The frame of Scott continuous nuclei on a preframe

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The Scott continuous nuclei on a preframe form a frame. In the case of a spectral frame, this gives the universal solution to the problem of adding boolean complements to the compact elements, to get a Stone frame. In the case of a stably continuous frame, this gives the universal solution to the problem of transforming the way-below relation into the well-inside relation, to get a compact regular frame. In the case of the Lawson dual $L^\wedge$ of a Hausdorff frame L (which is merely a preframe in general), it produces the compactly-generated reflection of the frame L. A Hausdorff frame L turns out to be compactly generated if and only if it is naturally isomorphic to its second Lawson dual $L^{\wedge\wedge}$. Hence for a compactly generated Hausdorff frame L, the frame of Scott continuous nuclei on the first Lawson dual $L^\wedge$ is isomorphic to L. The above results happen to be constructive in the sense of topos type theory. The talk will discuss this and a number of natural related open questions, in the language of locales, the objects of the opposite of the category of frames. In particular, what more is needed to get, constructively, a cartesian closed category of compactly generated locales? Non-constructively, this is not problematic: with choice, the category of compactly generated Hausdorff locales is equivalent to that of compactly generated Hausdorff spaces, and hence cartesian closed. For toposes that don't validate choice, I don't know of any non-trivial cartesian closed category of compactly generated locales. I will discuss the difficulties that arise in attempting to get such a category.