Lesson 5: Regression models for item description

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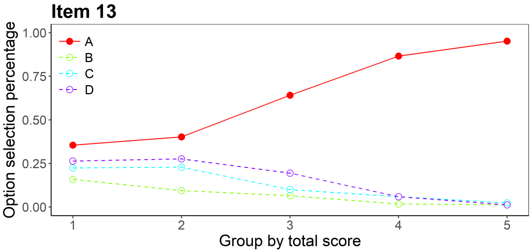
Traditional item analysis

Traditional item analysis describes item properties by

- Percentages of correct response
- Proportions of those who selected given distractor
- Differences of percentages for groups by total score
- Correlations of item score with total score

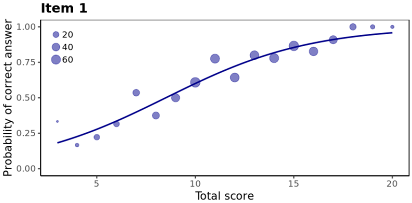


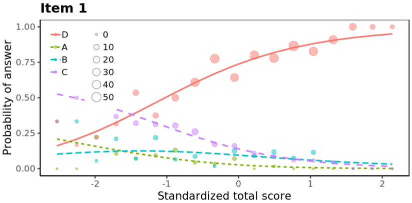
Traditional item analysis

- Difficulty
    - Ratio of correct answers
    - Average item score
    - Scaled average item score
- Discrimination
    - Upper-lower index (ULI)
    - Generalized ULI
    - Correlation Item – Test (RIT)
    - Correlation Item – Rest (RIR)
    - Cronbach's alpha without item
- Distractor analysis
- Analysis of non-reached items

Towards regression models for item description

Models describing mean item score or probability of correct answer with respect to total (or standardized total) score

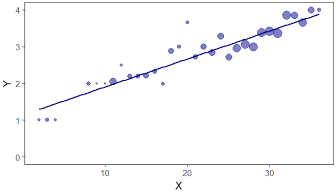




- Better description of item functioning
- Using few parameters per item only
- Possibility to test differences in item score
  - For different groups of respondents (gender, ethnicity)
  - For different types of items
- Possibility to account for specific data features
  - Hierarchical structure, etc.

Regression analysis

Statistical procedures for estimating the relationships among variables.



Simple linear regression:

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i, \qquad i = 1, \ldots, n.$$

- Dependent variable / response Y
- Independent/explanatory variables X (predictors/covariates/regressors)
- Unknown parameters β (intercept β_0)
- Random error ϵ

Regression analysis

Interpretation:

- β_0 is value of Y when $X = 0$ (intercept)
- β_1 describes change of Y with one-unit increase of X (slope)

Estimation procedure:

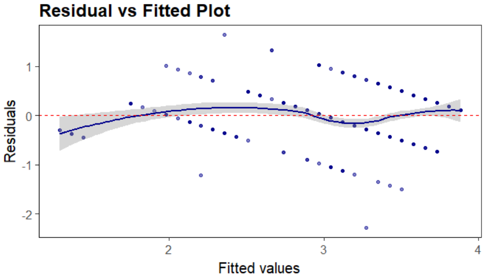
- Predicted value $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$
- Residuals $e_i = Y_i - \hat{Y}_i$
- Ordinary least squares estimation: Minimizes $RSS = \sum_{i=1}^{n} e_i^2$
- Model fit: R^2, F test of the overall fit, t tests
- Model selection:
    - Likelihood ratio test (LRT)
    - AIC, BIC (Akaike/Bayesian information criterion)

Assumptions of linear regression

- Representativeness of the sample
- The error is a random variable with a mean of zero conditional on the explanatory variables.
- The errors are uncorrelated
- The independent variables are measured with no error
- The predictors are linearly independent
- The error variance is constant across observations (homoscedasticity)

Task 1: How would you check these asumptions are fulfilled?
Task 2: Provide examples of cases when these assumptions don't hold.
Task 3: Which extensions may be applied in such cases?

Checking assumptions of linear regression



To learn more, see courses:

- NMSA407: Linear regression
- NMST432: Advanced regression models

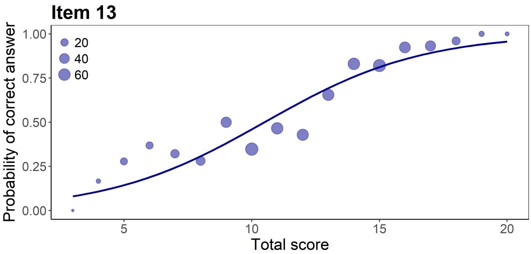
Logistic regression

For each item j, item properties are described by parameters b_{0j} and b_{1j} of logistic function

$$\pi_{ij} = P(Y_{ij} = 1 | X_i, b_{0j}, b_{1j}) = \frac{\exp(b_{0j} + b_{1j}X_i)}{1 + \exp(b_{0j} + b_{1j}X_i)}$$

Also can be written as:

$$\text{logit}(\pi_{ij}) = \log\left(\frac{\pi_{ij}}{1 - \pi_{ij}}\right) = b_{0j} + b_{1j}X_i$$



Logistic regression

Terminology/Interpretation:

- logit = log-odds = logarithm of the odds $\pi/(1-\pi)$ of answering the item correctly vs. uncorrectly
- b_0 is value of log-odds when $X = 0$ (intercept)
- b_1 is change of log-odds associated with one-unit increase of X (slope)

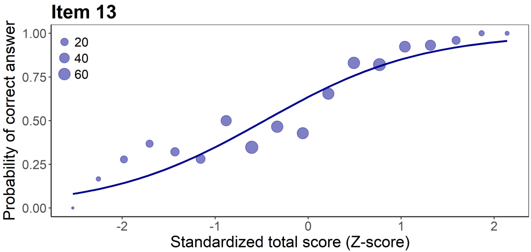
Notes:

- Logistic model belongs to larger class of Generalized linear models (GLM)
- In GLM, linear model is related to response variable via a link function
- Link functions: logit, probit (inverse of the cumulative distribution function), etc.

Logistic regression on Z-scores

Logistic regression on Z-score

$$\pi_{ij} = P(Y_{ij} = 1 | Z_i, b_{0j}, b_{1j}) = \frac{\exp(b_{0j} + b_{1j} Z_i)}{1 + \exp(b_{0j} + b_{1j} Z_i)}$$



Interpretation:

- b_0 is value of log-odds for average respondent $Z = 0$
- b_1 is change of log-odds associated with one-unit increase of Z, i.e., with 1 SD increase of X

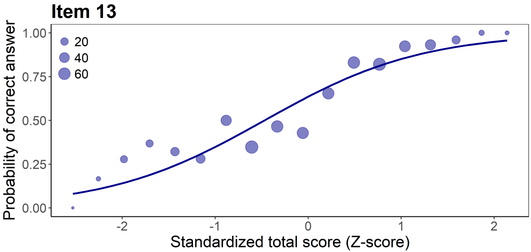
Logistic regression on Z-scores, IRT parametrization

Logistic regression on Z-score, with IRT parametrization

$$\pi_{ij} = P(Y_{ij} = 1 | Z_i, a_j, b_j) = \frac{\exp[a_j(Z_i - b_j)]}{1 + \exp[a_j(Z_i - b_j)]}$$

Also can be written as:

$$\text{logit}(\pi_{ij}) = \log\left(\frac{\pi_{ij}}{1 - \pi_{ij}}\right) = a_j(Z_i - b_j)$$



Logistic regression - IRT parametrization

Interpretation:

- Z_i standardized total score of person i (Z-score)
- b_j difficulty of item j, location of inflexion point, Z_i such that $P(Y_{ij} = 1|Z_i) = 0.5$
- a_j discrimination of item j, slope at $Z_i = b_j$, change of log-odds associated with one-unit increase of Z_i

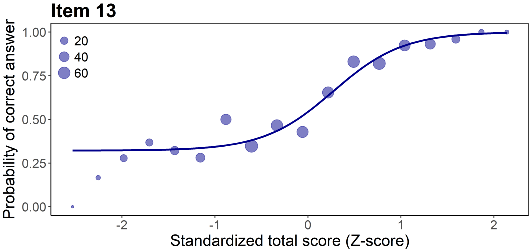
Nonlinear regression

$$\pi_{ij} = P(Y_{ij} = 1 | Z_i, a_j, b_j, c_i) = c_i + (1 - c_i)\frac{\exp[a_j(Z_i - b_j)]}{1 + \exp[a_j(Z_i - b_j)]}$$

b_j difficulty of item j

a_j discrimination of item j

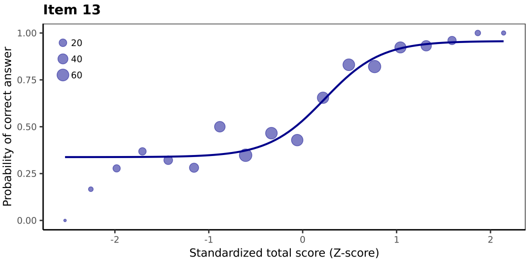
c_j probability of guessing of item j (lower asymptote)



Nonlinear regression

$$\pi_{ij} = P(Y_{ij} = 1|Z_i, a_j, b_j, c_i, d_i) = c_i + (d_i - c_i)\frac{\exp[a_j(Z_i - b_j)]}{1 + \exp[a_j(Z_i - b_j)]}$$

b_j difficulty of item j
a_j discrimination of item j
c_j probability of guessing of item j
d_j probability of innatention on item j (upper asymptote)

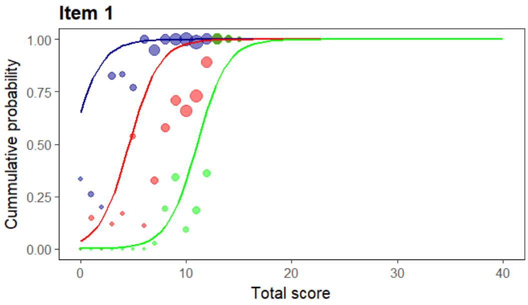


Ordinal regression - cumulative models

Cumulative probabilities are modelled using logistic regression:

$$\pi^*_{ijk} = P(Y_{ij} \geq k | Z_i, a_j, b_{jk}) = \frac{exp[a_j(Z_i - b_{jk})]}{1 + exp[a_j(Z_i - b_{jk})]}$$

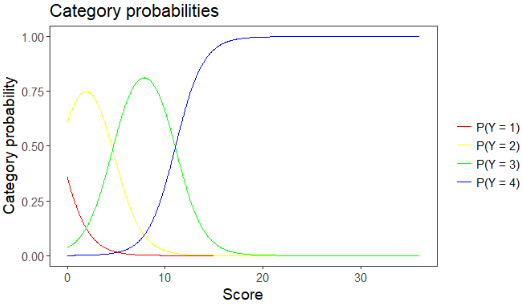
b_{jk} locations of inflection points of cumulative functions



Ordinal regression - cumulative models

Response category probabilities are given by difference of cumulative probabilities:

$$\pi_{ijk} = P(Y_{ij} = k | X_i, a_j, b_{jk}) = \pi^*_{ijk} - \pi^*_{ij(k+1)}$$



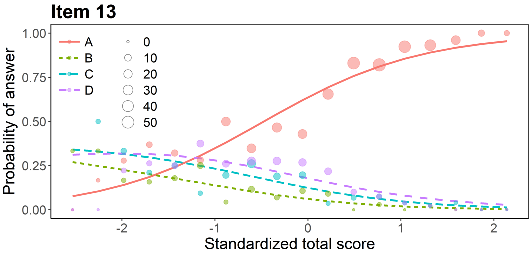
Multinomial Regression - divide-by-total models

Modells log odds of choosing distractor vs. correct answer (baseline)

$$\log \frac{\pi_{ijk}}{\pi_{ij0}} = a_{jk}(Z_i - b_{jk})$$

Response category probabilities are defined as the ratio between category-related functions and their sum:

$$\pi_{ijk} = P(Y_{ij} = k | Z_i, a_{j0}, \ldots, a_{jK_j}, b_{j0}, \ldots, b_{jK_k}) = \frac{\exp(a_{jk}(Z_i - b_{jk}))}{\sum_{r=0}^{K_j} \exp(a_{jr}(Z_i - b_{jr}))}$$



Conclusion

Regression models provide more flexible approach to item description than traditional item analysis

- Description of item functioning across whole ability (total scores) distribution using few parameters per item only
- Possibility to test differences in mean item score for given total score
  - In different groups of respondents (gender, ethnicity)
  - In different types of items
- Possibility to account for specific data features
  - Hierarchical structure
  - Correlations between some items, etc.

Thank you for your attention!

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Vocabulary

- Regression analysis
  - Dependent variable / response Y
  - Independent variables / predictors X
  - Unknown parameters β
  - Residuals $e_i = Y_i - \hat{Y}_i$
  - Ordinary least squares: minimizes RSS
  - Model fit R^2
  - Model selection: LRT, AIC, BIC
- Regression models
  - Linear regression
  - Generalized linear regression
    - Link function: logit, probit,...
  - Nonlinear regression
  - Ordinal regression
  - Multinomial regression