Strong Standard Completeness of **IUL** plus $\mathbf{t} \Leftrightarrow \mathbf{f}$ via a Structure Theorem for Finitely Generated Group-like FL_e-algebras à la Hahn

Sándor Jenei*

University of Pécs, Pécs, Hungary
jenei@ttk.pte.hu

Hahn's structure theorem [2] states that totally ordered Abelian groups can be embedded in the *lexicographic product* of *real groups*. Residuated lattices are semigroups only, and are algebraic counterparts of substructural logics [1]. Involutive commutative residuated chains (aka. involutive FL_e-chains) form an algebraic counterpart of the logic **IUL** [5]. The focus of our investigation is a subclass of them, called commutative *group-like* residuated chains, that is, totally ordered, involutive commutative residuated lattices such that the unit of the monoidal operation coincides with the constant that defines the involution. These algebras are algebraic counterparts of **IUL** plus $\mathbf{t} \Leftrightarrow \mathbf{f}$.

Group-like commutative residuated chains can be characterized as generalizations of totally ordered Abelian groups, hence their name, see Theorem 2. Thirdly, in quest for establishing a structural description for all commutative group-like residuated chains à la Hahn, so-called partial-lexicographic product constructions will be introduced. Roughly, only a cancellative subalgebra of a commutative group-like residuated chain is used as a first component of a lexicographic product, and the rest of the algebra is left unchanged. This results in group-like FL_e-algebras, see Theorem 1. The main theorem is about the structure of group-like FL_e-chains with a finite number of idempotents. Each such algebra is embeddable into a finite partial-lexicographic product of totally ordered Abelian groups, see Theorem 4. This result extends the famous structural description of totally ordered Abelian groups by Hahn, to, e.g, finitely generated group-like FL_e-chains. A corollary is the strong standard completeness of the logic **IUL** plus $\mathbf{t} \Leftrightarrow \mathbf{f}$.

Definition 1. *(Partial-lexicographic products)* Let $\mathbf{X} = (X, \wedge_X, \vee_X, *, \rightarrow_*, t_X, f_X)$ be a group-like FL_e-algebra and $\mathbf{Y} = (Y, \wedge_Y, \vee_Y, \star, \rightarrow_\star, t_Y, f_Y)$ be an involutive FL_e-algebra, with residual complement $'^*$ and $'^\star$, respectively. Add a top element $\top$ to Y, and extend $\star$ by $\top \star y = y \star \top = \top$ for $y \in Y \cup \{\top\}$, then add a bottom element $\bot$ to $Y \cup \{\top\}$, and extend $\star$ by $\bot \star y = y \star \bot = \bot$ for $y \in Y \cup \{\bot, \top\}$. Let $\mathbf{X}_1 = (X_1, \wedge_X, \vee_X, *, \rightarrow_*, t_X, f_X)$ be any cancellative subalgebra of $\mathbf{X}$. We define $\mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1}, \mathbf{Y}^{\bot\top})} = \left(X_{\Gamma(X_1, Y^{\bot\top})}, \leq, \circledast, \rightarrow_\circledast, (t_X, t_Y), (f_X, f_Y)\right)$, where $X_{\Gamma(X_1, Y^{\bot\top})} = (X_1 \times (Y \cup \{\bot, \top\})) \cup ((X \setminus X_1) \times \{\bot\})$, $\leq$ is the restriction of the lexicographic order of $\leq_X$ and $\leq_{Y \cup \{\bot,\top\}}$ to $X_{\Gamma(X_1, Y^{\bot\top})}$, $\circledast$ is defined coordinatewise, and the operation $\rightarrow_\circledast$ is given by $(x_1, y_1) \rightarrow_\circledast (x_2, y_2) = ((x_1, y_1) \circledast (x_2, y_2)')'$ where

$$(x, y)' = \begin{cases} (x'^*, y'^\star) & \text{if } x \in X_1 \\ (x'^*, \bot) & \text{if } x \notin X_1 \end{cases} .$$

Call $\mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1}, \mathbf{Y}^{\bot\top})}$ the *(type-I) partial-lexicographic product* of X, X_1, and Y, respectively.

Let $\mathbf{X} = (X, \leq_X, *, \rightarrow_*, t_X, f_X)$ be a group-like FL_e-chain, $\mathbf{Y} = (Y, \leq_Y, \star, \rightarrow_\star, t_Y, f_Y)$ be an involutive FL_e-algebra, with residual complement $'^*$ and $'^\star$, respectively. Add a top element $\top$ to

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Y, and extend $\star$ by $\top\star y = y\star\top = \top$ for $y \in Y\cup\{\top\}$. Further, let $\mathbf{X_1} = (X_1, \wedge, \vee, *, \to_*, t_X, f_X)$ be a cancellative, discrete, prime (that is, $(X \setminus X_1) * (X \setminus X_1) \subseteq X \setminus X_1$) subalgebra of $\mathbf{X}$. We define $\mathbf{X_{\Gamma(X_1,Y^\top)}} = \left(X_{\Gamma(X_1,Y^\top)}, \leq, \circledast, \to_\circledast, (t_X, t_Y), (f_X, f_Y)\right)$, where $X_{\Gamma(X_1,Y^\top)} = (X_1 \times (Y \cup \{\top\})) \cup ((X \setminus X_1) \times \{\top\})$, $\leq$ is the restriction of the lexicographic order of $\leq_X$ and $\leq_{Y\cup\{\top\}}$ to $X_{\Gamma(X_1,Y)}$, $\circledast$ is defined coordinatewise, and the operation $\to_\circledast$ is given by $(x_1, y_1) \to_\circledast (x_2, y_2) = ((x_1, y_1) \circledast (x_2, y_2)')'$ where

$$(x, y)' = \begin{cases} ((x'^{*}), \top) & \text{if } x \notin X_1 \text{ and } y = \top \\ (x'^{*}, y'^{\star}) & \text{if } x \in X_1 \text{ and } y \in Y \\ ((x'^{*})_\downarrow, \top) & \text{if } x \in X_1 \text{ and } y = \top \end{cases}.$$

[1] Call $\mathbf{X_{\Gamma(X_1,Y^\top)}}$ the *(type-II) partial-lexicographic product* of X, X_1, and Y, respectively.

Theorem 1. *$\mathbf{X_{\Gamma(X_1,Y^{\perp\top})}}$ and $\mathbf{X_{\Gamma(X_1,Y^\top)}}$ are involutive FL$_e$-algebras. If $\mathbf{Y}$ is group-like then also $\mathbf{X_{\Gamma(X_1,Y^{\perp\top})}}$ and $\mathbf{X_{\Gamma(X_1,Y^\top)}}$ are group-like.*

Theorem 2. *For a group-like FL$_e$-algebra $(X, \wedge, \vee, \circledast, \to_\circledast, t, f)$ the following statements are equivalent: $(X, \wedge, \vee, \circledast, t)$ is a lattice-ordered Abelian group if and only if $\circledast$ is cancellative if and only if $x \to_\circledast x = t$ for all $x \in X$ if and only if the only idempotent element in the positive cone of X is t.*

Theorem 3. *Any order-dense group-like FL$_e$-chain which has only a finite number of idempotents can be built by iterating finitely many times the partial-lexicographic product constructions using only totally ordered groups, as building blocks. More formally, let $\mathbf{X}$ be an order-dense group-like FL$_e$-chain which has $n \in \mathbf{N}$ idempotents in its positive cone. Denote $I = \{\perp\top, \top\}$. For $i \in \{1, 2, \ldots, n\}$ there exist totally ordered Abelian groups $\mathbf{G}_i$, $\mathbf{H}_1 \leq \mathbf{G}_1$, $\mathbf{H}_i \leq \mathbf{\Gamma}(\mathbf{H}_{i-1}, \mathbf{G}_i)$ ($i \in \{2, \ldots, n-1\}$), and a binary sequence $\iota \in I^{\{2,\ldots,n\}}$ such that $\mathbf{X} \simeq \mathbf{X}_n$, where $\mathbf{X}_1 := \mathbf{G}_1$ and $\mathbf{X}_i := \mathbf{X}_{i-1}{}_{\mathbf{\Gamma}(\mathbf{H}_{i-1}, \mathbf{G}_i{}^{\iota_i})}$ ($i \in \{2, \ldots, n\}$).*

We say that a group-like FL$_e$-chain is represented as a *finite* partial-lexicographic product of linearly ordered Abelian groups $\mathbf{G}_1 \ldots, \mathbf{G}_n$, if it arises via finitely many iterations of the type I and type II constructions using linearly ordered Abelian groups $\mathbf{G}_1 \ldots, \mathbf{G}_n$ in the way it is described in Theorem 3

Theorem 4. *Any group-like FL$_e$-chain, which has only a finite number of idempotents, can be embedded into the finite partial-lexicographic product of totally ordered Abelian groups.*

Lemma 1. *Any finitely generated group-like FL$_e$-chain has only a finite number of idempotents.*

Theorem 5. *The logic* **IUL** *extended by the axiom* $\mathbf{t} \Leftrightarrow \mathbf{f}$ *is strongly standard complete.*

References

[1] Galatos, N., Jipsen, P., Kowalski, T., Ono, H.: Residuated Lattices: An Algebraic Glimpse at Substructural Logics vol. 151, Studies in Logic and the Foundations of Mathematics. Elsevier, (2007)

[2] Hahn, H.: Über die nichtarchimedischen Grössensysteme, S.-B. Akad. Wiss. Wien. IIa, 116 (1907), 601–655.

[3] Jenei, S., Montagna, F.: A classification of certain group-like FL$_e$-chains, Synthese 192(7), (2015), 2095–2121

[4] Jenei, S., Montagna, F.: Erratum to "A classification of certain group-like FL$_e$-chains, Synthese 192(7), (2015), 2095–2121", Synthese, doi:10.1007/s11229-014-0409-2

[5] Metcalfe, G., Montagna. F.: Substructural fuzzy logics, J. Symb. Logic (2007), 72 (3) 834–864

[1] $x_\downarrow = \begin{cases} u & \text{if there exists } u < x \text{ such that there is no element in } X \text{ between } u \text{ and } x, \\ x & \text{if for any } u < x \text{ there exists } v \in X \text{ such that } u < v < x \text{ holds.} \end{cases}$