Trial and error mathematics: dialectical, *p*-dialectical, and *q*-dialectical systems

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Abstract. Formal systems represent mathematical theories in a rather static way, in which axioms of the represented theory have to be defined from the beginning, and no further modification is permitted. As is clear, this representation is not comprehensive of all aspects of real mathematical theories: for instance, when defining a new theory, a mathematician might choose axioms through some trial and error process, instead of fixing them, once for all, at the initial stage. One way of characterizating such cases is provided by the so-called experimental logics, firstly studied by Jeroslow in the 1970's [2], and also explored by Hajek [1]. Our approach is based on a different – yet related – notion, introduced by Magari [3] in the same period: dialectical systems.

The basic ingredients of a *dialectical system* are a number c, encoding a contradiction; a deduction operator H, that tells us how to derive consequences from a finite set of statements D; and a proposing function f, that proposes statements to be accepted or rejected as provisional theses of the system. We prove several results concerning dialectical systems and their expressiveness. Furthermore, we investigate two additional class of systems that enrich Magari's original proposal with a natural mechanism of revision: p-dialectical systems and q-dialectical systems.

We prove the following: dialectical, p-dialectical, and p-dialectical sets (i.e., the sets of statements that are eventually accepted by, respectively, dialectical, p-dialectical, and q-dialectical systems) are always Δ^0_2 sets; the three systems display the same computational power, in that dialectical, p-dialectical, and q-dialectical sets have the same Turing degrees (namely, the computably enumerable Turing degrees), and the same enumeration degrees (namely, the Π^0_1 enumeration degrees); yet, p-dialectical and q-dialectical sets form a class which is much larger than that of dialectical sets, since the first two inhabit each level of Ershov hierarchy, while dialectical sets are always ω-computably enumerable.

Finally we show that, if we restrict to the case of *systems with connectives*, i.e. systems in which the deduction operator H has to satisfy the rules of classical logic, then we obtain the following: if S is a system that does not derive the contradiction from the empty set of premises, then S is the *completion* of a given theory. We make use of this fact to study our systems as machines to build, in the limit, completions of first-order theories. We show that dialectical and q-dialectical completions coincide, while Peano Arithmetic has a p-dialectical completion which is neither a dialectical completion, nor a q-dialectical completion.

References

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