The Current State of the Foundations of Set Theory

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Abstract

Set-theorists have for many years had a pretty good system of axioms for mathematics, the ZFC axioms. Nearly all of the theorems of mathematics can be translated into set theory and then shown to follow from the ZFC axioms. But Goedel's incompleteness theorem tells us that no system of axioms, not even ZFC, is really complete: there always are statements that can be neither proved nor disproved in any formal system. The most famous example for ZFC is Cantor's continuum hypothesis (CH), stating that any two uncountable sets of real numbers have the same cardinality.

Goedel conjectured that one might resolve this incompleteness problem by adding axioms of large infinity to ZFC, now called large cardinal axioms, in order to resolve many of the natural problems of set theory like CH. Goedel was only partly right: Many natural questions concerning nicely definable sets of reals are resolved by large cardinal axioms as well as virtually any question about the consistency (freedom from contradiction) of statements of set theory. But many questions, including CH, remain untouched by large cardinal axioms.

Is the incompleteness of ZFC relevant for mathematics? In other words, are there questions that are important for areas of mathematics other than logic which are undecidable in ZFC? There is evidence for a positive answer: the Whitehead problem (Abelian group theory), the Kaplansky Conjecture (Banach algebras), the existence of outer automorphisms of the Calkin algebra (C* algebras), the Borel Conjecture (measure theory) are all undecidable in ZFC. But some will regard these examples as disguised versions of questions in abstract set theory, lying outside of "core mathematics". Whether the mathematicians of the future will need axioms beyond ZFC to resolve questions at the heart of mathematics remains a fascinating open question.

However there is no doubt that set-theorists themselves must go beyond ZFC if they wish to resolve questions at the heart of set theory. This problem has been approached in two distinct ways, through "intrinsic" or "extrinsic" evidence for new axioms of set theory. The former makes use of principles concerning sets that result from our intuitive understanding of the concept; only recently has it been discovered that such principles can lead to new axioms which go far beyond ZFC. The latter has until now been based on the choice of axioms which best facilitate the mathematical development of the subject. A new proposal is to expand this to the choice of axioms which best resolve questions outside set theory, such as those mentioned above, which are known to be undecidable in ZFC.