On blow-up in some mathematical models related to phase transitions and turbulence

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Introduction

Phase transitions and turbulence in the atmosphere have a great impact on distribution and dynamics of aerosols (Badger et al., 2006; Meneguz and Reeks, 2011).

Mathematical models of these phenomena are described by differential equations whose solutions (or their derivatives) tend to infinity as the time or other variables approach some finite values (Savin et al., 2006). This situation is known as *blow-up*. Therefore, determining conditions for blow-up and estimating blow-up time is an important theoretical and applied problem.

Recently, S. Pohozaev developed the nonlinear capacity method, which allows establish blow-up conditions for a broad range of problems (Pohozaev, 2009).

In the present work, the nonlinear capacity method is adapted to the case of operators with coefficients that tend to infinity near some unbounded sets, for example, straight lines or planes in the 3-dimensional space.

Assumptions

Let $S \subset R^n$ be a closed unbounded set. Take $\varepsilon > 0, x \in R^n$. Denote *ρ(x)*=dist(*x*,S) and $S^\varepsilon = \{x \in R^n : \rho(x) < \varepsilon\}$.

Let there exist positive constants α, c_1, c_2, ε_0, and R_0 such that for each $\varepsilon \in (0, \varepsilon_0)$ and $R > R_0$ there hold inequalities

$$c_1 \varepsilon^\alpha R^{n-\alpha} \le \mu(S^\varepsilon \cap B_R(0)) \le c_2 \varepsilon^\alpha R^{n-\alpha}, \quad (1)$$

where $B_R(0) = \{x \in R^n : |x| < R\}$.

Remark 1. Condition (1) on *S* is natural for a whole class of aforementioned sets.

Main Results

For the sake of definiteness, the results are formulated in a special case, namely for the evolutional inequality

$$u_t - \Delta u \ge u^q \rho^{-\alpha}(x) \quad (x \in R^n \setminus S, t \in R_+) \quad (2)$$

with initial conditions

$$u(x,0) = u_0(x) \quad (x \in R^n \setminus S). \quad (3)$$

The initial function $u_0(x)$ is assumed to satisfy the estimate

$$u_0(x) \ge c\,|x|^\lambda \rho^\mu(x) \quad (x \in R^n \setminus S) \quad (4)$$

with some constants $c>0$, $\lambda, \mu \in R$.

Remark 2. Condition (4) imposes limitations on the growth of initial functions as $|x| \to \infty$ and $\rho(x) \to 0$, which is natural in applications.

A typical result concerning inequality (2) can be formulated as follows.

Theorem. *Suppose that S satisfies* (1). *Let q* > 1, *and let* u_0 *satisfy* (4) *with* $\alpha > \max\{(\mu - \lambda)(q-1) + 2, 0\}$.

Then any solution of the Cauchy problem (2), (3) *blows up in finite time.*

Remark 3. Similarly, sufficient conditions for blow-up are established for other classes of differential inequalities such as:

- Semilinear and quasilinear elliptic inequalities.
- Elliptic inequalities with gradient terms.
- More general evolutional differential inequalities, etc.

Future Prospects

- To estimate blow-up time in terms of known data.
- To develop numerical simulation of blow-up for the class of problems under investigation.

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