm{ch}}^2 \rangle}$. Nuclear charge radii calculated only with the NN interaction are unrealistically compressed, whereas charge radii calculated with the NN + NNN interaction are qualitatively in better agreement with the available experimental data.

In the HF method, the value E_{HF} approximates the ground-state energy of a given nucleus. The calculated binding energy per nucleon is defined as:

$$\frac{BE}{A} = -\frac{E_{\mathrm{HF}}}{A}. \tag{5.6}$$

The binding energies per nucleon of the ^{16}O, ^{40}Ca, and ^{48}Ca are shown in Table 5.2. The repulsive character of the NNN force decreases significantly the binding energy

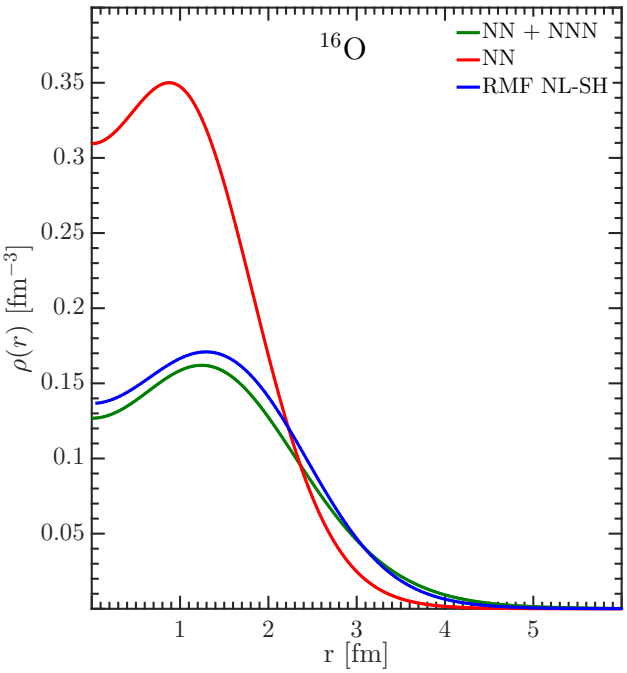

Fig. 5.2: The radial density distribution $\rho(r)$ of ^{16}O calculated only with the two-body (NN) interaction (red line), with the two-body plus three-body (NN + NNN) interaction (green line), and with the RMF model NL-SH [49] (blue line).

to unrealistic values. Generally, the binding energies calculated by the HF method from the realistic nucleon interactions do not reproduce experimental data [51]. In order to obtain realistic binding energies, we need to implement beyond mean-field correlations into our model.

Next, we study the convergence of the neutron single-particle energies $\varepsilon_i^{\mathrm{n}}$. The neutron single-particle energies calculated with the (NN + NNN) interaction are shown in Fig. 5.5. For all states which lie under the Fermi level, the convergence

Table 5.1: The charge radii $\langle r_{\mathrm{ch}} \rangle$ of the ^{16}O, ^{40}Ca, and ^{48}Ca calculated only with the NN interaction (NN) and the charged radii of the ^{16}O, ^{40}Ca, and ^{48}Ca calculated with the NN + NNN interaction (NN + NNN), compared to the experimental data (exp) taken from [52].

	$\langle r_{\mathrm{ch}} \rangle$ [fm]		
^{A}X	NN	NN + NNN	exp
^{16}O	2.24	2.96	2.70
^{40}Ca	2.62	3.68	3.48
^{48}Ca	2.59	3.60	3.47

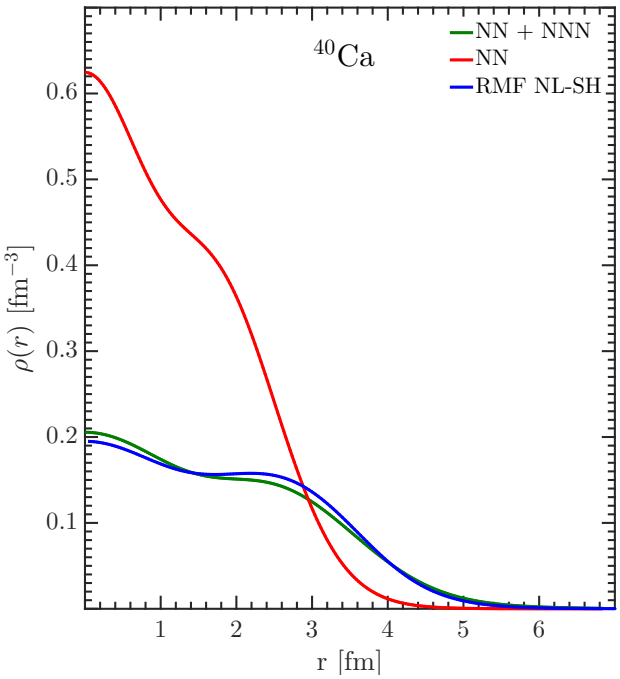

Fig. 5.3: The radial density distribution $\rho(r)$ of ^{40}Ca calculated only with the two-body (NN) interaction (red line), with the two-body plus three-body (NN + NNN) interaction (green line), and with the RMF model NL-SH [49] (blue line).

is reached. However, the convergence is much slower for all unoccupied levels. The convergence properties of the proton single-particle energies $\varepsilon_i^{\mathrm{p}}$ are analogical to the neutron ones. Therefore, we do not present them in this thesis.

In Figs. 5.6, 5.7, and 5.8, the neutron single-particle energies $\varepsilon_i^{\mathrm{n}}$ of the ^{16}O, ^{40}Ca, and ^{48}Ca calculated only with the two-body NN interaction are compared to the ones calculated with the two-body plus three-body (NN + NNN) interaction. The parameter N_{max} is fixed to 12. The empirical energies extracted from experimental data are shown as well.

Table 5.2: Binding energies per nucleon BE/A calculated with the NN interaction (NN) and with the NN + NNN interaction (NN + NNN) in ^{16}O, ^{40}Ca, and ^{48}Ca compared to the experimental values (exp).

	BE/A [MeV]		
^{A}X	NN	NN + NNN	exp
^{16}O	7.36	2.66	7.98
^{40}Ca	11.65	2.31	8.55
^{48}Ca	12.95	1.93	8.67

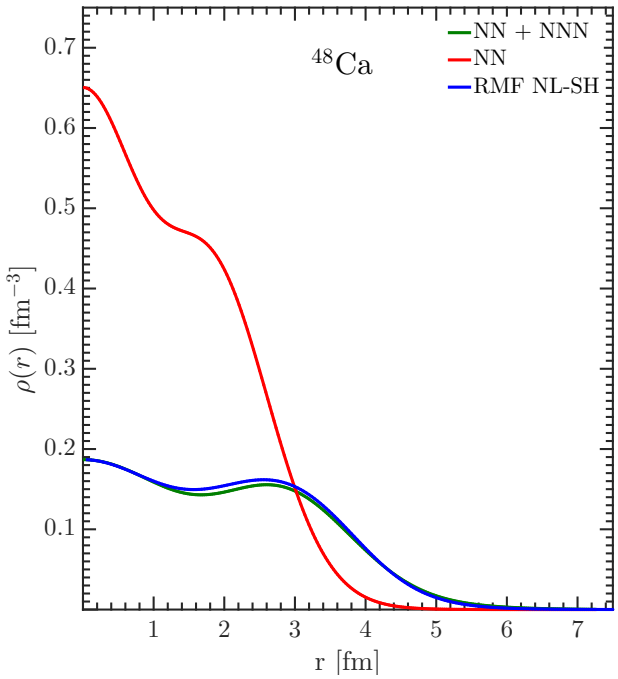

Fig. 5.4: The radial density distribution $\rho(r)$ of ^{48}Ca calculated only with the two-body (NN) interaction (red line), with the two-body plus three-body (NN + NNN) interaction (green line), and with the RMF model NL-SH [49] (blue line).

The empirical values of the binding energies are determined from the differences between binding energies of doubly-magic nuclei ^{16}O, ^{40}Ca, and ^{48}Ca and the corresponding neighboring odd-even nuclei. The single-particle energies of the unoccupied levels are calculated by the equations

$$\varepsilon^{\mathrm{n}}(^{16}\mathrm{O}) = BE(^{16}\mathrm{O}) - BE(^{17}\mathrm{O}), \tag{5.7a}$$
$$\varepsilon^{\mathrm{n}}(^{40}\mathrm{Ca}) = BE(^{40}\mathrm{Ca}) - BE(^{41}\mathrm{Ca}), \tag{5.7b}$$
$$\varepsilon^{\mathrm{n}}(^{48}\mathrm{Ca}) = BE(^{48}\mathrm{Ca}) - BE(^{49}\mathrm{Ca}), \tag{5.7c}$$

where $B(^{A}\mathrm{X})$ is the binding energy of the given nucleus. The single-particle energies of the occupied levels are obtained by

$$\varepsilon^{\mathrm{n}}(^{16}\mathrm{O}) = BE(^{15}\mathrm{O}) - BE(^{16}\mathrm{O}), \tag{5.8a}$$
$$\varepsilon^{\mathrm{n}}(^{40}\mathrm{Ca}) = BE(^{39}\mathrm{Ca}) - BE(^{40}\mathrm{Ca}), \tag{5.8b}$$
$$\varepsilon^{\mathrm{n}}(^{48}\mathrm{Ca}) = BE(^{47}\mathrm{Ca}) - BE(^{48}\mathrm{Ca}). \tag{5.8c}$$

The repulsive NNN interaction quenches the gaps between the major shells. The implementation of the NNN interaction yields results in better agreement with the available experimental data.

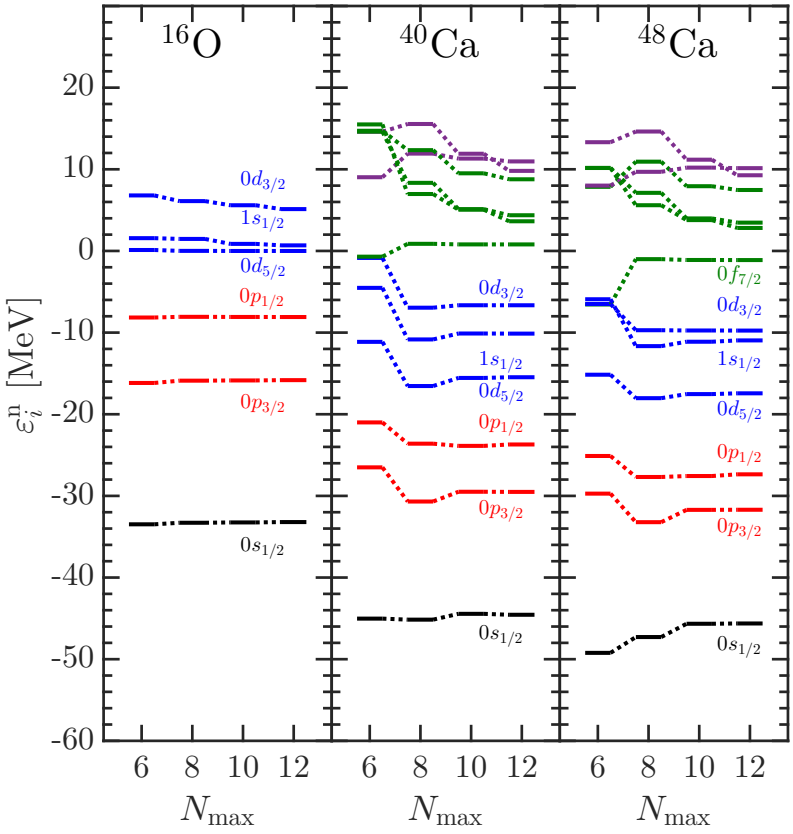

Fig. 5.5: The neutron single-particle energies $\varepsilon_i^{\mathrm{n}}$ of the nuclei ^{16}O, ^{40}Ca, and ^{48}Ca calculated with the two-body plus three-body (NN + NNN) interaction for $N_{\mathrm{max}} = 6, 8, 10, 12$.

We conclude that the effect of the three-body NNN interaction qualitatively improves the description of the radial density distributions, the charge radii, and nucleon single-particle energies in the studied nuclei (^{16}O, ^{40}Ca, and ^{48}Ca). These observables are crucial for the description of hypernuclear properties within the HF method in the p-n-Λ formalism and the NΛ TDA method.

The proton (neutron) density ρ_{p} (ρ_{n}) influences the Λ single-particle energies through the HF Eq. (4.4). The proton (neutron) single-particle energies $\varepsilon_i^{\mathrm{p}}$ ($\varepsilon_i^{\mathrm{n}}$) have main impact on the hypernuclear spectrum in the NΛ TDA Eq. (3.19a) and (3.19b).

The NNN interaction has a significant effect on the binding energies as well. The values of the binding energies calculated with the (NN + NNN) interaction underestimate the empirical data. However, our goal is to calculate the hypernuclear spectra which are more affected by the density distributions and neutron single-particle energies than by the binding energies of the nuclear cores. To reach satisfactory description of the nuclear ground-state properties, the HF ground state

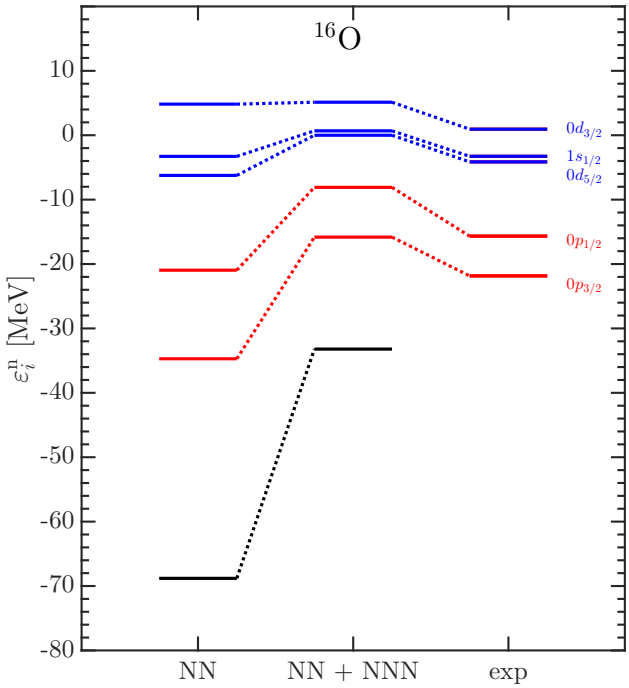

Fig. 5.6: The neutron single-particle energies $\varepsilon_i^{\mathrm{n}}$ of ^{16}O calculated with only two-body (NN) interaction and with two-body plus three-body (NN + NNN) interaction. The empirical data (exp) [53] are shown for comparison.

needs to be corrected by the many-body correlations [51]. The implementation of these correlations is beyond the scope of this thesis.

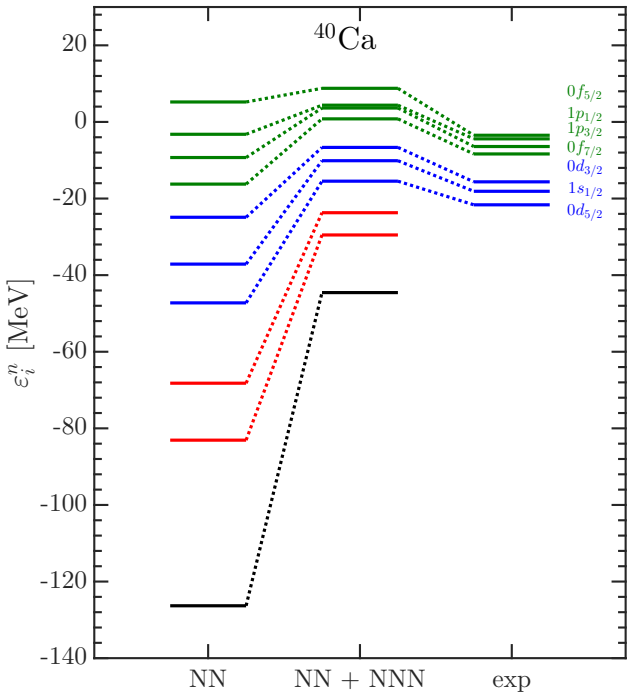

Fig. 5.7: The neutron single-particle energies $\varepsilon_i^{\mathrm{n}}$ of ^{40}Ca calculated with only two-body (NN) interaction and with two-body plus three-body (NN + NNN) interaction. The empirical data (exp) [53] are shown for comparison.

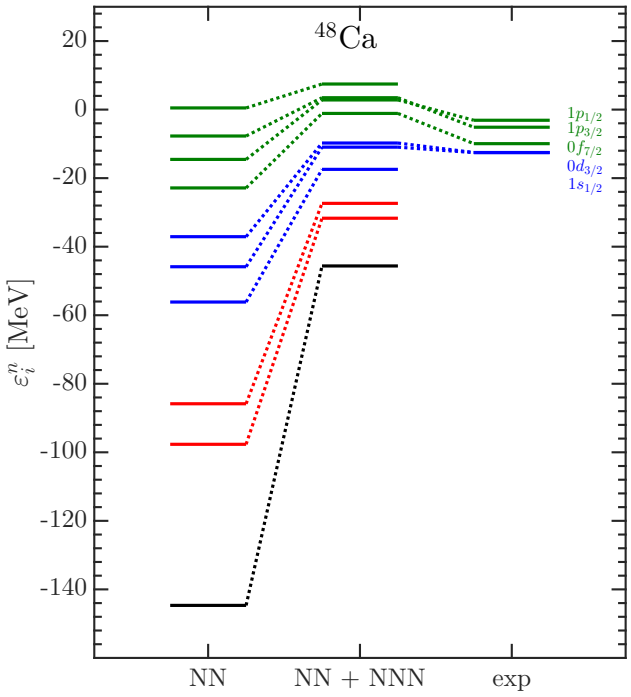

Fig. 5.8: The neutron single-particle energies $\varepsilon_i^{\mathrm{n}}$ of ^{48}Ca calculated with only two-body (NN) interaction and with two-body plus three-body (NN + NNN) interaction. The empirical data (exp) [53] are shown for comparison.

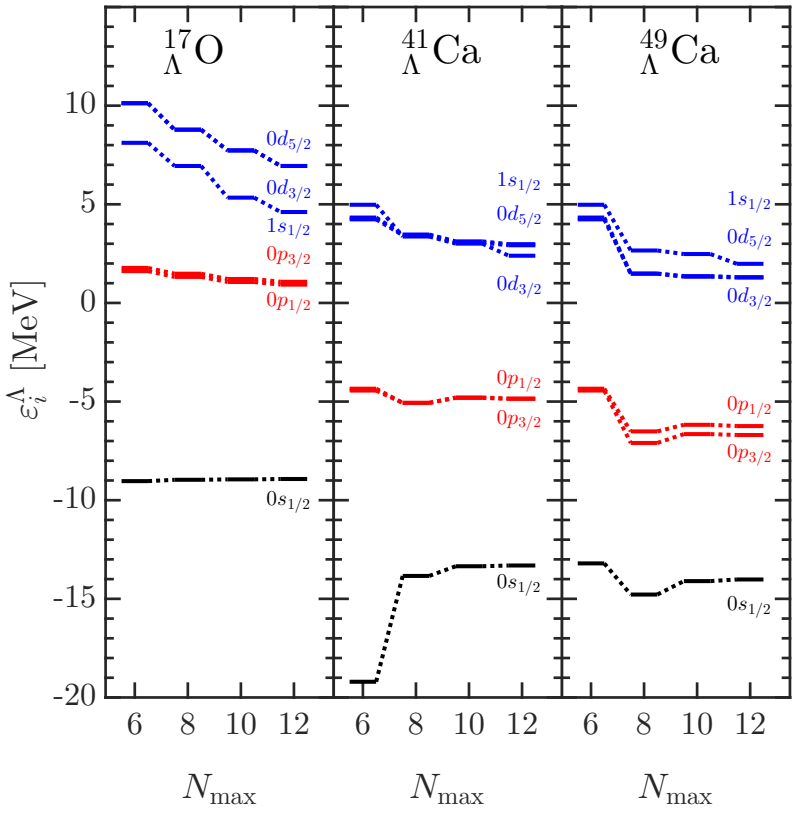

Fig. 5.9: The Λ single-particle energies ε_i^Λ in $^{17}_{\Lambda}$O, $^{41}_{\Lambda}$Ca, and $^{49}_{\Lambda}$Ca calculated with the (NN + NNN) interaction for $N_{\max} = 6, 8, 10$, and 12.

5.2 Calculations of $^{17}_{\Lambda}$O, $^{41}_{\Lambda}$Ca, and $^{49}_{\Lambda}$Ca

The hypernuclei $^{17}_{\Lambda}$O, $^{41}_{\Lambda}$Ca, and $^{49}_{\Lambda}$Ca consist of one Λ particle bound in the even-even nuclear cores ^{16}O, ^{40}Ca, and ^{48}Ca, respectively. We calculate the Λ single-particle energies in these hypernuclei by the HF method in the p-n-Λ formalism solved in the dynamical mode (see Chapter 4).

In Fig. 5.9, the Λ single-particle energies in $^{17}_{\Lambda}$O, $^{41}_{\Lambda}$Ca, and $^{49}_{\Lambda}$Ca are shown for $N_{\max} = 6, 8, 10$, and 12. The Λ single-particle states with negative energies represent the bound states of the Λ hyperon and reach quick convergence in all considered hypernuclei. The states with positive energies represent possible excitations of the Λ particle and their convergence is much slower.

In Figs. and 5.10, 5.11, and 5.12 there are Λ single-particle energies ε_i^Λ in the hypernuclei $^{17}_{\Lambda}$O, $^{41}_{\Lambda}$Ca, and $^{49}_{\Lambda}$Ca calculated with and without the NNN interaction. The results in $^{17}_{\Lambda}$O and $^{40}_{\Lambda}$Ca are compared to available empirical data. The Λ single-particle energies are systematically shifted upwards in energy with respect to empirical data. However, the relative energies between the major shells, denoted as

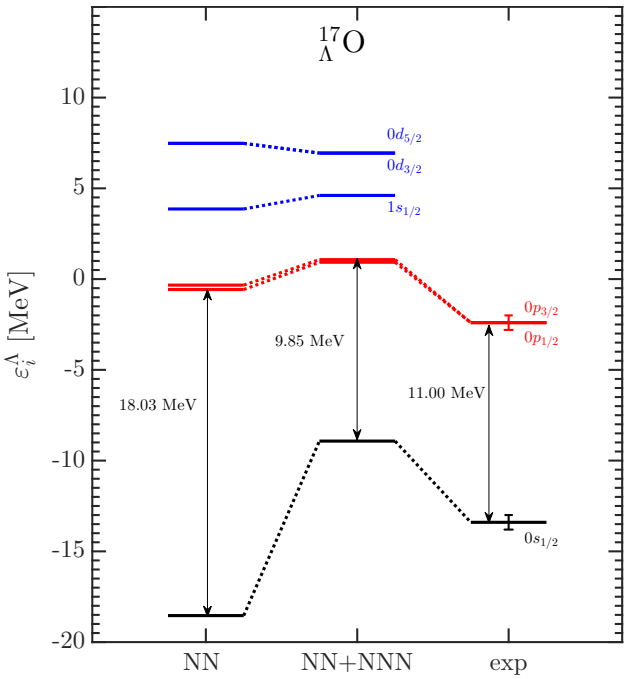

Fig. 5.10: The Λ single-particle energies ε_i^{Λ} of the $^{17}_{\Lambda}$O calculated with only two-body (NN) interaction and with two-body plus three-body (NN + NNN) interaction. The empirical data (exp) [54] are shown for comparison.

double arrows, are in qualitatively better agreement with the available data.

The YN interaction used in our model is derived only in the leading order and thus it is strongly dependent on the cutoff parameter λ [41]. In Fig. 5.13, the Λ single-particle energies of the $^{17}_{\Lambda}$O, $^{41}_{\Lambda}$Ca, and $^{49}_{\Lambda}$Ca for the cutoff parameter $\lambda = 550$ and 600 MeV for $N_{\max} = 12$ are shown. We observe that the Λ single-particle spectra of all hypernuclei are shifted upwards by using the YN interaction with the higher cutoff λ. The relative energies between the single-particle levels depend on λ as well. However, this dependence is very small in comparison to the influence of the NNN force which we discuss in this text.

We conclude that the effect of the NNN interaction qualitatively improves the description of the Λ single-particle energies in $^{17}_{\Lambda}$O, $^{41}_{\Lambda}$Ca, and $^{49}_{\Lambda}$Ca with respect to available experimental data. We do not compare the absolute values of calculated energies to the empirical data, since the Λ single-particle spectra shift upwards with higher values of the cutoff parameter λ. Instead, we compare the relative energetic gaps between the major shells. Here the effect of the NNN force is crucial. The relative gaps between the major shells are in much better qualitative agreement with empirical values if we include the NNN force in our calculations.

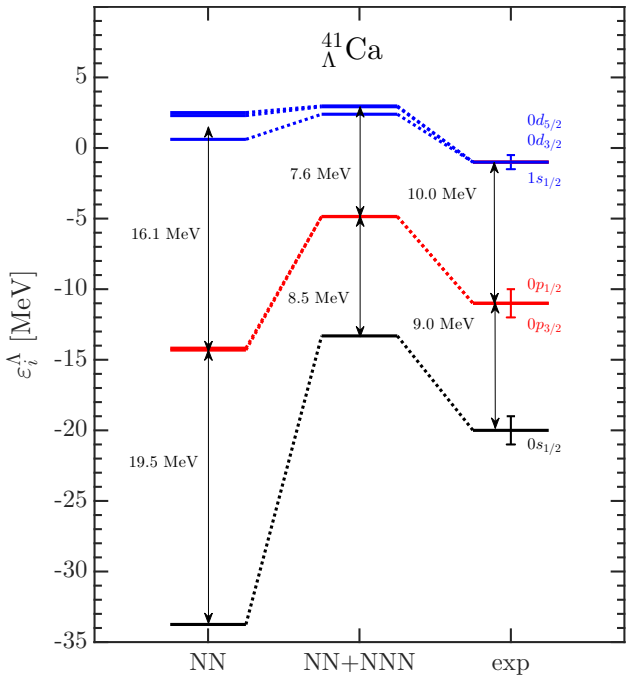

Fig. 5.11: The Λ single-particle energies ε_i^{Λ} of the $^{41}_{\Lambda}$Ca calculated with only two-body (NN) interaction and with two-body plus three-body (NN + NNN) interaction. The empirical data (exp) [55] are shown for comparison.

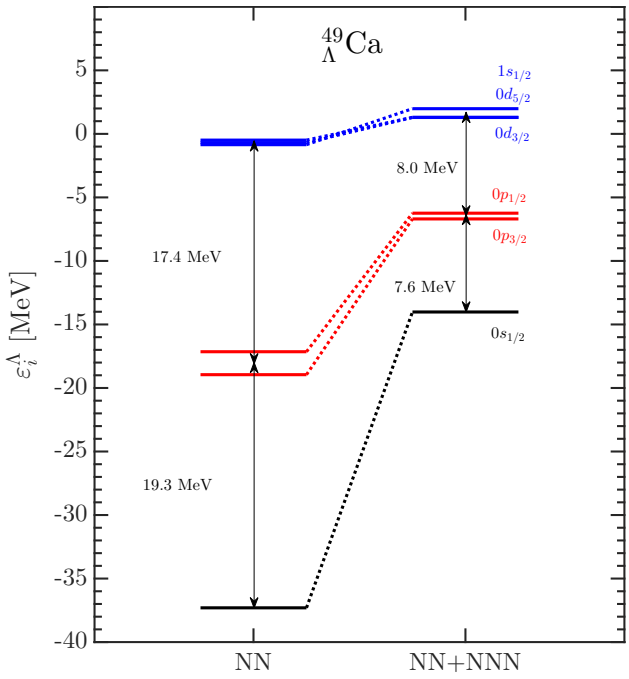

Fig. 5.12: The Λ single-particle energies ε_i^{Λ} of $^{41}_{\Lambda}$Ca calculated with only two-body (NN) interaction and with two-body plus three-body (NN + NNN) interaction.

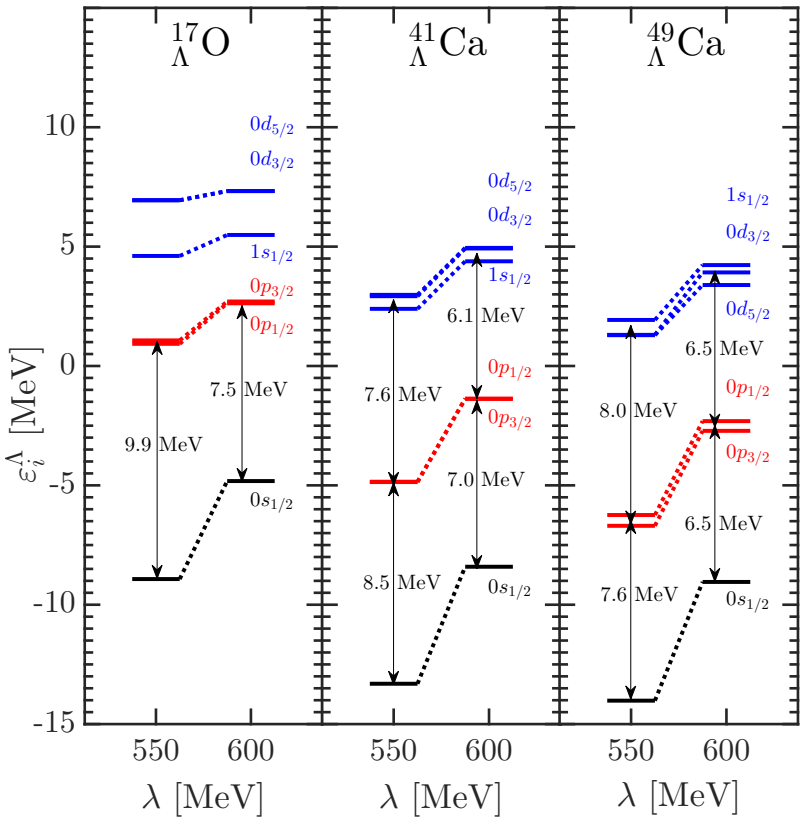

Fig. 5.13: The Λ single-particle energies ε_i^Λ of $^{17}_{\Lambda}$O, $^{41}_{\Lambda}$Ca, and $^{49}_{\Lambda}$Ca calculated with the (NN + NNN) interaction for cutoff parameter of the YN interaction λ = 550 and 600 MeV.

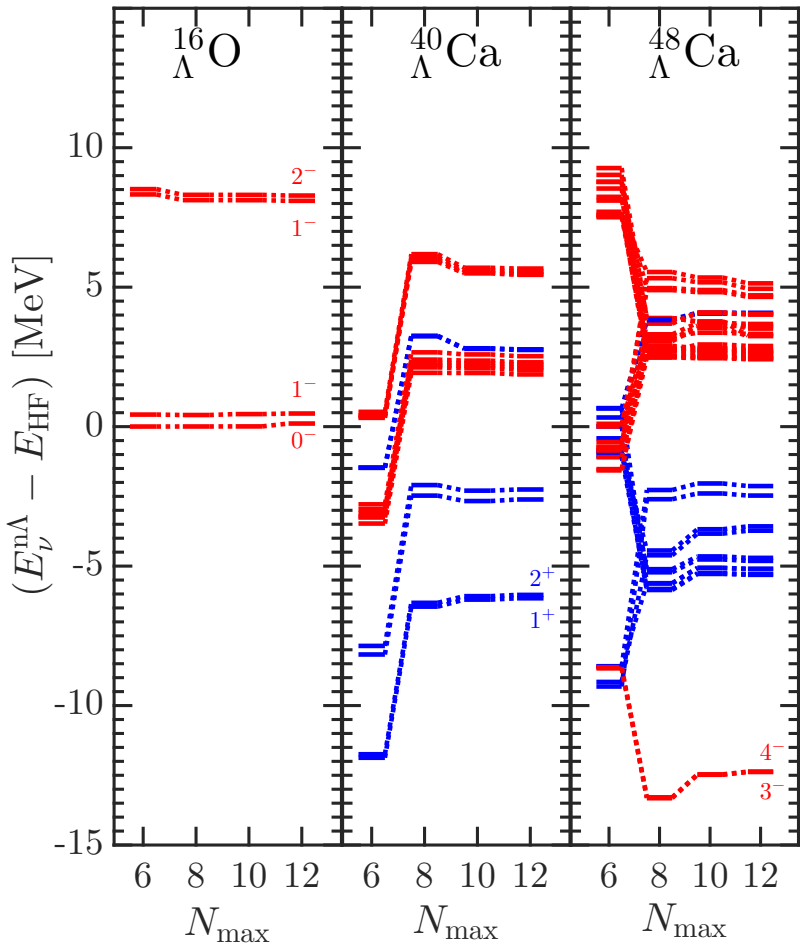

Fig. 5.14: The relative energies $(E_\nu^{n\Lambda} - E_{HF})$ of $^{16}_{\Lambda}$O, $^{40}_{\Lambda}$Ca, and $^{48}_{\Lambda}$Ca calculated for $N_{max} = 6, 8, 10$, and 12.

5.3 Calculations of $^{16}_{\Lambda}$O, $^{40}_{\Lambda}$Ca, and $^{48}_{\Lambda}$Ca

The NΛ TDA method is used for calculations of hypernuclei with Λ hyperon bound to even-odd nuclear core. Such hypernuclei are typically produced in experiments through (π^+, K^+) reactions [56]. In this section, we calculate the energy spectra of hypernuclei $^{16}_{\Lambda}$O, $^{40}_{\Lambda}$Ca, and $^{48}_{\Lambda}$Ca.

In Fig. 5.14, the relative energies $(E_\nu^{n\Lambda} - E_{HF})$ with respect to the ground-state energy E_{HF} of $^{16}_{\Lambda}$O, $^{40}_{\Lambda}$Ca, and $^{48}_{\Lambda}$Ca calculated with (NN + NNN) interaction for $N_{max} = 6, 8, 10$, and 12 are shown. We calculate the energies of hypernuclei with respect to the energies of the ground state E_{HF} of considered nuclei – ^{16}O, ^{40}Ca, and ^{48}Ca. The red lines in Fig. 5.14 represent the states with negative parity, the blue lines represent the states with positive parity. In $^{40}_{\Lambda}$Ca and $^{48}_{\Lambda}$Ca, the states are coupled in multiplets. The lowest energy levels shown in Fig. 5.14 exhibit quick

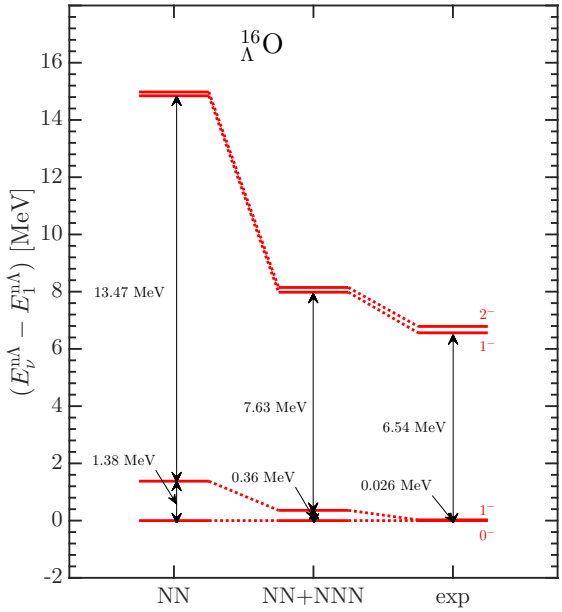

Fig. 5.15: The relative energies $(E_\nu^{n\Lambda} - E_1^{n\Lambda})$ $^{16}_{\Lambda}$O calculated with respect to the lowest energy level of $^{16}_{\Lambda}$O with only the two-body interactions (NN) and with the two-body plus three-body interactions (NN + NNN). The experimental data (exp) [57] are shown for comparison.

convergence.

In Fig. 5.15, the relative energies $(E_\nu^{n\Lambda} - E_1^{n\Lambda})$ of $^{16}_{\Lambda}$O calculated with respect to the lowest states 1^+ and 4^-, respectively, are shown for two-body (NN) interaction and two-body plus three-body (NN + NNN) interaction and compared to the experimental data [57]. The implementation of the NNN interaction yields results in qualitatively much better agreement with the experiment.

In Fig. 5.16, the relative energies $(E_\nu^{n\Lambda} - E_1^{n\Lambda})$ of $^{40}_{\Lambda}$Ca and $^{48}_{\Lambda}$Ca calculated with respect to the lowest state 1^+ and 4^-, respectively, are shown for two-body (NN) interaction and two-body plus three-body (NN + NNN) interaction. The implementation of the NNN force shows similar effect as in the case of $^{16}_{\Lambda}$O. The gaps between the multiplets of levels, as well as the gaps within each multiplet, are quenched.

In Fig. 5.17, the relative energies $(E_\nu^{n\Lambda} - E_{HF})$ with respect to the ground-state energy E_{HF} of $^{16}_{\Lambda}$O, $^{40}_{\Lambda}$Ca, and $^{48}_{\Lambda}$Ca calculated with the (NN + NNN) interaction are shown for the cutoff parameter $\lambda = 550$ and 600 MeV. The spectra are systematically shifted upwards in energy with higher λ. The relative distances among the levels or the multiplets of levels change with respect to λ. However, this effect is much smaller compared to the effect of the NNN force.

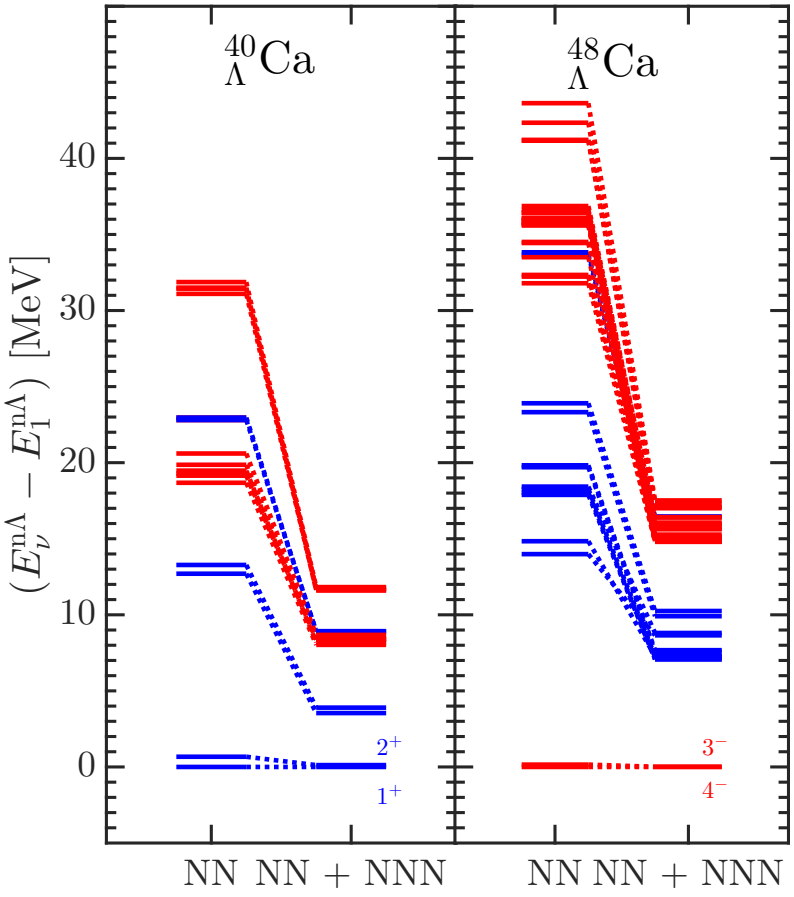

Fig. 5.16: The relative energies $(E_\nu^{n\Lambda} - E_1^{n\Lambda})$ of the $^{40}_{\Lambda}$Ca and $^{48}_{\Lambda}$Ca calculated with respect to the lowest levels of $^{40}_{\Lambda}$Ca and $^{48}_{\Lambda}$Ca, respectively, with only the two-body interactions (NN) and with the two-body plus three-body interactions (NN + NNN).

The implementation of the NNN interaction qualitatively improves the description of the energy spectra of $^{16}_{\Lambda}$O, $^{40}_{\Lambda}$Ca, and $^{48}_{\Lambda}$Ca. In $^{16}_{\Lambda}$O, the ordering of the states is in good agreement with experiment. The energies show qualitative improvement with the addition of the NNN force. In $^{40}_{\Lambda}$Ca and $^{48}_{\Lambda}$Ca, the levels form multiplets. Again, the gaps between multiplets and the single levels shrink when the NNN force is implemented.

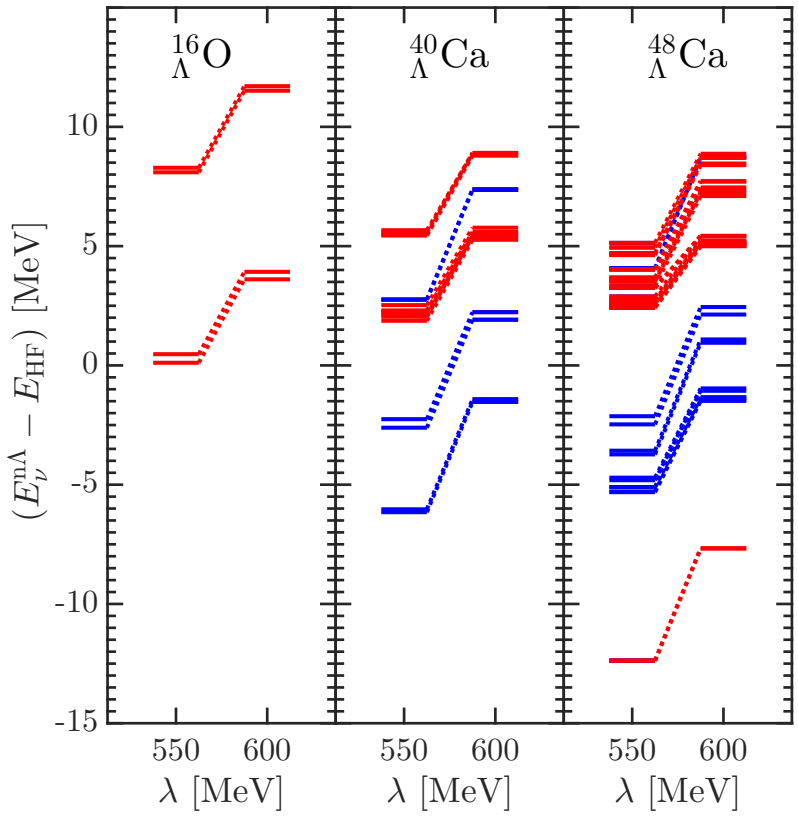

Fig. 5.17: The relative energies $(E_\nu^{n\Lambda} - E_{HF})$ of $^{16}_{\Lambda}$O, $^{40}_{\Lambda}$Ca, and $^{48}_{\Lambda}$Ca calculated for the cutoff parameter $\lambda = 550$ and 600 MeV.

Chapter 6

Conclusions

In this work, we studied the effect of the three-body NNN force on the properties of the single-Λ hypernuclei ${}^{17}_{\Lambda}$O, ${}^{41}_{\Lambda}$Ca, ${}^{49}_{\Lambda}$Ca, ${}^{16}_{\Lambda}$O, ${}^{40}_{\Lambda}$Ca, and ${}^{48}_{\Lambda}$Ca, and their corresponding doubly-magic cores. We derived the proton-neutron-Λ (p-n-Λ) formalism of the Hartree-Fock (HF) method that includes the three-body NNN and ΛNN interactions. This method was used for calculations of the hypernuclei with one Λ particle coupled to the even-even nuclear cores – ${}^{17}_{\Lambda}$O, ${}^{41}_{\Lambda}$Ca, and ${}^{49}_{\Lambda}$Ca. In addition, we derived the NΛ Tamm-Dancoff approximation (NΛ TDA). The NΛ TDA was used for description of the hypernuclei with one Λ particle coupled to the even-odd nuclear cores – ${}^{16}_{\Lambda}$O, ${}^{40}_{\Lambda}$Ca, and ${}^{48}_{\Lambda}$Ca. The principle of the NΛ TDA method is the annihilation of one neutron (proton) from the even-even nuclear system and coupling of this even-odd (odd-even) core with the Λ hyperon. Moreover, we laid out two possible extensions of both used methods – the HF method in the p-n-Λ formalism and the NΛ TDA method – towards more general formalism. In this formalism, the Λ particle and the NΛ TDA phonon operators would be coupled with general multiphonon excitations of the nuclear core generated by the Equation of Motion Phonon Method (EMPM).

We implemented the chiral N^2LO_{sat} NN and NNN potential as interaction among nucleons, and the ΛN-ΛN channel of the chiral LO YN potential that represented the two-body ΛN interaction. Although the whole theoretical formalism was derived with the three-body ΛNN force, we did not yet implement any version of the ΛNN interaction. The computer code for the HF method in the p-n-Λ formalism could run in the static or dynamical mode. In the static mode, the code solved first the HF equations for the bare nuclear core using only the NN and NNN interaction. The HF equation for the Λ particle was solved independently. The HF code in the static mode was used for the description of the nuclei ^{16}O, ^{40}Ca, and ^{48}Ca, and also as the starting point for the NΛ TDA calculations of the hypernuclei ${}^{16}_{\Lambda}$O, ${}^{40}_{\Lambda}$Ca,

and ${}^{48}_{\Lambda}$Ca. The HF code in the dynamical mode solved all HF equations together in each iteration. The dynamical mode was used for the description of hypernuclei ${}^{17}_{\Lambda}$O, ${}^{41}_{\Lambda}$Ca, and ${}^{49}_{\Lambda}$Ca.

We studied the nuclei ^{16}O, ^{40}Ca, and ^{48}Ca by using the HF code in the static mode. We studied the convergence of the radial density distributions, charge radii, nucleon single-particle energies, and the binding energies with respect to the size of the basis determined by the maximal oscillator shell number number $N_{\max}$. The maximal configuration space in our calculations was $N_{\max} = 12$, where we applied the constraint $N_{\max} = N_{\max}^{(12)} = N_{\max}^{(123)}$. We reached convergence for the nuclei ^{16}O and ^{40}Ca. We need to perform calculations with larger $N_{\max}$ in order to reach convergence for the ^{48}Ca. We showed the effect of the NNN force on the radial density distributions, charge radii, nucleon single-particle energies, and the binding energies of the studied nuclei by comparing the results calculated with and without the NNN interaction. We found that the inclusion of the NNN force significantly improves the description of the radial density distributions, radii, and relative distances among the nucleon single-particle energies in the studied nuclei. The binding energies calculated within our model underestimated the empirical data. This was caused by the fact, that we did not take into account many-body correlations. Their implementation was beyond the scope of this thesis. However, the most decisive influence on the description of the hypernuclear spectra had the nuclear density distributions and the nucleon single-particle energies which entered the HF and the NΛ TDA equations.

The single-Λ hypernuclei ${}^{17}_{\Lambda}$O, ${}^{41}_{\Lambda}$Ca, and ${}^{49}_{\Lambda}$Ca were calculated by the HF code in the dynamical code. These hypernuclei consisted of one Λ hyperon bound to the even-even nuclear core. We studied their Λ single-particle spectra. The states with negative energies ε^{Λ} quickly reached convergence with respect to the $N_{\max}$. The states with positive ε^{Λ} required bigger configuration space for reaching the convergence. The NNN force quenched the gaps between the major shells and yielded results in qualitatively much better agreement with the available empirical data. The YN force implemented in our model was strongly dependent on the cutoff parameter λ. We found that with the change of λ, the spectra systematically shifted in energy. The relative distances between between the Λ single-particle energies depended on λ as well, but the change was much smaller than the one caused by the studied effect of the NNN force.

The single-Λ hypernuclei ${}^{16}_{\Lambda}$O, ${}^{40}_{\Lambda}$Ca, and ${}^{48}_{\Lambda}$Ca were studied using the NΛ TDA method. We studied their energy spectra. We found that the spectra of all studied hypernuclei reached convergence with respect to the $N_{\max}$. Moreover, we explored that the NNN force improves the description of the spectrum of the ${}^{16}_{\Lambda}$O with respect

to the available experimental data. The NNN force quenched the relative distances between the energy states of ${}^{16}_{\Lambda}$O and yielded results in good qualitative agreement with the experiment. The energy states of the ${}^{40}_{\Lambda}$Ca and ${}^{48}_{\Lambda}$Ca formed multiplets. We found that the effect of the NNN force was quenching the relative distances among multiplets, as well as the distances among the levels within each multiplet. We explored, that the energy spectra of ${}^{16}_{\Lambda}$O, ${}^{40}_{\Lambda}$Ca, and ${}^{48}_{\Lambda}$Ca depended on the cutoff parameter of the YN potential λ. The change of λ resulted in systematic shift of the spectra in energy. The relative distances among levels and multiplets of levels changed as well. However, the change of the relative distances was not as significant as the one caused by the NNN force.

In our work, we discovered that the implementation of the three-body NNN force into the self-consistent mean-field model improves significantly the description of hypernuclear spectra. However, our method needs further improvements. The most straightforward improvement is the implementation of a YN force that does not depend on the cutoff parameter. It would also be desirable to implement the three-body ΛNN force and take into account the $\Lambda - \Sigma$ mixing. We plan to reach all of these goals by implementing the SRG chiral LO YN interaction.

Moreover, we would like to perform calculations of the HF method in the p-n-Λ formalism and the NΛ TDA method coupled with the multiphonon configurations of the nuclear core generated by the EMPM.

Appendix A

Matrix elements in spherical harmonic oscillator basis

The basis of the spherical harmonic oscillator is constructed from the single-particle states $|i\rangle$ which are denoted with the quantum numbers n, l, j, and m:

$$|i\rangle = |n_i l_i j_i m_i\rangle, \tag{A.1}$$

where n_i is the principal quantum number, l_i is the orbital angular momentum, j_i is the total angular momentum, and m_i is the projection of the total angular momentum. The numbers l_i, j_i and m_i satisfy the following relations

$$\left| l_i - \frac{1}{2} \right| \leq j_i \leq l_i + \frac{1}{2}, \tag{A.2a}$$

$$m_i = -j_i, -j_i + 1, \ldots, +j_i - 1, +j_i. \tag{A.2b}$$

The wave function $\psi_{n_i l_i j_i m_i}(\vec{r})$ of the state $|i\rangle$ is defined as follows:

$$\psi_{n_i l_i j_i m_i} = R_{n_i l_i}(r, b) \cdot \left[Y_{l_i}(\phi, \Omega) \otimes \chi_{\frac{1}{2}} \right]_{j_i m_i}. \tag{A.3}$$

The radial part $R_{n_i l_i}(r, b)$ of the wave function in (A.3) is defined as

$$R_{n_i l_i}(r, b) = b^{\frac{3}{2}} \sqrt{\frac{2n_i!}{(n_i + l_i + \frac{1}{2})!}} (br)^{l_i} L_{n_i}^{(l_i + \frac{1}{2})}(b^2 r^2) \mathrm{e}^{-\frac{b^2 r^2}{2}}, \tag{A.4}$$

where $L_{n_i}^{(l_i + \frac{1}{2})}$ are the generalized Laguerre polynomials.

Below we express the matrix elements of the kinetic operator, as well as all interaction terms used in the Hamiltonian (2.32) in the HO basis. The kinetic matrix elements $t_{ij}^{\mathrm{p,n}}$, t_{ij}^{Λ} in (2.32) are expressed as:

$$t_{ij}^{\mathrm{p,n}} = \left(1 - \frac{1}{(A-1)+\frac{M_\Lambda}{M}}\right)\left\langle n_i l_i j_i m_i \left| \frac{\widehat{\vec{P}}^2}{2M} \right| n_j l_j j_j m_j \right\rangle =$$
$$\left(1 - \frac{1}{(A-1)+\frac{M_\Lambda}{M}}\right)\left[\frac{1}{2}\hbar\omega\left(2n_i + l_i + \frac{3}{2}\right)\delta_{n_i n_j}\delta_{l_i l_j}\delta_{j_i j_j}\delta_{m_i m_j}\right.$$
$$+ \frac{1}{2}\hbar\omega\sqrt{n_i\left(n_i + l_i + \frac{1}{2}\right)}\delta_{n_i n_{j+1}}\delta_{l_i l_j}\delta_{j_i j_j}\delta_{m_i m_j}$$
$$\left. + \frac{1}{2}\hbar\omega\sqrt{n_j\left(n_j + l_j + \frac{1}{2}\right)}\delta_{n_i+1\ n_j}\delta_{l_i l_j}\delta_{m_i m_j}\right], \tag{A.5}$$

$$t_{ij}^{\Lambda} = \left(1 - \frac{1}{1+(A-1)\frac{M}{M_\Lambda}}\right)\left\langle n_i l_i j_i m_i \left| \frac{\widehat{\vec{P}}^2}{2M_\Lambda} \right| n_j l_j j_j m_j \right\rangle =$$
$$\left(1 - \frac{1}{1+(A-1)\frac{M}{M_\Lambda}}\right)\left[\frac{1}{2}\hbar\omega\left(2n_i + l_i + \frac{3}{2}\right)\delta_{n_i n_j}\delta_{l_i l_j}\delta_{j_i j_j}\delta_{m_i m_j}\right.$$
$$+ \frac{1}{2}\hbar\omega\sqrt{n_i\left(n_i + l_i + \frac{1}{2}\right)}\delta_{n_i n_{j+1}}\delta_{l_i l_j}\delta_{j_i j_j}\delta_{m_i m_j}$$
$$\left. + \frac{1}{2}\hbar\omega\sqrt{n_j\left(n_j + l_j + \frac{1}{2}\right)}\delta_{n_i+1\ n_j}\delta_{l_i l_j}\delta_{m_i m_j}\right]. \tag{A.6}$$

The matrix elements of the two-body potentials in (2.32) are defined as:

$$V_{ijkl}^{\mathrm{pp,nn}} = \left\langle ij \left| \left(\widehat{V}^{\mathrm{pp,nn}} - \frac{\widehat{\vec{P}}_1 \cdot \widehat{\vec{P}}_2}{(A-1)M + M_\Lambda} \right) \right| kl - lk \right\rangle, \tag{A.7a}$$

$$V_{ijkl}^{\mathrm{pn,p\Lambda,n\Lambda}} = \left\langle ij \left| \left(\widehat{V}^{\mathrm{pn,p\Lambda,n\Lambda}} - \frac{\widehat{\vec{P}}_1 \cdot \widehat{\vec{P}}_2}{(A-1)M + M_\Lambda} \right) \right| kl \right\rangle. \tag{A.7b}$$

The antisymmetrized matrix elements $\langle ij|\frac{\widehat{\vec{P}}_1\cdot\widehat{\vec{P}}_2}{(A-1)M+M_\Lambda}|kl-lk\rangle$, as well as the symmetrized matrix elements $\langle ij|\frac{\widehat{\vec{P}}_1\cdot\widehat{\vec{P}}_2}{(A-1)M+M_\Lambda}|kl\rangle$ are generated by the CENS code [58].

The matrix elements of the three-body NNN interaction in Hamiltonian (2.32)

are defined as:

$$V^{\mathrm{ppp}}_{ijklmn} = \langle ijk|\widehat{V}^{\mathrm{ppp}}|lmn - lnm + nlm - nml + mnl - mln\rangle, \tag{A.8a}$$

$$V^{\mathrm{nnn}}_{ijklmn} = \langle ijk|\widehat{V}^{\mathrm{nnn}}|lmn - lnm + nlm - nml + mnl - mln\rangle, \tag{A.8b}$$

$$V^{\mathrm{ppn}}_{ijklmn} = \langle ijk|\widehat{V}^{\mathrm{ppn}}|lmn - mln\rangle, \tag{A.8c}$$

$$V^{\mathrm{pnn}}_{ijklmn} = \langle ijk|\widehat{V}^{\mathrm{pnn}}|lmn - lnm\rangle, \tag{A.8d}$$

$$V^{\mathrm{pp}\Lambda}_{ijklmn} = \langle ijk|\widehat{V}^{\mathrm{pp}\Lambda}|lmn - mln\rangle, \tag{A.8e}$$

$$V^{\mathrm{nn}\Lambda}_{ijklmn} = \langle ijk|\widehat{V}^{\mathrm{nn}\Lambda}|lmn - mln\rangle, \tag{A.8f}$$

$$V^{\mathrm{pn}\Lambda}_{ijklmn} = \langle ijk|\widehat{V}^{\mathrm{pn}\Lambda}|lmn\rangle. \tag{A.8g}$$

The interaction matrix elements $\langle ij|\widehat{V}^{\mathrm{pp,nn}}|kl - lk\rangle$, $\langle ij|\widehat{V}^{\mathrm{pn}}|kl\rangle$, $V^{\mathrm{ppp}}_{ijklmn}, V^{\mathrm{nnn}}_{ijklmn}, V^{\mathrm{ppn}}_{ijklmn}, V^{\mathrm{pnn}}_{ijklmn}$ were provided to us by Petr Navrátil. The interaction matrix elements $\langle ij|\widehat{V}^{\mathrm{p}\Lambda}|kl\rangle$, $\langle ij|\widehat{V}^{\mathrm{n}_\Lambda}|kl\rangle$ were provided to us by Daniel Gazda.

Appendix B

J-scheme formalism

The formalism which uses the eigenstates as defined in (A.1) is called the M-scheme. If the studied system exhibits spherical symmetry, we can develop a formalism which disregards the projections of the total angular momenta and represents the eigenstates as sets of three quantum numbers

$$|i\rangle \rightarrow (n_i l_i j_i). \tag{B.1}$$

This formalism is called the J-scheme.

The matrix elements of the kinetic operator of protons, neutrons, and the Λ particle, respectively, are transformed into the J-scheme formalism as follows

$$t^{\mathrm{p}}_{n_i l_i j_i m_i, n_j l_j j_j m_j} = t^{\mathrm{p}}_{(n_i l_i j_i),(n_j l_j j_j)} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j}, \tag{B.2a}$$

$$t^{\mathrm{n}}_{n_i l_i j_i m_i, n_j l_j j_j m_j} = t^{\mathrm{n}}_{(n_i l_i j_i),(n_j l_j j_j)} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j}, \tag{B.2b}$$

$$t^{\Lambda}_{n_i l_i j_i m_i, n_j l_j j_j m_j} = t^{\Lambda}_{(n_i l_i j_i),(n_j l_j j_j)} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j}. \tag{B.2c}$$

Analogically, the matrix elements of the density matrices of protons, neutrons, and the Λ particle, respectively, read

$$\rho^{\mathrm{p}}_{n_i l_i j_i m_i, n_j l_j j_j m_j} = \rho^{\mathrm{p}}_{(n_i l_i j_i),(n_j l_j j_j)} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j} \tag{B.3a}$$

$$\rho^{\mathrm{n}}_{n_i l_i j_i m_i, n_j l_j j_j m_j} = \rho^{\mathrm{n}}_{(n_i l_i j_i),(n_j l_j j_j)} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j}, \tag{B.3b}$$

$$\rho^{\Lambda}_{n_i l_i j_i m_i, n_j l_j j_j m_j} = \rho^{\Lambda}_{(n_i l_i j_i),(n_j l_j j_j)} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j}. \tag{B.3c}$$

The transformations of the matrix elements of the two-body NN and ΛN interaction operators into the J-scheme are expressed as:

$$V^{\mathrm{pp}}_{n_il_ij_im_i,n_jl_jj_jm_j,n_kl_kj_km_k,n_ll_lj_lm_l} = \sum_J C^{Jm_i+m_j}_{j_im_i,j_jm_j} C^{Jm_k+m_l}_{j_km_k,j_lm_l} \delta_{m_k+m_l,m_i+m_j} \\ V^{J,\mathrm{pp}}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l)}, \tag{B.4a}$$

$$V^{\mathrm{pn}}_{n_il_ij_im_i,n_jl_jj_jm_j,n_kl_kj_km_k,n_ll_lj_lm_l} = \sum_J C^{Jm_i+m_j}_{j_im_i,j_jm_j} C^{Jm_k+m_l}_{j_km_k,j_lm_l} \delta_{m_k+m_l,m_i+m_j} \\ V^{J,\mathrm{pn}}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l)}, \tag{B.4b}$$

$$V^{\mathrm{nn}}_{n_il_ij_im_i,n_jl_jj_jm_j,n_kl_kj_km_k,n_ll_lj_lm_l} = \sum_J C^{Jm_i+m_j}_{j_im_i,j_jm_j} C^{Jm_k+m_l}_{j_km_k,j_lm_l} \delta_{m_k+m_l,m_i+m_j} \\ V^{J,\mathrm{nn}}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l)}, \tag{B.4c}$$

$$V^{\mathrm{p}\Lambda}_{n_il_ij_im_i,n_jl_jj_jm_j,n_kl_kj_km_k,n_ll_lj_lm_l} = \sum_J C^{Jm_i+m_j}_{j_im_i,j_jm_j} C^{Jm_k+m_l}_{j_km_k,j_lm_l} \delta_{m_k+m_l,m_i+m_j} \\ V^{J,\mathrm{p}\Lambda}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l)}, \tag{B.4d}$$

$$V^{\mathrm{n}\Lambda}_{n_il_ij_im_i,n_jl_jj_jm_j,n_kl_kj_km_k,n_ll_lj_lm_l} = \sum_J C^{Jm_i+m_j}_{j_im_i,j_jm_j} C^{Jm_k+m_l}_{j_km_k,j_lm_l} \delta_{m_k+m_l,m_i+m_j} \\ V^{J,\mathrm{n}\Lambda}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l)}, \tag{B.4e}$$

where the symbols

$$C^{Jm_i+m_j}_{j_im_i,j_jm_j} = \langle j_im_i, j_jm_j | J\ m_i{+}m_j\rangle, \quad C^{Jm_k+m_l}_{j_km_k,j_lm_l} = \langle j_km_k, j_lm_l | J\ m_k{+}m_l\rangle,$$

represent the Clebsch-Gordan coefficients. The transformation of the matrix elements of the three-body interactions requires the following relations

$$|n_1l_1j_1, n_2l_2j_2, n_3l_3j_3; J_{12}, JM\rangle = \\ = \sum_{m_1m_2m_3} \sum_{M_{12}} C^{J_{12}M_{12}}_{j_1m_1,j_2m_2} C^{JM}_{J_{12}M_{12},j_3m_3} |n_1l_1j_1m_1, n_2l_2j_2m_2, n_3l_3j_3m_3\rangle, \tag{B.5}$$

$$|n_1l_1j_1m_1, n_2l_2j_2m_2, n_3l_3j_3m_3\rangle = \\ = \sum_{J_{12}J} C^{J_{12}m_1+m_2}_{j_1m_1,j_2m_2} C^{Jm_1+m_2+m_3}_{J_{12}m_1+m_2,j_3m_3} |n_1l_1j_1, n_2l_2j_2, n_3l_3j_3; J_{12}, J\ m_1{+}m_2{+}m_3\rangle. \tag{B.6}$$

Here, the symbol J_{12} stands for the angular momentum which couples the angular momenta j_1 and j_2. The symbol J stands for the coupling of the angular momenta J_{12} and j_3. In addition, we can introduce the isospin quantum number t and its projection m_t. Each type of particles can be expressed with distinctive values of t and m_t – protons ($t = \frac{1}{2}, m_t = +\frac{1}{2}$), neutrons ($t = \frac{1}{2}, m_t = -\frac{1}{2}$), and the Λ hyperon

$(t = 0, m_t = 0)$. The equations (B.5) and (B.6) can be rewritten into the JT-coupled form

$$|n_1 l_1 j_1 t_1, n_2 l_2 j_2 t_2, n_3 l_3 j_3 t_3; J_{12} T_{12}, JMTM_T\rangle = \\ = \sum_{m_1 m_2 m_3} \sum_{M_{12}} \sum_{m_{t_1} m_{t_2} m_{t_3}} \sum_{M_{T_{12}}} C^{J_{12} M_{12}}_{j_1 m_1, j_2 m_2} C^{JM}_{J_{12} M_{12}, j_3 m_3} C^{T_{12} M_{T_{12}}}_{t_1 m_{t_1}, t_2 m_{t_2}} C^{TM_T}_{T_{12} M_{T_{12}}, t_3 m_{t_3}} \\ \times |n_1 l_1 j_1 m_1 t_1 m_{t_1}, n_2 l_2 j_2 m_2 t_2 m_{t_2}, n_3 l_3 j_3 m_3 t_3 m_{t_3}\rangle, \quad \text{(B.7)}$$

and

$$|n_1 l_1 j_1 m_1 t_1 m_{t_1}, n_2 l_2 j_2 m_2 t_2 m_{t_2}, n_3 l_3 j_3 m_3 t_3 m_{t_3}\rangle \\ = \sum_{J_{12} J} \sum_{T_{12} T} C^{J_{12} m_1 + m_2}_{j_1 m_1, j_2 m_2} C^{J m_1 + m_2 + m_3}_{J_{12} m_1 + m_2, j_3 m_3} C^{T_{12} m_{t_1} + m_{t_2}}_{t_1 m_{t_1}, t_2 m_{t_2}} C^{T m_{t_1} + m_{t_2} + m_{t_3}}_{T_{12} m_{t_1} + m_{t_2}, t_3 m_{t_3}} \\ \times |n_1 l_1 j_1 t_1, n_2 l_2 j_2 t_2, n_3 l_3 j_3 t_3; J_{12} T_{12}, J m_1 + m_2 + m_3 T m_{t_1} + m_{t_2} + m_{t_3}\rangle. \quad \text{(B.8)}$$

In analogy to Eqs. (B.4a)-(B.4e), we can introduce the relations between the three-body interaction matrix elements in the JT-scheme and in the M-scheme:

$$V^{\mathrm{ppp}}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l, n_m l_m j_m m_m, n_n l_n j_n m_n} \\ = \sum_{J_{12} J'_{12}} \sum_{J} C^{J'_{12} m_i + m_j}_{j_i m_i, j_j m_j} C^{J m_i + m_j + m_k}_{J'_{12} m_i + m_j, j_k m_k} C^{J_{12} m_l + m_m}_{j_l m_l, j_m m_m} C^{J m_l + m_m + m_n}_{J_{12} m_l + m_m, j_n m_n} \\ \times V^{T'_{12}=1, T_{12}=1, T=\frac{3}{2}, J'_{12}, J_{12}, J}_{(n_i l_i j_i), (n_j l_j j_j), (n_k l_k j_k), (n_l l_l j_l), (n_m l_m j_m), (n_n l_n j_n)}, \quad \text{(B.9)}$$

$$V^{\mathrm{nnn}}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l, n_m l_m j_m m_m, n_n l_n j_n m_n} \\ = \sum_{J_{12} J'_{12}} \sum_{J} C^{J'_{12} m_i + m_j}_{j_i m_i, j_j m_j} C^{J m_i + m_j + m_k}_{J'_{12} m_i + m_j, j_k m_k} C^{J_{12} m_l + m_m}_{j_l m_l, j_m m_m} C^{J m_l + m_m + m_n}_{J_{12} m_l + m_m, j_n m_n} \\ \times V^{T'_{12}=1, T_{12}=1, T=\frac{3}{2}, J'_{12}, J_{12}, J}_{(n_i l_i j_i), (n_j l_j j_j), (n_k l_k j_k), (n_l l_l j_l), (n_m l_m j_m), (n_n l_n j_n)}, \quad \text{(B.10)}$$

$$V^{\mathrm{ppn}}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l, n_m l_m j_m m_m, n_n l_n j_n m_n} \\ = \sum_{J_{12} J'_{12}} \sum_{J} C^{J'_{12} m_i + m_j}_{j_i m_i, j_j m_j} C^{J m_i + m_j + m_k}_{J'_{12} m_i + m_j, j_k m_k} C^{J_{12} m_l + m_m}_{j_l m_l, j_m m_m} C^{J m_l + m_m + m_n}_{J_{12} m_l + m_m, j_n m_n} \\ \left(\frac{2}{3} V^{T'_{12}=1, T_{12}=1, T=\frac{1}{2}, J'_{12}, J_{12}, J}_{(n_i l_i j_i), (n_j l_j j_j), (n_k l_k j_k), (n_l l_l j_l), (n_m l_m j_m), (n_n l_n j_n)} \right. \\ \left. + \frac{1}{3} V^{T'_{12}=1, T_{12}=1, T=\frac{3}{2}, J'_{12}, J_{12}, J}_{(n_i l_i j_i), (n_j l_j j_j), (n_k l_k j_k), (n_l l_l j_l), (n_m l_m j_m), (n_n l_n j_n)} \right), \quad \text{(B.11)}$$

$$
\begin{aligned}
&V^{\mathrm{pnn}}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l, n_m l_m j_m m_m, n_n l_n j_n m_n} \\
&= \sum_{J_{12} J'_{12}} \sum_{J} C^{J'_{12} m_i+m_j}_{j_i m_i, j_j m_j} C^{J m_i+m_j+m_k}_{J'_{12} m_i+m_j, j_k m_k} C^{J_{12} m_l+m_m}_{j_l m_l, j_m m_m} C^{J m_l+m_m+m_n}_{J_{12} m_l+m_m, j_n m_n} \\
&\quad \times \left(\frac{1}{2} V^{T'_{12}=0, T_{12}=0, T=\frac{1}{2}, J'_{12}, J_{12}, J}_{(n_i l_i j_i), (n_j l_j j_j), (n_k l_k j_k), (n_l l_l j_l), (n_m l_m j_m), (n_n l_n j_n)} \right. \\
&\qquad\qquad + \frac{1}{6} V^{T'_{12}=1, T_{12}=1, T=\frac{1}{2}, J'_{12}, J_{12}, J}_{(n_i l_i j_i), (n_j l_j j_j), (n_k l_k j_k), (n_l l_l j_l), (n_m l_m j_m), (n_n l_n j_n)} \\
&\qquad\qquad + \frac{1}{3} V^{T'_{12}=1, T_{12}=1, T=\frac{3}{2}, J'_{12}, J_{12}, J}_{(n_i l_i j_i), (n_j l_j j_j), (n_k l_k j_k), (n_l l_l j_l), (n_m l_m j_m), (n_n l_n j_n)} \\
&\qquad\qquad + \frac{1}{2\sqrt{3}} V^{T'_{12}=1, T_{12}=0, T=\frac{1}{2}, J'_{12}, J_{12}, J}_{(n_i l_i j_i), (n_j l_j j_j), (n_k l_k j_k), (n_l l_l j_l), (n_m l_m j_m), (n_n l_n j_n)} \\
&\qquad\qquad \left. + \frac{1}{2\sqrt{3}} V^{T'_{12}=0, T_{12}=1, T=\frac{1}{2}, J'_{12}, J_{12}, J}_{(n_i l_i j_i), (n_j l_j j_j), (n_k l_k j_k), (n_l l_l j_l), (n_m l_m j_m), (n_n l_n j_n)} \right), \qquad \text{(B.12)}
\end{aligned}
$$

$$
\begin{aligned}
&V^{\mathrm{pp}\Lambda}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l, n_m l_m j_m m_m, n_n l_n j_n m_n} \\
&= \sum_{J_{12} J'_{12}} \sum_{J} C^{J'_{12} m_i+m_j}_{j_i m_i, j_j m_j} C^{J m_i+m_j+m_k}_{J'_{12} m_i+m_j, j_k m_k} C^{J_{12} m_l+m_m}_{j_l m_l, j_m m_m} C^{J m_l+m_m+m_n}_{J_{12} m_l+m_m, j_n m_n} \\
&\qquad\qquad \times V^{T'_{12}=1, T_{12}=1, T=1, J'_{12}, J_{12}, J}_{(n_i l_i j_i), (n_j l_j j_j), (n_k l_k j_k), (n_l l_l j_l), (n_m l_m j_m), (n_n l_n j_n)}, \qquad \text{(B.13)}
\end{aligned}
$$

$$
\begin{aligned}
&V^{\mathrm{pn}\Lambda}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l, n_m l_m j_m m_m, n_n l_n j_n m_n} \\
&= \sum_{J_{12} J'_{12}} \sum_{J} C^{J'_{12} m_i+m_j}_{j_i m_i, j_j m_j} C^{J m_i+m_j+m_k}_{J'_{12} m_i+m_j, j_k m_k} C^{J_{12} m_l+m_m}_{j_l m_l, j_m m_m} C^{J m_l+m_m+m_n}_{J_{12} m_l+m_m, j_n m_n} \\
&\quad \times \left(\frac{1}{2} V^{T'_{12}=1, T_{12}=1, T=1, J'_{12}, J_{12}, J}_{(n_i l_i j_i), (n_j l_j j_j), (n_k l_k j_k), (n_l l_l j_l), (n_m l_m j_m), (n_n l_n j_n)} \right. \\
&\qquad\qquad \left. + \frac{1}{2} V^{T'_{12}=0, T_{12}=0, T=0, J'_{12}, J_{12}, J}_{(n_i l_i j_i), (n_j l_j j_j), (n_k l_k j_k), (n_l l_l j_l), (n_m l_m j_m), (n_n l_n j_n)} \right), \qquad \text{(B.14)}
\end{aligned}
$$

$$
\begin{aligned}
&V^{\mathrm{nn}\Lambda}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l, n_m l_m j_m m_m, n_n l_n j_n m_n} \\
&= \sum_{J_{12} J'_{12}} \sum_{J} C^{J'_{12} m_i+m_j}_{j_i m_i, j_j m_j} C^{J m_i+m_j+m_k}_{J'_{12} m_i+m_j, j_k m_k} C^{J_{12} m_l+m_m}_{j_l m_l, j_m m_m} C^{J m_l+m_m+m_n}_{J_{12} m_l+m_m, j_n m_n} \\
&\qquad\qquad \times V^{T'_{12}=1, T_{12}=1, T=1, J'_{12}, J_{12}, J}_{(n_i l_i j_i), (n_j l_j j_j), (n_k l_k j_k), (n_l l_l j_l), (n_m l_m j_m), (n_n l_n j_n)}. \qquad \text{(B.15)}
\end{aligned}
$$

B.1 Hartree-Fock equations in the j-scheme formalism

In this section we show the Hartree-Fock equations in the J-scheme formalism. We substitute matrix elements of the one-body, the two-body, and the three-body operators transformed into the J-scheme to respective Hartree-Fock equations in the

M-scheme (2.43), (2.44), and (2.45). We obtain the corresponding HF equations for protons, neutrons, and the Λ hyperon in the J-scheme:

$$
\begin{aligned}
&t^{\mathrm{p}}_{(n_i l_i j_i),(n_j l_j j_j)} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j} \\
&+ \sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{pp}}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\mathrm{p}}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_i l_k} \delta_{j_i j_k} \delta_{m_i m_j} \frac{(2J+1)}{(2j_i+1)} \\
&+ \sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{pn}}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\mathrm{n}}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_i l_k} \delta_{j_j j_k} \delta_{m_i m_j} \frac{(2J+1)}{(2j_i+1)} \\
&+ \sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{p}\Lambda}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\Lambda}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_i l_k} \delta_{j_l j_k} \delta_{m_i m_j} \frac{(2J+1)}{(2j_i+1)} \frac{1}{(2j_k+1)} \\
&+ \frac{1}{2} \sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{ab} J} \frac{(2J+1)}{(2j_i+1)} \delta_{j_i j_j} \delta_{l_m l_k} \delta_{j_m j_k} \delta_{l_l l_n} \delta_{j_l j_n} \delta_{m_i m_j} \times \\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{p}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+ \frac{1}{2} \Bigg[\frac{2}{3} \sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12} J} \frac{(2J+1)}{(2j_i+1)} \delta_{j_i j_j} \delta_{m_i m_j} \delta_{l_k l_m} \delta_{j_k j_m} \delta_{l_l l_n} \delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{n}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+ \frac{1}{3} \sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12} J} \frac{(2J+1)}{(2j_i+1)} \delta_{j_i j_j} \delta_{m_i m_j} \delta_{l_k l_m} \delta_{j_k j_m} \delta_{l_l l_n} \delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{n}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)} \Bigg] \\
&+ \frac{1}{2} \sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12} J} \frac{(2J+1)}{(2j_i+1)} \delta_{j_i j_j} \delta_{m_i m_j} \delta_{l_k l_m} \delta_{j_k j_m} \delta_{l_l l_n} \delta_{j_l j_n} \\
&\times V^{T'_{12}=0,T_{12}=0,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+ \frac{1}{6} \sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12} J} \frac{(2J+1)}{(2j_i+1)} \delta_{j_i j_j} \delta_{m_i m_j} \delta_{l_k l_m} \delta_{j_k j_m} \delta_{l_l l_n} \delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{3}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&-\frac{1}{2\sqrt{3}}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=0,T_{12}=1,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&-\frac{1}{2\sqrt{3}}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=1,T_{12}=0,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&+\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\frac{1}{(2j_n+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=1,J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\Lambda}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\frac{1}{(2j_n+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=0,J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{n}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\Lambda}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\frac{1}{(2j_n+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=1,J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{n}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\Lambda}_{(n_n l_n j_n),(n_l l_l j_l)}=\varepsilon^{\mathrm{p}}_i\delta_{ij},
\end{aligned}
\tag{B.16}
$$

$$
\begin{aligned}
&t^{\mathrm{n}}_{(n_i l_i j_i),(n_j l_j j_j)}\delta_{l_i l_j}\delta_{j_i j_j}\delta_{m_i m_j} \\
&+\sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{nn}}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\mathrm{n}}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_l l_k}\delta_{j_l j_k}\delta_{m_i m_j}\frac{(2J+1)}{(2j_i+1)} \\
&+\sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{pn}}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\mathrm{n}}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_l l_k}\delta_{j_j j_k}\delta_{m_i m_j}\frac{(2J+1)}{(2j_i+1)} \\
&+\sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{n}\Lambda}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\Lambda}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_l l_k}\delta_{j_l j_k}\delta_{m_i m_j}\frac{(2J+1)}{(2j_i+1)}\frac{1}{(2j_k+1)} \\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{ab} J} \frac{(2J+1)}{(2j_i+1)}\delta_{j_i j_j}\delta_{l_m l_k}\delta_{j_m j_k}\delta_{l_l l_n}\delta_{j_l j_n}\delta_{m_i m_j}\times \\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{n}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+\frac{1}{2}\Bigg[\frac{2}{3}\sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12} J} \frac{(2J+1)}{(2j_i+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{p}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+\frac{1}{3}\sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12} J} \frac{(2J+1)}{(2j_i+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{p}}_{(n_n l_n j_n),(n_l l_l j_l)}\Bigg] \\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12} J} \frac{(2J+1)}{(2j_i+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n} \\
&\times V^{T'_{12}=0,T_{12}=0,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+\frac{1}{6}\sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12} J} \frac{(2J+1)}{(2j_i+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{3}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&+\frac{1}{2\sqrt{3}}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=0,T_{12}=1,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&+\frac{1}{2\sqrt{3}}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=1,T_{12}=0,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&+\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\frac{1}{(2j_n+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=1,J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{n}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\Lambda}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\frac{1}{(2j_n+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=0,J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\Lambda}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\frac{1}{(2j_n+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=1,J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\Lambda}_{(n_n l_n j_n),(n_l l_l j_l)}=\varepsilon^{\mathrm{n}}_i\delta_{ij},
\end{aligned}
\tag{B.17}
$$

$$
\begin{aligned}
&t^{\Lambda}_{(n_i l_i j_i),(n_j l_j j_j)}\delta_{l_i l_j}\delta_{j_i j_j}\delta_{m_i m_j} \\
&+\sum_{J}\sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{p}\Lambda}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)}\rho^{\mathrm{p}}_{(n_l l_l j_l),(n_k l_k j_k)}\delta_{l_k l_l}\delta_{j_k j_l}\delta_{m_i m_j}\frac{(2J+1)}{(2j_i+1)} \\
&+\sum_{J}\sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{n}\Lambda}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)}\rho^{\mathrm{n}}_{(n_l l_l j_l),(n_k l_k j_k)}\delta_{l_k l_l}\delta_{j_k j_l}\delta_{m_i m_j}\frac{(2J+1)}{(2j_i+1)} \\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=1,J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{p}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=1,J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{n}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n} \\
&\times V^{T'_{12}=0,T_{12}=0,T=0,J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=1,J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)} = \varepsilon^{\Lambda}_i\delta_{ij}.
\end{aligned}
\tag{B.18}
$$

The HF energy corresponding to the minimized value of the energy functional (2.42) is expressed in the J-scheme as follows:

$$
\begin{aligned}
E_{\mathrm{HF}} = & \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j}} t^{\mathrm{p}}_{(n_i l_i j_i),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_j l_j j_j),(n_i l_i j_i)} (2j_i+1)\delta_{l_i l_j}\delta_{j_i j_j} \\
& \quad + \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j}} t^{\mathrm{n}}_{(n_i l_i j_i),(n_j l_j j_j)} \rho^{\mathrm{n}}_{(n_j l_j j_j),(n_i l_i j_i)} (2j_i+1)\delta_{l_i l_j}\delta_{j_i j_j} \\
& \quad + \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j}} t^{\Lambda}_{(n_i l_i j_i),(n_j l_j j_j)} \rho^{\Lambda}_{(n_j l_j j_j),(n_i l_i j_i)} \delta_{l_i l_j}\delta_{j_i j_j} \\
& + \frac{1}{2} \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j \\ n_k l_k j_k \\ n_l l_l j_l}} \sum_J (2J+1) V^{J,\mathrm{pp}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)} \rho^{\mathrm{p}}_{(n_k l_k j_k),(n_i l_i j_i)} \rho^{\mathrm{p}}_{(n_l l_l j_l),(n_j l_j j_j)} \delta_{l_i l_k}\delta_{j_i j_k}\delta_{l_j l_l}\delta_{j_i j_l} \\
& + \frac{1}{2} \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j \\ n_k l_k j_k \\ n_l l_l j_l}} \sum_J (2J+1) V^{J,\mathrm{nn}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)} \rho^{\mathrm{n}}_{(n_k l_k j_k),(n_i l_i j_i)} \rho^{\mathrm{n}}_{(n_l l_l j_l),(n_j l_j j_j)} \delta_{l_i l_k}\delta_{j_i j_k}\delta_{l_j l_l}\delta_{j_i j_l} \\
& + \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j \\ n_k l_k j_k \\ n_l l_l j_l}} \sum_J (2J+1) V^{J,\mathrm{pn}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)} \rho^{\mathrm{p}}_{(n_k l_k j_k),(n_i l_i j_i)} \rho^{\mathrm{n}}_{(n_l l_l j_l),(n_j l_j j_j)} \delta_{l_i l_k}\delta_{j_i j_k}\delta_{l_j l_l}\delta_{j_i j_l} \\
& + \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j \\ n_k l_k j_k \\ n_l l_l j_l}} \sum_J \frac{(2J+1)}{(2j_j+1)} V^{J,\mathrm{p}\Lambda}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)} \rho^{\mathrm{p}}_{(n_k l_k j_k),(n_i l_i j_i)} \rho^{\Lambda}_{(n_l l_l j_l),(n_j l_j j_j)} \delta_{l_i l_k}\delta_{j_i j_k}\delta_{l_j l_l}\delta_{j_i j_l} \\
& + \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j \\ n_k l_k j_k \\ n_l l_l j_l}} \sum_J \frac{(2J+1)}{(2j_j+1)} V^{J,\mathrm{n}\Lambda}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)} \rho^{\mathrm{n}}_{(n_k l_k j_k),(n_i l_i j_i)} \rho^{\Lambda}_{(n_l l_l j_l),(n_j l_j j_j)} \delta_{l_i l_k}\delta_{j_i j_k}\delta_{l_j l_l}\delta_{j_i j_l} \\
& + \frac{1}{6} \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j \\ n_k l_k j_k \\ n_l l_l j_l \\ n_m l_m j_m \\ n_n l_n j_n}} \sum_{J,J_{12}} (2J_{12}+1)(2J+1) V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l),(n_m l_m j_m),(n_n l_n j_n)} \times \\
& \qquad \times \rho^{\mathrm{p}}_{(n_l l_l j_l),(n_i l_i j_i)} \rho^{\mathrm{p}}_{(n_m l_m j_m),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_n l_n j_n),(n_k l_k j_k)} \delta_{l_i l_l}\delta_{j_i j_l}\delta_{l_j l_m}\delta_{j_j j_m}\delta_{l_k l_n}\delta_{j_k j_m} \\
& + \frac{1}{6} \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j \\ n_k l_k j_k \\ n_l l_l j_l \\ n_m l_m j_m \\ n_n l_n j_n}} \sum_{J,J_{12}} (2J_{12}+1)(2J+1) V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l),(n_m l_m j_m),(n_n l_n j_n)} \times \\
& \qquad \times \rho^{\mathrm{n}}_{(n_l l_l j_l),(n_i l_i j_i)} \rho^{\mathrm{n}}_{(n_m l_m j_m),(n_j l_j j_j)} \rho^{\mathrm{n}}_{(n_n l_n j_n),(n_k l_k j_k)} \delta_{l_i l_l}\delta_{j_i j_l}\delta_{l_j l_m}\delta_{j_j j_m}\delta_{l_k l_n}\delta_{j_k j_m}
\end{aligned}
$$

$$
+\frac{1}{2}\sum_{\substack{n_il_ij_i\\n_jl_jj_j\\n_kl_kj_k\\n_ll_lj_l\\n_ml_mj_m\\n_nl_nj_n}}\sum_{J,J_{12}}\Bigg\{\frac{2}{3}(2J_{12}+1)(2J+1)V^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}\times
$$

$$
\times\rho^{\mathrm{p}}_{(n_ll_lj_l),(n_il_ij_i)}\rho^{\mathrm{p}}_{(n_ml_mj_m),(n_jl_jj_j)}\rho^{\mathrm{n}}_{(n_nl_nj_n),(n_kl_kj_k)}\delta_{l_il_l}\delta_{j_ij_j}\delta_{l_jl_m}\delta_{j_jj_m}\delta_{l_kl_n}\delta_{j_kj_n}
$$

$$
+\frac{1}{3}(2J_{12}+1)(2J+1)V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}\times
$$

$$
\times\rho^{\mathrm{p}}_{(n_ll_lj_l),(n_il_ij_i)}\rho^{\mathrm{p}}_{(n_ml_mj_m),(n_jl_jj_j)}\rho^{\mathrm{n}}_{(n_nl_nj_n),(n_kl_kj_k)}\delta_{l_il_l}\delta_{j_ij_j}\delta_{l_jl_m}\delta_{j_jj_m}\delta_{l_kl_n}\delta_{j_kj_n}\Bigg\}
$$

$$
+\frac{1}{2}\sum_{\substack{n_il_ij_i\\n_jl_jj_j\\n_kl_kj_k\\n_ll_lj_l\\n_ml_mj_m\\n_nl_nj_n}}\sum_{J,J_{ab}}\Bigg\{\frac{2}{3}(2J_{12}+1)(2J+1)V^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}\times
$$

$$
\times\rho^{\mathrm{n}}_{(n_ll_lj_l),(n_il_ij_i)}\rho^{\mathrm{n}}_{(n_ml_mj_m),(n_jl_jj_j)}\rho^{\mathrm{p}}_{(n_nl_nj_n),(n_kl_kj_k)}\delta_{l_il_l}\delta_{j_ij_j}\delta_{l_jl_m}\delta_{j_jj_m}\delta_{l_kl_n}\delta_{j_kj_n}
$$

$$
+\frac{1}{3}(2J_{12}+1)(2J+1)V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}\times
$$

$$
\times\rho^{\mathrm{n}}_{(n_ll_lj_l),(n_il_ij_i)}\rho^{\mathrm{n}}_{(n_ml_mj_m),(n_jl_jj_j)}\rho^{\mathrm{p}}_{(n_nl_nj_n),(n_kl_kj_k)}\delta_{l_il_l}\delta_{j_ij_j}\delta_{l_jl_m}\delta_{j_jj_m}\delta_{l_kl_n}\delta_{j_kj_n}\Bigg\}
$$

$$
+\frac{1}{2}\sum_{\substack{n_il_ij_i\\n_jl_jj_j\\n_kl_kj_k\\n_ll_lj_l\\n_ml_mj_m\\n_nl_nj_n}}\sum_{J,J_{12}}\frac{(2J_{12}+1)(2J+1)}{(2j_k+1)}V^{T'_{12}=1,T_{12}=1,T=1,J_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}\times
$$

$$
\times\rho^{\mathrm{p}}_{(n_ll_lj_l),(n_il_ij_i)}\rho^{\mathrm{p}}_{(n_ml_mj_m),(n_jl_jj_j)}\rho^{\Lambda}_{(n_nl_nj_n),(n_kl_kj_k)}\delta_{l_il_l}\delta_{j_ij_l}\delta_{l_jl_m}\delta_{j_jj_m}\delta_{l_kl_n}\delta_{j_kj_n}
$$

$$
+\frac{1}{2}\sum_{\substack{n_il_ij_i\\n_jl_jj_j\\n_kl_kj_k\\n_ll_lj_l\\n_ml_mj_m\\n_nl_nj_n}}\sum_{J,J_{12}}\frac{(2J_{12}+1)(2J+1)}{(2j_k+1)}V^{T'_{12}=1,T_{12}=1,T=1,J_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}\times
$$

$$
\times\rho^{\mathrm{n}}_{(n_ll_lj_l),(n_il_ij_i)}\rho^{\mathrm{n}}_{(n_ml_mj_m),(n_jl_jj_j)}\rho^{\Lambda}_{(n_nl_nj_n),(n_kl_kj_k)}\delta_{l_il_l}\delta_{j_ij_l}\delta_{l_jl_m}\delta_{j_jj_m}\delta_{l_kl_n}\delta_{j_kj_n}
$$

$$
+\frac{1}{2}\sum_{\substack{n_il_ij_i\\n_jl_jj_j\\n_kl_kj_k\\n_ll_lj_l\\n_ml_mj_m\\n_nl_nj_n}}\sum_{J,J_{12}}\Bigg\{\frac{1}{2}\frac{(2J_{12}+1)(2J+1)}{(2j_j+1)}V^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=0,J_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}\times
$$

$$
\times\rho^{\mathrm{p}}_{(n_ll_lj_l),(n_il_ij_i)}\rho^{\Lambda}_{(n_ml_mj_m),(n_jl_jj_j)}\rho^{\mathrm{n}}_{(n_nl_nj_n),(n_kl_kj_k)}\delta_{l_il_l}\delta_{j_ij_j}\delta_{l_jl_m}\delta_{j_jj_m}\delta_{l_kl_n}\delta_{j_kj_n}
$$

$$
+\frac{1}{2}\frac{(2J_{12}+1)(2J+1)}{2j_j+1}V^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=1,J_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}\times
$$

$$
\times\rho^{\mathrm{p}}_{(n_ll_lj_l),(n_il_ij_i)}\rho^{\Lambda}_{(n_ml_mj_m),(n_jl_jj_j)}\rho^{\mathrm{n}}_{(n_nl_nj_n),(n_kl_kj_k)}\delta_{l_il_l}\delta_{j_ij_j}\delta_{l_jl_m}\delta_{j_lj_m}\delta_{l_kl_n}\delta_{j_kj_n}\Bigg\}. \tag{B.19}
$$

B.2 Tamm-Dancoff equations in the j-scheme formalism

The coefficients $C_{ph}^{\mu,\mathrm{p}}, C_{ph}^{\mu,\mathrm{n}}, r_{ph}^{\mu,\mathrm{p}\Lambda}$, and $r_{ph}^{\mu,\mathrm{n}\Lambda}$ in the TDA and NΛ TDA Eqs. (3.11), (3.19a), and (3.19b) are transformed into the j-scheme as:

$$C^{\mu,\mathrm{p}}_{n_p l_p j_p m_p, n_h l_h j_h m_h} = \sum_J (-1)^{j_h+m_h} C^{Jm_p-m_h}_{j_p m_p, j_h -m_h} C^{\mu,\mathrm{p},J}_{(n_p l_p j_p),(n_h l_h j_h)}, \tag{B.20a}$$

$$C^{\mu,\mathrm{n}}_{n_p l_p j_p m_p, n_h l_h j_h m_h} = \sum_J (-1)^{j_h+m_h} C^{Jm_p-m_h}_{j_p m_p, j_h -m_h} C^{\mu,\mathrm{n},J}_{(n_p l_p j_p),(n_h l_h j_h)}, \tag{B.20b}$$

$$r^{\mu,\mathrm{p}\Lambda}_{n_p l_p j_p m_p, n_h l_h j_h m_h} = \sum_J (-1)^{j_h+m_h} C^{Jm_p-m_h}_{j_p m_p, j_h -m_h} r^{\mu,\mathrm{p}\Lambda,J}_{(n_p l_p j_p),(n_h l_h j_h)}, \tag{B.20c}$$

$$r^{\mu,\mathrm{n}\Lambda}_{n_p l_p j_p m_p, n_h l_h j_h m_h} = \sum_J (-1)^{j_h+m_h} C^{Jm_p-m_h}_{j_p m_p, j_h -m_h} r^{\mu,\mathrm{n}\Lambda,J}_{(n_p l_p j_p),(n_h l_h j_h)}. \tag{B.20d}$$

The symbols $C^{Jm_p-m_h}_{j_p m_p, j_h -m_h}$ represent the Clebsch-Gordan coefficients. The TDA Eq. (3.11) is in the j-scheme expressed as:

$$\sum_{\substack{n_p l_p j_p \\ n_h l_h j_h}} \begin{pmatrix} \mathbb{A}^{\mathrm{pp},J'}_{\substack{(n_{p'} l_{p'} j_{p'}),(n_h l_h j_h) \\ (n_{h'} l_{h'} j_{h'}),(n_p l_p j_p)}} & \mathbb{A}^{\mathrm{pn},J'}_{\substack{(n_{p'} l_{p'} j_{p'}),(n_h l_h j_h) \\ (n_{h'} l_{h'} j_{h'}),(n_p l_p j_p)}} \\ \mathbb{A}^{\mathrm{pn},J'}_{\substack{(n_h l_h j_h),(n_{p'} l_{p'} j_{p'}) \\ (n_p l_p j_p),(n_{h'} l_{h'} j_{h'})}} & \mathbb{A}^{\mathrm{nn},J'}_{\substack{(n_{p'} l_{p'} j_{p'}),(n_h l_h j_h) \\ (n_{h'} l_{h'} j_{h'}),(n_p l_p j_p)}} \end{pmatrix} \begin{pmatrix} C^{\nu,\mathrm{p},J'}_{(n_p l_p j_p),(n_h l_h j_h)} \\ \\ C^{\nu,\mathrm{n},J'}_{(n_p l_p j_p),(n_h l_h j_h)} \end{pmatrix}$$
$$= (E_\nu - E_{\mathrm{HF}}) \begin{pmatrix} C^{\nu,\mathrm{p},J'}_{(n_{p'} l_{p'} j_{p'}),(n_{h'} l_{h'} j_{h'})} \\ \\ C^{\nu,\mathrm{n},J'}_{(n_{p'} l_{p'} j_{p'}),(n_{h'} l_{h'} j_{h'})} \end{pmatrix}. \tag{B.21}$$

The elements $\mathbb{A}^{\mathrm{pp},J'}_{\substack{(n_{p'}l_{p'}j_{p'}),(n_hl_hj_h)\\(n_{h'}l_{h'}j_{h'}),(n_pl_pj_p)}}$, $\mathbb{A}^{\mathrm{pn},J'}_{\substack{(n_{p'}l_{p'}j_{p'}),(n_hl_hj_h)\\(n_{h'}l_{h'}j_{h'}),(n_pl_pj_p)}}$, $\mathbb{A}^{\mathrm{pn},J'}_{\substack{(n_hl_hj_h),(n_{p'}l_{p'}j_{p'})\\(n_pl_pj_p),(n_{h'}l_{h'}j_{h'})}}$ and $\mathbb{A}^{\mathrm{nn},J'}_{\substack{(n_{p'}l_{p'}j_{p'}),(n_hl_hj_h)\\(n_{h'}l_{h'}j_{h'}),(n_pl_pj_p)}}$ in the Eq. (B.21) are defined as:

$$\mathbb{A}^{\mathrm{pp},J'}_{\substack{(n_{p'}l_{p'}j_{p'}),(n_hl_hj_h)\\(n_{h'}l_{h'}j_{h'}),(n_pl_pj_p)}} = (\varepsilon^{\mathrm{p}}_{n_pl_pj_p} - \varepsilon^{\mathrm{p}}_{n_hl_hj_h})\delta_{n_pn_{p'}}\delta_{l_pl_{p'}}\delta_{j_pj_{p'}}\delta_{n_hn_{h'}}\delta_{l_hl_{h'}}\delta_{j_hj_{h'}} + \sum_J(-1)^{J-j_p-j_{h'}}(2J+1)\left\{\begin{array}{ccc} j_{p'} & j_h & J\\ j_p & j_{h'} & J'\end{array}\right\}\overline{V}^{J',\mathrm{pp,gen}}_{\substack{(n_{p'}l_{p'}j_{p'}),(n_hl_hj_h),\\(n_{h'}l_{h'}j_{h'}),(n_pl_pj_p)}} \tag{B.22a}$$

$$\mathbb{A}^{\mathrm{pn},J'}_{\substack{(n_{p'}l_{p'}j_{p'}),(n_hl_hj_h)\\(n_{h'}l_{h'}j_{h'}),(n_pl_pj_p)}} = \sum_J(-1)^{J-j_p-j_{h'}}(2J+1)\left\{\begin{array}{ccc} j_{p'} & j_h & J\\ j_p & j_{h'} & J'\end{array}\right\}\overline{V}^{J',\mathrm{pn,gen}}_{\substack{(n_{p'}l_{p'}j_{p'}),(n_hl_hj_h),\\(n_{h'}l_{h'}j_{h'}),(n_pl_pj_p)}} \tag{B.22b}$$

$$\mathbb{A}^{\mathrm{pn},J'}_{\substack{(n_hl_hj_h),(n_{p'}l_{p'}j_{p'})\\(n_pl_pj_p),(n_{h'}l_{h'}j_{h'})}} = \sum_J(-1)^{J-j_{p'}-j_h}(2J+1)\left\{\begin{array}{ccc} j_h & j_{p'} & J\\ j_{h'} & j_p & J'\end{array}\right\}\overline{V}^{J',\mathrm{pn,gen}}_{\substack{(n_hl_hj_h),(n_{p'}l_{p'}j_{p'}),\\(n_pl_pj_p),(n_{h'}l_{h'}j_{h'})}} \tag{B.22c}$$

$$\mathbb{A}^{\mathrm{nn},J'}_{\substack{(n_{p'}l_{p'}j_{p'}),(n_hl_hj_h)\\(n_{h'}l_{h'}j_{h'}),(n_pl_pj_p)}} = (\varepsilon^{\mathrm{n}}_{n_pl_pj_p} - \varepsilon^{\mathrm{n}}_{n_hl_hj_h})\delta_{n_pn_{p'}}\delta_{l_pl_{p'}}\delta_{j_pj_{p'}}\delta_{n_hn_{h'}}\delta_{l_hl_{h'}}\delta_{j_hj_{h'}} + \sum_J(-1)^{J-j_p-j_{h'}}(2J+1)\left\{\begin{array}{ccc} j_{p'} & j_h & J\\ j_p & j_{h'} & J'\end{array}\right\}\overline{V}^{J',\mathrm{nn,gen}}_{\substack{(n_{p'}l_{p'}j_{p'}),(n_hl_hj_h).\\(n_{h'}l_{h'}j_{h'}),(n_pl_pj_p)}} \tag{B.22d}$$

The NΛ TDA Eqs. (3.19a), (3.19b) are in the j-scheme expressed as:

$$\sum_{\substack{n_pl_pj_p\\n_hl_hj_h}}\Bigg[(\varepsilon^{\Lambda}_{(n_pl_pj_p)} - \varepsilon^{\mathrm{p}}_{(n_hl_hj_h)})\delta_{n_pn_{p'}}\delta_{l_pl_{p'}}\delta_{j_pj_{p'}}\delta_{n_hn_{h'}}\delta_{l_hl_{h'}}\delta_{j_hj_{h'}} + \sum_J(-1)^{j_p+j_{p'}+j_h+j_{h'}+1}(2J+1)\left\{\begin{array}{ccc} j_h & j_{p'} & J\\ j_{h'} & j_p & J'\end{array}\right\} \times\overline{V}^{J',\mathrm{p\Lambda,gen}}_{\substack{(n_hl_hj_h),(n_{p'}l_{p'}j_{p'})\\(n_{h'}l_{h'}j_{h'}),(n_pl_pj_p)}}\Bigg] r^{\nu,\mathrm{p\Lambda},J'}_{(n_pl_pj_p),(n_hl_hj_h)} = (E^{\mathrm{p\Lambda}}_{\nu} - E_{\mathrm{HF}})r^{\nu,\mathrm{p\Lambda},J'}_{(n_{p'}l_{p'}j_{p'}),(n_{h'}l_{h'}j_{h'})}, \tag{B.23}$$

$$\sum_{\substack{n_pl_pj_p\\n_hl_hj_h}}\Bigg[(\varepsilon^{\Lambda}_{(n_pl_pj_p)} - \varepsilon^{\mathrm{n}}_{(n_hl_hj_h)})\delta_{n_pn_{p'}}\delta_{l_pl_{p'}}\delta_{j_pj_{p'}}\delta_{n_hn_{h'}}\delta_{l_hl_{h'}}\delta_{j_hj_{h'}} + \sum_J(-1)^{j_p+j_{p'}+j_h+j_{h'}+1}(2J+1)\left\{\begin{array}{ccc} j_h & j_{p'} & J\\ j_{h'} & j_p & J'\end{array}\right\} \times\overline{V}^{J',\mathrm{n\Lambda,gen}}_{\substack{(n_hl_hj_h),(n_{p'}l_{p'}j_{p'})\\(n_{h'}l_{h'}j_{h'}),(n_pl_pj_p)}}\Bigg] r^{\nu,\mathrm{n\Lambda},J'}_{(n_pl_pj_p),(n_hl_hj_h)} = (E^{\mathrm{p\Lambda}}_{\nu} - E_{\mathrm{HF}})r^{\nu,\mathrm{n\Lambda},J'}_{(n_{p'}l_{p'}j_{p'}),(n_{h'}l_{h'}j_{h'})}. \tag{B.24}$$

The expressions $\left\{ \begin{array}{ccc} j_h & j_{p'} & J \\ j_{h'} & j_p & J' \end{array} \right\}$ represent the 6j-symbols.

The generalized matrix elements are in the j-scheme defined as:

$$\begin{aligned}
&\overline{V}^{J',\mathrm{pp,gen}}_{\substack{(n_i l_i j_i),(n_j l_j j_j) \\ (n_k l_k j_k),(n_l l_l j_l)}} = \overline{V}^{J',\mathrm{pp}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)} \\
&+ \sum_J \sum_{\substack{n_m l_m j_m \\ n_n l_n j_n}} \frac{(2J+1)}{(2J'+1)} \\
&\Big\{ \overline{V}^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\mathrm{p}}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&+ \frac{1}{3} \overline{V}^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\mathrm{n}}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&+ \frac{2}{3} \overline{V}^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\mathrm{n}}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&+ \overline{V}^{T'_{12}=1,T_{12}=1,T=1,J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\Lambda}_{(n_n l_n j_n),(n_m l_m j_m)} \Big\} .
\end{aligned} \tag{B.25}$$

$$\begin{aligned}
&\overline{V}^{J',\mathrm{nn,gen}}_{\substack{(n_i l_i j_i),(n_j l_j j_j) \\ (n_k l_k j_k),(n_l l_l j_l)}} = \overline{V}^{J',\mathrm{nn}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)} \\
&+ \sum_J \sum_{\substack{n_m l_m j_m \\ n_n l_n j_n}} \frac{(2J+1)}{(2J'+1)} \\
&\Big\{ \overline{V}^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\mathrm{n}}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&+ \frac{1}{3} \overline{V}^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\mathrm{p}}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&+ \frac{2}{3} \overline{V}^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\mathrm{p}}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&+ \overline{V}^{T'_{12}=1,T_{12}=1,T=1,J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\Lambda}_{(n_n l_n j_n),(n_m l_m j_m)} \Big\} .
\end{aligned} \tag{B.26}$$

$$
\begin{aligned}
&\overline{V}^{J',\mathrm{pn,gen}}_{\substack{(n_il_ij_i),(n_jl_jj_j)\\(n_kl_kj_k),(n_ll_lj_l)}} = \overline{V}^{J',\mathrm{pn}}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l)} \\
&+\sum_J \sum_{\substack{n_ml_mj_m\\n_nl_nj_n}} \frac{(2J+1)}{(2J'+1)} \\
&\Bigg\{ \frac{1}{2}\overline{V}^{T'_{12}=0,T_{12}=0,T=\frac{1}{2},J',J',J}_{(n_il_ij_i),(n_jl_jj_j),(n_ml_mj_m),(n_kl_kj_k),(n_ll_lj_l),(n_nl_nj_n)} \overline{\rho}^{\mathrm{p}}_{(n_nl_nj_n),(n_ml_mj_m)} \\
&+ \frac{1}{6}\overline{V}^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J',J',J}_{(n_il_ij_i),(n_jl_jj_j),(n_ml_mj_m),(n_kl_kj_k),(n_ll_lj_l),(n_nl_nj_n)} \overline{\rho}^{\mathrm{p}}_{(n_nl_nj_n),(n_ml_mj_m)} \\
&+ \frac{1}{3}\overline{V}^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J',J',J}_{(n_il_ij_i),(n_jl_jj_j),(n_ml_mj_m),(n_kl_kj_k),(n_ll_lj_l),(n_nl_nj_n)} \overline{\rho}^{\mathrm{p}}_{(n_nl_nj_n),(n_ml_mj_m)} \\
&- \frac{1}{2\sqrt{3}}\overline{V}^{T'_{12}=0,T_{12}=1,T=\frac{1}{2},J',J',J}_{(n_il_ij_i),(n_jl_jj_j),(n_ml_mj_m),(n_kl_kj_k),(n_ll_lj_l),(n_nl_nj_n)} \overline{\rho}^{p}_{(n_nl_nj_n),(n_ml_mj_m)} \\
&- \frac{1}{2\sqrt{3}}\overline{V}^{T'_{12}=1,T_{12}=0,T=\frac{1}{2},J',J',J}_{(n_il_ij_i),(n_jl_jj_j),(n_ml_mj_m),(n_kl_kj_k),(n_ll_lj_l),(n_nl_nj_n)} \overline{\rho}^{p}_{(n_nl_nj_n),(n_ml_mj_m)} \\
&+ \frac{1}{3}\overline{V}^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J',J',J}_{(n_il_ij_i),(n_jl_jj_j),(n_ml_mj_m),(n_kl_kj_k),(n_ll_lj_l),(n_nl_nj_n)} \overline{\rho}^{n}_{(n_nl_nj_n),(n_ml_mj_m)} \\
&+ \frac{2}{3}\overline{V}^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J',J',J}_{(n_il_ij_i),(n_jl_jj_j),(n_ml_mj_m),(n_kl_kj_k),(n_ll_lj_l),(n_nl_nj_n)} \overline{\rho}^{n}_{(n_nl_nj_n),(n_ml_mj_m)} \\
&+ \frac{1}{2}\overline{V}^{T'_{12}=1,T_{12}=1,T=1,J',J',J}_{(n_il_ij_i),(n_jl_jj_j),(n_ml_mj_m),(n_kl_kj_k),(n_ll_lj_l),(n_nl_nj_n)} \overline{\rho}^{\Lambda}_{(n_nl_nj_n),(n_ml_mj_m)} \\
&+ \frac{1}{2}\overline{V}^{T'_{12}=0,T_{12}=0,T=0,J',J',J}_{(n_il_ij_i),(n_jl_jj_j),(n_ml_mj_m),(n_kl_kj_k),(n_ll_lj_l),(n_nl_nj_n)} \overline{\rho}^{\Lambda}_{(n_nl_nj_n),(n_ml_mj_m)} \Bigg\}. \qquad \text{(B.27)}
\end{aligned}
$$

$$
\begin{aligned}
&\overline{V}^{J',\mathrm{p}\Lambda,\mathrm{gen}}_{\substack{(n_il_ij_i),(n_jl_jj_j)\\(n_kl_kj_k),(n_ll_lj_l)}} = \overline{V}^{J',\mathrm{p}\Lambda}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l)} \\
&+\sum_J \sum_{\substack{n_ml_mj_m\\n_nl_nj_n}} \frac{(2J+1)}{(2J'+1)} \\
&\Bigg\{ \overline{V}^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=1,J',J',J}_{(n_il_ij_i),(n_jl_jj_j),(n_ml_mj_m),(n_kl_kj_k),(n_ll_lj_l),(n_nl_nj_n)} \overline{\rho}^{\mathrm{p}}_{(n_nl_nj_n),(n_ml_mj_m)} \\
&+ \frac{1}{2}\overline{V}^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=1,J',J',J}_{(n_il_ij_i),(n_jl_jj_j),(n_ml_mj_m),(n_kl_kj_k),(n_ll_lj_l),(n_nl_nj_n)} \overline{\rho}^{\mathrm{n}}_{(n_nl_nj_n),(n_ml_mj_m)} \\
&+ \frac{1}{2}\overline{V}^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=0,J',J',J}_{(n_il_ij_i),(n_jl_jj_j),(n_ml_mj_m),(n_kl_kj_k),(n_ll_lj_l),(n_nl_nj_n)} \overline{\rho}^{\mathrm{n}}_{(n_nl_nj_n),(n_ml_mj_m)} \Bigg\}. \qquad \text{(B.28)}
\end{aligned}
$$

$$
\begin{aligned}
&\overline{V}^{J',\mathrm{n}\Lambda,\mathrm{gen}}_{\substack{(n_i l_i j_i),(n_j l_j j_j)\\(n_k l_k j_k),(n_l l_l j_l)}} = \overline{V}^{J',\mathrm{n}\Lambda}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)} \\
&+ \sum_{J} \sum_{\substack{n_m l_m j_m \\ n_n l_n j_n}} \frac{(2J+1)}{(2J'+1)} \\
&\Big\{ \overline{V}^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=1,J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\mathrm{n}}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&+ \frac{1}{2} \overline{V}^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=1,J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\mathrm{p}}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&+ \frac{1}{2} \overline{V}^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=0,J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\mathrm{p}}_{(n_n l_n j_n),(n_m l_m j_m)} \Big\} .
\end{aligned}
\tag{B.29}
$$

All interaction matrix elements $\overline{V}$, as well as the density matrices $\overline{\rho}$ are expressed in the self-consistent basis. The transformation between the elements in the HO and in the self-consistent basis is the Eq. (2.30). The three-body terms with $T = 0$ or 1 represent the elements of the ΛNN interaction and they are not implemented in the NΛ TDA Eqs. (B.23) and (B.24).

Bibliography

[1] P. Ring and P. Schuck, *The nuclear many-body problem* (Berlin:Springer, 1980)

[2] M. Bender, P.-H. Heenen and P.-G. Reinhard, Rev. Mod. Phys. **75** (2003)

[3] T. Skyrme, Nuclear Physics **9** (1958) (4)

[4] J. Dechargé and D. Gogny, Phys. Rev. C **21** (1980)

[5] V. G. J. Stoks, R. A. M. Klomp, C. P. F. Terheggen and J. J. de Swart, Phys. Rev. C **49** (1994)

[6] R. B. Wiringa, V. G. J. Stoks and R. Schiavilla, Phys. Rev. C **51** (1995)

[7] R. Machleidt, F. Sammarruca and Y. Song, Phys. Rev. C **53** (1996)

[8] E. Epelbaum, H.-W. Hammer and U.-G. Meißner, Rev. Mod. Phys. **81** (2009)

[9] R. Machleidt and D. Entem, Physics Reports **503** (2011) (1)

[10] D. Entem and R. Machleidt, Physics Letters B **524** (2002) (1)

[11] S. Bogner, T. Kuo and A. Schwenk, Physics Reports **386** (2003) (1)

[12] S. Bogner, R. Furnstahl and A. Schwenk, Progress in Particle and Nuclear Physics **65** (2010) (1)

[13] H. Feldmeier, T. Neff, R. Roth and J. Schnack, Nuclear Physics A **632** (1998) (1)

[14] M. Hjorth-Jensen, T. T. Kuo and E. Osnes, Physics Reports **261** (1995) (3)

[15] B. D. DAY, Rev. Mod. Phys. **39** (1967)

[16] L. Faddeev, JETP (Sov. Phys.) **12** (1961) (1014)

[17] O. A. Yakubovsky, Sov. J. Nucl. Phys. **5** (1967) (937)

[18] B. R. Barrett, P. Navrtil and J. P. Vary, Progress in Particle and Nuclear Physics **69** (2013) (Supplement C)

[19] D. J. Dean and M. Hjorth-Jensen, Phys. Rev. C **69** (2004)

[20] H. Feldmeier, Nuclear Physics A **515** (1990) (1)

[21] V. Somà, C. Barbieri and T. Duguet, Phys. Rev. C **87** (2013)

[22] J. Carlson, Phys. Rev. C **36** (1987)

[23] A. Goodman, Adv. Nucl. Phys.; (United States) **11:1** (1979)

[24] H. Hergert, P. Papakonstantinou and R. Roth, Phys. Rev. C **83** (2011)

[25] D. Bianco, F. Knapp, N. L. Iudice, P. Vesel, F. Andreozzi, G. D. Gregorio and A. Porrino, Journal of Physics G: Nuclear and Particle Physics **41** (2014) (2)

[26] D. Bianco, F. Knapp, N. Lo Iudice, F. Andreozzi and A. Porrino, Phys. Rev. C **85** (2012)

[27] G. De Gregorio, F. Knapp, N. Lo Iudice and P. Vesely, Phys. Rev. C **94** (2016)

[28] G. De Gregorio, F. Knapp, N. Lo Iudice and P. Vesely, Phys. Rev. C **93** (2016)

[29] G. De Gregorio, F. Knapp, N. Lo Iudice and P. Veselý, Phys. Rev. C **95** (2017)

[30] D. Bianco, F. Knapp, N. Lo Iudice, F. Andreozzi, A. Porrino and P. Vesely, Phys. Rev. C **86** (2012)

[31] F. Knapp, N. Lo Iudice, P. Veselý, F. Andreozzi, G. De Gregorio and A. Porrino, Phys. Rev. C **90** (2014)

[32] F. Knapp, N. Lo Iudice, P. Veselý, F. Andreozzi, G. De Gregorio and A. Porrino, Phys. Rev. C **92** (2015)

[33] G. D. Gregorio, F. Knapp, N. L. Iudice and P. Vesel, Physica Scripta **92** (2017) (7)

[34] A. Ekström, G. R. Jansen, K. A. Wendt, G. Hagen, T. Papenbrock, B. D. Carlsson, C. Forssén, M. Hjorth-Jensen, P. Navrátil and W. Nazarewicz, Phys. Rev. C **91** (2015)

[35] M. Danysz and J. Pniewski, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science **44** (1953) (350)

[36] M. Agnello, Phys. Lett. B **622** (2005) (1)

[37] O. Hashimoto and H. Tamura, Progress in Particle and Nuclear Physics **57** (2006) (2)

[38] H. Tamura, M. Ukai, T. Yamamoto and T. Koike, Nuclear Physics A **881** (2012) (Supplement C), progress in Strangeness Nuclear Physics

[39] T. A. Rijken, M. M. Nagels and Y. Yamamoto, Progress of Theoretical Physics Supplement **185** (2010)

[40] J. Haidenbauer and U.-G. Meißner, Phys. Rev. C **72** (2005)

[41] H. Polinder, J. Haidenbauer and U.-G. Meiner, Nuclear Physics A **779** (2006) (Supplement C)

[42] J. Haidenbauer, S. Petschauer, N. Kaiser, U.-G. Meiner, A. Nogga and W. Weise, Nuclear Physics A **915** (2013)

[43] J. Pokorn, *Systematic Calculations of Hypernuclear Spectra*, candthesis, FJFI VUT v Praze (2016)

[44] A. Ekström, G. Baardsen, C. Forssén, G. Hagen, M. Hjorth-Jensen, G. R. Jansen, R. Machleidt, W. Nazarewicz, T. Papenbrock, J. Sarich and S. M. Wild, Phys. Rev. Lett. **110** (2013)

[45] M. Isaka, K. Fukukawa, M. Kimura, E. Hiyama, H. Sagawa and Y. Yamamoto, Phys. Rev. C **89** (2014)

[46] G. C. Wick, Phys. Rev. **80** (1950)

[47] D. J. Rowe, *Nuclear collective motion: models and theory* (World Scientific, 1970)

[48] Y. Akaishi, T. Harada, S. Shinmura and K. S. Myint, Phys. Rev. Lett. **84** (2000)

[49] M. Sharma, M. Nagarajan and P. Ring, Physics Letters B **312** (1993) (4)

[50] B. D. Serot and J. D. Walecka, Adv. Nucl. Phys. **16** (1984) (ITP-740-STANFORD)

[51] G. De Gregorio, J. Herko, F. Knapp, N. Lo Iudice and P. Veselý, Phys. Rev. C **95** (2017)

[52] I. Angeli, Atomic Data and Nuclear Data Tables **87** (2004) (2)

[53] D. Vautherin and D. M. Brink, Phys. Rev. C **5** (1972)

[54] M. Agnello, L. Benussi, M. Bertani, H. Bhang, G. Bonomi, E. Botta, M. Bregant, T. Bressani, S. Bufalino, L. Busso, D. Calvo, P. Camerini, B. Dalena, F. D. Mori, G. DErasmo, F. Fabbri, A. Feliciello, A. Filippi, E. Fiore, A. Fontana, H. Fujioka, P. Genova, P. Gianotti, N. Grion, V. Lucherini, S. Marcello, N. Mirfakhrai, F. Moia, P. Montagna, O. Morra, T. Nagae, H. Outa, A. Pantaleo, V. Paticchio, S. Piano, R. Rui, G. Simonetti, R. Wheadon and A. Zenoni, Physics Letters B **698** (2011) (3)

[55] R. Chrien, Nucl. Phys. A **478** (1998)

[56] T. Hasegawa, O. Hashimoto, S. Homma, T. Miyachi, T. Nagae, M. Sekimoto, T. Shibata, H. Sakaguchi, T. Takahashi, K. Aoki, H. Noumi, H. Bhang, M. Youn, Y. Gavrilov, S. Ajimura, T. Kishimoto, A. Ohkusu, K. Maeda, R. Sawafta and R. P. Redwine, Phys. Rev. C **53** (1996)

[57] H. Tamura, K. Hosomi, S. Bufalino, N. Chiga, P. Evtoukhovitch, A. Feliciello, R. Honda, T. Koike, Y. Ma, K. Miwa, A. Sasaki, Y. Sasaki, K. Shirotori, K. Tanida, Z. Tsamalaidze, M. Ukai, T. Yamamoto and S. Yang, Nuclear Physics A **914** (2013), xI International Conference on Hypernuclear and Strange Particle Physics (HYP2012)

[58] G. R. J. T. England, M. Hjorth-Jensen, *CENS, a Computational Environment for Nuclear Structure*, folk.uio.no/mhjensen/cp/software.html