Towards a Topological Theory of Knowledge, Inquiry and Correlations

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I accordance to the recent trend towards an interrogative epistemology, "To know is to know the answer to a question" (Schaffer). But this begs the question: *what is a question?*

At a first approximation, an *'exact' propositional question* (i.e. always having a proposition as its unique correct answer) *is a partition* of the state space. This representation suffers from three problems, none of which is addressed by its recent generalizations. I claim that a topological theory can address all of them

First, an answer to a non-propositional question (of the form *what, where, who*, etc) is not a set of worlds. Informationally, *a non-propositional exact question can be encoded as a variable*, taking various values ("answers") in different possible worlds. Reinterpreting the above quote in this way, we are lead to a paraphrase of Quine: *To know is to know the value of a variable.*

Second, questions are never investigated in isolation: we answer questions by reducing them to other questions. This means that the proper object of knowledge is uncovering correlations between questions. *To know is to know a functional dependence between variables.*

Third, when talking about *inexact* (or *empirical*) questions/variables, the exact value/answer might not be knowable, and instead only "feasible answers" can be known. It is reasonable to assume that the conjunction of two (correct) feasible answers is a (correct) feasible answer: this suggests modelling propositional questions as *topological bases* on the state space. I investigate this conception and show the importance of topological notions for understanding propositional knowledge and questions.

Combining further the three issues, we arrive at a conception of *an inexact (not necessarily propositional) question as a map from the state space into a topological space.* Here, the exact value of the variable (exact answer) is represented by the output of the map, while the open neighborhoods of this value represent the feasible answers (knowable approximations of the exact answer). A question Q epistemically solves question Q' if every feasible answer to Q' can be known if given some good enough feasible answer to Q. I argue that knowability in such an empirical context amounts to the *continuity* of the functional correlation. *To know is to know a continuous dependence between variables.*

I investigate a logic of *epistemic dependency*, that can express knowledge of functional dependencies between (the values of) variables, as well as dynamic modalities for learning new such dependencies. This dynamics captures the widespread view of knowledge acquisition as a process of *learning correlations* (with the goal of eventually tracking causal relationships in the actual world).

The frame of Scott continuous nuclei on a preframe

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The Scott continuous nuclei on a preframe form a frame. In the case of a spectral frame, this gives the universal solution to the problem of adding boolean complements to the compact elements, to get a Stone frame. In the case of a stably continuous frame, this gives the universal solution to the problem of transforming the way-below relation into the well-inside relation, to get a compact regular frame. In the case of the Lawson dual $L^\wedge$ of a Hausdorff frame L (which is merely a preframe in general), it produces the compactly-generated reflection of the frame L. A Hausdorff frame L turns out to be compactly generated if and only if it is naturally isomorphic to its second Lawson dual $L^{\wedge\wedge}$. Hence for a compactly generated Hausdorff frame L, the frame of Scott continuous nuclei on the first Lawson dual $L^\wedge$ is isomorphic to L. The above results happen to be constructive in the sense of topos type theory. The talk will discuss this and a number of natural related open questions, in the language of locales, the objects of the opposite of the category of frames. In particular, what more is needed to get, constructively, a cartesian closed category of compactly generated locales? Non-constructively, this is not problematic: with choice, the category of compactly generated Hausdorff locales is equivalent to that of compactly generated Hausdorff spaces, and hence cartesian closed. For toposes that don't validate choice, I don't know of any non-trivial cartesian closed category of compactly generated locales. I will discuss the difficulties that arise in attempting to get such a category.

Stone duality in the theory of formal languages

Mai Gehrke

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Formal languages are mathematical models of computing machines and have strong connections to logic and, in the setting of regular languages, to topological algebra a double connection which is governed by Stone duality. In this talk we survey this connection and its potential applications in Boolean circuit complexity and the semantic study of logics with binding of variables, such as first order logic.

Introduction to Effectus Theory

Bart Jacobs

Radboud University Nijmegen

An effectus is a special form of category that has been introduced recently by the speaker and his research group. It is intended as a general categorical model for Boolean, probabilistic, and quantum logic. This talk will introduce the notion of effectus, and will discuss its main properties. Also it will briefly describe the EfProb tool for probabilistic computation that has been developed on the basis of effectus-theoretic ideas: it works uniformly for discrete, continuous and quantum probability.

Sources:

B. Jacobs. New Directions in Categorical Logic, for Classical, Probabilistic and Quantum Logic. In: *Logical Methods in Computer Science* 11(3), pp. 1–76, Oct. 2015. http://arxiv.org/pdf/1205.3940

K. Cho, B. Jacobs, Bas Westerbaan, Bram Westerbaan. *An Introduction to Effectus Theory.* 2015. http://arxiv.org/abs/1512.05813

EfProb library https://efprob.cs.ru.nl/

A Concrete Category of Classical Proofs

Greg Restall

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I show that the cut-free proof terms defined in my paper *Proof Terms for Classical Derivations* form a well-behaved category. I show that this category is not cartesian—and that we'd be wrong to expect it to be. (It has no products or coproducts, nor any intial or final objects. Nonetheless, it is quite well behaved.) I show that the term category is star autonomous (so it fits well within the family of categories for multiplicative linear logic), with internal monoids and comonoids taking care of weakening and contraction. The category is enriched in the category of semilattices, as proofs are closed under the blend rule (also called mix in the literature).

(Co)algebraic foundations of automata learning

Alexandra Silva

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Automata learning is a technique that has many applications in verification and systems' modeling. In this talk I will review one of the classical algorithms, Angluin's algorithm using algebraic and coalgebraic tools. This will provide a new perspective on the correctness of the algorithm and, more interestingly, will unveil connections to testing and minimization. This talk is based on joint work with Matteo Sammartino and Gerco van Heerdt.

Geometric aspects of MV-algebras

Luca Spada

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The fact that every semisimple MV-algebra is isomorphic to a separating subalgebra of the algebra of continuous functions from X into $[0, 1]$, for X a compact Hausdorff space, has long been known. Considerable information on the structure of MV-algebras follow. E.g., *every* compact Hausdorff space can be obtained as the maximal spectrum of a suitable MV-algebra. Nonetheless, the result leaves at least two questions open:

1. Can the functions forming the aforementioned separating subalgebra be characterised in an intrinsic way?

2. How can the above result be extended to the whole class of MV-algebras?

In this talk I will report on a series of works (some in collaboration with Leonardo Cabrer and some in collaboration with Vincenzo Marra) that tackle the above questions.

The main point in Question 1 is that the topological structure alone is not sufficient to reconstruct the MV-algebra. It seems inevitable to move from pure topological spaces to *geometric* spaces, i.e., topological spaces equipped with a system of coordinates. The coordinatisation is crucial to describe the separating functions, but it turns out to be useful also in the characterisation of special classes of MV-algebras. E.g., finitely presented MV-algebras correspond to rational polyhedra, and the latter concept cannot be formalised in pure topological terms.

To address Question 2, one needs to deal with non-semisimple MV-algebras, that is, the ones containing infinitesimal elements. It turns out that infinitesimal elements correspond to regions of the space in which not only the *value* of a function matters, but also the *rate* at which this value is attained. We deal with this differential phenomenon considering the *pro-completion* of the category of rational polyhedra. This leads to a duality between the whole category of MV-algebras and a category whose objects are systems of rational polyhedra approximating a compact Hausdorff space.

Sketches for arithmetic universes as generalized spaces

Steve Vickers

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From topos theory comes the idea that continuity is a logical phenomenon, specifically one of geometric logic: a map is continuous if its construction can be carried out within the constructive constraints of geometric logic. This extends the notion of continuity far beyond ordinary topological spaces, as it applies also to maps valued in generalized spaces (toposes) such as the space of sets, and even to bundles as maps taking points to spaces (the fibres).

Categorically, geometric logic is interpreted in Grothendieck toposes. However, a problem is that it has extrinsic infinities (specifically: infinitary disjunctions) supplied by a choice of base topos. This would make it difficult to provide software support for geometric logic. In 1999 I conjectured that, by adding features of a finitary, intrinsic type theory, some infinite disjunctions could be replaced by existential quantification over infinite types such as the natural numbers. Categorically, Grothendieck toposes relative to a fixed base would be replaced by Joyal's arithmetic universes (AUs), base-independent.

I shall present my recent work on formalizing the AU-logic using finite sketches. These are restricted to "contexts", defined with the property that every non-strict model is uniquely isomorphic to a strict model. This allows us to reconcile the syntactic, dealt with strictly using universal algebra, with the semantic, in which non-strict models must be considered.

For each context T, there is a classifying AU analogous to a classifying topos, but base-independent, and I shall outline a purely finitary presentation of strict AU-functors and natural transformations between the classifying AUs.

Draft paper at <http://arxiv.org/abs/1608.01559>.

Topologizing filters on a ring of frictions RS^{-1} and congruence relation on $\mathrm{Fil}R$

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This paper introduce the notion of topologizing filters on rings of fractions RS^{-1} for a multiplicative subset S of a commutative ring R. It is shown that the mapping from $\mathrm{Id}R$ to $\mathrm{Id}RS^{-1}$ given by $I \mapsto IS^{-1}$ induces a map from $\mathrm{Fil}R$ to $\mathrm{Fil}RS^{-1}$. It is proved that for a multiplicative subset S of a commutative ring R the map $\hat{\varphi}_S : [FilR]^{du} \to [FilRS^{-1}]^{du}$ given by

$$\hat{\varphi}_S(\mathfrak{F}) \stackrel{\mathrm{def}}{=} \{AS^{-1} : A \in \mathfrak{F}\}$$

is an onto homomorphism of lattice ordered monoids. It is proved that for a multiplicative subset S of a commutative ring R, if the monoid operation on $\mathrm{Fil}R$ is commutative so is the monoid operation on $\mathrm{Fil}RS^{-1}$ and if every member of $\mathrm{Fil}R$ is idempotent then the same is true of every member of $\mathrm{Fil}RS^{-1}$. Moreover, such a map $\hat{\varphi}_S$ gives rise to a canonical congruence relation $\equiv_{\hat{\varphi}_S}$ on $\mathrm{Fil}R$ defined by $\mathfrak{F} \equiv_{\hat{\varphi}_S} \mathfrak{G} \Leftrightarrow \hat{\varphi}_S(\mathfrak{F}) = \hat{\varphi}_S(\mathfrak{G})$. The above result tells us that the homomorphism $\hat{\varphi}_S : [FilR]^{du} \to [FilRS^{-1}]^{du}$ restricts to a homomorphism from the Jansian topologizing filters of $\mathrm{Fil}R$ onto the Jansian topologizing filters of $\mathrm{Fil}RS^{-1}$.

It is proved that for a commutative ring R for which $\mathrm{Fil}R$ is commutative, then $\bigcap\{\equiv_{\hat{\varphi}_{S_P}} : P \in \mathrm{Spec}_\mathrm{m} R\}$ is the identity congruence on $\mathrm{Fil}R$, that is, for all $\mathfrak{F}, \mathfrak{G} \in FilR$,

$$\mathfrak{F} = \mathfrak{G} \Leftrightarrow \mathfrak{F} \equiv_{\hat{\varphi}_{S_P}} \mathfrak{G} \ \ \forall P \in \mathrm{Spec}_\mathrm{m} R.$$

As one of the main results of this paper it is shown that if R is a commutative ring for which $\mathrm{Fil}R$ is commutative, then the previous result yields the following subdirect decomposition:

$$[FilR]^{du} \cong [FilR]^{du} / \left(\bigcap_{P \in Spec_m R} \equiv_{\hat{\varphi}_{S_P}} \right) \hookrightarrow \prod_{P \in Spec_m R} ([FilR]^{du} / \equiv_{\hat{\varphi}_{S_P}}) \cong \prod_{P \in Spec_m R} [FilR_P]^{du}.$$

For an arbitrary ring R for which $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ is two-sided residuated, it is shown that R satisfies the DCC on left annihilator ideals, and the ACC on right annihilator ideals. It is well-known that a commutative noetherian ring has finitely many minimal prime ideals and as an extension of this, it is proved that if R is an arbitrary ring for which $[Fil\, R]^{du}$ is two-sided residuated, then R contains finitely many minimal prime ideals. It is also shown that for a Prüfer domain R for which $\mathrm{Fil}\, R$ is commutative, R_P is a (noetherian) rank 1 discrete valuation domain for every maximal ideal P of R.

This paper is concluded by proving that for a Prüfer domain R, $\mathrm{Fil}R$ is commutative if and only if R is noetherian and thus a Dedekind domain which extends a known result which says that a valuation domian for which $\mathrm{Fil}R$ is commutative is noetherian and thus rank 1 discrete.

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Logics for extended distributive contact lattices

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The notion of contact algebra is one of the main tools in the region based theory of space. It is an extension of Boolean algebra with an additional relation C called contact. The elements of the algebra are called regions and are considered as analogs of physical bodies. Boolean operations are considered as operations for constructing new regions from given ones. The unit element 1 symbolizes the region containing as its parts all regions, and the zero region 0 symbolizes the empty region. There are some problems related to the motivation of the operation of Boolean complementation. A question arises: if the region a in some universe represents a physical body, then what kind of body represents a^*? - it depends on the universe. Because of this in [1] this operation is dropped and the language of distributive lattices is extended by considering as non-definable primitives the relations of contact, nontangential inclusion $\ll$ and dual contact $\widehat{C}$. It has been obtained an axiomatization of the theory consisting of the universal formulas, true in all contact algebras. The structures, satisfying the axioms in question, are called extended distributive contact lattices (EDC-lattices). The well known RCC-8 system of mereotopological relations is definable in the language of EDC-lattices. In [1] are considered also some axiomatic extensions of EDC-lattices yielding representations in T_1 and T_2 topological spaces. In this abstract we consider several logics, corresponding to EDCL. We give completeness theorems with respect to both algebraic and topological semantics for these logics. It turns out that they are decidable.

We consider the quantifier-free first-order language with equality $\mathcal{L}$ which includes:
- constants: 0, 1;
- function symbols: $+$, $\cdot$;
- predicate symbols: $\leq$, C, $\widehat{C}$, $\ll$.

Every EDCL is a structure for $\mathcal{L}$.

We consider the logic L with rule MP and the following axioms:
- the axioms of the classical propositional logic;
- the axiom schemes of distributive lattice;
- the axioms for C, $\widehat{C}$, $\ll$ and the mixed axioms of EDCL - considered as axiom schemes [1].

We consider the following additional rules and an axiom scheme:

(R Ext $\widehat{O}$) $\dfrac{\alpha\to(a+p\neq 1\vee b+p=1) \text{ for all variables } p}{\alpha\to(a\leq b)}$, where α is a formula, a, b are terms

(R U-rich $\ll$) $\dfrac{\alpha\to(b+p\neq 1\vee aCp) \text{ for all variables } p}{\alpha\to(a\overline{\ll}b)}$, where α is a formula, a, b are terms

(R U-rich $\widehat{C}$) $\dfrac{\alpha\to(a+p\neq 1\vee b+q\neq 1\vee pCq) \text{ for all variables } p,\, q}{\alpha\to a\widehat{C}b}$, where α is a formula, a, b are terms

(R Ext C) $\dfrac{\alpha\to(p\neq 0\to aCp) \text{ for all variables } p}{\alpha\to(a=1)}$, where α is a formula, a is a term

(R Nor1) $\dfrac{\alpha\to(p+q\neq 1\vee aCp\vee bCq) \text{ for all variables } p,\, q}{\alpha\to aCb}$, where α is a formula, a, b are terms

(Con C) $p\neq 0\wedge q\neq 0\wedge p+q=1\to pCq$

To these rules correspond the additional axioms for EDC-lattices, considered in [1] - the axioms (Ext $\widehat{O}$), (U-rich $\ll$), (U-rich $\widehat{C}$), (Ext C), (Nor1).

Let L' be for example the extension of L with the rule (R Ext $\widehat{O}$) and the axiom scheme (Con C). Then we denote L' by $L_{ConC,Ext\widehat{O}}$ and call the axioms (Con C) and (Ext $\widehat{O}$) corresponding to L' additional axioms. In a similar way we denote any extension of L with some of the considered additional rules and axiom scheme and in a similar way we define its corresponding additional axioms.

Theorem 1 (Completeness theorem with respect to algebraic semantics). *Let L' be some extension of L with zero or more of the considered additional rules and axiom scheme. The following conditions are equivalent for any formula α:*
(i) α is a theorem of L';
(ii) α is true in all EDCL, satisfying the corresponding to L' additional axioms.

We consider the following logics, corresponding to the EDC-lattices, considered in [1]:
1) L;
2) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C}}$;
3) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC}$;
4) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ConC}$;
5) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},Nor1}$;
6) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,ConC}$;
7) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},Nor1,ConC}$;
8) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,Nor1}$;
9) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,ConC,Nor1}$.

To every of these logics we juxtapose a class of topological spaces:
1) the class of all T_0, semiregular, compact topological spaces;
2) the class of all T_0, semiregular, compact topological spaces;
3) the class of all T_0, compact, weakly regular topological spaces;
4) the class of all T_0, semiregular, compact, connected topological spaces;
5) the class of all T_0, semiregular, compact, κ - normal topological spaces;
6) the class of all T_0, compact, weakly regular, connected topological spaces;
7) the class of all T_0, semiregular, compact, κ - normal, connected topological spaces;
8) the class of all T_0, compact, weakly regular, κ - normal topological spaces;
9) the class of all T_0, compact, weakly regular, connected, κ - normal topological spaces.

Theorem 2 (Completeness theorem with respect to topological semantics). *Let L' be any of the considered logics. The following conditions are equivalent for any formula α:*
(i) α is a theorem of L';
(ii) α is true in all contact algebras over a topological space from the corresponding to L' class.

Theorem 3. *(i) The logics L, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C}}$,*
$L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC}$, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},Nor1}$,
$L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,Nor1}$ *have the same theorems and are decidable;*
(ii) The logics $L_{ConC,U-rich\ll}$, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ConC}$,
$L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ConC,Nor1}$, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,ConC}$,
$L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,ConC,Nor1}$ *have the same theorems and are decidable.*

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Injectivity of (Naturally) Ordered Projection Algebras

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A projection algebra is a set A with an action of the monoid $M = (\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}, \min, \infty)$, where $\mathbb{N}$ is the set of natural numbers, and $n \leq \infty$, for all $n \in \mathbb{N}$. Computer scientists use this notion as a convenient means for algebraic specification of process algebras. In this paper, we study injectivity of ordered projection algebras. We characterize injective cyclic naturally ordered projection algebras as complete posets, also we compare some kinds of weak injectivity such as ideal injectivity and $\mathbb{N}$-injectivity with regular injectivity.

1 Introduction

Algebraic and categorical properties of Projection algebras (or spaces) have been introduced and studied as an algebraic version of ultrametric spaces as well as algebraic structures, for example, in [5, 7, 8, 2]. Computer scientists use this notion as a convenient means for algebraic specification of process algebras (see [6]).

A projection algebra is in fact a set A with an action of the monoid $M = (\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}, \min, \infty)$, where $\mathbb{N}$ is the set of natural numbers, and $n \leq \infty$, for all $n \in \mathbb{N}$. By an *ordered projection algebra* we mean a projection algebra A which is also a poset such that the order is compatible with the action. A *naturally ordered projection algebra* is an ordered projection algebra with the order $a \leq b$ if and only if $a = nb$, for some $n \in \mathbb{N}$. We denote the category of projection algebras by **PRO**, the category of ordered projection algebras by **O-PRO**, and the subcategory of naturally ordered projection algebras by **O-PRO$_{\text{nat}}$**.

2 Regular injectivity and Ideal injectivity

In this section, we study injectivity of ordered projection algebras with respect to order emdedding projection maps, so called *regular injectivity*. Also, we compare it with injectivity with respect to embedding of the form $I \to \mathbb{N}^\infty$ for an ideal I of $\mathbb{N}^\infty$, so called *I-injectivity*, and with *ideal injective* which is I-injectivity, for all ideals I of $\mathbb{N}^\infty$.

Theorem 2.1. *Every ordered projection algebras is $\downarrow k$-injective, for $k \in \mathbb{N}$.*

Corollary 2.2. *For ordered projection algebras, ideal injectivity coincides with $\mathbb{N}$-injectivity.*

Theorem 2.3. *For ordered projection algebras, the following are equivalent:*
  (1) *Ideal injectivity in **O-PRO**.*
  (2) *$\mathbb{N}$-injectivity in **O-PRO**.*
  (3) *$\mathbb{N}$-injectivity in **PRO**.*
  (4) *Injectivity in **PRO**.*

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[*]Speaker

Theorem 2.4. *A continuously complete naturally ordered projection algebra A is injective in* **O-PRO**$_\text{nat}$. *But, the converse is not generally true.*

Theorem 2.5. *For a projection algebra A with natural order, the following are equivalent:*
  (1) *A is a complete poset.*
  (2) *A is a continuously complete ordered projection algebra.*
  (3) *A is an infinite countable bounded chain.*
  (4) *A is an infinite countable complete chain.*
  (5) *A is a cyclic projection algebra.*

Corollary 2.6. *A naturally ordered projection algebra satisfying one of the equivalent conditions of the above theorem is injective in* **PRO**.

Proposition 2.7. *There is no non trivial regular injective projection algebra with natural order in* **O-PRO**.

Theorem 2.8 (Baer). *Let A be an ordered projection algebra.*
  (1) *If A is injective as an object of* **PRO** *then A is injective with respect to ordered projection algebras with natural order.*
  (2) *If A is injective as an object of* **PRO** *with respect to one fixed projection algebras (with natural order) then A is injective with respect to all projection algebras (with natural order).*
  (3) *If A is regular injective with respect to embeddings into cyclic* **O-PRO** *then A is regular injective with respect to all projection algebras with natural order.*

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Quasi injectivity of partially ordered acts

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1 Introduction and Preliminaries

It is well-known that injective objects play a fundamental role in many branches of mathematics. The question whether a given category has injective objects has been investigated for many categories. As for posets, Banaschewski [1] proves that complete posets are exactly $\mathcal{E}$-injective posets (injective with respect to order-embeddings), and Sikorski [4] shows the same result for injective Boolean algebras (see also [2, 3]).

In this paper, we study quasi injectivity in the category of (right) actions of a partially ordered monoid on partially ordered sets (**Pos**-S) with respect to embeddings ($\mathcal{E}$-quasi injectivity). First, we study the relation between $\mathcal{E}$-injectivity, $\mathcal{E}$-quasi injectivity, and completeness in **Pos**-S. Then, we show when a θ-extension of an $\mathcal{E}$-quasi injective S-poset A ($A \oplus \{\theta\}$, which is obtained by adjoining a zero top element θ to A) is K_θ-quasi injective. Note that an S-poset $A \oplus \{\theta\}$ is called K_θ-quasi injective if for every sub S-poset B of $A \oplus \{\theta\}$, any S-poset map $f : B \to A \oplus \{\theta\}$, with $f^{-1}(\theta) = \overleftarrow{f}(\theta) = \{b \in B : f(b) = \theta\} \neq \emptyset$, can be extended to $\bar{f} : A \oplus \{\theta\} \to A \oplus \{\theta\}$. Finally, we study the relation between $\mathcal{E}$-injectivity, $\mathcal{E}$-quasi injectivity, and completeness in some useful subcategories of **Pos**-S.

Definition 1.1. Let $\mathcal{M}$ be a class of monomorphisms in a category $\mathcal{C}$. An object $A \in \mathcal{C}$ is called

1. $\mathcal{M}$-*injective* if for each $\mathcal{M}$-*morphism* $m : B \to C$ and any morphism $f : B \to A$ there exists a morphism $\bar{f} : C \to A$ such that $\bar{f}m = f$,

2. $\mathcal{M}$-*quasi injective* if for each $\mathcal{M}$-morphism $m : B \to A$ and any morphism $f : B \to A$ there exists a morphism $\bar{f} : A \to A$ which extends f,

3. $\mathcal{M}$-*absolute retract* if it is a retract of each of its $\mathcal{M}$-extensions; that is, for each $\mathcal{M}$-morphism $m : A \to C$ there exists a morphism $\bar{f} : C \to A$ such that $\bar{f}m = id_A$, in which case $\bar{f}$ is said to be a *retraction*.

2 Main Results

Remark 2.1. It is clear that every $\mathcal{E}$-injective S-poset is $\mathcal{E}$-quasi injective. But the converse is not necessarily true. (In Theorem 2.4 (below) we give conditions under which the converse is also true.)

Remark 2.2. 1. Every complete poset with identity action is $\mathcal{E}$-quasi injective in the category **Pos**-S.

2. The action on an $\mathcal{E}$-quasi injective S-poset need not be identity.

3. An $\mathcal{E}$-quasi injective S-poset is not necessarily complete as a poset.

4. There exists complete S-poset which is not $\mathcal{E}$-quasi injective.

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[*]Speaker

Proposition 2.3. *There exists no pomonoid S over which all S-posets are $\mathcal{E}$-quasi injective.*

Theorem 2.4. *An $\mathcal{E}$-quasi injective S-poset A is $\mathcal{E}$-injective if and only if A has a zero element and $A \times \bar{A}^{(S)}$ is $\mathcal{E}$-quasi injective ($\bar{A}$ is the Dedekind-MacNeille completion of A).*

Definition 2.5. Let A be an S-act. A subset B of A is called *consistent* if for each $a \in A$ and $s \in S$, $as \in B$ implies $a \in B$. We call a consistent subact an *S-filter*.

Theorem 2.6. *Let A be an $\mathcal{E}$-quasi injective S-poset. Also, assume that $f : B \to A \oplus \{\theta\}$ is an S-poset map, where B is a sub S-poset of $A \oplus \{\theta\}$ and $\overleftarrow{f}(\theta) \neq \emptyset$. Then there exists an S-filter $\tilde{A}$ of $A \oplus \{\theta\}$ which is upward closed in $A \oplus \{\theta\}$, $\overleftarrow{f}(\theta) \subseteq \tilde{A}$, and $\tilde{A} \cap \{b \in B : f(b) \neq \theta\} = \emptyset$ if and only if there exists an S-poset map $\bar{f} : A \oplus \{\theta\} \to A \oplus \{\theta\}$ which extends f.*

Corollary 2.7. *Let A be an $\mathcal{E}$-quasi injective S-poset. If for each S-poset map $f : B \to A \oplus \{\theta\}$, where B is a sub S-poset of $A \oplus \{\theta\}$ and $\overleftarrow{f}(\theta) \neq \emptyset$, we have*

$$\tilde{A} = \{a \in A \oplus \{\theta\} : \exists s \in S, as \in\, \uparrow \overleftarrow{f}(\theta)\}$$

is an S-filter of $A \oplus \{\theta\}$, then $A \oplus \{\theta\}$ is a K_θ-quasi injective S-poset.

Corollary 2.8. *Suppose S is a pomonoid with any one of the following properties*:
(1) $\forall s \in S$, $\exists t \in S$, $st \leq e$ (e is the identity element of S).
(2) $\forall s \in S$, $s^2 \leq e$.
(3) S is a pogroup.
(4) $\top_S = e$ (S has the top element $\top_S$).
(5) S is a right zero semigroup with an adjoined identity.
If A is an $\mathcal{E}$-quasi injective S-poset then $A \oplus \{\theta\}$ is K_θ-quasi injective in $\mathbf{Pos}$-S.

Definition 2.9. An S-poset A is called *strong reversible* if for every $s \in S$ there exists $t \in S$ such that $ast = ats = a$ for all $a \in A$. Also, an S-poset A is *square reversible* if for every $a \in A$ and $s \in S$, we have $as^2 = a$. So, we have the category $\mathbf{SR}$-$\mathbf{Pos}$-S ($\mathbf{SQ}$-$\mathbf{Pos}$-S) of all strong (square) reversible S-posets and S-poset maps between them.

Theorem 2.10. *A strong (square) reversible S-poset is $\mathcal{E}$-quasi injective in $\mathbf{SR}$-$\mathbf{Pos}$-S ($\mathbf{SQ}$-$\mathbf{Pos}$-S) if it is complete.*

Theorem 2.11. *Let A be a strong (square) reversible S-poset. Then the following are equivalent*:
(i) *A is $\mathcal{E}$-injective in $\mathbf{SR}$-$\mathbf{Pos}$-S ($\mathbf{SQ}$-$\mathbf{Pos}$-S).*
(ii) *A is $\mathcal{E}$-absolute retract in $\mathbf{SR}$-$\mathbf{Pos}$-S ($\mathbf{SQ}$-$\mathbf{Pos}$-S).*
(iii) *A is complete.*

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Many-Valued Arrow Logic with Scalar Multiplication

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In order to reinforce the link between linear algebra, modal logic and many-valued logic we present an extension of Basic Arrow Logic (BAL) based on the introduction of a new modal operator of scalar multiplication and on a redefinition of the basic arrow operators. As we know since the work of Venema [11] and Marx [6], the novelty of BAL lies on the introduction of three modal operators, namely, identity ($\int$), converse ($-$), and composition ($\circ$). Along with other important extensions, like presented in [2], [8], [10], what results in this case is Many-valued Arrow Logic with Scalar multiplication (MALS).

The motivations of presenting MALS as an extension of BAL comes from two different sides. The first is due to van Benthem's infinitary operator "$\mathcal{M}, x \models \varphi^*$", presented in [3]. This modal operator is defined as a finite composition of a formula φ ("x can be C-decomposed into some finite sequence of arrows satisfying φ in $\mathcal{M}$" [3, p. 18]), we may think of this operator as a kind of scalar multiplication but, the definition do not specify nothing about how to interpret them like that. MALS make explicit the definition of scalar multiplication validating all vector spaces' axioms (some intuitions of our work are presents in [4, p. 289]). The second aspect is related with many valued logics, in specific with the evident similarities with the logic FDE [5], and its informational interpretation [9]. In this case our proposal is to define a kind of non-classical vector algebra, invalidating some intuitive properties like consistency, and showing that a non-classical vector algebra is still significant. In MALS this approach can be realized defining the operators in a more general way, following the work of Priest [7].

The plan of the talk is as follows. First we may introduce Arrow Logic with Scalar multiplication (ALS), later we may define Many-valued Arrow Logic (MAL). This two logics are also extensions of BAL and if we join the two we have MALS, this will be done in a third place. As a result, we may obtain MALS as a union of ALS and MAL, that means that, we may interpret – with a 4-valued semantics– composition as vector addition, converse as subtraction, and scalar multiplication (following van Benthem) as n-composition of φ where n ranges over the scalar magnitude.

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Structure Theorem for a Class of Group-like Residuated Chains à la Hahn

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Hahn's structure theorem [2] states that totally ordered Abelian groups can be embedded in the *lexicographic product* of *real groups*. Residuated lattices [7] are semigroups only, and are algebraic counterparts of substructural logics [1]. Involutive commutative residuated chains (aka. involutive FL_e-chains) form an algebraic counterpart of the logic **IUL** [6]. The focus of our investigation is a subclass of them, called commutative *group-like* residuated chains. Commutative, group-like residuated chains are totally ordered, involutive commutative residuated lattices such that the unit of the monoidal operation coincides with the constant that defines the involution. The latest postulate forces the structure to resemble totally ordered Abelian groups in many ways. Firstly, similar to lattice-ordered Abelian groups, for complete, densely ordered, group-like FL_e-chains the monoidal operation can be recovered from its restriction solely to its positive cone (see [3, Theorem 1]). Secondly, group-like commutative residuated chains can be characterized as generalizations of totally ordered Abelian groups by weakening the strictly-increasing nature of the partial mappings of the group multiplication by nondecreasing behaviour, see Theorem 1. Thirdly, in quest for establishing a structural description for commutative group-like residuated chains à la Hahn, "partial-lexicographic product" constructions will be introduced. Roughly, only a cancellative subalgebra of a commutative group-like residuated chain is used as a first component of a lexicographic product, and the rest of the algebra is left unchanged. This results in group-like FL_e-algebras, see Theorem 2. The main theorem is about the structure of order-dense group-like FL_e-chains with a finite number of idempotents: Each such algebra can be constructed by iteratively using the partial-lexicographic product constructions using totally ordered Abelian groups as building blocks, see Theorem 3. This result extends the famous structural description of totally ordered Abelian groups by Hahn [2], to order-dense group-like commutative residuated chains with finitely many idempotents. The result is quite surprising.

Theorem 1. *For a group-like FL_e-algebra $(X, \wedge, \vee, \circledast, \to_\circledast, t, f)$ the following statements are equivalent: $(X, \wedge, \vee, \circledast, t)$ is a lattice-ordered Abelian group if and only if $\circledast$ is cancellative if and only if $x \to_\circledast x = t$ for all $x \in X$ if and only if the only idempotent element in the positive cone of X is t.*

Definition 1. *(Partial-lexicographic products)* Let $\mathbf{X} = (X, \wedge_X, \vee_X, *, \to_*, t_X, f_X)$ be a group-like FL_e-algebra and $\mathbf{Y} = (Y, \wedge_Y, \vee_Y, \star, \to_\star, t_Y, f_Y)$ be an involutive FL_e-algebra, with residual complement $'^*$ and $'^\star$, respectively. Add a top element $\top$ to Y, and extend $\star$ by $\top \star y = y \star \top = \top$ for $y \in Y \cup \{\top\}$, then add a bottom element $\bot$ to $Y \cup \{\top\}$, and extend $\star$ by $\bot \star y = y \star \bot = \bot$ for $y \in Y \cup \{\bot, \top\}$. Let $\mathbf{X_1} = (X_1, \wedge_X, \vee_X, *, \to_*, t_X, f_X)$ be any cancellative subalgebra of $\mathbf{X}$ (by Theorem 1, $\mathbf{X_1}$ is a lattice ordered group). We define $\mathbf{X}_{\Gamma(\mathbf{X_1}, \mathbf{Y}^{\bot\top})} = \left(X_{\Gamma(X_1, Y^{\bot\top})}, \leq, \circledast, \to_\circledast, (t_X, t_Y), (f_X, f_Y)\right)$, where $X_{\Gamma(X_1, Y^{\bot\top})} = (X_1 \times (Y \cup \{\bot, \top\})) \cup ((X \setminus X_1) \times \{\bot\})$, $\leq$ is the restriction of the lexicographic order of $\leq_X$ and $\leq_{Y \cup \{\bot, \top\}}$

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to $X_{\Gamma(X_1, Y^{\perp\top})}$, $\circledast$ is defined coordinatewise, and the operation $\to_{\circledast}$ is given by $(x_1, y_1) \to_{\circledast} (x_2, y_2) = ((x_1, y_1) \circledast (x_2, y_2)')'$ where

$$(x, y)' = \begin{cases} (x'^{*}, y'^{*}) & \text{if } x \in X_1 \\ (x'^{*}, \perp) & \text{if } x \notin X_1 \end{cases}.$$

Call $\mathbf{X}_{\Gamma(\mathbf{X_1}, \mathbf{Y}^{\perp\top})}$ the *(type-I) partial-lexicographic product* of X, X_1, and Y, respectively.

Let $\mathbf{X} = (X, \leq_X, *, \to_*, t_X, f_X)$ be a group-like FL_e-chain, $\mathbf{Y} = (Y, \leq_Y, \star, \to_\star, t_Y, f_Y)$ be an involutive FL_e-algebra, with residual complement $'^{*}$ and $'^{*}$, respectively. Add a top element $\top$ to Y, and extend $\star$ by $\top \star y = y \star \top = \top$ for $y \in Y \cup \{\top\}$. Further, let $\mathbf{X_1} = (X_1, \wedge, \vee, *, \to_*, t_X, f_X)$ be a cancellative, discrete, prime[1] subalgebra of $\mathbf{X}$ (by Theorem 1, $\mathbf{X_1}$ is a discrete lattice ordered group). We define $\mathbf{X}_{\Gamma(\mathbf{X_1}, \mathbf{Y}^\top)} = (X_{\Gamma(X_1, Y^\top)}, \leq, \circledast, \to_{\circledast}, (t_X, t_Y), (f_X, f_Y))$, where $X_{\Gamma(X_1, Y^\top)} = (X_1 \times (Y \cup \{\top\})) \cup ((X \setminus X_1) \times \{\top\})$, $\leq$ is the restriction of the lexicographic order of $\leq_X$ and $\leq_{Y \cup \{\top\}}$ to $X_{\Gamma(X_1, Y)}$, $\circledast$ is defined coordinatewise, and the operation $\to_{\circledast}$ is given by $(x_1, y_1) \to_{\circledast} (x_2, y_2) = ((x_1, y_1) \circledast (x_2, y_2)')'$ where

$$(x, y)' = \begin{cases} ((x'^{*}), \top) & \text{if } x \notin X_1 \text{ and } y = \top \\ (x'^{*}, y'^{*}) & \text{if } x \in X_1 \text{ and } y \in Y \\ ((x'^{*})_\downarrow, \top) & \text{if } x \in X_1 \text{ and } y = \top \end{cases}.$$

[2] Call $\mathbf{X}_{\Gamma(\mathbf{X_1}, \mathbf{Y}^\top)}$ the *(type-II) partial-lexicographic product* of X, X_1, and Y, respectively.

Theorem 2. $\mathbf{X}_{\Gamma(\mathbf{X_1}, \mathbf{Y}^{\perp\top})}$ *and* $\mathbf{X}_{\Gamma(\mathbf{X_1}, \mathbf{Y}^\top)}$ *are involutive FL_e-algebras. If $\mathbf{Y}$ is group-like then also $\mathbf{X}_{\Gamma(\mathbf{X_1}, \mathbf{Y}^{\perp\top})}$ and $\mathbf{X}_{\Gamma(\mathbf{X_1}, \mathbf{Y}^\top)}$ are group-like.*

Theorem 3. *Any order-dense group-like FL_e-chain which has only a finite number of idempotents can be built by iterating finitely many times the partial-lexicographic product constructions using only totally ordered groups, as building blocks. More formally, let $\mathbf{X}$ be an order-dense group-like FL_e-chain which has $n \in \mathbf{N}$ idempotents in its positive cone. Denote $I = \{\perp\top, \top\}$. For $i \in \{1, 2, \ldots, n\}$ there exist totally ordered Abelian groups $\mathbf{G}_i$, $\mathbf{H}_1 \leq \mathbf{G}_1$, $\mathbf{H}_i \leq \Gamma(\mathbf{H}_{i-1}, \mathbf{G}_i)$ $(i \in \{2, \ldots, n-1\})$, and a binary sequence $\iota \in I^{\{2, \ldots, n\}}$ such that $\mathbf{X} \simeq \mathbf{X}_n$, where $\mathbf{X}_1 := \mathbf{G}_1$ and $\mathbf{X}_i := \mathbf{X}_{i-1 \Gamma(\mathbf{H}_{i-1}, \mathbf{G}_i^{\iota_i})}$ $(i \in \{2, \ldots, n\})$.*

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[1] Call a subalgebra $(X_1, \wedge, \vee, \circledast, \to_{\circledast}, t_X, f_X)$ of an FL_e-algebra $(X, \leq_X, \circledast, \to_{\circledast}, t_X, f_X)$ *prime* if $(X \setminus X_1) * (X \setminus X_1) \subseteq X \setminus X_1$.

[2] $x_\downarrow = \begin{cases} u & \text{if there exists } u < x \text{ such that there is no element in } X \text{ between } u \text{ and } x, \\ x & \text{if for any } u < x \text{ there exists } v \in X \text{ such that } u < v < x \text{ holds.} \end{cases}$

Diagrammatic duality

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Dualities between algebras and representation spaces are a staple topic at the interface of algebra and logic. While dualities are often produced with the aid of dualizing objects as codomains for both algebra and space homomorphisms, other techniques are also useful. As an addition to the palette of alternative techniques, diagrammatic duality is a method for obtaining new dualities founded on existing ones. Whenever algebras of a certain class are equivalent to diagrams in a category of known dualizable algebras, **diagrammatic duality** furnishes representation spaces for the algebras in the class by examining dual diagrams in the category of representation spaces for the known dualizable algebras. We present two examples: Nelson algebras, and algebras from an arbitrary variety.

A *diagram* in a category $\mathbf{C}$ is a graph map $F: D \to \mathbf{C}$ from a quiver (or directed graph) D to (the underlying graph of) $\mathbf{C}$. The diagram is *proper* if its domain D is small. For a quiver D and category $\mathbf{C}$, let $\left(\mathbf{C}^D\right)_0$ be the class of diagrams $F: D \to \mathbf{C}$. For given diagrams $F: D \to \mathbf{C}$ and $G: D \to \mathbf{C}$, let $\left(\mathbf{C}^D\right)(F, G)$ be the class of natural transformations $\tau : F \to G$. Then $\mathbf{C}^D$ forms a category. If D is a proper diagram, then categories of the form $\mathbf{C}^D$ are known as *diagram categories*.

Suppose that $\mathfrak{C}$ and $\mathfrak{A}$ are categories of algebras and homomorphisms. Suppose that there is an equivalence $\mathfrak{C} \cong \mathfrak{C}_\mathfrak{A}$ between $\mathfrak{C}$ and a subcategory $\mathfrak{C}_\mathfrak{A}$ of a diagram category $\mathfrak{A}^V$ with given domain diagram V. Then the objects of $\mathfrak{C}$ are known as *diagrammatic algebras* (relative to $\mathfrak{A}$). In this context it is often convenient to abuse notation and suppress the distinction between $\mathfrak{C}$ and $\mathfrak{C}_\mathfrak{A}$, merely stating that a $\mathfrak{C}$-algebra C is equivalent to a diagram $\gamma: V \to \mathfrak{A}$.

A *duality* denotes a dual equivalence $D : \mathfrak{A} \rightleftarrows \mathfrak{X} : E$ in which $\mathfrak{A}$ is a category of *algebras* (in the sense of modern universal algebra) and homomorphisms, while $\mathfrak{X}$ is a concrete category of objects known as *spaces*. For an algebra A, the image AD is called the *representation space* of A. For a space X, the image XE is called the *algebra represented* by X. The functor D is called the *dual space functor*. The functor E is called the *represented algebra functor*.

For **Esakia duality** [1], take $\mathfrak{A}$ to be the category **Heyt** of Heyting algebras. Take $\mathfrak{X}$ to be the category **Esakia** of *Esakia spaces*, partially ordered Stone spaces where the downset $C^\geq$ of each clopen subset C is clopen. For a Heyting algebra H, the representation space $H^D = \mathbf{Heyt}(H, 2)$ carries the induced order and subspace topology from the product 2^H. An Esakia space S represents the algebra $S^E = \mathbf{Esakia}(S, 2)$, a Heyting subalgebra of 2^S.

For **Lindenbaum-Tarski duality** [2, p. xiv], [8], take $\mathfrak{A}$ to be the category **Set** of sets (algebras without operations). Take $\mathfrak{X}$ to be the category **CABA** of complete atomic Boolean algebras and homomorphisms preserving all joins and meets. Consider the set $2 = \{0, 1\}$, possibly endowed with Boolean algebra structure. For a set A, the representation space A^D is defined to be the set 2^A or $\mathcal{P}(A)$ of (characteristic functions of) subsets of A, with the singletons as atoms. For a complete atomic Boolean algebra B, the represented algebra $B^E := \mathbf{CABA}(B, 2)$ is naturally isomorphic to the set of atoms of B.

Nelson algebras

Nelson algebras [6], also known as "N-lattices" [4], provide algebraic semantics for Nelson's *constructive logic with strong negation* [3, 5]. Consider an algebra $(B, \vee, \wedge, \rightarrow, \sim, 0, 1)$ equipped with three binary operations $\vee, \wedge, \rightarrow$, and with $\sim$ as a unary operation (*strong negation*). Suppose that $(B, \vee, \wedge, 0, 1)$ is a bounded distributive lattice, with $\leq$ as the lattice ordering. Then the algebra is a *Nelson algebra* if the following conditions are satisfied: the reduct $(B, \vee, \wedge, \sim, 0, 1)$ is a De Morgan algebra; a reflexive, transitive relation $\preceq$ is defined on B by setting $x \preceq y$ iff $(x \rightarrow y) \rightarrow (x \rightarrow y) = x \rightarrow y$; the lattice order relation $x \leq y$ on B is equivalent to $x \preceq y$ and $\sim x \preceq \sim y$; the equivalence relation χ, defined on B by $x \chi y$ iff $x \preceq y$ and $y \preceq x$, is a congruence on the reduct $(B, \vee, \wedge, \rightarrow)$ such that the quotient $(B, \vee, \wedge, \rightarrow, 0, 1)^\chi$ is a Heyting algebra; and for all $x, y \in B$, one has $(x \wedge \sim x, 0) \in \chi$ and $(\sim (x \rightarrow y), x \wedge \sim y) \in \chi$. Now a congruence α on a Heyting algebra H is *Boolean* if the quotient H^α is a Boolean algebra. Then the category of Nelson algebras is equivalent to the category of pairs (H, α), where H is a Heyting algebra and α is a Boolean congruence on H [6, Th. 4.1]. Thus Nelson algebras are diagrammatic relative to (the category **Heyt** of) Heyting algebras: Consider the quiver V given as $a: h \rightarrow b$. Then a Nelson algebra B is equivalent to a diagram $\beta: V \rightarrow$ **Heyt** sending the arrow a to the natural projection of the Boolean congruence α_B from the Heyting algebra B^χ. By this means, Nelson algebras have a diagrammatic duality based on Esakia duality for Heyting algebras.

Classical universal algebras

Consider a *type* $\tau: \Omega \rightarrow \mathbb{N}$, with *operator domain* Ω. Then a *τ-algebra* (A, τ) is a set A with an *operation* $\omega: A^{\omega\tau} \rightarrow A$ corresponding to each *operator* or element ω of Ω. Let $\underline{\tau}$ denote the category of τ-algebras and homomorphisms between them [7, §§IV1.1–2]. The $\underline{\Omega}$-*cospan* is a quiver Ω_∞ with edge set Ω. Its vertex set is the disjoint union $\Omega + \top$ of Ω with a singleton $\top = \{\infty\}$. The tail map is the identity function on Ω, while the head map is the unique function $\Omega \rightarrow \top$. A τ-algebra A is equivalent to a diagram $\alpha: \Omega_\infty \rightarrow$ **Set** with edge map $\alpha_1: \omega \mapsto (\omega: A^{\omega\tau} \rightarrow A)$. Thus the edge map sends each operator to the corresponding operation on the set A. It follows that algebras from any variety are diagrammatic relative to sets, and thus possess a diagrammatic duality based on Lindenbaum-Tarski duality for sets.

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Topological spaces of monadic MV-algebras

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The finitely valued propositional calculi, which have been described by Łukasiewicz and Tarski in [1], are extended to the corresponding predicate calculi. The predicate Łukasiewicz (infinitely valued) logic QL is defined in the following standard way. The existential (universal) quantifier is interpreted as supremum (infimum) in a complete MV-algebra. Then the valid formulas of predicate calculus are defined as all formulas having value 1 for any assignment. The functional description of the predicate calculus is given by Rutledge in [2]. Scarpellini in [3] has proved that the set of valid formulas is not recursively enumerable.

Let L and L_m denote a first-order language and propositional language, respectively, based on $\cdot, +, \rightarrow, \neg, \exists$. We fix a variable x in L, associate with each propositional letter p in L_m a unique monadic predicate $p^*(x)$ in L and define by induction a translation $\Psi : Form(L_m) \rightarrow Form(L)$ by putting: i) $\Psi(p) = p^*(x)$ if p is propositional variable, ii) $\Psi(\alpha \circ \beta) = \Psi(\alpha) \circ \Psi(\beta)$, where $\circ = \cdot, +, \rightarrow$, iii) $\Psi(\exists \alpha) = \exists x \Psi(\alpha)$.

Monadic MV-algebras were introduced and studied by Rutledge in [2] as an algebraic model for the predicate calculus QL of Łukasiewicz infinite-valued logic, in which only a single individual variable occurs. Rutledge followed P.R. Halmos' study of monadic Boolean algebras. In view of the incompleteness of the predicate calculus the result of Rutledge in [2], showing the completeness of the monadic predicate calculus, has been of great interest.

The characterization of monadic MV-algebras as pair of MV-algebras, where one of them is a special kind of subalgebra (m-relatively complete subalgebra), is given in [4]. MV-algebras were introduced by Chang in [5] as an algebraic model for infinitely valued Łukasiewicz logic. An MV-algebra is an algebra $A = (A, \oplus, \odot, ^*, 0, 1)$ where $(A, \oplus, 0)$ is an abelian monoid, and the following identities hold for all $x, y \in A$: $x \oplus 1 = 1$, $x^{**} = x$, $0^* = 1$, $x \oplus x^* = 1$, $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$, $x \odot y = (x^* \oplus y^*)^*$. An algebra $A = (A, \oplus, \odot, ^*, \exists, 0, 1)$ (for short $(A, \exists)$) is said to be a monadic MV-algebra (MMV-algebra for short) if $A = (A, \oplus, \odot, ^*, 0, 1)$ is an MV-algebra and in addition $\exists$ satisfies the following identities: $x \leq \exists x$, $\exists(x \vee y) = \exists x \vee \exists y$, $\exists(\exists x)^* = (\exists x)^*$, $\exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$, $\exists(x \odot x) = \exists x \odot \exists x$, $\exists(x \oplus x) = \exists x \oplus \exists x$.

A topological space X is said to be an MV-space iff there exists an MV-algebra A such that $Spec(A)$ (= the set of prime filters of the MV-algebra A equipped with spectral topology) and X are homeomorphic. Any MV-space is a Priestley space (X, R) such that $R(x)$ (= $\{y \in X : xRy\}$ is a chain for any $x \in X$ and a morphism between MV-spaces is a strongly isotone map (or an MV-morphism), i. e. a continuous map $\varphi : X \rightarrow Y$ such that $\varphi(R(x)) = R(\varphi(x))$ for all $x \in X$.

Define on A the binary relation $\stackrel{\sim}{\cong}$ by the following stipulation: $x \stackrel{\sim}{\cong} y$ iff $supp^*(x) = supp^*(y)$, where $supp^*(x)$ is defined as the set of all prime filters of A containing the element x [6]. Then, $\stackrel{\sim}{\cong}$ is a congruence with respect to $\otimes$ and $\vee$. The resulting set $\beta^*(A)(= A/\stackrel{\sim}{\cong})$ of equivalence classes is a bounded distributive lattice (which we also call the Belluce lattice of A) $(\beta^*(A), \vee, \wedge, 0, 1)$, where $\beta^*(x) \wedge \beta^*(y) = \beta^*(x \otimes y)$, $\beta^*(x) \vee \beta^*(y) = \beta^*(x \oplus y) = \beta^*(x \vee y)$, $\beta^*(1) = 1$, $\beta^*(0) = 0$, $\beta^*(x)$ is the equivalence class containing the element x.

Q-distributive lattices were introduced by Cignoli in [7]. A Q-distributive lattice is an algebra $(A, \vee, \wedge, \exists, 0, 1)$ such that $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $\exists$ is a

quantifier on A, where: $\exists 0 = 0, a \wedge \exists a = a, \exists(a \wedge \exists b) = \exists a \wedge \exists b, \exists(a \vee b) = \exists a \vee \exists b$.

A $Q-space$ is a triplet (X, R, E) such that (X, R) is a Priestley space and E is an equivalence relation on X which satisfies the following two conditions: 1) $E(U) \in \mathcal{P}(X)$ for each $U \in \mathcal{P}(X)$, and 2) the equivalence classes for E are closed in X (recall that $E(U)$ is the union of the equivalence classes which intersect U and $\mathcal{P}(X)$ is the set of the clopen increasing subsets of X).

Let (X, R, E) and (Y, S, F) be Q-spaces. A $Q - mapping$ from (X, R, E) into (Y, S, F) is a continuous and order-preserving function $f : X \to Y$ such that $E(f^{-1}(V)) = f^{-1}(F(V))$ for each $V \in \mathcal{P}(Y)$. Let $\mathcal{QD}$ and $\mathcal{QD}^*$ be the category of Q-lattices and Q-spaces respectively. There exist contravariant functors $Q^* : \mathcal{QD} \to \mathcal{QD}^*$ and $Q : \mathcal{QD}^* \to \mathcal{QD}$ that define a dual equivalence between $\mathcal{QD}$ and $\mathcal{QD}^*$ [7].

We define a covariant functor γ from the category **MMV** of monadic MV-algebras into the category of Q-distributive lattices $\mathcal{QD}$ in the following way. Let $(A, \exists) \in$ **MMV** and define a relative congruence relation $\widetilde{\cong}_E$ with respect to $\odot, \vee$ and $\exists$ on the $(A, \exists)$: for every $x, y \in A$ $x\widetilde{\cong}_E y$ if and only if $supp(x) = supp(y)$ and $supp(\exists x) = supp(\exists y)$. Let $\gamma : A \to A/\widetilde{\cong}$ be a natural mapping. The resulting set $\gamma(A)(= A/\widetilde{\cong}_E)$ of equivalence classes is a Q-distributive lattice. For each $x \in A$ let us denote by $\gamma(x)$ the equivalence class of x. Let $f : A \to B$ be an MMV-homomorphism. Then $\gamma(f)$ is a Q-mapping from $\gamma(A)$ to $\gamma(B)$ defined as follows: $\gamma(f)(\gamma(x)) = \gamma(f(x))$.

Theorem 1. *If $(A, \exists) \in$ **MMV***, then $\gamma(A, \exists) \in \mathcal{QD}$, and γ is a covariant functor from the category **MMV** into the category of Q-distributive lattices $\mathcal{QD}$.*

(X, R, E) is named MQ-space if (X, R) is an MV-space, (X, R, E) is a Q-space and: $R(E(x)) = E(R(x)), E(R^{-1}(x)) = R^{-1}(E(x)), R^{-1}(x) \cap E(x) = R(x) \cap E(x) = \{x\}$.

Let $\mathcal{MQ}$ be the category the objects of which are MQ-spaces and morphisms strongly isotone Q-mappings. Strongly isotone Q-mappings we name MQ-mappings.

Theorem 2. *There exists a contravariant functor MQ^* from **MMV** into $\mathcal{MQ}$: $MQ^*(A, \exists) = Q^*(\gamma(A, \exists)) = (\mathcal{F}(A), E(\exists))$, where $\mathcal{F}(A)$ is the prime spectrum of $\gamma(A, \exists)$ with the patch topology and the inclusion relation and $E(\exists) = \{(F, G) \in \mathcal{F}(A)^2 | F \cap \exists\gamma(A, \exists) = G \cap \exists\gamma(A, \exists)\}$.*

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Undefinability of Standard Sequent Calculi
for 3-valued Paraconsistent Logics

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Distinguishing between a "strong sense" and a "weak sense" of propositional connectives when partially defined predicates are present in a language is an idea due to Kleene [12]. Each of these meanings is made explicit by introducing 3-valued truth tables, which have become widely known as *strong Kleene tables* and *weak Kleene tables*. By labelling the elements as $0, n, 1$, the *strong tables* for conjunction, disjunction and negation are displayed below:

$\wedge$	0	n	1		$\vee$	0	n	1		$\neg$	
0	0	0	0		0	0	n	1		1	0
n	0	n	n		n	n	n	1		n	n
1	0	n	1		1	1	1	1		0	1

The *weak tables* basically differ for the behavior of the third value n and are given by:

$\wedge$	0	n	1		$\vee$	0	n	1		$\neg$	
0	0	n	0		0	0	n	1		1	0
n	n	n	n		n	n	n	n		n	n
1	0	n	1		1	1	n	1		0	1

Each set of tables naturally gives rise to two options for building a three-valued logic, according to the choice of 1 (only) as designated value, or 1 together with the third value n. Therefore, four logics populate the Kleene family:[1]

- Strong Kleene logic [12, §64] and the Logic of Paradox, LP [13], obtained out of the strong Kleene tables by choosing 1 and $1, n$, respectively, as designated values;
- Bochvar's logic [6] and Paraconsistent Weak Kleene logic, PWK [11, 14], given by the weak Kleene tables choosing 1 and $1, n$, respectively, as designated values.

In the present paper we focus on a family of paraconsistent logics including both the Logic of Paradox [13] (LP) and Paraconsistent Weak Kleene logic, PWK [11, 14], which has been recently studied under different perspectives [8, 7].
Different types of sequent calculi has been introduced for LP [1], [2], [3], [5]. On the other hand, to the authors' best knowledge, the only attempt to provide a sequent calculus for PWK is [9]. All the existing sequent calculi for these paraconsistent three-valued logics present non-standard features, for instance non standard axioms [4], logical rules introducing more than one connective [4], [3] or logical rules that can be applied only in presence of certain linguistic conditions (this is the case in [9]). In our approach a *standard Gentzen calculus* for a logic L is a calculus (on multisets) having the following properties:

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[1] Here we treat the expression "Kleene family" informally and we do not intend to be exhaustive. There are other logics that could also be considered within the family of Kleene logics, defined by using two or more of these matrices (see for instance [10]).

1. Axioms shall be only of the form $\alpha \Rightarrow \alpha$, for any propositional variable α.
2. The premises of logical rules must contain only *subformulas* of the conclusion and each logical rule must introduce exactly one connective at time.
3. Logical rules must have no linguistic restrictions.
4. Sequents shall be interpreted in the object language, that is: $\Gamma \Rightarrow \Delta$ means that the formula $\bigvee_{i=1}^{n} \delta_i$, with $\delta_i \in \Delta$ follows from the formula $\bigwedge_{j=1}^{m} \gamma_j$, with $\gamma_j \in \Gamma$.
5. Only standard structural rules, i.e. contraction, weakening and cut are (possibly) allowed.

Furthermore, by *quasi-standard* we mean a calculus where condition 4 above is replaced by the usual metalinguistic interpretation of the comma in the sequents.

The main result of this work consists of proving the impossibility of providing *standard*, as well as *quasi-standard* sequent calculi for a family of logics including both LP and PWK.

PWK has been extensively studied with the tools of Abstract Algebraic Logic in [7]. We wonder whether the above mentioned negative result might have an algebraic counterpart.

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Standard completeness for mianorm-based logics with n-contraction, n-mingle, and n-potency axioms

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1 Introduction

The aim of this paper is to introduce standard completeness results for substructural fuzzy logics based on mianorms (binary monotonic identity aggregation operations on the real unit interval $[0, 1]$) with n-contraction, n-mingle, and n-potency axioms. For this, we note that Baldi [1] introduced Wang's $\mathbf{C}_n\mathbf{UL}$ (Uninorm logic $\mathbf{UL}$ with n-potency) as $\mathbf{UL}$ with both the n-contraction axiom and the n-mingle axiom. However, micanorm- and mianorm-based logics with each of these axioms have not yet been investigated. Furthermore, we can divide n-contraction, n-mingle, and n-potency axioms into right and left ones in the context of non-commutative logic. Here, we introduce such logic systems and their standard completeness via Yang's construction in the style of Jenei–Montagna (see [3, 4]).

2 Logic systems, Algebras, and Standard completeness

Let φ^n and $^n\varphi$ stand for $((...(\varphi \& \varphi) \& \cdots \& \varphi) \& \varphi$, n φ's, and $\varphi \& (\varphi \& \cdots \& (\varphi \& \varphi)...))$, n φ's, respectively. We introduce the following extensions of $\mathbf{MIAL}$ (Mianorm logic, $= \mathrm{SL}^\ell$).

Definition 1. *Let $2 \leq n$. $\boldsymbol{C}_n^r\boldsymbol{MIAL}$ is $\boldsymbol{MIAL}$ plus (right n-contraction, c_n^r) $\varphi^{n-1} \rightarrow \varphi^n$; $\boldsymbol{C}_n^l\boldsymbol{MIAL}$ is $\boldsymbol{MIAL}$ plus (left n-contraction, c_n^l) $^{n-1}\varphi \rightarrow {}^n\varphi$; $\boldsymbol{M}_n^r\boldsymbol{MIAL}$ is $\boldsymbol{MIAL}$ plus (right n-mingle, m_n^r) $\varphi^n \rightarrow \varphi^{n-1}$; $\boldsymbol{M}_n^l\boldsymbol{MIAL}$ is $\boldsymbol{MIAL}$ plus (left n-mingle, m_n^l) $^n\varphi \rightarrow {}^{n-1}\varphi$; $\boldsymbol{P}_n^r\boldsymbol{MIAL}$ is $\boldsymbol{MIAL}$ plus (right n-potency, p_n^r) $\varphi^{n-1} \leftrightarrow \varphi^n$; and $\boldsymbol{P}_n^l\boldsymbol{MIAL}$ is $\boldsymbol{MIAL}$ plus (left n-potency, p_n^l) $^{n-1}\varphi \leftrightarrow {}^n\varphi$.*

Definition 2. *$Ls = \{\boldsymbol{C}_n^r\boldsymbol{MIAL}, \boldsymbol{C}_n^l\boldsymbol{MIAL}, \boldsymbol{M}_n^r\boldsymbol{MIAL}, \boldsymbol{M}_n^l\boldsymbol{MIAL}, \boldsymbol{P}_n^r\boldsymbol{MIAL}, \boldsymbol{P}_n^l\boldsymbol{MIAL}\}\}$.*

An $\mathcal{A}$-*evaluation* is a function $v : Fm \rightarrow \mathcal{A}$ satisfying: $v(\sharp(\varphi_1, \ldots, \varphi_m)) = \sharp^{\mathcal{A}}(v(\varphi_1), \ldots, v(\varphi_m))$, where $\sharp \in \{\rightarrow, \rightsquigarrow, \wedge, \vee, \&, \top, \bot, \overline{1}, \overline{0}\}$ and $\sharp^{\mathcal{A}} \in \{\backslash, /, \wedge, \vee, *, \top, \bot, t, f\}$. A formula φ is *valid* in $\mathcal{A}$ if $v(\varphi) \geq t$ for each $\mathcal{A}$-evaluation v. An $\mathcal{A}$-evaluation v is an $\mathcal{A}$-*model* of T if $v(\varphi) \geq t$ for each $\varphi \in T$.

Definition 3. *For $\boldsymbol{L}$ an extension of $\boldsymbol{MIAL}$, a $\boldsymbol{MIAL}$-algebra $\mathcal{A}$ is an $\boldsymbol{L}$-algebra if all axioms of L are valid in $\mathcal{A}$. Especially, for all $x \in A$ and $2 \leq n$, A $\boldsymbol{C}_n^r\boldsymbol{MIAL}$-algebra is a $\boldsymbol{MIAL}$-algebra satisfying $(c_n^{r\,\mathcal{A}})$ $x^{n-1} \leq x^n$; A $\boldsymbol{C}_n^l\boldsymbol{MIAL}$-algebra is a $\boldsymbol{MIAL}$-algebra satisfying $(c_n^{l\,\mathcal{A}})$ $^{n-1}x \leq {}^n x$; An $\boldsymbol{M}_n^r\boldsymbol{MIAL}$-algebra is a $\boldsymbol{MIAL}$-algebra satisfying $(m_n^{r\,\mathcal{A}})$ $x^n \leq x^{n-1}$; An $\boldsymbol{M}_n^l\boldsymbol{MIAL}$-algebra is a $\boldsymbol{MIAL}$-algebra satisfying $(m_n^{l\,\mathcal{A}})$ $^n x \leq {}^{n-1}x$; A $\boldsymbol{C}_n^r\boldsymbol{MIAL}$-algebra is a $\boldsymbol{MIAL}$-algebra satisfying $(p_n^{r\,\mathcal{A}})$ $x^{n-1} = x^n$; A $\boldsymbol{C}_n^l\boldsymbol{MIAL}$-algebra is a $\boldsymbol{MIAL}$-algebra satisfying $(p_n^{l\,\mathcal{A}})$ $^{n-1}x = {}^n x$. For convenience, we call all these algebras L-algebras.*

Theorem 4. *(Strong completeness) Let T be a theory over L ($\in Ls$) and φ a formula. $T \vdash_L \varphi$ iff for every linearly ordered L-algebra $\mathcal{A}$ and an $\mathcal{A}$-evaluation v, if v is an $\mathcal{A}$-model of T, then $v(\varphi) \geq t$.*

Proposition 5. *For every finite or countable, linearly ordered L-algebra $\boldsymbol{A} = (A, \leq_A, \top, \bot, t, f, \wedge, \vee, *, \backslash, /)$, there is a countable ordered set X, a binary operation $\circ$ on X, and a map h from A into X such that (I) X is densely ordered and has a maximum Max, a minimum Min, and special elements e and ∂; (II) $(X, \circ, \preceq, e)$ is a linearly ordered, monotonic groupoid with unit; (III) $\circ$ is conjunctive and left-continuous with respect to (w.r.t.) the order topology on $(X, \preceq)$; (IV) h is an embedding of the structure $(A, \leq_A, \top, \bot, t, f, \wedge, \vee, *)$ into $(X, \preceq, Max, Min, e, \partial, min, max, \circ)$, and, for all $m, n \in A$, $h(m \backslash n)$ and $h(n / m)$ are the residuated pair of $h(m)$ and $h(n)$ in $(X, \preceq, Max, Min, e, \partial, max, min, \circ)$; and (V) $\circ$ satisfies right and left n-contraction, n-mingle, and n-potency properties corresponding to $*$.*

Theorem 6. *(Strong standard completeness) For $L \in Ls$, $T \vdash_L \varphi$ iff for every standard L-algebra and evaluation v, if $v(\psi) \geq e$ for all $\psi \in T$, then $v(\varphi) \geq e$.*

Remark 7. *[(i)]*

1. *For $L \in Ls$, L_e is L plus $(e, exchange)$ $(\varphi \& \psi) \to (\psi \& \varphi)$. By almost the same construction, we can prove standard completeness for L_e. But this construction does not work for L_a, L plus $(a, associativity)$ $(\varphi \& \psi) \& \chi \leftrightarrow \varphi \& (\psi \& \chi)$, since the operation $\circ$ for L_a does not satisfy associativity (see Theorem 7 (v) in [3]).*

2. *The operation $\circ$ in Wang's construction in the style of Jenei–Montagna satisfy associativity (see Theorem 4.3 in [2]). But this construction does not work for n-contraction and n-potency, for given $n = 2$, since $(m, x) \npreceq (m, x) \circ (m, x)$ (see p. 212 in [2]).*

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Sasaki projections and related operations

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Orthomodular lattices were introduced as event structures of quantum mechanics. They admit the modeling of events which are not simultaneously observable. They are not distributive. Therefore the computation in orthomodular lattices is much more advanced than in Boolean algebras. The use of the lattice operations was mostly exhausted and there seems not to be much space for breaking results concerning their properties. Therefore we considered other operations in our previous work. We have shown that there are no other useful associative operations. Among the non-associative ones, Sasaki projection (and its dual) satisfy the most equations which are weakenings of associativity. We collected many of the known properties of Sasaki projection, added new ones, and concentrated on the question of which elements of an orthomodular lattice have a common complement. Here we extend these results.

An *orthomodular lattice* (abbr. OML) is a bounded lattice with an antitone involution $^\perp$ (*orthocomplementation*) satisfying $x \vee x^\perp = \mathbf{1}$, $x \wedge x^\perp = \mathbf{0}$, and $x \leq y \implies y = x \vee (x^\perp \wedge y)$ (*orthomodular law*). A prototypical example of an orthomodular lattice is the lattice of all closed linear subspaces of a Hilbert space with $x^\perp$ being the closure of $\{\boldsymbol{u} \mid \boldsymbol{u} \perp \boldsymbol{v} \text{ for all } \boldsymbol{v} \in x\}$. Without the use of the inner product, only some properties of subspaces can be expressed in algebraic terms of OMLs. Sasaki [11] showed that the orthogonal projection of a subspace y to a subspace x can be expressed without the use of the inner product as the *Sasaki projection* ϕ_x,

$$\phi_x(y) = x \wedge (x^\perp \vee y).$$

Sasaki projections are also studied in [1, 2, 3, 9, 10] and generalized in the context of *synaptic algebras* by D. Foulis and S. Pulmannová in [5].

Throughout this abstract, L denotes an orthomodular lattice and $\Phi(L) = \{\phi_x \mid x \in L\}$. The fundamental observation [10] is that kernels of congruences in L are exactly the subsets I satisfying $\phi_x(y) \in I$ whenever $x \in I$ or $y \in I$. (The meet, $x \wedge y$, does not possess this property.)

Sasaki projections preserve the join [3, 4], i.e., $\phi_x(y \vee z) = \phi_x(y) \vee \phi_x(z)$, therefore they are monotonic. Each monotonic mapping $\theta \colon L \to L$ has a unique *dual*, which is a monotonic mapping $\bar{\theta} \colon L \to L$ defined by

$$\bar{\theta}(y) = (\theta(y^\perp))^\perp.$$

The composition of two Sasaki projections, $\phi_x \phi_y$, is a Sasaki projection iff x and y *commute*, i.e., $x = (x \wedge y) \vee (x \wedge y')$. All finite compositions of Sasaki projections on L form a monoid $S(L)$.

For $x_1, \ldots, x_n \in L$, we study the compositions $\xi = \phi_{x_n} \cdots \phi_{x_2} \phi_{x_1}, \xi^* = \phi_{x_1} \phi_{x_2} \cdots \phi_{x_n} \in S(L)$. They form an *adjoint pair*, i.e., $\xi^*(y) = \min\{z \in L \mid \bar{\xi}(z) \geq y\}$, thus each of them uniquely determines the other and the mapping $^* \colon S(L) \to S(L)$ is correctly defined (although the representations of ξ, ξ^* as compositions of Sasaki projections are not unique).

Elements x and y of an OML L are said to be *strongly perspective* if they have a common (relative) complement in the interval $[\mathbf{0}, x \vee y]$. Following [2], we ask when $\xi(\mathbf{1}), \xi^*(\mathbf{1})$ are

strongly perspective. For $n = 2$, this is always the case. We have found a constructive proof for $n = 3$ and a counterexample for $n = 4$ [8]. Chevalier and Pulmannová [2] have given a non-constructive proof for *complete modular* OMLs and arbitrary n; however, a *constructive* proof for $n > 3$ is not known.

Let S be a semigroup with an absorbing element 0 and an involution $^*\colon S \to S$ such that for any $\theta, \eta \in S$, $(\theta\eta)^* = \eta^*\theta^*$. We call S a *Baer *-semigroup* if, for each $\theta \in S$, there is a greatest element, θ', of the right ideal $\{\eta \in S \mid \theta\eta = 0\}$ and $\pi = \theta'$ is a projection, i.e., $\pi = \pi^2 = \pi^*$ [2, 3, 4]. We denote by $P(S)$ the set $\{\theta' \mid \theta \in S\}$.

The theory of Baer *-semigroups can be directly applied to the monoid $S(L)$. Projections of $S(L)$ are exactly the Sasaki projections, $P(S(L)) = \Phi(L)$. The order on $\Phi(L)$ is defined by $\theta \leq \eta \iff \theta\eta = \theta$. For each $\theta \in S(L)$, we define $\theta' := \phi_{\overline{\theta^*}(\mathbf{0})}$. It is the unique element such that for any $\eta \in S(L)$

$$\theta\eta = \phi_{\mathbf{0}} \iff \eta(y) \leq \theta'(y) \text{ for all } y \in L. \tag{1}$$

The mapping $'$ is an orthocomplementation which equips $\Phi(L)$ with the structure of an OML. The mapping $\Phi\colon L \to \Phi(L)$, $x \mapsto \phi_x$, is an isomorphism. We derived many of the relevant results using the tools of OML computations.

Chevalier and Pulmannová [2] proved that $\xi^*\xi(\mathbf{1}) = \xi^*(\mathbf{1})$. Their proof heavily used methods of Baer *-semigroups, a direct proof using techniques of OMLs is still not known.

We bring arguments why Sasaki operations form a promising alternative to lattice operations (join and meet) in the study of orthomodular lattices. Only the lattice operations "satisfy more equations" than Sasaki operations. The potential of using Sasaki operations in the algebraic foundations of orthomodular lattices is still not sufficiently exhausted. Besides, they have a natural physical interpretation.

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Projective WS5-Algebras

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The logic WS5 plays an important role in extending Glivenko's Theorem to MIPC (see [2]). The algebraic models for WS5 are monadic Heyting algebras in which the open elements form a Boolean algebra. We study the variety $\mathcal{M}$ of such algebras from the standpoint of projectivity. We give a description of $\mathbf{F}_{\mathcal{M}}(1)$, and we prove a criterion of projectivity of finitely-presented algebra from any of subvarieties of $\mathcal{M}$.

Free Single-Generated Algebra

An algebra $\langle A; \wedge, \vee, \rightarrow, \mathbf{0}, \mathbf{1}, \square \rangle$, where $\langle A; \wedge, \vee, \rightarrow, \mathbf{0}, \mathbf{1} \rangle$ is a Heyting algebra and $\square$ satisfies the following identities:

$$
\begin{array}{ll}
(M0) & \square \mathbf{1} \approx \mathbf{1}; \\
(M1) & \square x \rightarrow x \approx \mathbf{1}; \\
(M2) & \square(x \rightarrow y) \rightarrow (\square x \rightarrow \square y) \approx \mathbf{1}; \\
(M3) & \square x \rightarrow \square \square x \approx \mathbf{1}; \\
(M4) & \neg \square \neg \square x \approx \square x.
\end{array}
$$

is called an *m-algebra*. It is clear that the set of all m-algebras forms a variety that we denote by $\mathcal{M}$. All necessary information about monadic Heyting algebras (including m-algebras) can be found in [1]. An element a of an m-algebra is *open*, if $\mathsf{a} = \square \mathsf{a}$. Recall that an m-algebra is subdirectly irreducible (s.i. for short), if it has exactly two open elements: $\mathbf{0}$ and $\mathbf{1}$.

For any element a of any m-algebra $\mathbf{A}$, we define the degrees of a as follows: $\mathsf{a}^0 := \mathbf{0}$, $\mathsf{a}^1 := \neg \mathsf{a}$, $\mathsf{a}^2 := \mathsf{a}$ and for all $k \geq 0$ $\mathsf{a}^{2k+3} := \mathsf{a}^{2k+1} \rightarrow \mathsf{a}^{2k}$, $\mathsf{a}^{2k+4} := \mathsf{a}^{2k+1} \vee \mathsf{a}^{2k+2}$, and we let $\mathsf{a}^\omega := \mathbf{1}$.

For $n > 1$ we denote by $\mathbf{Z}_n$ a single-generated s.i. m-algebra of cardinality n. The Heyting reduct of $\mathbf{Z}_n$ (H-reduct for short) is a single-generated Heyting algebra with n elements in which $\square \mathbf{1} = \mathbf{1}$ and $\square \mathsf{a} = \mathbf{0}$ for all $\mathsf{a} < \mathbf{1}$. Every algebra $\mathbf{Z}_n$ consists of degrees of its generator that we denote by g_n. $\mathbf{Z}_2$ is a two-element m-algebra with generator $\mathsf{g}_2 = \mathbf{0}$, while by $\mathbf{Z}_1$ we denote a two-element m-algebra with generator $\mathsf{g}_1 = \mathbf{1}$.

Let

$$\mathbf{P} = \prod_{i>0} \mathbf{Z}_i \text{ and } \mathbf{Z} \text{ be a subalgebra of } \mathbf{P} \text{ generated by element } \mathsf{g} = (\mathsf{g}_1, \mathsf{g}_2, \dots),$$

that is, by the element g such that $\pi_i(\mathsf{g}) = \mathsf{g}_i, i > 0$, where π_i is a i-th projection.

Proposition 1. $\mathbf{Z}$ *is isomorphic to* $\mathbf{F}_{\mathcal{M}}(1)$.

An element $\mathsf{a} \in \mathbf{P}$ is called *leveled*, if there are $0 < k < \omega$ and $0 < m \leq \omega$ such that $\pi_j(\mathsf{a}) = \mathsf{g}_j^m$ for all $j \geq k$. Let $\mathbf{L}$ be a set of all leveled elements of $\mathbf{P}$. The following theorem gives a convenient intrinsic description of $\mathbf{F}_{\mathcal{M}}(1)$.

Theorem 2. $\mathbf{L} = \mathbf{Z}$, *hence* $\mathbf{F}_{\mathcal{M}}(1)$ *is isomorphic to a subalgebra of* $\mathbf{P}$ *consisting of all leveled elements.*

As one can see from the following corollary, the structure of $\mathbf{F}_{\mathcal{M}}(1)$ is much more complex than the structure of free single-generated Heyting algebra.

Corollary 3. *The following holds*

(a) *H-reduct of $\mathbf{F}_{\mathcal{M}}(1)$ is not finitely generated as Heyting algebra;*
(b) *$\mathbf{F}_{\mathcal{M}}(1)$ contains infinite ascending and descending chains of open elements;*
(c) *$\mathbf{F}_{\mathcal{M}}(1)$ is atomic and it has infinite set of atoms;*
(d) *$\mathbf{Z}_2$ is the only s.i. subalgebra of $\mathbf{F}_{\mathcal{M}}(1)$.*

Projective Algebras

In the following theorem we use the notations from [1]: $\varphi(\mathbf{A})$ denotes the H-reduct of $\mathbf{A}$, $\psi(\mathbf{A})$ denotes a relatively complete subalgebra of $\varphi(\mathbf{A})$ defining modal operations, and $\psi(\mathcal{V}) = \{\psi(\mathbf{A}) \mid \mathbf{A} \in \mathcal{V}\}$.

Theorem 4. *(comp. [3, Corollary 5.5]) Let $\mathcal{V} \subseteq$ MHA be a variety of monadic Heyting algebras. If $\mathbf{A} \in \mathcal{V}$ is such an algebra that $\varphi(\mathbf{A}) = \psi(\mathbf{A})$ and algebra $\psi(\mathbf{A})$ is projective in $\psi(\mathcal{V})$, then $\mathbf{A}$ is projective in $\mathcal{V}$.*

Corollary 5. *If $\mathbf{A}$ is at most countable m-algebra and each element of $\mathbf{A}$ is open, then $\mathbf{A}$ is projective in $\mathcal{M}$.*

Proposition 6. *Each projective in MHA algebra has $\mathbf{Z}_2$ as a homomorphic image.*

Let $\mathcal{V}$ be a variety of m-algebras and $\mathbf{A} \in \mathcal{V}$. Then $\mathbf{A} \in \mathcal{V}$ is *finitely presented* in $\mathcal{V}$ if $\mathbf{A} \cong \mathbf{F}_{\mathcal{V}}(n)/\theta$ for some n, where θ is a principal congruence on $\mathbf{F}_{\mathcal{V}}(n)$.

The following theorem extends the criterion of projectivity [4, Theorem 5.2] from finite to finitely-presented m-algebras.

Theorem 7. *Let $\mathcal{V}$ be a variety of m-algebras and $\mathbf{A} \in \mathcal{V}$ be finitely presented in $\mathcal{V}$. Then $\mathbf{A}$ is projective in $\mathcal{V}$ if and only if $\mathbf{Z}_2$ is a homomorphic image of $\mathbf{A}$.*

Corollary 8. *Let $\mathcal{V}$ be a variety of m-algebras. Then every finitely presented subalgebra of $\mathbf{F}_{\mathcal{V}}(\omega)$ is projective in $\mathcal{V}$. In particular, every finite subalgebra of $\mathbf{F}_{\mathcal{V}}(\omega)$ is projective.*

Corollary 9. *Let $\mathcal{V}$ be a variety of m-algebras and $\mathbf{A} \in \mathcal{V}$ be given by defining relation $t(x_1, \ldots, x_n) = \mathbf{1}$. Then $\mathbf{A}$ is projective in $\mathcal{V}$ if and only if the term t is satisfiable in $\mathbf{Z}_2$.*

Corollary 10. *Let $\mathcal{V}$ be a variety of m-algebras. Then the problem whether a given finite set of equations defines in $\mathcal{V}$ a projective finitely presented algebra is decidable.*

Corollary 11. *$\mathbf{Z}_2$ is the only projective s.i. m-algebra.*

Corollary 12. *Let $\mathcal{V}$ be a variety of m-algebras. Then the problem whether a given finite set of equations defines in $\mathcal{V}$ a projective finitely presented algebra is decidable.*

Theorem 13. *For every finitely generated m-algebra $\mathbf{A}$ the following is equivalent*

(a) *$\mathbf{A}$ has $\mathbf{Z_2}$ as a homomorphic image;*
(b) *$\mathbf{A}$ does not contain an element a such that $\Box\,\mathsf{a} = \Box\neg\mathsf{a}$;*
(c) *quasi-identity $\rho \coloneqq (\neg\,\Box\,x \wedge \neg\,\Box\,\neg x) \approx \mathbf{1} \Rightarrow \mathbf{0}$ holds on $\mathbf{A}$.*

Corollary 14. *The quasivariety $\mathcal{Q}$ defined by quasi-identity ρ is primitive and $\mathcal{Q}$ contains every primitive quasivariety of m-algebras as a subquasivariety.*

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The Convolution Algebra

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A relational structure, or frame, $\mathfrak{X} = (X, (R_i)_I)$ is a set X with a family $(R_i)_I$ of relations on X where we assume that R_i is $n_i + 1$-ary. The complex algebra $\mathfrak{X}^+ = (\mathcal{P}(X), (\Diamond_i)_I, (\Box_i)_I)$ of this frame is the algebra consisting of the power set $\mathcal{P}(X)$ of X with n_i-ary operations $\Diamond_i$ and $\Box_i$ for each $i \in I$. We follow the Jónsson and Tarski method of defining $\Diamond_i$ through the relational image of R_i with $\Box_i$ as its dual. When $\mathcal{P}(X)$ is viewed as 2^X, these operations are given by

$$\Diamond_i(f_1, \ldots, f_{n_i})(x) = \bigvee \{f(x_1) \wedge \cdots \wedge f(x_n) : (x_1, \ldots, x_n, x) \in R_i\}$$

$$\Box_i(f_1, \ldots, f_{n_i})(x) = \bigwedge \{f(x_1) \vee \cdots \vee f(x_n) : (x_1, \ldots, x_n, x) \in R_i\}$$

Replacing 2 with a complete lattice L leads to an obvious generalization of this construction to what we call the convolution algebra $L^{\mathfrak{X}}$. The name is given to reflect that the operations $(\Diamond_i)_I$ and $(\Box_i)_I$ are obtained via a form of convolution.

We consider basic properties of this convolution algebra. Among our results, we show that when L is a non-trivial complete Heyting algebra that the operations $\Diamond_i$ are complete operators and that $L^{\mathfrak{X}}$ and $\mathfrak{X}^+$ satisfy the same equations in the signature $\wedge, \vee, 0, 1, (\Diamond_i)_I$. The dual result holds when L is a non-trivial complete dual Heyting algebra. When L is non-trivial, complete, and completely distributive, $L^{\mathfrak{X}}$ and $\mathfrak{X}^+$ satisfy the same equations in $\wedge, \vee, 0, 1, (\Diamond_i)_I, (\Box_i)_I$.

Frames of a given type $\tau' = (n_i + 1)_I$ form a category $\mathrm{FRM}_{\tau'}$ with the morphisms being p-morphisms. Then considering the category LAT of complete lattices with morphisms being maps that preserve finite meets and arbitrary joins, and ALG_τ the category of algebras of type $\tau = (n_i)_I$, there is a bifunctor

$$\mathrm{CONV} : \mathrm{LAT} \times \mathrm{FRM}_{\tau'} \longrightarrow \mathrm{ALG}_\tau$$

that is covariant in the first argument and contravariant in the second. Here we are considering the restriction to the $(\Diamond_i)_I$ fragment. Modifications to the morphisms of LAT provide versions for $(\Box_i)_I$ fragment, and to the full language. Various results are shown related to the behavior of this bifunctor with respect to one-one and onto maps, and with respect to products and coproducts in its two components.

Several examples are considered. These include monadic Heyting algebras; versions of intuitionistic relation algebras obtained from $H^{\mathfrak{G}}$ where H is a complete Heyting algebra and $\mathfrak{G}$ is a group; and the convolution algebra $\mathrm{I}^{\mathfrak{I}}$ where I is the real unit interval and the relational structure $\mathfrak{I} = (\mathrm{I}, \wedge, \vee, 0, 1, \neg, \triangle, \triangledown)$ consists of I with its max and min operations, bounds, negation, and a t-norm $\triangle$ and co-norm $\triangledown$. This algebra $\mathrm{I}^{\mathfrak{I}}$ is the truth value object used in type-2 fuzzy sets.

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Two systems of point-free affine geometry

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Our presentation is devoted to two systems of geometry which are point-free, in the sense that the notion of *a point* is absent from their basic notions.

The first system, created by the Polish mathematician Aleksander Śniatycki in [3], is based on the notions of *region*, *parthood* and *half-plane*. Śniatycki describes structures of the form:

$$\langle \mathbb{R}, +, \cdot, -, \mathbb{H}, \mathbf{0}, \mathbf{1} \rangle$$

such that $\mathbb{R}$ is a non-empty set whose elements are called *regions*, $\mathbb{H} \subseteq \mathbb{R}$ is a set whose elements are called *half-planes*, and:

$$\langle \mathbb{R}, +, \cdot, -, \mathbf{0}, \mathbf{1} \rangle \text{ is a complete Boolean algebra.} \tag{H0}$$

The specific half-plane postulates (which we formulate here in an abbreviated form) are:

$$h \in \mathbb{H} \longrightarrow -h \in \mathbb{H}, \tag{H1}$$

$$x_1, x_2, x_3 \in \mathbb{R}^+ \longrightarrow \big(\exists_{h \in \mathbb{H}} \forall_{i \in \{1,2,3\}} \left(x_i \cdot h \neq \mathbf{0} \neq x_i \cdot -h \vee \right.$$
$$\exists_{h_1, h_2, h_3 \in \mathbb{H}} \left(\forall_{i \in \{1,2,3\}} x_i \cdot -h_i = \mathbf{0} \wedge \right. \tag{H2}$$
$$\left. ((x_1 + x_2) \cdot h_3) + ((x_1 + x_3) \cdot h_2) + ((x_2 + x_3) \cdot h_1) = \mathbf{0} \right),$$

$$\forall_{h_1, h_2, h_3 \in \mathbb{H}} \left(h_1 \cdot (h_2 + h_3) = \mathbf{0} \longrightarrow h_2 \cdot -h_3 = \mathbf{0} \vee h_3 \cdot -h_2 = \mathbf{0} \right), \tag{H3}$$

$$\forall_{h_1, h_2, h_3, h_4 \in \mathbb{H}} \big(h_1 \cdot h_2 \cdot ((h_3 \cdot -h_4) + (h_4 \cdot -h_3)) = \mathbf{0} \longrightarrow$$
$$(h_3 = h_4) \vee (h_1 \cdot h_2 \cdot h_3 = \mathbf{0}) \vee (h_1 \cdot h_2 \cdot -h_3 = \mathbf{0}) \big). \tag{H4}$$

Śniatycki demonstrates that the standard notions of *line*, *point*, *incidence* relation between lines and points and *betweenness* relation on points are definable in his structures, and that the set of axioms he puts forward is sufficiently strong to prove all the axioms of a system of affine geometry (by which, for the purpose of this talk, we may understand the part of geometry which is expressed by *incidence* and *betweenness* only). In this sense, the theory with axioms (H0)–(H4) may be considered as a system of *point-free* affine geometry.

The second system, which comes from [2], pursues the old idea of Alfred Whitehead's [4] of establishing geometry by combining the mereological notions of *region* and *parthood* with that of *oval* as primitives. Indeed, taking the convex opens subsets of the Cartesian plane as paradigms of oval regions we construct geometry in which the notion of *oval* (treated as a point-free counterpart of the notion of *convex* set) is assumed as basic. Via this (and two other notions, of *region* and *parthood*) we introduce *lines* and *half-planes*, and formulate the following axioms which have very natural geometrical interpretation ($\mathbb{O}$ is the set of ovals, $\mathbb{O}^+$ is the set of non-zero ovals):

$$\langle \mathbb{R}, \leqslant \rangle \text{ is a complete atomless Boolean lattice.} \tag{00}$$

$$\mathbb{O} \text{ is an algebraic closure system in } \langle \mathbb{R}, \leqslant \rangle \text{ containing } \mathbf{0}. \tag{01}$$

$$\mathbb{O}^+ \text{ is dense in } \langle \mathbb{R}^+, \leqslant \rangle. \tag{02}$$

$$\text{The sides of a line form a partition of } \mathbf{1}. \tag{03}$$

For any $a, b, c \in \mathbb{O}^+$ which are not aligned there is a line which separates a from $\mathrm{hull}(b+c)$. (04)

If distinct lines L_1 and L_2 both cross an oval a, then they split a into at least three parts. (05)

$$\text{No half-plane is part of any stripe or angle.} \tag{06}$$

In the axioms above hull is the closure operator arising from $\mathbb{O}$, alignment of regions may be geometrically interpreted as being crossed by one line (which is a pair of maximal ovals), stripe is the product of two «parallel» half-planes, and angle may be interpreted in the traditional Euclidean way as the intersection of two half-planes which are not «parallel».

About the theory composed of axioms (00)–(06) we prove that its definitional extension with the notion of *half-plane* is strong enough to prove all Śniatycki's axioms, and therefore is suitable for reconstruction of affine geometry. So this theory deserves the name of *point-free* affine geometry as well.

In our talk we would like to:

(i) present geometrical interpretation of (cryptic at first sight) axioms of Śniatycki's,

(ii) describe the steps in construction of our system from [2] and justify our choice of axioms from point of view of pursuing affine geometry,

(iii) and sketch the proof of axioms of Śniatycki's from (00)–(06).

Acknowledgments

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Axiomatizing a Reflexive Real-valued Modal Logic

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Many-valued modal logics extend the Kripke frame setting of classical modal logic with a many-valued semantics at each world to model modal notions such as necessity, belief, and spatio-temporal relations in the presence of uncertainty, possibility, or vagueness (see, e.g., [1, 2, 4]). In [3] a many-valued modal logic defined over serial frames with connectives interpreted locally as abelian group operations over the real numbers was introduced and the completeness of an axiomatization established. In this work we extend this result to reflexive frames, thereby taking a first step towards a more general theory of modal logics based on abelian groups. This logic can be viewed as a modal extension of the multiplicative fragment of abelian logic (see, e.g., [5]) and can be axiomatized by adding an axiom expressing reflexivity to the axiom system provided for the logic in [3]. We give a sound and complete axiom system for this logic, where we prove completeness using both a sequent calculus and a labelled tableau system.

Let us denote by Fm the set of formulas defined inductively over a countably infinite set Var of propositional variables using the binary connective $\to$ and modal connective $\Box$. We define

$$\overline{0} := p_0 \to p_0, \quad \neg\varphi := \varphi \to \overline{0}, \quad \varphi \& \psi := \neg\varphi \to \psi, \quad \text{and} \quad \Diamond\varphi := \neg\Box\neg\varphi,$$

and let $0\varphi = \overline{0}$ and $(n+1)\varphi = \varphi \& (n\varphi)$ for all $n \in \mathbb{N}$.

A *frame* is a pair $\mathfrak{F} = \langle W, R \rangle$ such that W is a non-empty set of *worlds* and $R \subseteq W \times W$ is an *accessibility relation* on W. $\mathfrak{F}$ is called *reflexive* if the accessibility relation is reflexive, that is, if for all $x \in W$, Rxx. A KT($\mathbb{R}$)-*model* is a triple $\mathfrak{M} = \langle W, R, V \rangle$ such that $\langle W, R \rangle$ is a reflexive frame and $V \colon \text{Var} \times W \to [-r, r]$ for some $r \in \mathbb{R}_+$ is a *valuation* that extends to $V \colon \text{Fm} \times W \to \mathbb{R}$ via

$$
\begin{aligned}
V(\varphi \to \psi, x) &= V(\psi, x) - V(\varphi, x) \\
V(\Box\varphi, x) &= \bigwedge\{V(\varphi, y) : Rxy\}.
\end{aligned}
$$

A formula $\varphi \in \text{Fm}$ will be called *valid* in a KT($\mathbb{R}$)-model $\mathfrak{M} = \langle W, R, V \rangle$ if $V(\varphi, x) \geq 0$ for all $x \in W$. If φ is valid in all KT($\mathbb{R}$)-models, then φ is said to be KT($\mathbb{R}$)-*valid*, written $\models_{\text{KT}(\mathbb{R})} \varphi$.

The proposed axiom system KT($\mathbb{R}$) for this logic is given in Fig. 1.

(B)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	
(C)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$	$\dfrac{\varphi \quad \varphi \to \psi}{\psi}$ (mp)
(I)	$\varphi \to \varphi$	
(A)	$((\varphi \to \psi) \to \psi) \to \varphi$	$\dfrac{\varphi}{\Box\varphi}$ (nec)
(K)	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	
(T)	$\Box\varphi \to \varphi$	$\dfrac{n\varphi}{\varphi}$ (con$_n$) $\quad (n \geq 2)$
(D$_n$)	$\Box(n\varphi) \to n\Box\varphi \quad (n \geq 2)$	

Figure 1: The axiom system KT($\mathbb{R}$)

$$\frac{}{\Delta \Rightarrow \Delta} \ (\text{ID})$$

$$\frac{\Gamma \Rightarrow \Delta \quad \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Sigma, \Delta} \ (\text{MIX}) \qquad\qquad \frac{n\Gamma \Rightarrow n\Delta}{\Gamma \Rightarrow \Delta} \ (\text{SC}_n) \quad (n \geq 2)$$

$$\frac{\Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta} \ (\to\Rightarrow) \qquad\qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta} \ (\Rightarrow\to)$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \Box\varphi \Rightarrow \Delta} \ (\Box\Rightarrow) \qquad\qquad \frac{\Gamma \Rightarrow n[\varphi]}{\Box\Gamma \Rightarrow n[\Box\varphi]} \ (\Box_n) \quad (n \geq 0)$$

Figure 2: The sequent calculus GKT($\mathbb{R}$)

That any formula derivable in this system is KT($\mathbb{R}$)-valid is easily shown. To prove the converse, we first introduce the sequent calculus in Fig. 2, where a sequent $\Gamma \Rightarrow \Delta$ is defined to be an ordered pair of finite multisets of formulas, $k\Gamma$ denotes $\Gamma, \ldots, \Gamma$ (k times), and $\Box\Gamma$ denotes the multiset of boxed formulas $[\Box\varphi : \varphi \in \Gamma]$. A sequent can be translated into a formula via the interpretation (where $\varphi_1 \& \ldots \& \varphi_n = \bar{0}$ for $n = 0$):

$$\mathcal{I}(\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_m) := (\varphi_1 \& \ldots \& \varphi_n) \to (\psi_1 \& \ldots \& \psi_m).$$

We then prove that $\Gamma \Rightarrow \Delta$ is derivable in GKT($\mathbb{R}$) if and only if $\mathcal{I}(\Gamma \Rightarrow \Delta)$ is derivable in KT($\mathbb{R}$). Completeness is then established via an intermediate labelled tableau calculus in which derivability is equivalent to KT($\mathbb{R}$)-validity. This tableau calculus reduces the problem of proving completeness to solving linear inequations over $\mathbb{R}$. We hence obtain the main result:

Theorem 1. *The following are equivalent for any formula φ:*

(1) φ *is* KT($\mathbb{R}$)-*valid.*

(2) φ *is derivable in* KT($\mathbb{R}$)

(3) $\Rightarrow \varphi$ *is derivable in* GKT($\mathbb{R}$).

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A duality for involutive bisemilattices

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It is a common trend in mathematics to study (natural) dualities for general algebraic structures and, in particular, for those arising from mathematical logic. The first step towards this direction traces back to the pioneering work by Stone for Boolean algebras [12]. Later on, Stone duality has been extended to the more general case of distributive lattices by Priestley [8], [9]. The two above mentioned are the prototypical examples of natural dualities and will be both recalled and constructively used in the present work.

A natural duality, in the sense of [2], is built using a schizophrenic object living in two different categories and has an intrinsic value: it is a way of describing the very same mathematical object from two different perspectives, the target category and its dual.

The starting point of our analysis is the duality established by Gierz and Romanowska [4] between distributive bisemilattices and compact totally disconnected partially ordered left normal bands with constants, which we refer to as GR spaces. Such duality is natural; however, its relevance mainly lies in the use of the technique of Płonka sums [6], [7], as an essential tool for proving the duality [11], [10].

Our aim is to provide a duality between the categories of involutive bisemilattices and certain topological spaces, here christened as GR spaces with involution. The former consists of a class of algebras introduced and extensively studied in [1] as algebraic semantics (although not equivalent[1]) for paraconsistent weak Kleene logic. Involutive bisemilattices are strictly connected to Boolean algebras as they are representable as Płonka sums of Boolean algebras.

The present work consists of two main results. On one hand, taking advantage of the Płonka sums representation in terms of Boolean algebras and Stone duality, we are able to describe the dual space of an involutive bisemilattice as a strongly inverse system of Stone spaces (the use of this terminology is borrowed from [5]). On the other hand, we generalize Gierz and Romanowska duality by considering GR spaces with involution as an additional operation: the duality cannot be constructed using the usual techniques for natural dualities. As a byproduct of our analysis we get a topological description of *strongly inverse systems* of Stone spaces.

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Finite MTL-algebras

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Abstract

We obtain a duality between the category of finite MTL-algebras and the category of finite Labeled Trees. In addition we proof that the forest product of MTL-algebras is essentialy a sheaf of MTL-chains over an Alexandrov space.

MTL-logic was introduced by Esteva and Godo in [5] as the basic fuzzy logic of left-continuous t-norms. Furthermore, a new class of algebras was defined, the variety of MTL-algebras. This variety constitutes an equivalent algebraic semantics for MTL-logic. MTL-algebras are essentially integral commutative residuated lattices with bottom satisfying the prelinearity equation:

$$(x \to y) \vee (y \to x) \approx 1$$

We call $f\mathcal{MTL}$ to the algebraic category of finite MTL-algebras.

A totally ordered MTL-algebra (MTL-chain) is archimedean if for every $x \leq y < 1$, there exists $n \in \mathbb{N}$ such that $y^n \leq x$.

A *forest* is a poset X such that for every $a \in X$ the set

$$\downarrow a = \{x \in X \mid x \leq a\}$$

is a totally ordered subset of X. A *p-morphism* is a morphism of posets $f : X \to Y$ satisfying the following property: given $x \in X$ and $y \in Y$ such that $y \leq f(x)$ there exists $z \in X$ such that $z \leq x$ and $f(z) = y$. Let $fa\mathcal{MTL}$ be the algebraic category of finite archimedean MTL-algebras and $fa\mathcal{MTL}c$ the full subcategory of finite archimedean MTL-chains. Let $\mathfrak{C}$ be its skeleton. A *labeled forest* is a function $l : F \to \mathfrak{C}$, where F is a forest. Consider two labeled forests $l : F \to \mathfrak{C}$ and $m : G \to \mathfrak{C}$. A morphism $l \to m$ is a pair $(\varphi, \mathcal{F})$ such that $\varphi : F \to G$ is a p-morphism and $\mathcal{F} = \{f_x\}_{x \in F}$ is a family of morphisms $f_x : m\varphi(x) \to l(x)$ of MTL algebras. We call $f\mathcal{LF}$ to the category of labeled forests and their morphisms.

Definition 1. Let $\mathbf{F} = (F, \leq)$ a forest and let $\{\mathbf{M}_i\}_{i \in \mathbf{F}}$ a collection of MTL-chains such that, up to isomorphism, all they share the same neutral element 1 and the same minimum element 0. If $(\bigcup_{i \in \mathbf{F}})^F$ denotes the set of functions $h : F \to \bigcup_{i \in \mathbf{F}} \mathbf{M}_i$ such that $h(i) \in \mathbf{M}_i$ for all $i \in \mathbf{F}$, the *forest product* $\bigotimes_{i \in \mathbf{F}} A_i$ is the algebra $\mathbf{M}$ defined as follows:

(1) The elements of $\mathbf{M}$ are the $h \in \left(\bigcup_{i \in \mathbf{F}} \mathbf{M}_i\right)^F$ such that, for all $i \in \mathbf{F}$ if $h(i) \neq 0_i$ then for all $j < i$, $h(j) = 1$.

(2) The monoid operation and the lattice operations are defined pointwise.

(3) The residual is defined as follows:

$$(h \to g)(i) = \begin{cases} h(i) \to_i g(i), & \text{if for all } j < i, \ h(j) \leq_j g(j) \\ \\ 0_i & \text{otherwise} \end{cases}$$

where de subscript $_i$ denotes the realization of operations and of order in $\mathbf{M}_i$.

In every poset $\mathbf{P}$ the collection $\mathcal{D}(\mathbf{P})$ of lower sets of $\mathbf{P}$ defines a topology over P called the *Alexandrov topology* on $\mathbf{P}$. Let $\mathbf{Shv}(\mathcal{D}(\mathbf{P}))$ the category of sheaves over $\mathcal{D}(\mathbf{P})$.

Lemma 1. Let $\mathbf{F}$ a forest and $\{\mathbf{M}_i\}_{i\in\mathbf{F}}$ a collection of MTL-chains as in Definition 1. Then, the assignment $\mathcal{P} : \mathcal{D}(\mathbf{F})^{op} \to \mathbf{Set}$, $\mathcal{P}(U) = \bigotimes_{i\in\mathbf{U}} A_i$ is a sheaf of MTL-algebras in $\mathbf{Shv}(\mathcal{D}(\mathbf{P}))$. Moreover, $\mathcal{P}$ is a sheaf of MTL-chains in $\mathbf{Shv}(\mathcal{D}(\mathbf{P}))$.

Let M be a finite MTL-algebra. A submultiplicative monoid F of M is called a filter if is an upset respect to the order of M. A filter F of M is *prime* if $0 \notin F$ and $x \vee y \in F$ entails $x \in F$ or $y \in F$, for every $x, y \in M$. The set of prime filters of a MTL-algebra M ordered by the inclusion will be noted as $\mathrm{Spec}(M)$. Let $\mathcal{I}(M)$ be the poset of idempotent elements of M; $\mathcal{J}(\mathcal{I}(M))^*$ the subposet of non zero join irreducible elements of $\mathcal{I}(M)$ and $m(M)$ the set of minimal elements of $\mathcal{J}(\mathcal{I}(M))^*$. Let $e \in \mathcal{J}(\mathcal{I}(M))^*$ and a_e the smallest $a \in m(M)$ such that $a \leq e$.

Lemma 2. For every finite MTL-algebra M the following holds:

i) $\mathcal{J}(\mathcal{I}(M))^*$ is a finite forest.

ii) For every $e \in \mathcal{J}(\mathcal{I}(M))^*$, $M/\uparrow a_e$ is a finite archimedean MTL-chain, so the function $l_M : \mathcal{J}(\mathcal{I}(M))^* \to \mathfrak{C}$ defined as $l_M(e) = M/\uparrow a_e$ becomes a finite labeled forest.

In this work we pretend to transform the results obtained in Lemmas 1 and 2 in functorial assignments in order to obtain a categorical equivalence between the categories $f\mathcal{LF}$ and $f\mathcal{MTL}$.

It is worth to mention that from the well known equivalence between the topos of sheaves over a topological space X and the topos of local homeos over X the Lemma 1 can be stated as: *The forest product of MTL algebras is isomorphic to the algebra of global sections of a bundle over an Alexandrov space.*

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A Loomis-Sikorski theorem and functional calculus for a generalized Hermitian algebra

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This contribution is based on the joint work with David J. Foulis and Anna Jenčová [6].

Generalized Hermitian (GH-) algebras, which were introduced in [9] incorporate several important algebraic and order theoretic structures including effect algebras [8], MV-algebras [4], orthomodular lattices [10], Boolean algebras [14], and Jordan algebras [12]. Apart from their intrinsic interest, all of the latter structures host mathematical models for quantum-mechanical notions such as observables, states, properties, and experimentally testable propositions [5, 15] and thus are pertinent in regard to the quantum-mechanical theory of measurement [2].

It turns out that GH-algebras are special cases of the more general synaptic algebras introduced in [7]. Thus, in this paper, it will be convenient for us to treat GH-algebras as special kinds of synaptic algebras . In most of the paper, we focus on commutative GH-algebras. A commutative GH-algebra A can be shown to be isomorphic to a lattice ordered Banach algebra $C(X, \mathbb{R})$, under pointwise operations and partial order, of all continuous real-valued functions on a basically disconnected compact Hausdorff space X.

As indicated by the title, one of our purposes in this paper is to formulate and prove an analogue for commutative GH-algebras of the classical Loomis-Sikorski representation theorem for Boolean σ-algebras [11, 14], and its extension for σ MV-algebras and Dedekind σ-complete ℓ-groups [1, 3, 13].

A real observable ξ for a physical system $\mathcal{S}$ is understood to be a quantity that can be experimentally measured, and that when measured yields a result in a specified set $\mathbb{R}_\xi$ of real numbers. A state ρ for $\mathcal{S}$ assigns to ξ an expectation, i.e., the long-run average value of a sequence of independent measurements of ξ in state ρ. If f is a function defined on $\mathbb{R}_\xi$, then $f(\xi)$ is defined to be the observable that is measured by measuring ξ to obtain, say, the result $\lambda \in \mathbb{R}_\xi$, and then regarding the result of this measurement of $f(\xi)$ to be $f(\lambda)$.

We use our Loomis-Sikorski theorem to show that each element a in a GH-algebra A corresponds to a real observable ξ_a. Moreover, we obtain an integral formula for the expectation of the observable ξ_a in state ρ,and we provide a continuous functional calculus for A.

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Gelfand duality for compact pospaces

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Let X be a compact Hausdorff space. It is well known that X can be characterized by its ring of real continuous functions, by its set of regular open subsets or more simply by its set of open subsets. These characterizations lead to dualities between the category **KHaus**, of compact Hausdorff space and respectively the categories **$C^\star$-alg** (or equivalently **ubal**), of commutative $C^\star$-algebras, **DeV** of de Vries algebras and **KRFrm** of compact regular frames. We thus get a square of dualities. (see [1], [2] and [6]).

Later, G.Bezhanishvili and J.Harding extended in [1] a part square to dualities between the categories **StKSp** of stably compact spaces, **RPrFrm** of regular proximity frames and **StKFrm** of stably compact frames.

We thus get the square of dualities extended this way.
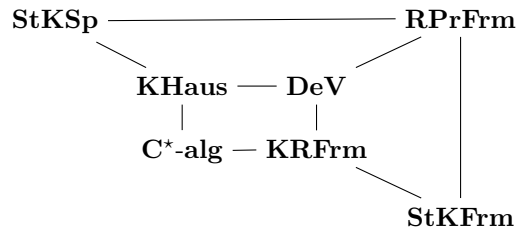
Our aim is to complete the outside triangle, looking for a category generalizing the $C^\star$-algebras.

Using the equivalences between **StKSp** and the category **KPSp** of compact po-spaces (see [4]), an essential fact, due to G.Hansoul in [5] leads us to consider a category of ordered semi-ring. Indeed, we can see that the Nachbin-Stone-Cech compactification of a completely regular ordered po-space X can be realized through its semi-ring of increasing, continuous and real, positive functions, denoted $I(X, \mathbb{R}^+)$.

Following the definitions of G.Bezhanishvili, P.Morandi and B.Olberding in [2], we define the bounded Archimedean ℓ-semi-algebras this way.

Definition 1. 1. An ℓ-semi-ring is an algebra $(A, +, ., 0, 1, \leq)$ with the following axioms :

    (a) $(A, +, 0)$ and $(A, ., 1)$ are commutative monoids.

    (b) $(A, +, .)$ is distributive.

    (c) $a \leq b \Leftrightarrow a + c \leq b + c$.

    (d) $a \geq 0$ and $a \leq b \Rightarrow a.c \leq b.c$

    (e) $(A, \leq)$ is a lattice.

2. An ℓ-semi-ring A is bounded if for all $a \in A$, there is $n \in \mathbb{N}$ such that $a \leq n.1$.

3. An ℓ-semi-ring A is Archimedean if for all $a, b, c, d \in A$, whenever $n.a + b \leq n.c + d$, then $a \leq c$.

4. An ℓ-semi-ring A is an ℓ-semi-algebra if it is an $\mathbb{R}^+$-algebra such that for all $a, b \in A$ and $r \in \mathbb{R}^+$, $r.a \leq r.b$.

5. $(a \vee b) + c = (a + c) \vee (b + c)$ and $(a \wedge b) + c = (a + c) \wedge (b + c)$.

We now denote **sbal** the category of bounded Archimedean ℓ-semi-algebras, and defining the morphisms in the natural way.

In order to get the missing duality, we define the $\sim$-relation on $A \times A$, with A an sbal, such as

$$(a, b) \sim (c, d) \Leftrightarrow a + d = b + c,$$

allowing us to construct the functor $\cdot^b : \textbf{sbal} \longrightarrow \textbf{bal}$ which sends A to $A \times A/\sim$. In particular, this functor enable us to easily transfer structures from rings to semi-rings.

With all these tools, we propose the following functors between **KPSp** and **sbal** : the first functor, denoted I, sends a compact po-space X to the set $I(X, \mathbb{R}^+)$ and a continuous increasing function $f : X \longrightarrow Y$ between compact po-spaces to

$$f_\star : I(Y, \mathbb{R}^+) \longrightarrow I(X, \mathbb{R}^+) : g \longmapsto g \circ f.$$

On the other side, the second functor, denoted χ, maps a sbal A to its set of ℓ-congruences, denoted X_A and a morphism $\alpha : A \longrightarrow B$ between sbals to

$$\alpha^\star : X_B \longrightarrow X_A$$

such that, if $\theta \in X_B$, $(a, b) \in \alpha^\star(\theta)$ if and only if $(\alpha(a), \alpha(b)) \in \theta$.

Definition 2. An ℓ-semi-ring A admits difference with constants if for all $a \in A$ and $r \in \mathbb{R}^+$ $a \leq r.1$ implies there is $b \in A$ such that $a = b + r.1$. It is uniformly complete if it is complete for the norm $||a|| = \inf\{\lambda \in \mathbb{R} : a \leq \lambda.1\}$. We then denote **usbal** the full subcategory of **sbal** whose objects are the uniformly complete bounded Archimedean ℓ-semi-algebras with difference with constants.

Theorem 3. *The functors χ and I establish a dual equivalence between* **usbal** *and* **KPSp**

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Algorithmic Correspondence, Canonicity and Completeness for Possibility Semantics

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Possibility semantics.   Possibility semantics for modal logic is a generalization of standard Kripke semantics. In this semantics, a possibility frame has a refinement relation which is a partial order between states, in addition to the accessibility relation for modalities. From an algebraic perspective, full possibility frames are dually equivalent to complete Boolean algebras with complete operators which are not necessarily atomic, while filter-descriptive possibility frames are dually equivalent to Boolean algebras with operators.

In recent years, the theoretic study of possibility semantics has received more attention. In [23], Yamamoto investigates the correspondence theory in possibility semantics in a frame-theoretic way and prove a Sahlqvist-type correspondence theorem over full possibility frames, which are the possibility semantic counterpart of Kripke frames, using insights from the algebraic understanding of possibility semantics. In [15, Theorem 7.20], it is shown that all inductive formulas are filter-canonical and hence every normal modal logic axiomatized by inductive formulas is sound and complete with respect to its canonical full possibility frame. However, the correspondence result for inductive formulas is still missing, as well as the correspondence result over filter-descriptive possibility frames (see [15, page 103]) and soundness and completeness with respect to the corresponding elementary class of full possibility frames. The present paper aims at giving a closer look at the aforementioned unsolved problems using the algebraic and order-theoretic insights from a current ongoing research project, namely *unified correspondence*.

Unified correspondence.   Correspondence and completeness theory have a long history in modal logic, and they are referred to as the "three pillars of wisdom supporting the edifice of modal logic" [22, page 331] together with duality theory. Dating back to [20, 21], the Sahlqvist theorem gives a syntactic definition of a class of modal formulas, the *Sahlqvist class*, each member of which defines an elementary (i.e. first-order definable) class of Kripke frames and is canonical.

Recently, a uniform and modular theory which subsumes the above results and extends them to logics with a *non-classical* propositional base has emerged, and has been dubbed *unified correspondence* [5]. It is built on duality-theoretic insights [9] and uniformly exports the state-of-the-art in Sahlqvist theory from normal modal logic to a wide range of logics which include, among others, intuitionistic and distributive and general (non-distributive) lattice-based (modal) logics [6, 8], non-normal (regular) modal logics based on distributive lattices of arbitrary modal signature [19], hybrid logics [12], many valued logics [16] and bi-intuitionistic and lattice-based modal mu-calculus [1, 3, 2].

The breadth of this work has stimulated many and varied applications. Some are closely related to the core concerns of the theory itself, such as understanding the relationship between different methodologies for obtaining canonicity results [18, 7], the phenomenon of pseudocorrespondence [10], and the investigation of the extent to which the Sahlqvist theory of classes of

normal distributive lattice expansions can be reduced to the Sahlqvist theory of normal Boolean algebra expansions, by means of Gödel-type translations [11]. Other, possibly surprising applications include the dual characterizations of classes of finite lattices [13], the identification of the syntactic shape of axioms which can be translated into structural rules of a proper display calculus [14] and of internal Gentzen calculi for the logics of strict implication [17], and the epistemic interpretation of lattice-based modal logic in terms of categorization theory in management science [4]. These and other results (cf. [9]) form the body of a theory called unified correspondence [5], a framework within which correspondence results can be formulated and proved abstracting away from specific logical signatures, using only the order-theoretic properties of the algebraic interpretations of logical connectives.

Methodology. Our contribution is methodological: we analyze the correspondence phenomenon in possibility semantics using the dual algebraic structures, namely complete (not necessarily atomic) Boolean algebras with complete operators, where the atoms are not always available. For the correspondence over full possibility frames, our strategy is to identify two different Boolean algebras with operators as the dual algebraic structures of the possibility frame, namely the Boolean algebra of regular open subsets $\mathbb{B}_{\mathsf{RO}}$ (when viewing the possibility frame as a possibility frame itself) and the Boolean algebra of arbitrary subsets $\mathbb{B}_{\mathsf{Full}}$ (when viewing the possibility frame as a bimodal Kripke frame), where a canonical order-embedding map $e : \mathbb{B}_{\mathsf{RO}} \to \mathbb{B}_{\mathsf{Full}}$ can be defined. The embedding e preserves arbitrary meets, therefore a left adjoint $c : \mathbb{B}_{\mathsf{Full}} \to \mathbb{B}_{\mathsf{RO}}$ of e can be defined, which sends a subset X of the domain W of possibilities to the smallest regular open subset containing X. This left adjoint c plays an important role in the dual characterization of the interpretations of the expanded language, which form the ground of the regular open translation, i.e. the counterpart of standard translation in possibility semantics. When it comes to canonicity, we use the fact that filter-canonicity is equivalent to constructive canonicity [15, Theorem 5.46, 7.20], and prove a topological Ackermann lemma, which justifies the soundness of propositional variable elimination rules and forms the basis of the correspondence result with respect to the class of filter-descriptive frames as well as the canonicity and completeness result with respect to the corresponding class of full possibility frames.

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On Two Approaches to Concrete Dualities and Their Relationships

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There are two categorical approaches to the unification of the dualities between various kinds of algebraic structures and of topological structures. The present study investigates how these essentially different approaches are related, and applies the presented results to the categories of certain types of universal algebras (including infinitary operations).

1 Introduction

Duality of algebraic structures (e.g., Boolean algebras, distributive lattices and Heyting algebras) with topological structures (e.g., Stone spaces, Priestly spaces and Heyting spaces) has been a major issue in topology, algebra and logic [1, 2, 6, 8]. There are two general and categorical approaches [2, 8] to the duality issue: The first approach operates with concrete categories $\mathbf{C}$ and $\mathbf{D}$ over the category $\mathbf{Set}$ of sets and functions. Schizophrenic object is the key concept of this approach [8] defined as a triple $\left(\widetilde{C}, s, \widetilde{D}\right)$ provided that $\widetilde{C}$ is a $\mathbf{C}$-object, $\widetilde{D}$ is a $\mathbf{D}$-object, s is a bijective function from the underlying set of $\widetilde{C}$ to the underlying set of $\widetilde{D}$, and two additional conditions are satisfied. Such a schizophrenic object determines an adjoint situation

$$(\gamma, \alpha) : S \dashv T : \mathbf{C}^{op} \to \mathbf{D}. \tag{1}$$

If we consider the full subcategory $Fix(\alpha)$ of $\mathbf{C}$ with those $\mathbf{C}$-objects A for which the Ath component α_A of α is an isomorphism in $\mathbf{C}^{op}$, and similarly, the full subcategory $Fix(\gamma)$ of $\mathbf{D}$ with respect to γ, then the adjoint situation (1) restricts to a duality between $Fix(\alpha)$ and $Fix(\gamma)$, which is the main result of the first approach describing many existing dualities, e.g., Stone, Priestley and localic dualities.

In the second approach [2], $\mathbf{C}$ is taken as an abstract category, which is not necessarily a concrete category over $\mathbf{Set}$. As a formulation of fixed-basis fuzzy topological spaces in the category $\mathbf{C}$ with set-indexed products, $\mathbf{C}$-$\mathcal{M}$-L-spaces are defined in this approach to be pairs $\left(X, \tau \xrightarrow{m} L^X\right)$ consisting of a set X and an $\mathcal{M}$-morphism $\tau \xrightarrow{m} L^X \in \mathcal{M}$, where L is an arbitrarily fixed object of $\mathbf{C}$, L^X is an Xth power of L and $\mathcal{M}$ is a class of $\mathbf{C}$-monomorphisms.

$\mathbf{C}$-$\mathcal{M}$-L-spaces and $\mathbf{C}$-$\mathcal{M}$-L-continuous functions form a category $\mathbf{C}$-$\mathcal{M}$-L-$\mathbf{Top}$, which, under the assumption of $\mathbf{C}$ being essentially $(\mathcal{E}, \mathcal{M})$-structured, relates to $\mathbf{C}$ with the adjoint situation

$$(\eta, \varepsilon) : L\Omega_{\mathcal{M}} \dashv LPt_{\mathcal{M}} : \mathbf{C}^{op} \to \mathbf{C}\text{-}\mathcal{M}\text{-}L\text{-}\mathbf{Top}.$$

This adjoint situation gives rise to a duality between the full subcategory $\mathbf{SPA}(\mathbf{C})$ of $\mathbf{C}$ with L-spatial objects and the full subcategory $\mathbf{SOBTop}(\mathbf{C})$ of $\mathbf{C}$-$\mathcal{M}$-L-$\mathbf{Top}$ with L-sober objects, where L-spatiality of a $\mathbf{C}$-object A means $\varepsilon_A \in Iso\left(\mathbf{C}^{op}\right)$ and L-sobriety of a $\mathbf{C}$-$\mathcal{M}$-L-space W refers to $\eta_W \in Iso\left(\mathbf{C}\text{-}\mathcal{M}\text{-}L\text{-}\mathbf{Top}\right)$. The equivalence $\mathbf{SPA}(\mathbf{C})^{op} \sim \mathbf{SOBTop}(\mathbf{C})$ is the central

result of the second approach, named as "Fundamental Categorical Duality Theorem", and produces many existing and new dualities [2, 3, 4, 5].

2   Relations Between Two Approaches

As a comparison of the two approaches, the former one is more familiar, and has been utilized by several authors [1, 6, 7], while the latter one has been used only by this author. Although the two approaches are primarily different from each other, we aim in this study to ascertain how they are interrelated. We will particularly show that for categories $\mathbf{C}$ and $\mathbf{D}$ with the properties fulfilling the requirements in both approaches, there exists an adjunction

$$S^* \dashv T^* : \mathbf{SOBTop}\,(\mathbf{C}) \to \mathbf{D},$$

and be interested in the situation whenever this adjunction turns into an equivalence. We also wish to give applications of the presented results to the categories of certain types of universal algebras (possibly with infinitary operations), e.g., the category of sup-lattices with morphisms all sup-preserving maps [9].

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Q-sup-algebras and their representation

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The topic of sets with fuzzy order relations valued in complete lattices with additional structure has been quite active in the recent decade, and a number of papers have been published (see [3, 5, 6] among many others).

Based on a quantale-valued order relation and subset membership, counterparts to common order-theoretic notions can be defined, like monotone mappings, adjunctions, joins and meets, complete lattices, or join-preserving mappings, and one can consider a category formed from the latter two concepts. An attempt for systematic study of such categories of fuzzy complete lattices with quantale valuation ("Q-sup-lattices") with fuzzy join-preserving mappings has been made by the second author in his recent paper [8].

With some theory of Q-sup-lattices available, new concepts of algebraic structures in this category can easily be built. In this paper, we shall deal with general algebras with finitary operations, building on existing results obtained for algebras based on crisp sup-lattices ('sup algebras' as in [1, 7]). We can see [9] that our fuzzy structures behave in strong analogy to their crisp counterparts.

We also highlight an important fact: that concepts based on a fuzzy order relation (in the sense of the quantale valuation as studied in this work) *should not be treated as generalizations of their crisp variants* – they are rather standard crisp concepts of order theory, satisfying certain additional properties. This fact also reduces the work needed to carry out proofs. Thus, even with the additional properties imposed, the theory of fuzzy-ordered structures develops consistently with its crisp counterpart.

The connection between fuzzy and crisp order concepts has also been justified by I. Stubbe in a general categorial setting of modules over quantaloids [4], and in the recent work of S. A. Solovyov in the quantale-fuzzy setting [3] where categories of quantale-valued sup-lattices are proved to be isomorphic to well-investigated categories of quantale modules. This isomorphism will enable us to make direct transfer of some of the fundamental constructions and properties known for quantale modules, to our framework. The bridge between these two worlds allows us to open a space for surprising interpretations.

With this paper we hope to contribute to the theory of quantales and quantale-like structures. It considers the notion of Q-sup-algebra and shows a representation theorem for such structures generalizing the well-known representation theorems for quantales, sup-algebras and quantale algebras [2].

Theorem 1. *If $(A, \bigsqcup_A, \Omega)$ is a Q-sup-algebra, then*

1. *Q^A can be equipped with a structure of a Q-sup-algebra.*

2. *There is a nucleus j on Q^A such that $A \cong Q_j^A$.*

In addition, we present some important properties of the category of *Q*-sup-algebras.

Theorem 2. *The category of Q-sup-algebras is a monadic construct.*

Corollary 3. *The category of Q-sup-algebras is complete, cocomplete, wellpowered, extremally co-wellpowered, and has regular factorizations. Moreover, monomorphisms are precisely those morphisms that are injective functions.*

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Bicategory of Theories as an Approach to Model Theory

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First-order categorical logic (FOCL for short) originated as a categorical foundation for model theory (in Makkai & Reyes [6]). Some classical model-theoretic phenomena can be efficiently described in terms of FOCL. However, most concepts in modern model theory remain to be under categorical consideration. Our present aim is to set up a framework suitable for comprehensive categorical analysis of model theory. In this talk, we work on the notion of "category of theories," whose importance will be discussed below.

From the viewpoint of FOCL, classical first-order theories give rise to *Boolean pretoposes*, i.e. categories equipped with logical operations and quotients of equivalence relations. They are called *classifying pretoposes* of theories. As Harnik [5] pointed out, a construction of classifying pretoposes can be given via Shelah's eq-construction. Moreover, any Boolean pretopos arises (up to categorical equivalence) as a classifying pretopos of some classical theory.

Since mathematical objects often constitute a category, it is natural to ask what morphisms between theories are. It has been observed that *interpretations* between theories (in the model-theoretic sense) induce pretopos functors, i.e. functors preserving pretopos structures, between corresponding classifying pretoposes. So, among categorical logicians, there exists a common sense that

the (2-)category $\mathfrak{BPretop}_*$ of Boolean pretoposes, pretopos functors
and natural isomorphisms can be regarded as a "category of theories,"

while no purely syntactic definition of "category of theories" is widely accepted.

Our approach is as follows: once we have defined *homotopies* between interpretations, we obtain a *bicategory* $\mathfrak{Th}$ which consists of theories, interpretations and homotopies. We show that the construction of classifying pretoposes gives a pseudofunctor $\mathfrak{Th} \to \mathfrak{BPretop}_*$. In fact, it is a biequivalence, and hence our definition of $\mathfrak{Th}$ is consistent with the above consensus.

We also make a close observation on (internal) equivalences in $\mathfrak{Th}$. In the model-theoretic context, these equivalences are called *bi-interpretations*. Via the biequivalence above, existence of a bi-interpretation between two theories coincides with *Morita equivalence*, i.e. categorical equivalence between corresponding classifying pretoposes. We give another characterization of bi-interpretability (and Morita equivalence) by using the notion of *Morita extension*, recently introduced by Barrett & Halvorson [2], which is a slight generalization of definitional extension admitting sort definitions. We also give a simple proof for Tsementzis' syntactic characterization of Morita equivalence [7].

Future directions. We believe that this framework will promote more extensive uses of categories in modern model theory. We indicate the following lines of research:

- Using preceding works on dualities in first-order logic (e.g. Caramello's [3] and Forssell's [4, 1]), we will consider relationships between various mathematical objects associated with theories. Examples of such mathematical objects include classifying (pre)toposes, categories of models and topological groupoids of models and isomorphisms.

- Certain model-theoretic constructions of theories, e.g. elementary diagrams and nonforking extensions of types, can be interpreted as categorical constructions in the bicategory

$\mathfrak{Th}$ of theories or other related 2-categories. Moreover, we also expect that category theory will give new constructions of theories.

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Multiplicative derivations of commutative residuated lattices

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1 Introduction

For a theory of algebras with two operations $+$ and $\cdot$, we have an interesting method, *derivations*, to develop the structure theory, as an analogy of derivations of analysis. The notion of derivation of algebras was firstly applied to the theory of ring ([4]), and after that it was also applied to other algebras, such as lattices ([2, 6]) and MV-algebras ([1, 7]). We here aplied the derivation theory to the (commutative) residuated lattices which are very basic algebtras corresponding to fuzzy logic. For a residuated lattice L, a map $d : L \to L$ is called a derivation in [3] if it satisfies the condition: For all $x, y \in L$,

$$d(x \odot y) = (dx \odot y) \vee (x \odot dy).$$

Let L be a commutative residuated lattice and d be a good ideal derivation and F a d-filter of L, which are defined later. We show that

(1) The set $\mathrm{Fix}_d(L)$ of all fixed points of d forms a residuated lattice and $L/\ker d \cong \mathrm{Fix}_d(L)$.

(2) A map $d/F : L/F \to L/F$ defined by $(d/F)(x/F) = dx/F$ is also a good ideal derivation of L/F.

(3) The quotient residuated lattices $\mathrm{Fix}_{d/F}(L/F)$ and $\mathrm{Fix}_d(L)/d(F)$ are isomorphic, namely,

$$\mathrm{Fix}_{d/F}(L/F) \cong \mathrm{Fix}_d(L)/d(F).$$

2 Derivations of residuated lattices

Let $L = (L, \wedge, vee, \to 0, 1)$ be a (commutative) residuated lattice and $B(L)$ be the set of all complemented elements of L. We define derivations of residuated lattices according to [3]. A map $d : L \to L$ is called a *multiplicative derivation* (or simply *derivation*) of L if it satisfies the condition

$$d(x \wedge y) = (dx \odot y) \vee (x \odot dy) \quad (\forall x, y \in L)$$

A derivation d is called *ideal* if $x \leq y$ then $dx \leq dy$ and $dx \leq x$ for all $x, y \in L$. Moreover, a derivation d is said to be *good* if $d1 \in B(L)$.

Theorem 1 ([3]). *Let d be a derivation of L and $d1 \in B(L)$. Then the following are equivalent: for all $x, y \in L$,*

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(1) d *is an ideal derivation;*

(2) $dx \leq d1$*;*

(3) $dx = x \odot d1$*;*

(4) $d(x \wedge y) = dx \wedge dy$*;*

(5) $d(x \vee y) = dx \vee dy$*;*

(6) $d(x \odot y) = dx \odot dy$*.*

For a derivation d of L, we consider a subset $\mathrm{Fix}_d(L) = \{x \in L \,|\, dx = x\}$ of the set of all fixed elements of L for d.

Proposition 1. *For a good ideal derivation d, we have $\mathrm{Fix}_d(L) = d(L)$.*

We have

Theorem 2. $\mathrm{Fix}_d(L) = (\mathrm{Fix}_d(L), \wedge, \vee, \odot, \mapsto, 0, d1)$ *is a residuated lattice, where operations on* $\mathrm{Fix}_d(L)$ *are defined as follows:*

$$dx \wedge dy = d(x \wedge y) \qquad\qquad dx \vee dy = d(x \vee y)$$
$$dx \odot dy = d(x \odot y) \qquad\qquad dx \mapsto dy = d(dx \rightarrow dy).$$

A filter F is called a *d-filter* if $x \in F$ implies $dx \in F$ for all $x \in L$. It is easy to show that a quotient structure L/F is also a residuated lattice for a filter F. Moreover, we have the following.

Proposition 2. *Let d be a good ideal derivation and F be a d-filter of L. A map $d/F : L/F \rightarrow L/F$ defined by $(d/F)(x/F) = dx/F$ for all $x/F \in L/F$ is a good ideal derivation of L/F.*

Therefore, the quotient structure $(d/F)(L/F) = \mathrm{Fix}_{d/F}(L/F)$ is a residuated lattice. Since F is a d-filter of L, $d(F)$ is also a filter of $d(L)$ and thus $d(L)/d(F)$ forms a residuated lattice. It is natural to ask what the relation between two residuated lattices $(d/F)(L/F)$ and $d(L)/d(F)$ is. Next result is an answer.

Theorem 3. *Let d be a good ideal derivation and F be a d-filter of L. Then we have $(d/F)(L/F) = \mathrm{Fix}_{d/F}(L/F)$ is isomorphic to $d(L)/d(F)$, that is,*

$$(d/F)(L/F) = \mathrm{Fix}_{d/F}(L/F) \cong d(L)/d(F).$$

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An Ordering Condition for Groups

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Ordering conditions for groups provide useful tools for the study of various relationships between group theory, universal algebra, and topology (see, e.g., [2, 4, 3, 1]). In this work, we establish a new "algorithmic" ordering condition for extending partial orders on groups to total orders. We then use this condition to show that the problem of extending a finite subset of a free group to a total order corresponds to the problem of checking validity of a certain inequation in the variety of representable lattice-ordered groups (or, equivalently, the class of totally ordered groups). As a direct consequence, we obtain a new proof that free groups are orderable.

Let us fix a group $\mathbf{G} = \langle G, \cdot, ^{-1}, e \rangle$. Recall that a *partial order* of $\mathbf{G}$ is a partial order $\leq$ on G satisfying also for $a, b, c, d \in G$,

$$a \leq b \implies cad \leq cbd.$$

Its positive cone $P_{\leq} = \{a \in G : e < a\}$ is a normal subsemigroup of $\mathbf{G}$ (a subset of G closed under $\cdot$ and conjugation by elements of G) that omits e. Conversely, if P is a normal subsemigroup of $\mathbf{G}$ omitting e, then $\mathbf{G}$ is partially ordered by

$$a \leq^P b \iff ba^{-1} \in P \cup \{e\}.$$

Hence partial orders of $\mathbf{G}$ can be identified with normal subsemigroups of $\mathbf{G}$ not containing e. For $S \subseteq G$, the normal subsemigroup of $\mathbf{G}$ generated by S, denoted by $\langle\langle S \rangle\rangle$, is a partial order of $\mathbf{G}$ if and only if $e \notin \langle\langle S \rangle\rangle$. A partial order $\leq$ of $\mathbf{G}$ is a *(total) order* if $G = P_{\leq} \cup P_{\leq}^{-1} \cup \{e\}$.

Now, for finite subsets $S \subseteq G$, we define a relation $\vdash_{\mathbf{G}} S$ inductively by the clauses

(i) $\vdash_{\mathbf{G}} S \cup \{a, a^{-1}\}$;

(ii) $\vdash_{\mathbf{G}} S \cup \{ab\}$, whenever $\vdash_{\mathbf{G}} S \cup \{a\}$ and $\vdash_{\mathbf{G}} S \cup \{b\}$;

(iii) $\vdash_{\mathbf{G}} S \cup \{ab\}$, whenever $\vdash_{\mathbf{G}} S \cup \{ba\}$.

The following theorem describes our new condition for extending a finite subset of $\mathbf{G}$ to an order, noting that the equivalence of (1) and (2) is a reformulation of an ordering theorem for groups due to Fuchs [2].

Theorem 1. *The following are equivalent for a finite $S \subseteq G$:*

(1) *S does not extend to a total order of $\mathbf{G}$.*

(2) *There exist $a_1, \ldots, a_m \in G \setminus \{e\}$ such that for all $\delta_1, \ldots, \delta_m \in \{-1, 1\}$,*

$$e \in \langle\langle S \cup \{a_1^{\delta_1}, \ldots, a_m^{\delta_m}\}\rangle\rangle.$$

(3) $\vdash_{\mathbf{G}} S$.

We now consider a non-trivial free group $\mathbf{F}$, which may be viewed as an algebra of reduced group terms obtained by cancelling all the occurrences of xx^{-1} and $x^{-1}x$. For convenience, we deliberately confuse group terms t with their counterparts in $\mathbf{F}$. We consider also the variety $\mathcal{RG}$ of representable lattice-ordered groups (in an algebraic language with operations $\wedge, \vee, \cdot, ^{-1}, e$) generated by the class of totally ordered groups. Using Theorem 1, we then obtain the following correspondence between extending a finite subset of $\mathbf{F}$ to an order and the validity of a corresponding inequation in $\mathcal{RG}$.

Theorem 2. *The following are equivalent for any $t_1, \ldots, t_n \in F$:*

(1) $\{t_1, \ldots, t_n\}$ *does not extend to a total order of* $\mathbf{F}$.

(2) $\vdash_{\mathbf{F}} \{t_1, \ldots, t_n\}$.

(3) $\mathcal{RG} \models e \leq t_1 \vee \ldots \vee t_n$.

This result is then used to obtain a new proof of the orderability of free groups, first proved in [5]. In fact, it is sufficient to observe that $\mathcal{RG} \not\models e \leq x$ for any generator x, and hence, by Theorem 2, there exists an order of $\mathbf{F}$ where x is positive.

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Sahlqvist via Translation

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Unified correspondence. In recent years, Sahlqvist theory has significantly broadened its scope, extending the benefits it originally imparted to modal logic to a wide range of logics which includes, among others, intuitionistic and distributive and general (non-distributive) lattice-based (modal) logics [6, 8], non-normal (regular) modal logics based on distributive lattices of arbitrary modal signature [18], hybrid logics [12], many valued logics [15] and bi-intuitionistic and lattice-based modal mu-calculus [1, 3, 2].

The breadth of this work has stimulated many and varied applications. Some are closely related to the core concerns of the theory itself, such as understanding the relationship between different methodologies for obtaining canonicity results [17, 7], and the phenomenon of pseudo-correspondence [10]. Other, possibly surprising applications include the dual characterizations of classes of finite lattices [13], the identification of the syntactic shape of axioms which can be translated into structural rules of a proper display calculus [14] and of internal Gentzen calculi for the logics of strict implication [16], and the epistemic interpretation of lattice-based modal logic in terms of categorization theory in management science [4]. These and other results (cf. [9]) form the body of a theory called unified correspondence [5], a framework within which correspondence results can be formulated and proved abstracting away from specific logical signatures, using only the order-theoretic properties of the algebraic interpretations of logical connectives.

Focus of the present talk. Notwithstanding the new insights and the connections with various areas of logic brought about by these developments, a natural question to ask is whether, just for the sake of Sahlqvist theory, it is possible to obtain Sahlqvist-type results for nonclassical logics by means of a reduction to a setting of normal modal logic via some suitable translations, such as the Gödel-Tarski. The present talk reports on the results of [11], in which this question is investigated for logics algebraically captured by normal distributive lattice expansions (DLEs). Our conclusions are that, while the most general Sahlqvist-type *correspondence* result for DLE-inequalities can indeed be obtained straightforwardly via translation, the proof of *canonicity* can be obtained as straightforwardly only in the special setting of normal *bi-Heyting algebra* expansions.

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Constructive canonicity for lattice-based fixed point logics

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The present contribution lies at the crossroads of at least three active lines of research in nonclassical logics: the one investigating the semantic and proof-theoretic environment of fixed point expansions of logics algebraically captured by varieties of (distributive) lattice expansions [1, 19, 24, 2, 16]; the one investigating constructive canonicity for intuitionistic and substructural logics [17, 25]; the one uniformly extending the state-of-the-art in Sahlqvist theory to families of nonclassical logics, and applying it to issues both semantic and proof-theoretic [7], known as 'unified correspondence'.

We prove the algorithmic canonicity of two classes of μ-inequalities in a constructive metatheory of normal lattice expansions. This result simultaneously generalizes Conradie and Craig's canonicity results for μ-inequalities based on a bi-intuitionistic bi-modal language [3], and Conradie and Palmigiano's constructive canonicity for inductive inequalities [4] (restricted to normal lattice expansions). Besides the greater generality, the unification of these strands smoothes the existing proofs for the canonicity of μ-formulas and inequalities. Specifically, the two canonicity results proven in [3], namely, the tame and proper canonicity, fully generalize to the constructive setting and normal LEs. Remarkably, the rules of the algorithm ALBA used for this result have exactly the same formulation as those of [4], with no additional rule added specifically to handle the fixed point binders. Rather, fixed points are accounted for by certain restrictions on the application of the rules, concerning the order-theoretic properties of the term functions associated with the formulas to which the rules are applied.

The contributions reported on in the proposed talk pertain to unified correspondence theory [7], a line of research which applies duality-theoretic insights to Sahlqvist theory (cf. [11]), with the aim of uniformly extending the benefits of Sahlqvist theory from modal logic to a wide range of logics which include, among others, intuitionistic and distributive and general (non-distributive) lattice-based (modal) logics [8, 10], non-normal (regular) modal logics based on distributive lattices of arbitrary modal signature [23], hybrid logics [14], many valued logics [20] and bi-intuitionistic and lattice-based modal mu-calculus [3, 5].

The breadth of this work has stimulated many and varied applications. Some are closely related to the core concerns of the theory itself, such as understanding the relationship between different methodologies for obtaining canonicity results [22, 9], the phenomenon of pseudo-correspondence [12], and the investigation of the extent to which the Sahlqvist theory of classes of normal distributive lattice expansions can be reduced to the Sahlqvist theory of normal Boolean algebra expansions, by means of Gödel-type translations [13]. Other, possibly surprising applications include the dual characterizations of classes of finite lattices [15], the identification of the syntactic shape of axioms which can be translated into structural rules of a proper display calculus [18] and of internal Gentzen calculi for the logics of strict implication [21], and the epistemic interpretation of lattice-based modal logic in terms of categorization

theory in management science [6]. These and other results (cf. [11]) form the body of a theory called unified correspondence [7], a framework within which correspondence results can be formulated and proved abstracting away from specific logical signatures, using only the order-theoretic properties of the algebraic interpretations of logical connectives.

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Analogies between small scale topology and large scale topology from the nonstandard perspective

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Large scale topology is the study of the large scale (asymptotic) behaviour of various spaces. It is well-known that there are many analogies between small scale topology and large scale topology. Our contribution is to study these analogies in the light of nonstandard analysis.

Let $\mathbb{U}$ be a transitive universe that satisfies sufficiently many axioms of ZFC and has all standard objects we need. We fix an enlargement $*\colon \mathbb{U} \hookrightarrow {}^{*}\mathbb{U}$ of $\mathbb{U}$. A formula is said to be Π_1^{st} if it is of the form $\forall x \in \mathbb{U}.\varphi\,(x, \vec{a})$, where φ is a $\in$-formula and $\vec{a}$ is parameters from $\mathbb{U}$. A formula is said to be Σ_1^{st} if it is of the form $\exists x \in \mathbb{U}.\varphi\,(x, \vec{a})$, where φ and $\vec{a}$ are the same as above.

Let X be a topological space with a topology $\mathcal{O}_X$. The *monad* of $x \in X$ is the Π_1^{st}-set $\mu_X\,(x) := \bigcap_{x \in U \in \mathcal{O}_X} {}^{*}U$. The monad map $\mu_X \colon X \to \mathcal{P}\,({}^{*}X)$ uniquely determines the topology $\mathcal{O}_X$. Next, let X be a uniform space with a uniformity $\mathcal{U}_X$. The *infinite closeness relation* on ${}^{*}X$ is the Π_1^{st}-equivalence relation defined by $\approx_X := \bigcap_{E \in \mathcal{U}_X} {}^{*}E$. Like topological spaces, the infinite closeness relation $\approx_X$ uniquely determines the uniformity $\mathcal{U}_X$ ([1]). Thus we can consider small scale topology as the study of Π_1^{st}-sets.

Let X be a bornological space with a bornology $\mathcal{B}_X$. In our setting, a bornology on X is defined to be a nonempty cover of X that is closed under taking subsets and finite *nondisjoint* unions. Bornology is a minimal framework in which we can discuss boundedness. For more details, see Hogbe-Nlend [2]. The *galaxy* of $x \in X$ is defined as the Σ_1^{st}-set $G_X\,(x) := \bigcup_{x \in B \in \mathcal{B}_X} {}^{*}B$. We show that the galaxy map $G_X \colon X \to \mathcal{P}\,({}^{*}X)$ uniquely determines the bornology $\mathcal{B}_X$. Next, let X be a coarse space with a coarse structure $\mathcal{E}_X$. The *finite closeness relation* on ${}^{*}X$ is defined as the Σ_1^{st}-equivalence relation $\sim_X := \bigcup_{E \in \mathcal{E}_X} {}^{*}E$. We show that the finite closeness relation $\sim_X$ uniquely determines the coarse structure $\mathcal{E}_X$. Similarly to small scale, we can think of large scale topology as the study of Σ_1^{st}-sets. In this sense, large scale topology is the logical dual of small scale topology.

Many small scale concepts topology have nonstandard characterisations in terms of monad and infinite closeness (see Robinson [3] and Stroyan and Luxemburg [4]). For example,

- a map $f \colon X \to Y$ between topological spaces is continuous at $x \in X$ if and only if ${}^{*}f\,(\mu_X\,(x)) \subseteq \mu_Y\,(f\,(x))$;

- a map $f \colon X \to Y$ between uniform spaces is uniformly continuous if and only if for every $x, y \in {}^{*}X$, if $x \approx_X y$, then ${}^{*}f\,(x) \approx_Y {}^{*}f\,(y)$;

- a family $\mathcal{F}$ of maps between uniform spaces X, Y is uniformly equicontinuous if and only if for any $f \in {}^{*}\mathcal{F}$ and $x, y \in {}^{*}X$, if $x \approx_X y$, then $f\,(x) \approx_Y f\,(y)$.

As the large scale analogues, we obtain the following nonstandard characterisations of large scale concepts in terms of galaxy and finite closeness:

- a map $f \colon X \to Y$ between bornological spaces is bornological at $x \in X$ if and only if ${}^{*}f\,(G_X\,(x)) \subseteq G_Y\,(f\,(x))$;

- a map $f\colon X \to Y$ between coarse spaces is bornologous if and only if for every $x, y \in {}^{*}X$, if $x \sim_X y$, then ${}^{*}f(x) \sim_Y {}^{*}f(y)$;

- a family $\mathcal{F}$ of maps between coarse spaces X, Y is uniformly equibounded if and only if for any $f \in {}^{*}\mathcal{F}$ and $x, y \in {}^{*}X$, if $x \sim_X y$, then $f(x) \sim_Y f(y)$.

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$\mathbf{Alt}_n$ in a Strictly Positive Context

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A *strictly positive term* (or *$\mathcal{SP}$-term*) is a modal formula constructed from propositional variables $p_0, p_1, \ldots$, constants $\top$ and $\bot$, conjunction $\wedge$, and the unary diamond operator $\diamond$. An *$\mathcal{SP}$-implication* takes the form $\sigma \to \tau$, where σ, τ are $\mathcal{SP}$-terms, and an *$\mathcal{SP}$-logic* is a set of $\mathcal{SP}$-implications. (An $\mathcal{SP}$-implication $\sigma \to \tau$ can be regarded as an algebraic equation $\sigma \wedge \tau \equiv \sigma$, while $\sigma \equiv \tau$ as a shorthand for '$\sigma \to \tau$ and $\tau \to \sigma$'.) In various contexts, $\mathcal{SP}$-logics were investigated in [3, 7, 2, 1, 8, 6, 5, 4].

We consider two consequence relations. For an $\mathcal{SP}$-logic $\mathcal{L}$ and $\mathcal{SP}$-implication φ, we write $\mathcal{L} \models_{\mathsf{Kr}} \varphi$ if φ is valid in all Kripke frames for $\mathcal{L}$, and we write $\mathcal{L} \models_{\mathsf{SLO}} \varphi$ if φ is valid in all *bounded meet-semilattices with normal monotone operators* (or SLOs) that validate $\mathcal{L}$. We call $\mathcal{L}$ (*Kripke*) *complete* in case $\mathcal{L} \models_{\mathsf{Kr}} \varphi$ iff $\mathcal{L} \models_{\mathsf{SLO}} \varphi$, for all φ. Since $\mathcal{SP}$-implications are Sahlqvist formulas, $\mathcal{L} \models_{\mathsf{Kr}} \varphi$ iff $\mathcal{L} \models_{\mathsf{BAO}} \varphi$, where BAO stands for Boolean algebras with operators. Thus, completeness is equivalent to (purely algebraic) *conservativity* of $\models_{\mathsf{BAO}}$ over $\models_{\mathsf{SLO}}$. Completeness of an $\mathcal{SP}$-logic $\mathcal{L}$ also means that its $\mathcal{SP}$-implications *axiomatise* the $\mathcal{SP}$-*fragment* of $\mathcal{L}$ regarded as a standard modal logic. A simple example of an incomplete $\mathcal{SP}$-logic is $\mathcal{L} = \{\diamond p \to p\}$; indeed, for $\varphi = (p \wedge \diamond \top \to \diamond p)$, we have $\mathcal{L} \models_{\mathsf{Kr}} \varphi$ and $\mathcal{L} \not\models_{\mathsf{SLO}} \varphi$.

A classical method of showing completeness of a modal logic $\mathcal{L}$ is to prove its canonicity, which can be done by establishing that every BAO for $\mathcal{L}$ is embeddable into the full complex BAO $\mathfrak{F}^+$ of some Kripke frame $\mathfrak{F}$ for $\mathcal{L}$. We call an $\mathcal{SP}$-theory $\mathcal{L}$ *complex* if every SLO for $\mathcal{L}$ is embeddable into the SLO-type reduct of $\mathfrak{F}^+$ of some Kripke frame $\mathfrak{F}$ for $\mathcal{L}$. Examples of complex, and so complete $\mathcal{SP}$-logics include $\{p \to \diamond p\}$ (reflexivity), $\{\diamond\diamond p \to \diamond p\}$ (transitivity), $\{q \wedge \diamond p \to \diamond(p \wedge \diamond q)\}$ (symmetry), $\{\diamond p \wedge \diamond q \to \diamond(p \wedge q)\}$ (functionality), and their unions. By Sahlqvist's theorem, all $\mathcal{SP}$-logics have first-order correspondents. A number of general results linking complexity of $\mathcal{SP}$-logics to the form of their correspondents have been obtained in [4].

On the other hand, there are many $\mathcal{SP}$-logics that define standard frame properties, but are not complex. In this note, we aim to develop a new method for proving completeness of such logics. First, we axiomatise the $\mathcal{SP}$-fragment of the (Kripke complete) modal logic $\mathbf{Alt}_n$ whose Kripke frames are *n-functional*, i.e., satisfy $\forall x, y_0, \ldots, y_n \left(\bigwedge_{i \leq n} R(x, y_i) \to \bigvee_{i \neq j} (y_i = y_j) \right)$. We set $\mathbf{Alt}_n^+ = \{\varphi_{fun}^n\}$, where $P = \{p_0, \ldots, p_n\}$ and

$$\varphi_{fun}^n = \Big(\bigwedge_{Q \subseteq P, |Q| = n} \diamond \bigwedge Q \ \to \ \diamond \bigwedge P \Big).$$

Note that Kripke frames for φ_{fun}^n are exactly n-functional frames. Here we sketch the proof of

Theorem 1. *For any $n \geq 1$, the $\mathcal{SP}$-logic $\mathbf{Alt}_n^+$ is complete, though not complex if $n \geq 2$.*

To prove that $\mathbf{Alt}_n^+$ ($n \geq 2$) is not complex, one can show that the SLO on the right (where $\diamond \top = \top$, $\diamond \bot = \bot$, and the arrows define $\diamond$ in other cases) validates φ_{fun}^n but is not embeddable into $\mathfrak{F}^+$, for any n-functional $\mathfrak{F}$.
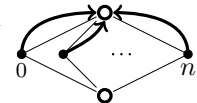
To show completeness, we require *n-terms* that are defined by induction: (*i*) all propositional variables, $\bot$ and $\top$ are n-terms; (*ii*) if $\tau_1, \ldots, \tau_n$ are n-terms, then so is $\diamond(\tau_1 \wedge \cdots \wedge \tau_n)$.

Lemma 2. *For any $\mathcal{SP}$-term ϱ, there is conjunction ϱ' of n-terms with $\mathbf{Alt}_n^+ \models_{\mathsf{SLO}} (\varrho \equiv \varrho')$.*

The proof is by induction on the modal depth d of ϱ. The basis $d = 0$ is trivial. Suppose now that ϱ is of depth $d > 0$. Then $\varrho = \bigwedge P_\varrho \wedge \diamond \varrho_1 \wedge \cdots \wedge \diamond \varrho_k$, where P_ϱ is a set of

propositional variables, $\bot$ and $\top$, and each ϱ_i is of depth $\leq d-1$. By IH, $\mathbf{Alt}_n^+ \models_{\mathsf{SLO}} (\varrho_i \equiv \bigwedge A_i)$, for some set A_i of n-terms. Then $\mathbf{Alt}_n^+ \models_{\mathsf{SLO}} \left(\varrho \equiv (\bigwedge P_\varrho \wedge \bigwedge_{i=1}^k \Diamond \bigwedge A_i) \right)$. If $|A_i| \leq n$, then we are done. So fix some i and suppose that $|A_i| = k > n$. Then we always have $\models_{\mathsf{SLO}} \left((\Diamond \bigwedge A_i) \rightarrow (\bigwedge_{Q \subseteq A_i, |Q|=n} \Diamond \bigwedge Q) \right)$. We show that

$$\mathbf{Alt}_n^+ \models_{\mathsf{SLO}} \big(\bigwedge_{Q \subseteq A_i, |Q|=n} \Diamond \bigwedge Q \ \rightarrow \ \Diamond \bigwedge A_i \big). \tag{1}$$

Indeed, by a syntactic argument, we have $\mathbf{Alt}_n^+ \models_{\mathsf{SLO}} \varphi_{fun}^m$, for every $m > n$, from which we obtain (1) as a substitution instance of φ_{fun}^k.

Lemma 3. *For any $\mathcal{SP}$-term σ and any n-term τ, $\mathbf{Alt}_n^+ \models_{\mathsf{Kr}} \sigma \rightarrow \tau$ implies $\models_{\mathsf{Kr}} \sigma \rightarrow \tau$.*

The proof is by induction on the modal depth d of τ. The basis is again trivial. Now assume inductively that the lemma holds for d and the depth of τ is $d+1$. Let $\sigma = \bigwedge P_\sigma \wedge \Diamond \sigma_1 \wedge \ldots \wedge \Diamond \sigma_k$, where P_σ is some set of propositional variables, $\bot$, $\top$, and each σ_i is an $\mathcal{SP}$-term. Suppose $\tau = \Diamond(\tau_1 \wedge \ldots \wedge \tau_n)$, where each τ_i is either a variable, $\top$, $\bot$, or of the form $\Diamond(\tau_1^i \wedge \cdots \wedge \tau_n^i)$.

Suppose $\not\models_{\mathsf{Kr}} \sigma \rightarrow \tau$. Then, for every j $(1 \leq j \leq k)$, there is i $(1 \leq i \leq n)$ such that $\not\models_{\mathsf{Kr}} \sigma_j \rightarrow \tau_i$, and so $\bigcup_{i=1}^n K_i = \{1, \ldots, k\}$, for $K_i = \{1 \leq j \leq k \mid \not\models_{\mathsf{SLO}} \sigma_j \rightarrow \tau_i\}$. It is not hard to see that, for any i with $K_i \neq \emptyset$, we have $\not\models_{\mathsf{Kr}} (\bigwedge_{j \in K_i} \sigma_j) \rightarrow \tau_i$. By IH, for any such i, there is a Kripke model $\mathfrak{M}_i$ based on an n-functional frame with root r_i where $\bigwedge_{j \in K_i} \sigma_j$ holds, but τ_i does not. Now take a fresh node r, make $\bigwedge P_\sigma$ true in r, and connect r to r_i of each $\mathfrak{M}_i$. The constructed model is based on an n-functional frame and refutes $\sigma \rightarrow \tau$ at r, showing that $\mathbf{Alt}_n^+ \not\models_{\mathsf{Kr}} \sigma \rightarrow \tau$ as required. That $\mathbf{Alt}_n^+$ is complete follows now from Lemmas 2, 3 and the completeness of the empty $\mathcal{SP}$-logic [7].

Using a similar (but more involved) technique, we can also show (see [4] for details) that the $\mathcal{SP}$-logic $\mathbf{S4.3}^+ = \{p \rightarrow \Diamond p, \Diamond\Diamond p \rightarrow \Diamond p, \Diamond(p \wedge q) \wedge \Diamond(p \wedge r) \rightarrow \Diamond(p \wedge \Diamond q \wedge \Diamond r)\}$ is complete, has exactly the same frames as $\mathbf{S4.3}$, and is decidable in polynomial time. However, this does not generalise to $\mathbf{K4.3}$ whose class of Kripke frames is not $\mathcal{SP}$-definable [4]. Svyatlovski has recently shown that the $\mathcal{SP}$-logic $\mathcal{L}_s = \{\Diamond\Diamond p \rightarrow \Diamond p, \Diamond(p \wedge \Diamond q) \wedge \Diamond(p \wedge \Diamond r) \rightarrow \Diamond(p \wedge \Diamond q \wedge \Diamond r)\}$ is complete, tractable, and, for any $\mathcal{SP}$-implication φ, we have $\mathcal{L}_s \models \varphi$ iff φ is valid in all frames for $\mathbf{K4.3}$ (although $\mathcal{L}_s$ has non-$\mathbf{K4.3}$ frames).

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A Duality for Boolean Contact Algebras

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1 Introduction

The well-known de Vries duality, established by H. de Vries in 1962, states that the category of compact Hausdorff spaces is dually equivalent to that of de Vries algebras [4]. The notion of Boolean *contact* algebra (BCA) was developed independently in the context of region-based theory of space. Düntsch and Winter established in [5] a representation theorem for BCAs, showing that every BCA is isomorphic to a dense subalgebra of the regular closed sets of a T_1 weakly regular space. It appears that BCAs are a direct generalization of de Vries algebras, and that the representation theorem for complete BCAs generalizes de Vries duality for objects. During a conference, Vakarelov raised the question of dualizing morphisms. We answer this question using concepts similar to those of modal logic's neighborhood semantics.

2 de Vries Duality and the Representation Theorem

A *de Vries algebra* (DVA), is a complete Boolean algebra B endowed with a binary relation $\prec$ satisfying the following axioms:

DV1 $0 \prec 0$;

DV2 $a \prec b \Rightarrow a \leq b$;

DV3 $a \leq b \prec c \Rightarrow a \prec c$;

DV4 $a \prec b, c \prec d \Rightarrow a \wedge c \prec b \wedge d$;

DV5 $a \prec b \Rightarrow -b \prec -a$;

DV6 $a \prec b \neq 0 \Rightarrow \exists c \neq 0$ such that $a \prec c \prec b$,

where $-a$ denotes the Boolean complement of a. A filter x of B is a *round filter* if for each $b \in x$ there is some $a \in x$ such that $a \prec b$; maximal round filters are called *ends*. Then the set $\mathcal{E}(B)$ of all ends, equipped with the topology having the sets $r_B(a) = \{x \in \mathcal{E}(B) : x \ni a\}$ as a basis, is a compact Hausdorff space. This leads to a dual equivalence between the category of compact Hausdorff spaces and the category of de Vries algebras with suitable morphisms [4].

Boolean contact algebras were studied independently as a formalization of Whiteheadean vision of space. A *Boolean contact algebra* (BCA) is a Boolean algebra B endowed with a binary relation $\mathcal{C}$ satisfying the following axioms:

C1 $a \, \mathcal{C} \, b \Rightarrow a \neq 0$;

C2 $a \neq 0 \Rightarrow a \, \mathcal{C} \, a$;

C3 $a \, \mathcal{C} \, b \Rightarrow b \, \mathcal{C} \, a$;

C4 $a \, \mathcal{C} \, b, b \leq c \Rightarrow a \, \mathcal{C} \, c$;

C5 $a \, \mathcal{C} \, (b \vee c) \Rightarrow a \, \mathcal{C} \, b$ or $a \, \mathcal{C} \, c$;

C6 $a \not\leq b \Rightarrow \exists c \in B$ such that $a \, \mathcal{C} \, c$ and $c \perp b$,

where $\perp$ denotes the complement of the relation $\mathcal{C}$.

The relations $\prec$ and $\perp$ are linked by $a \prec b \Leftrightarrow a \perp -b$. The axioms DV1-DV5 are then equivalent to C1-C5. However, the axiom C6 is weaker than DV6.

Due to the lack of the axiom DV6, round filters and ends do not work anymore. Those have to be replaced by the notions of clan and cluster. A non-empty subset Γ of B is a *clan* if its complement is an ideal and if $a, b \in \Gamma \Rightarrow a \, \mathcal{C} \, b$. A maximal clan is called a *cluster*. The set clust(B) of clusters of B is then equipped with the topology having the sets $\eta_B(a) = \{\Gamma \in \mathrm{clust}(B) : \Gamma \ni a\}$ as a basis for closed sets. This topological space appears to be T_1 and weakly regular.

The representation theorem for BCAs, due to Düntsch and Winter [5], states that η_B is a dense embedding from B to the algebra RC(clust(B)) of regular closed sets of clust(B) endowed with the contact relation $F \, \mathcal{C} \, G \Leftrightarrow F \cap G \neq \emptyset$.

3   A Duality for Morphisms

Let $\beta : B \to B'$ be a map between two complete BCAs satisfying

CM1 $\beta(a \vee b) = \beta(a) \vee \beta(b)$;

CM2 $\beta(1) = 1$;

CM3 $a \perp b \Rightarrow \beta(a) \perp \beta(b)$.

In the presence of the axiom DV6, it is not difficult to define a dual morphism between clust(B') and clust(B), as the inverse image of any cluster is a clan, which is contained in a unique cluster. If β additionally satisfies

CM4 $\beta(a) = \bigwedge\{\beta(b) : a \perp -b\}$,

one easily recovers de Vries duality. However, in general, a clan may be contained in several clusters. We then define a morphism from clust(B') to clust(B) to be a map N from clust(B') to clan(RC(clust(B))), defined as follows

$$N(\Gamma') = \{F \in \mathrm{RC}(\mathrm{clust}(B)) : \beta(\eta_B^{-1}(F)) \in \Gamma'\}.$$

While this definition may seem unnatural, it is quite similar to the accessibility relation in neighborhood semantics.

This leads to two dualities: one involving the category of BCAs with their natural morphisms (satisfying CM1-CM3) and another one extending de Vries duality.

2

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Matthew effects via dependence and independence logic
(work in progress)

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Matthew effect. Introduced by Merton [5], the term *Matthew effect* was used in reference to the self-reinforcing process whereby reputationally rich academics tend to get richer over time. The author defined this phenomenon as 'the accruing of large increments of peer recognition to scientists of great repute for particular contributions in contrast to the minimizing or withholding of such recognition for scientists who have not yet made their mark'. Its recurring appearance in social life led to its recognition as a powerful engine of social, economic, and cultural inequality, to the extent that it can be considered a social law.

There is an extensive literature on the Matthew effect in the fields of sociology, economics, and management [1]. Yet, the effect is not precisely and unequivocally defined in the literature: as a result, researchers are hardly able to compare or integrate theoretical models and empirical findings. This motivates our present attempt to formalize the Matthew effect through mathematical logic.

Dependence and independence logic. The logical framework that we propose for the formalization is the framework of *Dependence and independence logic* introduced by Väänänen [6] and by Grädel and Väänänen [2]. This framework aims at characterizing the notions of *dependence* and *independence* found in social and natural sciences, such as the dependencies involved in Matthew effects. The logics extend first-order logic with new atomic formulas, called *dependence and independence atoms*, that specify explicitly the dependence and independence relations between variables. To evaluate formulas concerning dependency statements the logics adopt an innovative new semantics introduced by Hodges [3, 4]. This new semantics, called *team semantics*, defines the satisfaction relation with respect to *sets* of assignments (called *teams*), instead of single assignments as in the standard Tarskian semantics of first-order logic. Teams can be easily conceived as tables or data sets. The flexible and multidisciplinary interpretations of teams results in a rapid development of applications of the logics in recent years.

Formalization of Matthew effects. In this work, we describe three distinct types of Matthew effect, namely *direct Matthew effect*, *mediated Matthew effect* and *complete Matthew effect*, and we give formal definitions for them via independence logic. Consider the signature $\mathscr{L}$ that contains the equality symbol =, the constant symbols r for each real number $r \in \mathbb{R}$, the function symbols $+, -, \cdot, \div, (\cdot)^r$ for each $r \in \mathbb{R}$, relation symbols $\leq, \geq, <, >$ and other relevant non-logical symbols. We assume that the context of the Matthew effects in question is captured by a first-order $\mathscr{L}$-model M. The domain of an intended model M of a Matthew effect scenario consists of the set of all possible values of all data sets (e.g. real numbers, names of products, names of artists, etc.).

Given a data set and a system of equations that corresponds to a statistical analysis of the data set. We use $x, y, w, \ldots$ to denote the variables in the data set, and we reserve the letter t for the time variable. We write $x_{(t)}$ for the value of the variable x at time t. For the formal definitions, following [7], we view the properties being defined as new atomic formulas and only give their corresponding team semantics.

- y is **(positively) dependent on** x, denoted $x \nearrow y \mid_{\vec{w}}^t$, if there exists an equation in the system such that for some threshold $\gamma \in \mathbb{R}$, $x_{(t-1)} \geq \gamma \Longrightarrow y_{(t)} = \beta_0 + \delta x_{(t-1)} + \beta_1 w_{1(t-1)} + \cdots + \beta_m w_{m(t-1)} + \epsilon$, where $\delta > 0$ is a constant, $w_1, \ldots, w_m$ are dependent variables, $\beta_0, \beta_1, \ldots, \beta_m$ are nonzero parameters and ϵ is an error term. In other words, if $x \nearrow y \mid_{\vec{w}}^t$, then, after x reaches the threshold γ, when all the other

relevant variables $\vec{w}$ are held constant, we have $y_{(t)} - y_{(t-1)} = \delta \cdot (x_{(t-1)} - x_{(t-2)})$ for some $\delta > 0$. Formally, we introduce a new atomic formula $x \curvearrowright y \mid^{t}_{\beta,\gamma,\vec{w}}$, and define $x \curvearrowright y \mid^{t}_{\vec{w}} := \exists^1 \beta \exists^1 \gamma (\beta > 0 \wedge x \curvearrowright y \mid^{t}_{\beta,\gamma,\vec{w}})$.

- y is subject to a **(positive) direct Matthew effect**, denoted $\mathsf{ME}y \mid^{t}_{\vec{w}}$, if y is positively dependent on itself. Formally, we define $\mathsf{ME}y \mid^{t}_{\vec{w}} := \exists^1 \delta \exists^1 \gamma (\delta > 0 \wedge y \curvearrowright y \mid^{t}_{\delta,\gamma,\vec{w}})$.

- y is subject to a **(positive) x-mediated Matthew effect**, denoted $\mathsf{MME}y(x) \mid^{t}_{\vec{w}}$, if after some threshold γ, x is positively dependent on y, and y is positively dependent on x. Formally, define $\mathsf{MME}y(x) \mid^{t}_{\vec{w}} := \exists^1 \delta_1 \exists^1 \delta_2 \exists^1 \gamma_1 \exists^1 \gamma_2 (\delta_1 > 0 \wedge \delta_2 > 0 \wedge (y \curvearrowright x \mid^{t}_{\delta_1,\gamma_1,\vec{w}}) \wedge (x \curvearrowright y \mid^{t}_{\delta_2,\gamma_2,\vec{w}}))$.

- x and y are subjects to a **(positive) complete Matthew effect**, denoted $\mathsf{CME}(x,y) \mid^{t}_{\vec{w}}$, if y is subject to a positive x-mediated Matthew effect, y is subject to a positive direct Matthew effect, and x is subject to a positive direct Matthew effect. Formally, define $\mathsf{CME}(x,y) \mid^{t}_{\vec{w}} := \mathsf{MME}x(y) \mid^{t}_{\vec{w}} \wedge \mathsf{ME}x \mid^{t}_{\vec{w}} \wedge \mathsf{ME}y \mid^{t}_{\vec{w}}$.

Results. It is clear from its defining clause of team semantics that the auxiliary new atomic formula $x \curvearrowright y \mid^{t}_{\beta,\gamma,\vec{w}}$ we introduced is a Π_1 atom in the sense of [7], and therefore both definable and negatable in $\mathcal{I}$. Since first-order atomic formulas are negatable in $\mathcal{I}$ and the class of negatable formulas of $\mathcal{I}$ is closed under $\wedge$ and $\exists^1$ [7], we conclude that the formula $x \curvearrowright y \mid^{t}_{\vec{w}} = \exists^1 \delta \exists^1 \gamma (\delta > 0 \wedge x \curvearrowright y \mid^{t}_{\delta,\gamma,\vec{w}})$ is negatable and definable in $\mathcal{I}$. Similarly, the defining formulas $\mathsf{ME}y \mid^{t}_{\vec{w}}$, $\mathsf{MME}y(x) \mid^{t}_{\vec{w}}$ and $\mathsf{CME}(x,y) \mid^{t}_{\vec{w}}$ of the different types of Matthew effects are all definable and negatable in $\mathcal{I}$. This means that the completeness theorem of independence logic applies to the formulas defining different Matthew effects, and therefore many properties of Matthew effects can be derived formally in the system of [7]. Simple examples of such properties include: $\mathsf{MME}y(x) \mid^{t}_{\vec{w}} \vdash \mathsf{MME}x(y) \mid^{t}_{\vec{w}}$ (mediated Matthew effects are always *reciprocal* for the two variables involved) and $\mathsf{ME}y \mid^{t}_{\vec{w}} \vdash \mathsf{MME}y(y) \mid^{t}_{\vec{w}}$ (a direct Matthew effect is a special case of the mediated Matthew effect). More interesting properties will be explored in our future work.

Further research. Future research will be directed at formalizing the Matthew effect in a real-world context [1]. The authors analyze differentials in the recognition received by U.S. biomedical scientists who are awarded the prestigious Howard Hughes Medical Institute (HHMI) appointment, relative to scientists of comparable quality who are not awarded the HHMI affiliation. The empirical analysis reveals that HHMI-appointed scientists tend to earn greater recognition, especially if there is greater uncertainty about the quality of their output. This suggests important boundary conditions to the Matthew effect, which will be taken into account in our formal approach.

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Twist products and dualities

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Twist products were originally introduced by Kalman in [6] in the context of lattices enriched with an involution, and have subsequently been employed in different guises by numerous authors (see, e.g., [1, 7, 8] for a sample of the rapidly-growing literature on twist products). Different versions of the twist product construction have often been employed to provide representations of various classes of algebras. In the best cases, twist product constructions participate as one functor witnessing an equivalence between two categories of algebras. On the other hand, such categories sometimes admit topological dualities, such as the Esakia duality for Heyting algebras or Urquhart's duality for algebras associated with relevance logics [9]. Despite its proliferation in algebraic studies, the manner in which the twist product construction manifests on the duals of algebras remains relatively unexplored.

In the present work, we provide a case study illustrating the twist product of dual structures by examining two dualities for the class of bounded Sugihara monoids. These algebras are involutive, idempotent, distributive, bounded commutative residuated lattices, and were shown in [4, 5] to be equivalent to a category of enriched Gödel algebras. One of the functors witnessing this equivalence is a variant of the twist product, and we render this variant as a construction on the topological duals of bounded Sugihara monoids. We call a structure $(X, \leq, X_0, \mathcal{T})$ a *Sugihara space* if

1. $(X, \leq, \mathcal{T})$ is an Esakia space,

2. $(X, \leq)$ is a forest (i.e., for all $x \in X$, the up-set of x is a chain), and

3. $X_0 \subseteq X$ is a clopen collection of $\leq$-minimal elements.

The category of bounded Sugihara monoids is dually equivalent to the category of Sugihara spaces as defined with the appropriate morphisms, and this duality is anchored in the Davey-Werner duality for Kleene algebras [2]. On the other hand, as the equivalent algebraic semantics for the relevance logic **R**-mingle with sentential constant t, the bounded Sugihara monoids also admit a duality in terms of Urquhart's *relevant spaces* [9]. We call the relevant spaces corresponding to bounded Sugihara monoids *Sugihara relevant spaces*, and illustrate a construction that, given a Sugihara space $\mathbf{X}$, produces a Sugihara relevant space $\mathbf{X}^{\bowtie}$. This construction has a much more pictorial character than its analogue on the algebraic side of the duality. Given a Sugihara space $\mathbf{X} = (X, \leq, X_0, \mathcal{T})$, the construction proceeds by producing a copy $-X = \{-x : x \in X \setminus X_0\}$ of those elements outside the designated subset X_0, and defining a new ordering relation $\leq^{\bowtie}$ on $X \cup -X$ by

1. If $x, y \in X$, then $x \leq^{\bowtie} y$ if and only if $x \leq y$,

2. If $-x, -y \in -X$, then $-x \leq^{\bowtie} -y$ if and only if $y \leq x$,

3. If $-x \in -X$ and $y \in X$, then $-x \leq^{\bowtie} y$ if and only if x is $\leq$-comparable to y.

In other words, the underlying poset of $\mathbf{X}^{\bowtie}$ is constructed from $\mathbf{X}$ by doubling the elements $X \setminus X_0$ and reflecting them accross the designated subset X_0.

Having their basis in the Routley-Meyer semantics, Sugihara relevant spaces incorporate a ternary accessibility relation in their signature. This ternary relation realizes the monoid operation of a given bounded Sugihara monoid on its dual space. Capturing the behavior of this relation using only the information encoded in a Sugihara space is a key difficulty in obtaining a dual space analogue of the twist product. It turns out that the appropriate ternary relation R on $X \cup -X$ may be defined in terms of simple conditions on meets and joins of elements of $X \cup -X$, and gives a much simpler presentation of the monoid multiplication than on the algebraic side of the duality. In more detail, define the *absolute value* of an element of $X \cup -X$ by $|x| = x$ if $x \in X$ and $|-x| = x$ if $-x \in -X$. Further, define a partial binary operation $\cdot$ on $X \cup -X$ by

$$
x \cdot y = \begin{cases}
x \vee y & \text{if } x, y \in X \text{ or } x \parallel y, \text{ provided the join exists} \\
z & \text{if } x \perp y, x \notin X \text{ or } y \notin X, \text{ and } |x| \neq |y|, \text{ where} \\
& z \text{ is whichever of } x, y \text{ has greater absolute value} \\
x \wedge y & \text{if } x \perp y, \text{ and either } x, y \notin X \text{ or } |x| = |y| \\
\text{undefined} & \text{otherwise}
\end{cases}
$$

where $\perp$ denotes the relation of comparability. The appropriate ternary relation R on $X \cup -X$ is defined by $Rxyz$ if and only if $x \cdot y$ exists and $x \cdot y \leq z$. With this construction, we obtain the following representation theorem.

Theorem 1. *Up to isomorphism, every Sugihara relevant space is of the form $\mathbf{X}^{\bowtie}$ for some Sugihara space $\mathbf{X}$.*

Among other things, the above representation theorem explicates the connection between Dunn's Kripke-style semantics for $\mathbf{R}$-mingle using a binary accessibility relation [3], and the more usual Routley-Meyer semantics for $\mathbf{R}$-mingle using a ternary accessibility relation.

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Definability and conceptual completeness for regular logic

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Regular theories consist of sequents $\varphi \vdash_{\boldsymbol{x}} \psi$, where φ, ψ are built from atomic formulae by $\wedge$ and $\exists$. Their algebraic counterpart is the notion of regular category, i.e one with finite limits and regular epi - mono factorizations (sufficient for expressing $\exists$) that are stable under pullback ($\exists$ is compatible with substitution of terms). The effectivization $\mathcal{C}_{ef}$ of a regular category $\mathcal{C}$ is the process of universally turning it into an effective (=Barr-exact) one, i.e making every equivalence relation the kernel pair of its coequalizer. It was described in [4] as a full subcategory of the category of sheaves for the subcanonical Grothendieck topology on $\mathcal{C}$ whose coverings are singleton families consisting of regular epis. $\mathcal{C}_{ef}$ has as objects quotients in $\mathrm{Sh}(\mathcal{C}, J)$ of equivalence relations coming from $\mathcal{C}$. Regular functors $F \colon \mathcal{C} \to \mathcal{D}$ between regular categories preserve finite limits and regular epis. Such a functor is covering if for every object $D \in \mathcal{D}$ there is $C \in \mathcal{C}$ and a regular epi $FC \to D$. Regular categories with regular functors are organized in a 2-category REG. $\zeta_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}_{ef}$ is the obvious inclusion (restriction of the Yoneda embedding) and we omit it from our notation when it acts on morphisms coming from $\mathcal{C}$. The action of effectivization on a regular functor $F \colon \mathcal{C} \to \mathcal{D}$ is $F^* = F_{ef} \colon \mathcal{C}_{ef} \to \mathcal{D}_{ef}$ (so that $F^* \cdot \zeta_{\mathcal{C}} \cong \zeta_{\mathcal{D}} \cdot F$). Abusively we may write composites such as $F^* q \cdot Fu$, relying on the latter isomorphism and consistently omitting $\zeta_{\mathcal{C}}$. Our main technical result is the following

Lemma 1. *If $F \colon \mathcal{C} \to \mathcal{D}$ is a full on subobjects regular functor then $F^* = F_{ef} \colon \mathcal{C}_{ef} \to \mathcal{D}_{ef}$ is also full on subobjects.*

Proof: For a subobject $\sigma \colon S \to F^* X$ the presentation
$$FC_1 \underset{Fc_1}{\overset{Fc_0}{\rightrightarrows}} FC_0 \xrightarrow{F^*e} F^* X$$
of $F^* X$, arises from the obvious presentation of X in $\mathcal{C}_{ef}$. We pull back the subobject S along $F^* e$ obtaining by our assumption a subobject $Fi \colon FR_0 \to FC_0$, for a subobject $i \colon R_0 \to C_0$, and a regular epimorphism $s \colon FR_0 \to S$.

Let the equivalence relation $(r_0, r_1) \colon R_1 \to R_0 \times R_0$ arise as the intersection of $(c_0, c_1) \colon C_1 \to C_0 \times C_0$ with the subobject $R_0 \times R_0 \to C_0 \times C_0$. Its coequalizer
$$\zeta_{\mathcal{C}} R_1 \underset{r_1}{\overset{r_0}{\rightrightarrows}} \zeta_{\mathcal{C}} R_0 \xrightarrow{q} Q$$
in $\mathcal{C}_{ef}$ gives $S \cong F^* Q$. Indeed we find that $s \cdot Fr_0 = s \cdot Fr_1$, hence a regular epi $r \colon F^* Q \to S$ with $r \cdot F^* q = s$. It is also a mono:

Consider arrows $u_0\ u_1\colon \zeta_{\mathcal{D}}D \to F^*Q$, such that $r \cdot u_0 = r \cdot u_1$. Since F^*q is a regular epi the generalized elements u_0, u_1 are locally in $\zeta_{\mathcal{D}}D_i$, i.e there is a covering $d'\colon D' \to D$, $i = 0,1$ and factorizations of $u_i \cdot d' = F^*q \cdot v_i$.

Diagram chasing gives $F^*e \cdot Fi \cdot v_0 = F^*e \cdot Fi \cdot v_1$. The universal properties of (Fc_0, Fc_1) as kernel pair of its coequalizer and of the pullback diagram arising by applying F to the intersection defining (r_0, r_1) give a factorization $\gamma\colon D' \to FC_1$ such that $Fi \cdot v_i = Fc_i \cdot \gamma$, $i = 0,1$ and, respectively, an $\alpha\colon D' \to FR_1$ such that $(v_0, v_1) = (Fr_0, Fr_1) \cdot \alpha$. Hence $u_0 \cdot d' = u_1 \cdot d'$ and d' is an epi, so $u_0 = u_1$. ∎

Regular functors that are full on subobjects and covering correspond to extensions of theories (of their domain categories) by adding new axioms but no new symbols. Regular functors that are covering, faithful and full on subobjects are full as well. By Lemma 1 and results in [1], such a functor induces a functor with the same properties at the level of effectivizations. By [5], 1.4.9, the induced functor between effectivizations is an equivalence. Hence we have the following strengthening of [6] 2.4.4

Proposition 1. *A regular functor $F\colon \mathcal{C} \to \mathcal{D}$ to an effective category is the effectivization of $\mathcal{C}$ iff it is covering, faithfull and full on subobjects.*

Combining these with results from [2], [3] D3.5.12, we get

Theorem 1. *For $F\colon \mathcal{C} \to \mathcal{D}$ in REG, the induced functor between the categories of models $- \cdot F\colon \mathrm{REG}(\mathcal{D}, \mathrm{Set}) \to \mathrm{REG}(\mathcal{C}, \mathrm{Set})$ is fully faithful iff F is full on subobjects and covering.*

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Expansions of Heyting algebras

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It is well-known that congruences on a Heyting algebra are in one-to-one correspondence with filters of the underlying lattice. If an algebra $\mathbf{A}$ has a Heyting algebra reduct, it is of natural interest to characterise which filters correspond to congruences on $\mathbf{A}$. Such a characterisation was given by Hasimoto [1]. When the filters can be sufficiently described by a single unary term, many useful properties are uncovered. The traditional example arises from boolean algebras with operators. In this setting, an algebra $\mathbf{B} = \langle B; \vee, \wedge, \neg, \{f_i \mid i \in I\}, 0, 1 \rangle$ is a *boolean algebra with (dual) operators* (BAO for short) if $\langle B; \vee, \wedge, \neg, 0, 1 \rangle$ is a boolean algebra, and for each $i \in I$, the operation f_i is a unary map satisfying $f_i 1 = 1$ and $f_i(x \wedge y) = f_i x \wedge f_i y$. If $\mathbf{B}$ is of finite type, then congruences on $\mathbf{B}$ are determined by filters closed under the map d, defined by

$$dx = \bigwedge \{f_i x \mid i \in I\}.$$

This is easily generalised to the case that each f_i is of any finite arity. The reader is warned that, conventionally, the definition of an operator is dual to the definition given here. However, when the algebra of interest is a Heyting algebra, it turns out that meet-preserving operations are more natural than join-preserving operations. Hasimoto gave a construction which generalises the term above to Heyting algebras equipped with an arbitrary set of arbitrarily many operations (note that Hasimoto uses the word "operator" for an arbitrary unary operation). The construction does not apply in all cases, and even when it does, it does not guarantee that the result is a term function on the algebra. Having said that, natural constraints exist which guarantee both that the construction applies, and produces a term function. As is the case for BAOs, we will restrict our attention to unary operations here and observe that everything is easily generalised to operations of arbitrary arity. In this talk we provide some general conditions which guarantee such a term function. Moreover, provided that the Heyting algebra also includes a dual pseudocomplement operation, we prove that a variety of these algebras is a discriminator variety if and only if it is semisimple, alongside an equational characterisation.

Definition 1.1. We will say that an algebra $\mathbf{A} = \langle A; M, \vee, \wedge, \rightarrow, 0, 1 \rangle$ is an *expanded Heyting algebra* (EHA for short) if $\langle A; \vee, \wedge, \rightarrow, 0, 1 \rangle$ is a Heyting algebra, and M is an arbitrary set of unary operations on A. Let $x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$. We say that a filter $F \subseteq A$ is a *normal filter (of $\mathbf{A}$)* provided that, for every $f \in M$, if $x \leftrightarrow y \in F$ then $fx \leftrightarrow fy \in F$. It is easily verified that the set of normal filters of $\mathbf{A}$ forms a complete lattice, and so we will let $\mathbf{Fil}(\mathbf{A})$ denote the lattice of normal filters of $\mathbf{A}$. For all $F \in \mathrm{Fil}(\mathbf{A})$, let $\theta(F)$ be the equivalence relation defined by

$$\theta(F) = \{(x, y) \mid x \leftrightarrow y \in F\}.$$

Theorem 1.2 (Hasimoto [1])**.** *Let $\mathbf{A}$ be an EHA, let F be a normal filter on $\mathbf{A}$, and let α be a congruence on $\mathbf{A}$. Then $\theta(F)$ is a congruence on $\mathbf{A}$. Moreover, the map $\theta \colon \mathbf{Fil}(\mathbf{A}) \rightarrow \mathbf{Con}(\mathbf{A})$, defined by $F \mapsto \theta(F)$, is an isomorphism with its inverse given by $\alpha \mapsto 1/\alpha$.*

Definition 1.3. Let $\mathbf{A}$ be an EHA and let t be a unary term in the language of $\mathbf{A}$. We say that t is a *normal filter term (on $\mathbf{A}$)* if $t^{\mathbf{A}}$ is order-preserving, and, whenever F is a filter of $\mathbf{A}$, then F is a normal filter of $\mathbf{A}$ if and only if F is closed under $t^{\mathbf{A}}$.

Henceforth we will not be careful to distinguish between terms and term functions. The map d for BAOs seen before is an example of a normal filter term. An easy description of congruences via a normal filter term allows for a deeper investigation of congruence-related properties. In particular, we can characterise equationally definable principal congruences (EDPC) in a very straightforward manner.

Theorem 1.4. *Let $\mathcal{V}$ be a variety of EHAs and assume that t is a normal filter term on $\mathcal{V}$. Let $dx = x \wedge tx$. Then $\mathcal{V}$ has EDPC if and only if there exists $n \in \omega$ such that $\mathcal{V} \models d^{n+1}x = d^n x$.*

Our main result involves dually pseudocomplemented Heyting algebras. A *dual pseudocomplement operation* is an operation $\sim$ such that $x \vee y = 1$ if and only if $y \geq \sim x$. If $\mathbf{A}$ is an EHA and there exists $\sim \in M$ such that $\sim$ is a dual pseudocomplement operation, we say that $\mathbf{A}$ is a *dually pseudocomplemented EHA*, and if $M = \{\sim\}$ then $\mathbf{A}$ is a *dually pseudocomplemented Heyting algebra*. Sankappanavar [5] characterised congruences for dually pseudocomplemented Heyting algebras, which is expressed in our terminology by saying that the term $\neg \sim$ is a normal filter term (where $\neg x = x \rightarrow 0$). Our main result is as follows.

Theorem 1.5 (T., [8]). *Let $\mathcal{V}$ be a variety of dually pseudocomplemented EHAs and assume $\mathcal{V}$ has a normal filter term t. Let $dx = x \wedge tx$. Then the following are equivalent:*

1. *$\mathcal{V}$ is semisimple.*

2. *There exists $n \in \omega$ such that $\mathcal{V} \models d^{n+1}x = d^n x$ and $\mathcal{V} \models x \leq d\sim d^n \neg x$.*

3. *$\mathcal{V}$ is a discriminator variety.*

Note that the second condition implies EDPC by Theorem 1.4. The argument is based on an argument by Kowalski & Kracht [4] proving the same characterisation for BAOs, which now follows as a corollary of the above theorem. The present author also proved the same characterisation for double-Heyting algebras [7], which also follows. We will also see some new cases for which the characterisation applies to. On the other hand, certain classes of residuated lattices have a suspiciously similar characterisation (Kowalski [2], Kowalski & Ferreirim [3], Takamura [6]), using a similar proof technique, but this theorem does not apply to them. It is believed that this is no coincidence, and further research will attempt to unite these results.

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Types and models
in core fuzzy predicate logics

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1 Introduction

The rudiments for the development of the model theory of predicate core fuzzy logics were laid down in [5] with follow up work in places like [3, 2, 4]. A great deal still remains to be done, though. The aim of this talk is to explore the construction of models realizing many and few types in the setting of these logics as well as applications. This kind of problems are well-known from the classical case (cf.[1, 6]).

2 Quick preliminaries

We more or less follow the notation of [3] below. In particular, recall that we write models for our (function symbol-free) predicate language L as structures of the form $(\mathbf{B}, \mathbf{M})$ where $\mathbf{B}$ is an algebra belonging to some variety (which is an extension of the variety of the so called MTL-algebras) corresponding to the logic under consideration and $\mathbf{M}$ is a structure with a domain M and appropriate assignments of truth values to the predicates of the language and of individuals of M to its constants. We write $(\mathbf{B}, \mathbf{M}) \models \phi$ when $||\phi||_{\mathbf{M}}^{\mathbf{B}} = 1$.

Moreover, we are only interested in models where: $||\exists x \phi(x)||_{\mathbf{M}}^{\mathbf{B}} = 1$ means that $||\phi[d]||_{\mathbf{M}}^{\mathbf{B}} = 1$ for some element d of its domain of individuals (call them $\exists$-Henkin models). Henceforth, by a *model* we will always mean one such model.

A *tableau* is going to be a pair (T, U) such that T and U are theories. A tableau is *satisfied* by a model $(\mathbf{B}, \mathbf{M})$, if we have that both $(\mathbf{B}, \mathbf{M}) \models T$ and, for all $\phi \in U$, $(\mathbf{B}, \mathbf{M}) \not\models \phi$. We may define the expression $(T, U) \models \phi$ as meaning that for any model that satisfies (T, U), the model must make ϕ true as well. A tableau (T, U) is said to be *consistent* if $T \vdash \bigvee U_0$ for no finite $U_0 \subseteq U$. In particular, $\bigvee \emptyset$ we define as $\perp$ (semantically, of course, $\perp$ is the l. u. b. of $\emptyset$).

The following result is what we need for our purposes here instead of Theorem 4 from [5].

Theorem 1. (Model Existence Theorem) *Let (T, U) be a consistent tableau. Then there is a model satisfying (T, U).*

Corollary 1. (Tableaux Compactness) *Let (T, U) be a tableau. If every (T_0, U_0), with $|T_0|, |U_0|$ finite and $T_0 \subseteq T$ and $U_0 \subseteq U$, has a model satisfying it, then (T, U) is satisfied in some model.*

3 Models realizing many types

Let $(\mathbf{B}, \mathbf{M})$ be a model. If (p, p') is a tableau in some variable x and parameters in some $A \subseteq M$, we will call p a type of $(\mathbf{B}, \mathbf{M})$ in A if the tableaux $(Th_A(\mathbf{B}, \mathbf{M}) \cup p, \overline{Th_A}(\mathbf{B}, \mathbf{M}) \cup p')$ is satisfiable —where $Th_A(\mathbf{B}, \mathbf{M})$ is the collection of formulas with constants for the elements in A that hold in $(\mathbf{B}, \mathbf{M})$. We will denote the set of all such types by $S^{(\mathbf{B}, \mathbf{M})}(A)$. A model

$(\mathbf{B}, \mathbf{M})$ is κ-saturated if for any $A \subseteq M$ suh that $|A| < \kappa$, all $(p, p') \in S^{(\mathbf{B},\mathbf{M})}(A)$ are realized in $(\mathbf{B}, \mathbf{M})$.

Theorem 2. *For any* $(\mathbf{B}, \mathbf{M})$ *there is a* κ^+*-saturated L-elementary extension (in the sense of [5, 3])* $(\mathbf{C}, \mathbf{N})$ *of* $(\mathbf{B}, \mathbf{M})$.

4 Models realizing few types

A pair of sets of formulas (p, p') is a type of a tableau (T, U) if the tableau $(T \cup p, U \cup p')$ is satisfiable.

A type (p, p') of (T, U) is *non-isolated* if for any formulas ϕ, ϕ' such that $(T \cup \{\phi\}, U \cup \{\phi'\})$ is satisfiable, there are $\psi \in p, \psi' \in p'$ such that $(T \cup \{\phi\}, U \cup \{\phi'\}) \nvDash \psi$ or $(T \cup \{\phi, \psi'\}, U \cup \{\phi'\})$ is satisfiable.

Theorem 3. (Omitting types) *Let* (T, U) *be a tableau realized by some model and* (p, p') *a non-isolated n-type of* (T, U). *Then there is model satisfying* (T, U) *which omits* (p, p').

Theorem 4. (Omitting countably many types) *Let* (T, U) *be a tableau realized by some model and* $(p_i, p'_i)(i < \omega)$ *a sequence of non-isolated n-types of* (T, U). *Then there is model satisfying* (T, U) *which omits* $(p_i, p'_i)(i < \omega)$.

These omitting types results differ from those in [7] since we are working with tableaux rather than simply theories.

5 Applications

Now we finish with an example of an application of the countable omitting types theorem.

Proposition 1. *Suppose we have binary symbols in our language* $<$ *and* R. *Let* $(\mathbf{B}, \mathbf{M})$ *be a countable model of the theory* (Γ, Δ) *where*

$$\Gamma = \{\forall x, y(x < y \lor R(x, y) \lor y < x)\} \cup \{\forall x, y, z(R(x, y) \land R(yz) \to R(x, z))\} \cup \{\forall z(\forall x \exists y > x \exists v < z(\psi(v, y)) \to \exists v < z \forall x \exists y > x(\psi(v, y)))\} \cup \{\forall x_0, \dots x_n \exists y(\bigwedge_{i \le n} x_i < y) : n < \omega\}$$

and

$$\Delta = \emptyset$$

Then there is an L-elementary extension $(\mathbf{A}, \mathbf{N})$ *of* $(\mathbf{B}, \mathbf{M})$, *such that if* $b \in N \setminus M$ *is such that* $R(b, c)$ *does not hold in* $(\mathbf{A}, \mathbf{N})$ *for any* $c \in M$, *then, given* $a \in M$, $a < b$ *must hold in* $(\mathbf{A}, \mathbf{N})$ *(this model might be called an end extension of* $(\mathbf{B}, \mathbf{M})$ *relative to R).*

6 Acknowledgments

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Existentially Closed Brouwerian Semilattices

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In algebraic logic some attention has been paid to the class of existentially closed structures in varieties coming from the algebraization of common propositional logics. In fact, there are relevant cases where such classes are elementary: this includes, besides the easy case of Boolean algebras, also Heyting algebras [GZ02], diagonalizable algebras [GZ02] and some universal classes related to temporal logics [GvG16]. This is also true for the variety of Brouwerian semilattices, i.e. the algebraic structures corresponding to the implication-conjunction fragment of intuitionistic logic. Said variety is amalgamable and locally finite, hence by well-known results [Whe76], it has a model completion (whose models are the existentially closed structures). However, very little is known about the related axiomatizations, with the remarkable exception of the case of the locally finite amalgamable varieties of Heyting algebras recently investigated in [DJ10] and the simpler cases of posets and semilattices studied in [AB86]. We use a methodology similar to [DJ10] (relying on classifications of minimal extensions) in order to investigate the case of Brouwerian semilattices. We obtain the finite axiomatization reported below, which is similar in spirit to the axiomatizations from [DJ10] (in the sense that we also have kinds of 'density' and 'splitting' conditions). The main technical problem we must face for this result (making axioms formulation slightly more complex and proofs much more involved) is the lack of joins in the language of Brouwerian semilattices. This investigation also revealed some properties of existentially closed Brouwerian semilattices, namely the nonexistence of meet-irreducible elements, of the minimum and of the joins of incomparable elements, which are suggested and in fact implied by the 'density' and 'splitting' conditions.

Statement of the main result

A *Brouwerian semilattice* is a poset $(P, \leq)$ having a greatest element, inf's of pairs and relative pseudo-complements. We denote the greatest element with 1, the inf of $\{a, b\}$ is called 'meet' of a and b and denoted with $a \wedge b$. The relative pseudo-complement of a and b is denoted with $a \rightarrow b$. We recall that $a \rightarrow b$ is characterized by the the following property: for every $c \in P$ we have

$$c \leq a \rightarrow b \quad \text{iff} \quad c \wedge a \leq b$$

Brouwerian semilattices can also be defined in an alternative way as algebras over the signature $1, \wedge, \rightarrow$, subject to the following equations

- $a \wedge a = a$

- $a \wedge b = b \wedge a$

- $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

- $a \wedge 1 = a$

- $a \wedge (a \rightarrow b) = a \wedge b$

- $b \wedge (a \rightarrow b) = b$

- $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$

- $a \to a = 1$

In case this equational axiomatization is adopted, the partial order $\leq$ is recovered via the definition $a \leq b$ iff $a \wedge b = a$. See [Köh81] for relevant information on Brouwerian semilattices.

By a result due to Diego-McKay, Brouwerian semilattices are locally finite (meaning that all finitely generated Brouwerian semilattices are finite); since they are also amalgamable, it follows [Whe76] that the theory of Brouwerian semilattices has a model completion. We prove that such a model completion is given by the above set of axioms for the theory of Brouwerian semilattices together with the three additional axioms (Density1, Density2, Splitting) below.

We use the abbreviation $a \ll b$ for $a \leq b$ and $b \to a = a$.

[**Density 1**] For every c there exists an element b different from 1 such that $b \ll c$.

[**Density 2**] For every c, a_1, a_2, d such that $a_1, a_2 \neq 1$, $a_1 \ll c$, $a_2 \ll c$ and $d \to a_1 = a_1$, $d \to a_2 = a_2$ there exists an element b different from 1 such that:

$$a_1 \ll b$$
$$a_2 \ll b$$
$$b \ll c$$
$$d \to b = b$$

[**Splitting**] For every a, b_1, b_2 such that $1 \neq a \ll b_1 \wedge b_2$ there exist elements a_1 and a_2 different from 1 such that:

$$b_1 \geq a_1 = a_2 \to a$$
$$b_2 \geq a_2 = a_1 \to a$$
$$a_2 \to b_1 = b_2 \to b_1$$
$$a_1 \to b_2 = b_1 \to b_2$$

Proofs of this and other results can be found in the preliminary manuscript at the following link: http://arxiv.org/abs/1702.08352

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Locally tabular polymodal logics

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A logic L is *locally tabular* if, for any finite l, there exist only finitely many pairwise nonequivalent formulas in L built from the variables $p_1, ..., p_l$. Equivalently, a logic L is locally tabular if the variety of its algebras is locally finite, i.e., every finitely generated L-algebra is finite. This is a very strong property: if a logic is locally tabular, then it has the finite model property (thus it is Kripke complete); every extension of a locally tabular logic is locally tabular (thus it has the finite model property); every finitely axiomatizable extension of a locally tabular logic is decidable.

According to the classical results by Segerberg and Maksimova [4, 3], a unimodal logic containing K4 is locally tabular iff it is of finite height. The notion of finite height can also be defined for logics, in which the master modality is expressible ('pretransitive' logics). Recently [5], it was shown that every locally tabular unimodal logic is a pretransitive logic of finite height. Also, in [5], two semantic criteria of local tabularly for unimodal logics were proved. In this note we formulate the analogs of these facts for the polymodal case and discuss some of their corollaries.

Fix some $n > 0$ and the n-modal language with the modalities $\Diamond_0, \ldots, \Diamond_{n-1}$.

Necessary condition. For a Kripke frame $F = (W, (R_i)_{i<n})$, put $R_F = \cup_{i<n} R_i$. Let $\sim_F$ be the equivalence relation $R_F^* \cap R_F^{*-1}$, where R_F^* denotes the transitive reflexive closure of R_F. A *cluster* in F is an equivalence class under $\sim_F$. For clusters C, D, put $C \leq_F D$ iff xR_F^*y for some $x \in C, y \in D$. The poset $(W/\sim_F, \leq_F)$ is called the *skeleton of* F.

A poset is of *finite height* $\leq h$ if every of its chains contains at most h elements. The *height of a frame* F, in symbols, $ht(F)$, is the height of its skeleton.

For a binary relation R put $R^{\leq m} = \cup_{i \leq m} R^i$. R is called *m-transitive*, if $R^{m+1} \subseteq R^{\leq m}$. A frame F is *m-transitive* if R_F is m-transitive.

Put $\Diamond^0 \varphi = \varphi$, $\Diamond^{i+1} \varphi = \Diamond^i(\Diamond_0 \varphi \vee \ldots \vee \Diamond_{n-1} \varphi)$, $\Diamond^{\leq m} \varphi = \vee_{i \leq m} \Diamond^i \varphi$, $\Box^{\leq m} \varphi = \neg \Diamond^{\leq m} \neg \varphi$.

Proposition 1. *A frame* F *is m-transitive iff* $F \vDash \Diamond^{m+1} p \to \Diamond^{\leq m} p$.

Put $B_1^{[m]} = p_1 \to \Box^{\leq m} \Diamond^{\leq m} p_1$, $B_{i+1}^{[m]} = p_{i+1} \to \Box^{\leq m}(\Diamond^{\leq m} p_{i+1} \vee B_i^{[m]})$.

Proposition 2. *For an m-transitive frame* F, $F \vDash B_h^{[m]}$ *iff* $ht(F) \leq h$.

A logic L is called *m-transitive* if $L \vdash \Diamond^{m+1} p \to \Diamond^{\leq m} p$. L is *pretransitive* if it is m-transitive for some $m \geq 0$. An m-transitive logic L is of *finite height* $\leq h$ if $L \vdash B_h^{[m]}$.

Theorem 1. *Every locally tabular logic is a pretransitive logic of finite height.*

Being a pretransitive logic of finite height is not sufficient for local tabularity. For example, products of transitive logics are pretransitive, in particular, the logic $S5 \times S5$ is a pretransitive logic of height 1. It is known to be not locally tabular (still, it is pre-locally tabular [1]).

First criterion. As usual, *a partition* $\mathcal{A}$ of a set W is a set of non-empty pairwise disjoint sets such that $W = \cup\mathcal{A}$. A partition $\mathcal{B}$ *refines* $\mathcal{A}$, if each element of $\mathcal{A}$ is the union of some elements of $\mathcal{B}$.

Definition 1. Let F $= (W, (R_i)_{i<n})$ be a Kripke frame. A partition $\mathcal{A}$ of W is F-*tuned*, if for every $U, V \in \mathcal{A}$, and every $i < n$

$$\exists u \in U \; \exists v \in V \; uR_iv \quad \Rightarrow \quad \forall u \in U \; \exists v \in V \; uR_iv.$$

A class of frames $\mathcal{F}$ is *ripe*, if there exists $f : \mathbb{N} \to \mathbb{N}$ such that for every F $\in \mathcal{F}$ and every finite partition $\mathcal{A}$ of F there exists an F-tuned refinement $\mathcal{B}$ of $\mathcal{A}$ with $|\mathcal{B}| \leq f(|\mathcal{A}|)$.

Theorem 2. *A logic L is locally tabular iff L is the logic of a ripe class of frames.*

Let L_u be the extension of L with the universal modality. Trivially, if a partition is tuned for a frame $(W, (R_i)_{i<n})$, then it is tuned for the frame $(W, (R_i)_{i<n}, W \times W)$.

Corollary 1. *If L is locally tabular, then L_u is locally tabular.*

Let L_t be the tense counterpart of L. From Theorem 2 and the filtration technique proposed in [2, Theorem 2.4], we have

Corollary 2. *If L is locally tabular, then L_t has the finite model property.*

Second criterion. For a class $\mathcal{F}$ of frames let $cl\mathcal{F}$ be the class of restrictions on clusters occurring in frames from $\mathcal{F}$: $cl\mathcal{F} = \{F{\upharpoonright}C \mid F \in \mathcal{F}$ and C is a cluster in F$\}$. $\mathcal{F}$ has the *ripe cluster property* if $cl\mathcal{F}$ is ripe. A class $\mathcal{F}$ of frames is of *finite height* if there exists $h \in \mathbb{N}$ such that $ht(F) \leq h$ for all F $\in \mathcal{F}$.

Theorem 3. $\mathcal{F}$ *is ripe iff* $\mathcal{F}$ *is of finite height and has the ripe cluster property.*

A logic has the *ripe cluster property* if the class of all its frames has. Note that if a pretransitive logic is canonical, then its extensions with formulas of finite height are canonical, thus are Kripke complete. From Theorems 2 and 3, we obtain

Corollary 3. *Suppose L_0 is a canonical pretransitive logic with the ripe cluster property. Then for any logic $L \supseteq L_0$, L is locally tabular iff L is of finite height.*

It is known that a unimodal logic $L \supseteq$ K4 is locally tabular iff its 1-generated free algebra $\mathfrak{A}_L(1)$ is finite [3]. It allows us to formulate another corollary of Theorem 3:

Corollary 4. *Suppose L_0 is a canonical pretransitive logic with the ripe cluster property. Then for any logic $L \supseteq L_0$, L is locally tabular iff $\mathfrak{A}_L(1)$ is finite.*

Problem. *Does this equivalence hold for every modal logic?*

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Topology as faithful communication through relations

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Most topological concepts can be presented in a predicative and constructive framework, such as that of basic pairs (see [6]). A basic pair $(X, \Vdash, S)$ consists of a set X, a set S and a relation $\Vdash$ from X to S. X represents points, while S is a set of indexes for a basis of neighbourhoods of a topology on X. Elements of S are shortly called observables. An observable a in S is read as an index for the subset $\mathsf{ext}\, a$ of X of those x for which $x \Vdash a$. The presence of S makes the structure of a basic pair *symmetric*. Adding the two axioms

B1) $\mathsf{ext}\, a \cap \mathsf{ext}\, b = \bigcup \{ \mathsf{ext}\, c \mid \mathsf{ext}\, c \subseteq \mathsf{ext}\, a \cap \mathsf{ext}\, b \}$

B2) $(\forall x \in X)(\exists a \in S)(x \Vdash a)$

one obtains a predicative and constructive account of topological spaces.

We here add the idea that basic topological concepts, such as closure, interior and continuity, can be characterized as those which can be communicated faithfully between the side of points and the side of observables. This interpretation introduces a new intuitive point of view on topology which can shed light on unexpected links. The foundational framework assumed here is in the common core between the most relevant classical and constructive, predicative and impredicative, foundations, as in [4, 3].

Communication

Suppose an individual A wants to communicate with another individual B, but suppose A and B do not share the same language. However A and B both have their own collection of messages M_A and M_B which they use to represent information. Some messages in M_A are equivalent in the sense that they have the same meaning, and the same for messages in M_B. Hence A is equipped with a pair $(M_A, \sim_A)$ and B with a pair $(M_B, \sim_B)$. If we want A and B to communicate, then

1. B needs a decoding procedure Δ to transform every message in M_A into one of its own messages in M_B. This decoding procedure is good if it translates equivalent messages in M_A into equivalent messages in M_B.

2. Conversely A needs a decoding procedure ∇ to transform every message in M_B into a message in M_A. This decoding procedure is good if it translates equivalent messages in M_B into equivalent messages in M_A.

We can say that a message m in M_A is (faithfully) communicable if it satisfies the following requirement: if A communicates m to B, B translates it obtaining $\Delta(m)$ and then sends $\Delta(m)$ back to A, then the translation $\nabla(\Delta(m))$ by A of $\Delta(m)$ is equivalent to m, that is $\nabla(\Delta(m)) \sim_A m$.

Communication of subsets: interior and closure

Let $(X, \Vdash, S)$ be a basic pair. For all $a \in S$ and $x \in X$, we put $x \in \mathsf{ext}\, a$ if and only $a \in \Diamond x$ if and only if $x \Vdash a$. For all subsets D of X, we put $a \in \Diamond D$ if and only if $\mathsf{ext}\, a \between D$ and $a \in \Box D$ if and only if $\mathsf{ext}\, a \subseteq D$. For all subsets U of S, we put $x \in \mathsf{ext}\, U$ if and only if $\Diamond x \between U$ and $x \in \mathsf{rest}\, U$ if and only if $\Diamond x \subseteq U$.

Considering X and S as individuals with $(\mathcal{P}(X), =)$ and $(\mathcal{P}(S), =)$ as collections of messages respectively, we prove that for a subset D of X

1. D is open if and only if D is $(\Box, \mathsf{ext})$-communicable;

2. D is closed if and only if D is $(\Diamond, \mathsf{rest})$-communicable.

Communication of relations: continuity

It is natural (see e. g. [1], [2], [5]) to define a *continuous* relation from a basic pair $(X, \Vdash, S)$ to another one $(Y, \Vdash, T)$ as a relation r from X to Y such that for all $b \in T$ and $x \in X$

$$x \in \mathsf{r}^{-}\mathsf{ext}b \rightarrow (\exists a \in S)(x \Vdash a \wedge \mathsf{ext}a \subseteq \mathsf{r}^{-}\mathsf{ext}b).$$

One can take (X, Y) and (S, T) as individuals. Their collections of messages are $\mathsf{Rel}(X, Y)$ and $\mathsf{Rel}(S, T)$ equipped with suitable equivalence relations. In this context we prove that there exist natural decoding procedures σ and ρ such that a relation r is continuous if and only if r is (σ, ρ)-communicable.

Communication of elements and functions: convergence

We will finally discuss the notions of convergent subset and convergent relation (see [6]) as notions of communicable element and communicable function respectively.

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Completenes and Cocompletenes of the category of Cuntz Semigroups

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The category of Cuntz Semigroups, denoted Cu, is a category of ordered commutative monoids with a rich ordered structure. Its introduction and main motivation arises from the Classification program of separable, nuclear C*-algebras, since a certain invariant for an algebra A, namely the *Cuntz semigroup of A* which is denoted $Cu(A)$, has the structure of an object in this category.

After its introduction back in 1978 by J. Cuntz [Cun78], this semigroup received more attention when its non trivial ordered structure was used as a counterexample to existing conjectures regarding classification of C*-algebras (see [Tom08]). Shortly after, and with the aim of using $Cu(A)$ as a classification invariant, Coward, Elliott and Ivanescu [CEI08] proved that $Cu(A)$ had the structure of an ω-domain (with further compatibility properties) and introduced the category Cu as the target for the functor $Cu(-)$, proving in particular that the category has sequential limits and the functor $Cu(-)$ is sequentially continuous. This is important since many examples of C*-algebras are built as inductive limits of this kind.

In this note, we improve this result by proving that the category Cu is both complete and cocomplete. As for the functor $Cu(-)$, not all (co)limits are preserved, but there are some positive results, including a construction of ultraproducts in Cu.

Our approach is to to develop the constructions in some structurally simpler categories, and then use either a reflection or a coreflection functor to define them in Cu. This approach has already been carried out with success in the category Cu for the construction of tensor products (see [APT14]), and resembles the way the tensor product of C*-algebras A, B is carried out: one first develops the *algebraic* tensor product $A \hat{\otimes} B$, then defines there a pseudo-norm, and finally makes the completion.

In our case, our simpler categories will be categories of ordered semigroups with an auxiliary relation. In these, the constructions (products, limits, etc..) will extend the set theoretic constructions, and only the appropriate auxiliary relation will have to be chosen. Let us make this concrete:

In [APT14] we introduced a category of ordered semigroups with an auxiliary relation, and satisfying certain axioms, which we termed PreW. The category Cu is then, in a natural way, a full subcategory of PreW when the *way below relation* is chosen as the auxiliary relation. Moreover it was proved ([APT14, Theorem 3.1.10]) that Cu is a reflexive full subcategory of PreW by providing a reflector functor $\gamma \colon \text{Prew} \to \text{Cu}$, that is based on the round ideal completion (see [Law97]).

Hence, we obtain a category whose objects are structurally simpler, and from which Cu-semigroups can be obtained though a completion process. It is interesting to note, as observed

by K. Keimel in [Kei16], that similar notions had already been around in the field of Domain Theory with different names.

In a similar way, we introduced in [APT17] a different category of ordered semigroups with an auxiliary relation, and again certain properties, which we termed $\mathcal{Q}$. Whereas in PreW we *mainly* relaxed continuity notions, now certain *interpolation* notions are relaxed. Again Cu can naturally be viewed as a full subcategory of $\mathcal{Q}$, and it turns out that a functor $\tau \colon \mathcal{Q} \to \mathrm{Cu}$ can be defined which is now a coreflector, and we have ([APT14, Theorem 3.1.10]) that Cu is a coreflective full subcategory of $\mathcal{Q}$. In this case, we are not aware of a similar or equivalent notion in Domain Theory for the functor τ.

This categories, PreW and $\mathcal{Q}$ can be viewed in a larger category $\mathcal{P}$ (not as full subcategories though), and the constructions γ and τ are naturally equivalent when restricted to the intersection. Moreover, the functor τ, exactly as it is defined, can be extended to $\mathcal{P}$ and then, its restriction to PreW is naturally equivalent to γ. This is clarified in the following diagram.
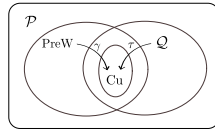
As mentioned above, doing the necessary constructions in either PreW or $\mathcal{Q}$, and using the fact that Cu is respectively a reflexive or coreflexive full subcategory, we obtain:

The Category of Cuntz semigroups is both complete and cocomplete.

With respect to the question of which of these (co)limit constructions are preserved by the functor, one can not expect a general affirmative answer. There examples of certain pullbacks and certain inverse limits which are not preserved. Nevertheless our techniques allow us to give a positive answer under certain hypothesis. For instance, using the reflector γ above, we prove that $\mathrm{Cu}(-)$ preserves arbitrary inductive limits (see [APT14]).

As a dual example, we can use the coreflector τ to prove that Cu preserves products. Moreover, given an ultrafilter ω in a set I, a notion of ultraproduct can be defined in Cu (as well as for C*-algebras), and prove that, if $(A_i)_{i \in I}$ is a family of C*-algebras then

$$\mathrm{Cu}(\prod_\omega A_i) \cong \prod_\omega \mathrm{Cu}(A_i).$$

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The Cuntz semigroup
and the category Cu

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The Cuntz semigroup of a C*-algebra is an important invariant in the structure and classification theory of C*-algebras. There has been a huge effort towards the classification of these objects over the last 30 years or so, using invariants of K-Theoretical nature. In general, this semigroup captures more information than K-theory but is often more delicate to handle. The aim of this talk is to introduce it and discuss various examples as well as its connections with domain theory ([6]).

Very briefly, if $\mathcal{H}$ is a (complex) Hilbert space, let us denote by $\mathbb{B}(\mathcal{H})$ the algebra of bounded, linear operators on $\mathcal{H}$. A C*-algebra A is any norm-closed, involutive subalgebra of $\mathbb{B}(\mathcal{H})$. These objects can also be described abstractly, as follows:

Definition. A C*-algebra is a complex Banach algebra A, equipped with an involution * such that $\|a^*a\| = \|a\|^2$ for any $a \in A$. Homomorphisms between C*-algebras are complex algebra maps that respect the involution. (They are automatically continuous.)

The classical definition of the Cuntz semigroup goes back to 1978, and is based on the notion of comparison for positive elements in C*-algebras, as introduced by Cuntz himself ([5]). We review this construction below

Definition (The Cuntz semigroup). Let A be a C*-algebra. For positive elements $a, b \in A$, say that a is *Cuntz subequivalent* to b (and write $a \precsim b$) provided there is a sequence (x_n) in A such that $a = \lim_{n \to \infty} x_n b x_n^*$. We say that a and b are *Cuntz equivalent* provided $a \precsim b$ and $b \precsim a$. In symbols, we write $a \sim b$.

Denote by $M_\infty(A) = \cup_{n=1}^\infty M_n(A)$, a directed union of all matrix algebras, and put $W(A) := M_\infty(A)_+/\sim$. We denote the class of an element in $W(A)$ by $[a]$, and we define

$$[a] + [b] = [\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)], \ [a] \le [b] \text{ if and only if } a \precsim b$$

With these operations, $W(A)$ becomes a partially ordered, abelian semigroup, termed the *classical Cuntz semigroup*. The *complete Cuntz semigroup* is constructed in a similar fashion, by replacing A by its tensor product with the compact operators. Namely, it is defined as $\mathrm{Cu}(A) := W(A \otimes \mathbb{K}(\mathcal{H}))$.

In its classical formulation, the Cuntz semigroup may be equipped with an auxiliary relation that makes it into a predomain, in the sense of [7]. This is defined as follows: $[a] \prec [b]$ if and only if $a \precsim (b - \epsilon)_+$ for some $\epsilon > 0$. In the case of the complete Cuntz semigroup, this relation agrees with the sequential way-below relation as in domain theory. In fact, $\mathrm{Cu}(A)$ is an ω-domain, a result that was established by Coward, Elliott, and Ivanescu (see [4]). More concretely, they introduced a category of semigroups, termed Cu and showed the following:

Theorem. *For any C*-algebra A, the Cuntz semigroup $\mathrm{Cu}(A)$ is an object in* Cu. *Moreover, the assignment $A \mapsto \mathrm{Cu}(A)$ is a sequentially continuous functor from the category of C*-algebras to the category* Cu.

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The continuity of this functor is very important as many examples in the theory arise as inductive limits. Hence, any valuable invariant must be continuous. That was not the case with the functor $A \mapsto W(A)$. However, this fact can be remedied by looking at the right domain and codomain categories where all the objects belong to. We briefly explain how the category Cu is constructed:

Definition. A Cu-*semigroup*, also called *abstract Cuntz semigroup*, is a positively ordered semigroup S that satisfies the following axioms (O1)-(O4):

(O1) Every increasing sequence $(a_n)_n$ in S has a supremum $\sup_n a_n$ in S.

(O2) For every element $a \in S$ there exists a sequence $(a_n)_n$ in S with $a_n \ll a_{n+1}$ for all $n \in \mathbb{N}$, and such that $a = \sup_n a_n$.

(O3) If $a' \ll a$ and $b' \ll b$ for $a', b', a, b \in S$, then $a' + b' \ll a + b$.

(O4) If $(a_n)_n$ and $(b_n)_n$ are increasing sequences in S, then $\sup_n(a_n + b_n) = \sup_n a_n + \sup_n b_n$.

Morphisms in the category are called Cu-morphisms, that is, maps that preserve addition, order, the zero element, the way-below relation and suprema of increasing sequences. Other maps of interest are the so-called *generalized* Cu-*morphism*, that is, maps as above that do not necessarily preserve the way-below relation.

One of our main results (further developed in [2] and [3]) is the following:

Theorem ([1]). *The following conditions hold true:*

(i) *There exists a category* W *that admits arbitrary inductive limits and such that the assignment $A \mapsto W(A)$ defines a continuous functor from the category C^*_{loc} of local C^*-algebras to the category* W.

(ii) *The category* Cu *is a full, reflective subcategory of* W. *Therefore,* Cu *admits arbitrary inductive limits.*

(iii) *There is a diagram, that commutes up to natural isomorphisms:*

$$
\begin{array}{ccc}
C^*_{\mathrm{loc}} & \xrightarrow{\ \mathrm{W}\ } & \mathrm{W} \\
\gamma \big\downarrow \updownarrow & & \updownarrow \big\downarrow \gamma \\
C^* & \xrightarrow{\ \mathrm{Cu}\ } & \mathrm{Cu}
\end{array}
$$

where $\gamma\colon \mathrm{W} \to \mathrm{Cu}$ is the reflection functor and $\gamma\colon C^*_{\mathrm{loc}} \to C^*$ is the completion functor that assigns to a local C^*-algebra its completion (which is a C^*-algebra). In particular, the assignment $A \mapsto \mathrm{Cu}(A)$ is also a continuous functor from the category of C^*-algebras to the category Cu *(with respect to arbitrary limits, thus extending the results in [4]).*

(iv) *The category* Cu *is symmetric monoidal.*

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A proper Multi-type display calculus for Semi De Morgan Logic

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Semi De Morgan algebras form a variety of normal distributive lattice expansions [7] introduced by H.P. Sankappanavar [16] as a common abstraction of De Morgan algebras and distributive pseudocomplemented lattices. A fully selfextensional logic SDM naturally arises from semi De Morgan algebras, which has been studied from a duality-theoretic perspective [13], from the perspective of canonical extensions [15], and from a proof-theoretic perspective [14]. Related to the proof theoretic perspective, the G3-style sequent calculus introduced in [14] is shown to be cut-free. However, the proof of cut elimination is quite involved, due to the fact that, along with the standard introduction rules for conjunction and disjunction, this calculus includes also introduction rules under the scope of structural connectives. These difficulties can be explained by the fact that the axiomatization of SDM is not analytic inductive in the sense of [10, Definition 55], due to the presence of the following axioms

$$(a \wedge b)'' = a'' \wedge b'' \qquad a' = a'''.$$

In order to address these difficulties, an analytic calculus for SDM is introduced in [9], which is sound, complete, conservative, and enjoys cut elimination and subformula property proved by means of a general Belnap-style method.

This calculus is a proper multi-type display calculus according to the definition of [12, Definition A.1]. The methodology of multi-type calculi has been introduced in [8, 3], motivated by proof-theoretic semantic considerations [5], and further developed in [6, 4, 1, 11].

Our main insights come from algebra. Specifically, we introduce an equivalent representation of semi De Morgan algebras as the following heterogeneous algebras (in the sense of [2]): structures $\mathbb{H} = (\mathbb{L}, \mathbb{D}, f, h)$ such that:

$\mathbb{L}$ is a bounded distributive lattice,

$\mathbb{D}$ is a De Morgan algebra,

$h : \mathbb{L} \to \mathbb{D}$ is a surjective lattice homomorphism,

$f : \mathbb{D} \to \mathbb{L}$ is a finitely meet-preserving order embedding which preserves the bottom element,

$h(f(\alpha)) = \alpha$ for every $\alpha \in \mathbb{D}$.

We show that any semi De Morgan algebra $\mathbb{A}$ gives rise to one such heterogeneous algebra $\mathbb{A}^+$, and conversely any heterogeneous algebra $\mathbb{H}$ as above gives rise to one semi De Morgan algebra $\mathbb{H}_+$, so that

$$\mathbb{A} \cong (\mathbb{A}^+)_+ \qquad \mathbb{H} \cong (\mathbb{H}_+)^+.$$

This equivalence motivates a reformulation of the logic SDM into the multi-type language canonically interpreted in the heterogeneous algebras defined above. In this reformulation, all the axioms are analytic inductive. This makes it possible to obtain a proper multi-type display calculus for SDM by suitably generalizing the method introduced in [10].

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Effect algebras as colimits of finite Boolean algebras *

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Effect algebras [1] are positive, cancellative, unital partial abelian monoids. The category of effect algebras is denoted by **EA**.

Let us denote the initial segment of natural numbers $\{1, \ldots, n\}$ by $[n]$. Note that $[0] = \emptyset$. Let **FinBool** be the full subcategory of the category **FinBool** of Boolean algebras spanned by the set of objects $\{2^{[n]} : n \in \mathbb{N}\}$. **FinBool** is a small, full subcategory of the category of effect algebras.

It was proved by Staton and Uijlen in [6] that every effect algebra A can be faithfully represented by a presheaf $P(A)$ on the category **FinBool**. Explicitely, for an effect algebra A the presheaf $P(A) :$ **FinBool** $\to$ **Set** maps every object $2^{[n]}$ to the homset $\mathbf{EA}(2^{[n]}, A)$ and every arrow $f : 2^{[n]} \to 2^{[m]}$ the mapping $P(A)(f) : P(2^{[m]}) \to P(2^{[n]})$ defined as the precomposition by f. This determines a functor $P : \mathbf{EA} \to [\mathbf{FinBool}^{op}, \mathbf{Set}]$.

The *category of tests* of an effect algebra A is the category of elements of the presheaf $P(A)$, in symbols $el(P(A))$. We note that every object of $el(P(A))$ is just a morphism of effect algebras $g : 2^{[n]} \to A$ (a *finite observable*) and these are in a one-to-one correspondence with finite sequences $(a_i)_{i \in [n]} \subseteq A$ with $\Sigma_{i \in [n]} a_i = 1$, that are called *tests* [2, 3]. The morphisms then correspond to refinements of tests.

It is clear that for every effect algebra A, there is a functor $D_A : el(P(A)) \to \mathbf{EA}$ that maps every $g : 2^{[n]} \to A$ to its domain $2^{[n]}$. As proved in [6], **FinBool** is a dense subcategory of **EA**. This implies that every effect algebra A is a colimit of its D_A. Moreover, since **EA** is cocomplete [4], we may apply a general argument [5, Theorem I.5.2] to prove that there is a reflection $[\mathbf{FinBool}^{op}, \mathbf{Set}] \to \mathbf{EA}$ left adjoint to P.

Recall, that an effect algebra satisfies the *Riesz decomposition property* (abbreviated by RDP) if and only if, for all u, v_1, v_2 such that $u \leq v_1 + v_2$ there are u_1, u_2 such that $u = u_1 + u_2$, $u_1 \leq v_1$, $u_2 \leq v_2$. Every Boolean algebra and every effect algebra arising from an MV-algebra satisfies the RDP.

Theorem 1. *An effect algebra A satisfies the RDP if and only if every span in $el(P(A))$ can be extended to a commutative square.*

Recall, that an effect algebra is an *orthoalgebra* if and only if, for every element a, the existence of $a + a$ implies that $a = 0$.

Theorem 2. *An effect algebra is an orthoalgebra if and only if for every parallel pair of morphisms in $f_1, f_2 : g \to g'$ in $el(P(A))$ there is a coequalizing morphism $q : g' \to h$ such that $q \circ f_1 = q \circ f_2$.*

Theorem 3. *An effect algebra A is a Boolean algebra if and only if $el(P(A))$ is filtered.*

Let A be an effect algebra. For every Boolean algebra B, a morphism $f : B \to A$ gives rise to a morphism $el(P(f)) : el(P(B)) \to el(P(A))$ in **Cat**. Since $el(P(B))$ is filtered, every such f gives rise to an *ind-object* of the category $el(P(A))$.

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Intermediate logics admitting structural hypersequent calculi

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A recent trend in proof theory of non-classical logics is to develop systematic and effective procedures to obtain well-behaved proof calculi for uniformly defined classes of non-classical logics. Such procedures will, given a certain kind of specifying data for a logic L, produce an analytic proof calculi with respect to which L is sound and complete. Many procedures fitting this general template already exist, e.g., in the context of sequent calculi for substructural logics [7]; hypersequent calculi for substructural logics [6, 8]; hypersequent calculi for modal logics [11, 12]; labelled sequent calculi for modal and intermediate logics [13, 10] and display calculi for extensions of bi-intuitionistic logic [9].

So far less attention has be given to obtaining negative results demarcating the classes of logics for which such procedures may succeed. See, however, [6, 7, 8] for examples of such negative results. Ideally we would like, given a uniform procedure for obtaining proof calculi of a certain type, a complete classification of the logics for which this procedure may successfully be applied.

We focus on the case of intermediate logics, i.e., consistent extensions of propositional intuitionistic logic **IPC**. For these logics Ciabattoni et al. [6, 8] have isolated a class of axioms, called $\mathcal{P}_3$, which may effectively be translated into so-called structural hypersequent calculi with the property that adding them to the hypersequent calculus **HLJ** for **IPC** preserves cut-admissibility.[1] However, since the class $\mathcal{P}_3$ is not closed under provable equivalence, semantic notions must be introduced in order to determine the class of intermediate logics which can be axiomatised by $\mathcal{P}_3$-formulas and therefore be given cut-free structural hypersequent calculi.

Our contribution consists in introducing criteria for when a given intermediate logic admits a structural hypersequent calculi for which the cut-rule is admissible. These criteria are presented in terms of the algebraic semantics as well as the Kripke semantics. Concretely, we provide the following algebraic characterisation of intermediate logics for which a structural cut-free hypersequent calculus may be provided.

Theorem 1. *An intermediate logic L admits a cut-free structural hypersequent calculus precisely when the corresponding variety of Heyting algebras $\mathbb{V}(L)$ is closed under taking bounded meet-semilattices of its subdirectly irreducible members.*

We note that the requirement that $\mathbf{A} \in \mathbb{V}(L)$, whenever $\mathbf{A}$ is a bounded meet-semilattice of some subdirectly irreducible $\mathbf{B} \in \mathbb{V}(L)$ is a strengthening of the stability condition explored by Bezhanishvili et al. [2, 3, 4, 1]. Our findings may thus be seen as further corroborating the connection between proof-theory and stable logics [5].

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[1] In fact, this result by Ciabattoni et al. holds in the more general setting of substructural logics.

Furthermore, we show that any intermediate logic with a structural hypersequent calculus is necessarily sound and complete with respect to an elementary class of Kripke frames. In fact the first-order frame conditions determining such intermediate logics may be classified. These are certain positive Π_2-sentence in the language of Kripke frames, the modal analogue of which are found in the work of Lahav [11] where they are used to construct analytic hypersequent calculi for modal logics.

Finally, our criteria also allow us to show that certain well-known intermediate logics, such as $\mathbf{BD}_n$, for $n \geq 2$, cannot be axiomatised over $\mathbf{HLJ}$ by structural hypersequent rules.

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Infinitary Propositional Logics
and Subdirect Representation

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In this talk we focus on some aspects of the algebraic approach to infinitary propositional logics (for a related development of fuzzy logics and abstract algebraic logic see e.g. [2, 5]). By finitarity, we mean the property of a logic saying that any deduction from a set of premises can be carried out in finitely many steps; infinitary logics are, thus, logics in which some derivations need infinitely many steps (or, equivalently, logics that need some inference rules with infinitely-many premises in their Hilbert-style presentation). Although the majority of logics studied in the literature are finitary, there are prominent natural examples of infinitary ones like the infinitary Łukasiewicz logic of the standard $[0, 1]$ chain or, analogously, the infinitary product logic. For the purpose of this talk, we will refer to them as main examples of infinitary logics.

In [6] we proposed a new hierarchy of infinitary logics based on their completeness properties. Every finitary logic is well-known (see e.g. [5]) to be complete w.r.t. the class of all its relatively subdirectly irreducible models (*RSI-completeness*) and hence also w.r.t. *finitely* relatively subdirectly irreducible models (*RFSI-completeness*). However, not even the later is true for infinitary logics in general. We studied an intermediate (syntactical) notion between finitarity and RFSI-completeness, namely the property that every theory of the logic is the intersection of finitely $\cap$-irreducible theory, i.e. finitely $\cap$-irreducible theories form a basis for all theories. This property is called the *intersection prime extension property* (IPEP) and had already been important in the study of generalized implication and disjunctive connectives (see [1, 4, 3]). The hierarchy also includes a natural stronger extension property that refers to $\cap$-irreducible theories. Their relations are depicted in the figure below. For example both infinitary product and Łukasiewicz logic are shown to have the CIPEP.

Finitary $\rightleftarrows$ IPEP $\rightleftarrows$ RFSI-complete

CIPEP $\rightleftarrows$ RSI-complete
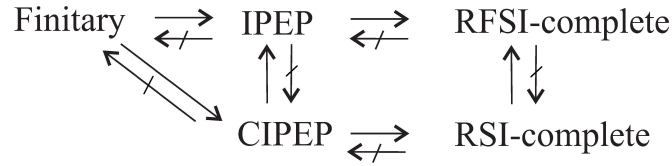
A natural matricial semantics for a propositional logic is that given by its reduced models (which happen to be based on the expected algebras in prominent cases, that is, Boolean algebras for classical logic, Heyting algebras for intuitionistic logic, etc.). For finitary logics such semantics has a powerful property: If L is finitary, then each member of the class of its reduced models $\mathbf{MOD}^*(\mathrm{L})$ can be represented as a subdirect product of reduced subdirectly irreducible models, i.e. $\mathbf{MOD}^*(\mathrm{L}) = \mathbf{P}_{\mathrm{SD}}(\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}})$. This property can be seen as a generalization to matrices of the well-known Birkhoff's representation theorem.

We will discuss the following transferred versions of the syntactical properties: $\mathcal{L}$ has the τ-IPEP whenever for each matrix model $\langle \boldsymbol{A}, F \rangle$ the filter F is the intersection of a collection of (finitely) $\cap$-irreducible L-filters on the algebra $\boldsymbol{A}$, and analogously for τ-CIPEP with $\cap$-irreducible L-filters. Then we can prove the following characterization theorem:

Theorem 1. *For any logic* L *the following are equivalent*

1. L *is protoalgebraic and has the τ-CIPEP.*

2. L *is protoalgebraic and the* CIPEP *holds on any free algebra* $\boldsymbol{Fm}_{\mathrm{L}}(\kappa)$.

3. *Each member of* $\mathbf{MOD}^*(\mathrm{L})$ *is a subdirect product of subdirectly irreducible members.*

An analogous theorem can be proved for τ-IPEP using finitely subdirectly irreducible models. Also the characterization can written in algebraic terms: κ-generalized quasivarieties are those classes of algebras axiomatized by quasirules with less than κ premises and they are subdirectly representable if and only if an algebraic analog of τ-(C)IPEP holds: The identity congruence on any algebra in the generalized quasivariety can be written as the intersection of a family of $\cap$-irreducible congruences.

We will prove the following about the two mentioned examples of infinitary logics:

1. The infinitary product logic has the CIPEP, but not the τ-IPEP.

2. The infinitary Łukasiewicz logic enjoys the subdirect representation property.

From the first one (which is proved using the fact the each Archimedean product algebras is embeddable into the standard product algebra), we conclude that neither CIPEP nor IPEP imply in general their transferred counterparts. For the proof of the second claim the essential step is to show that any natural extension (variant of the logic with a larger set of variables) is still strongly complete w.r.t. the standard semantics; this involves a topological argument, which is only possible because the connectives of Łukasiewicz logic are continuous w.r.t. the standard interval topology. Observe that the second claim implies that, unlike the other example, the infinitary Łukasiewicz logic has the τ-CIPEP.

Moreover, we will build a variant with rational constants of the infinitary Łukasiewicz logic and show that it has τ-IPEP, i.e. its models are subdirect products of chains, but it is not RSI-complete. Putting all these facts together we will conclude that τ-CIPEP and τ-IPEP are distinct properties and are also different from all the remaining properties seen in the figure, and hence we will obtain a finer hierarchy for infinitary propositional logics.

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Reduced Rickart Rings and Skew Nearlattices

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1   Introduction

Sussman and Subrahmayam proved in [8] and [7] that a certain kind of reduced ring (called *m-domain ring* in [7]) can be decomposed into a collection of disjoint subsets which are closed with respect to multiplication. In [6] it is shown that reduced Rickart rings and m-domain rings are the same thing. This talk is about the order structure of a reduced Rickart ring's decomposition into disjoint semigroups.

Cīrulis proved in [3] that every right normal skew nearlattice can be regarded as a structure called *strong semilattice of semigroups*, and in [5] he shows that any reduced Rickart ring admits a structure of right normal skew nearlattice. It turns out that this strong semilattice of semigroups arises from the semigroup decomposition of [7].

1.1   Reduced Rickart rings

A ring is called *reduced* if it has no nonzero nilpotent elements. It can be easily checked that for all elements x, y of a reduced ring R, $xy = 0$ if and only if $yx = 0$.

The *Abian partial order* on a reduced ring is defined as $x \leq y$ if and only if $xy = xx$. It was proved in [2] that this relation on an arbitrary ring is a partial order if and only if the ring is reduced.

Definition 1.1. [1] A unitary ring R is called a *right Rickart ring* iff for every $a \in R$ there is an idempotent $e \in R$ such that, for all $x \in R$,

$$ax = 0 \quad \text{iff} \quad ex = x.$$

Dually, it is called *left Rickart* iff for every $a \in R$ there is an idempotent $f \in R$ such that, for all $x \in R$, $xa = 0$ iff $xf = x$. A *Rickart ring* is a ring wich is both right and left Rickart.

In a reduced (right or left) Rickart ring R the idempotents e and f from Definition 1.1 are unique and coincide.

1.2   Skew nearlattices

A meet-semilattice is called *nearlattice* if it is finitely bounded complete (i.e., whenever a finite subset has an upper bound, it also has a least upper bound). Skew nearlattices are a generalization of nearlattices. Instead of a meet operation they have an associative and idempotent operation that might not be commutative.

Definition 1.2 ([4, 5]). Let S be a finitely bounded complete poset and let $\vee$ denote its join operation. If $*$ is an associative operation on S such that, for all $x, y \in S$, $x \vee y = y$ if and only if $x * y = x$, then the partial algebra $\langle S, *, \vee \rangle$ is called a *(right) skew nearlattice* (see [4]).

For any skew nearlattice $\langle S, *, \vee \rangle$, the reduct $\langle S, * \rangle$ obviously is a band (i.e., an idempotent semigroup). A band $\langle S, * \rangle$ is called *singular* iff $x * y = y$ for all $x, y \in S$ ([3]). A skew nearlattice is called *singular* if the underlying band is singular.

Example 1.3. It was proved in [5] that, given a reduced Rickart ring R equipped with an operation $a \overleftarrow{\wedge} b := a''b$, the partial algebra $\langle R, \vee, \overleftarrow{\wedge} \rangle$ is a skew nearlattice ($\vee$ denotes the join with respect to the natural order of the semigroup $\langle R, \overleftarrow{\wedge} \rangle$, which coincides with the Abian order). The operation $\overleftarrow{\wedge}$ is therefore called *skew meet*.

Definition 1.4 ([4])**.** Let T be a meet-semilattice and let $\{A_s \mid s \in T\}$ be a family of disjoint semigroups such that, for all $s, t \in T$, the inequality $s \leq t$ implies that there is a semigroup homomorphism $f_s^t \colon A_t \to A_s$ such that the homomorphisms f_t^t are the identity maps, and for all $r, s, t \in T$, if $r \leq s \leq t$, then $f_s^t f_r^s = f_r^t$.

On the union $A = \bigcup_{s \in T} A_s$ of all the semigroups we define an operation $\overleftarrow{\sqcap}$: If $x \in A_s$ and $y \in A_t$, and $\cdot$ denotes the multiplication of the semigroup $A_{s \wedge t}$, then $x \overleftarrow{\sqcap} y := f_{s \wedge t}^s(x) \cdot f_{s \wedge t}^t(y)$. Then we call the algebra $\langle A, \overleftarrow{\sqcap} \rangle$ a *strong semilattice of the semigroups* $\{A_s\}_{s \in T}$.

2 Skew nearlattices in a reduced Rickart ring

Let U be the set of non-zero-divisors of a reduced Rickart ring R. As shown in [6], we can apply the results on m-domain rings from [7] to R. Therefore we know that the ring R can be decomposed into semigroups of the form Ue (with the usual ring multiplication), where e is an idempotent. Then the set Ue equipped with the skew meet operation $\overleftarrow{\wedge}$, the corresponding partial join operation $\vee$ and the ring multiplication $\cdot$ forms a multiplicative singular skew nearlattice $\langle Ue, \vee, \overleftarrow{\wedge}, \cdot \rangle$ (i.e., $\langle Ue, \vee, \overleftarrow{\wedge} \rangle$ is a singular skew nearlattice and $\langle Ue, \cdot \rangle$ is a monoid).

If the semigroups of a strong semilattice of semigroups happen to be multiplicative skew nearlattices and the corresponding semigroup homomorphisms are actually homomorphisms of multiplicative skew nearlattices, then we call this a *strong semilattice of multiplicative skew nearlattices*. Now the whole ring admits such a structure:

Theorem 2.1. *If R is a reduced Rickart ring whose skew meet operation is denoted by $\overleftarrow{\wedge}$, and $\cdot$ is the ring multiplication, then $\langle R, \overleftarrow{\wedge}, \cdot \rangle$ is a strong semilattice of the multiplicative skew nearlattices $\langle Ue, \vee, \overleftarrow{\wedge}, \cdot \rangle$.*

There arises the question how much of the structure of a reduced Rickart ring can be "reconstructed" from its strong semilattice of multiplicative skew nearlattices. Given a strong semilattice of multiplicative skew nearlattices that satisfies some additional conditions, we can define a binary operation and constants 0 and 1 on the union of the skew nearlattices such that the resulting algebra is a reduced *Baer semigroup*, i.e., a reduced semigroup with zero such that, for every $a \in S$, there are idempotents $e, f \in S$ such that $ax = 0$ if and only if $ex = x$, and $xa = 0$ if and only if $xf = x$. A Baer semigroup is what is left of a Rickart ring if we "forget" about the addition.

Furthermore, the skew nearlattice of Example 1.3 can be shown to be isomorphic to a skew nearlattice of partial functions.

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States of free product algebras and their integral representation

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In his monograph [6], Hájek established theoretical basis for a wide family of fuzzy (thus, many-valued) logics which, since then, has been significantly developed and further generalized, giving rise to a discipline that has been named as Mathematical Fuzzy logic (MFL). Hájek's approach consists in fixing the real unit interval as standard domain to evaluate atomic formulas, while the evaluation of compound sentences only depends on the chosen operation which provides the semantics for the so called *strong conjunction* connective. His general approach to fuzzy logics is grounded on the observation that, if strong conjunction is interpreted by a continuous t-norm [7], then any other connective of a logic has a natural standard interpretation.

Among continuous t-norms, the so called Łukasiewicz, Gödel and product t-norms play a fundamental role. Indeed, Mostert-Shields' Theorem [7] shows that a t-norm is continuous if and only if it can be built from the previous three ones by the construction of ordinal sum. In other words, a t-norm is continuous if and only if it is an ordinal sum of Łukasiewicz, Gödel and product t-norms. These three operations determine three different algebraizable propositional logics (bringing the same names as their associated t-norms), whose equivalent algebraic semantics are the varieties of MV, Gödel and product algebras respectively.

Within the setting of MFL, *states* were first introduced by Mundici [8] as maps averaging the truth-value in Łukasiewicz logic. In his work, states are functions mapping any MV-algebra **A** in the real unit interval $[0, 1]$, satisfying a normalization condition and the additivity law. Such functions suitably generalize the classical notion of finitely additive probability measures on Boolean algebras, besides corresponding to convex combinations of valuations in Łukasiewicz propositional logic.

One of the most important results of MV-algebraic state theory is Kroupa-Panti theorem [9, §10], a representation theorem showing that every state of an MV-algebra is the Lebesgue integral with respect to a regular Borel probability measure. Moreover, the correspondence between states and regular Borel probability measures is one-to-one.

Many attempts of defining states in different structures have been made (see for instance [5, §8] for a short survey). In particular, in [2], the authors provide a definition of state for the Lindenbaum algebra of Gödel logic that results in corresponding to the integration of the truth value functions induced by Gödel formulas, with respect to Borel probability measures on the real unit cube $[0, 1]^n$. Moreover, such states correspond to convex combinations of finitely many truth-value assignments.

The aim of this contribution is to introduce and study states for the Lindenbaum algebra of product logic, the remaining fundamental many-valued logic for which such a notion is still lacking.

Recall that up to isomorphism (see [1, Theorem 3.2.5]) every element of the free n-generated product algebra $\mathscr{F}_{\mathbb{P}}(n)$ is a *product logic function*, i.e. $[0, 1]$-valued function defined on $[0, 1]^n$ associated to a

product logic formula built over n propositional variables.

Definition 1. A *state of* $\mathscr{F}_{\mathbb{P}}(n)$ will be a map $s : \mathscr{F}_{\mathbb{P}}(n) \to [0,1]$ satisfying the following conditions:

S1. $s(1) = 1$ and $s(0) = 0$,

S2. $s(f \wedge g) + s(f \vee g) = s(f) + s(g)$,

S3. If $f \leq g$, then $s(f) \leq s(g)$,

S4. If $f \neq 0$, then $s(f) = 0$ implies $s(\neg\neg f) = 0$.

By the previous definition, it follows that states of a free product algebra are lattice valuations (axioms S1–S3) as introduced by Birkhoff in [3].

It is worth noticing that product logic functions in $\mathscr{F}_{\mathbb{P}}(n)$ are not continuous, unlike the case of free MV-algebras, and the free n-generated product algebra is not finite, unlike the case of free Gödel algebras. However, there is a *finite* partition of their domain in σ-locally compact subsets, depending on the Boolean skeleton of $\mathscr{F}_{\mathbb{P}}(n)$, upon which the restriction of each product function is continuous. By exploiting this fact, we are able to prove the following integral representation theorem, where we show that our states interestingly represent an axiomatization of the Lebesgue integral as an operator acting on product logic formulas.

Theorem 2 (Integral representation). *For a $[0,1]$-valued map s on $\mathscr{F}_{\mathbb{P}}(n)$, the following are equivalent:*

(i) s is a state,

(ii) there is a unique regular Borel probability measure μ such that, for every $f \in \mathscr{F}_{\mathbb{P}}(n)$,

$$s(f) = \int_{[0,1]^n} f \, d\mu.$$

Moreover, and quite surprisingly since in the axiomatization of states the product t-norm operation is only indirectly involved via a condition concerning double negation, we prove that every state belongs to the convex closure of product logic valuations. Indeed, in particular, extremal states will result to correspond to the homomorphisms of $\mathscr{F}_{\mathbb{P}}(n)$ into $[0,1]$, that is to say, to the valuations of the logic. Indeed, let $\delta : \mathscr{S}(n) \to \mathscr{M}(n)$ be the map that associates to every state its corresponding measure via Theorem 2.

Theorem 3. *The following are equivalent for a state $s : \mathscr{F}_{\mathbb{P}}(n) \to [0,1]$*

1. s is extremal;

2. $\delta(s)$ is a Dirac measure;

3. s is a product homomorphism.

Thus, since the state space $\mathscr{S}(n)$ is a convex subset of $[0,1]^{\mathscr{F}_{\mathbb{P}}(n)}$, via Krein-Milman Theorem we obtain the following:

Corollary 4. *For every $n \in \mathbb{N}$, the state space $\mathscr{S}(n)$ is the convex closure of the set of product homomorphisms from $\mathscr{F}_{\mathbb{P}}(n)$ into $[0,1]$.*

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Lawson Topology as the Space of Located Subsets

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This work is a contribution to the constructive theory of Lawson topology, which plays a fundamental role in the theory of topological semilattices [4]. A constructive construction of Lawson topologies (and more generally patch topologies) in the setting of point-free locale theory has already been given by Escardó [3] using the frames of perfect nuclei. There is also a predicative construction by Coquand and Zhang [2] based on entailment relations. However, a geometric notion behind those constructions has not been articulated.

The aim of this work is to clarify the spatial (or geometric) notion behind the Lawson topology on a continuous lattice from a constructive point of view. Our main observation is that the Lawson topology on a continuous lattice has a clear geometric meaning that is of fundamental importance in constructive mathematics, which can be roughly put as follows:

Theorem 1. *The Lawson topology on a continuous lattice is the space of its located subsets.*

In the rest of this abstract, we make the statement of the above theorem precise.

A predicative notion of continuous lattice, *continuous cover*, is given by a triple $(S, \lhd, \mathsf{wb})$ of a set S, a covering relation $\lhd \subseteq S \times \mathcal{P}(S)$, and a base of the way-below relation $\mathsf{wb} \colon S \to \mathcal{P}(S)$ such that

1. $a \in U \implies a \lhd U$;

2. $a \lhd U \ \& \ (\forall b \in U)\, b \lhd V \implies a \lhd V$;

3. $a \lhd \mathsf{wb}(a)$;

4. $a \lhd U \implies (\forall b \in \mathsf{wb}(a))\,(\exists \{a_0, \ldots, a_{n-1}\} \subseteq U)\, a \lhd \{a_0, \ldots, a_{n-1}\}$.

Then, define a subset $V \subseteq S$ to be *located* if

1. $a \lhd \{a_0, \ldots, a_{n-1}\} \ \& \ a \in V \implies (\exists i < n)\, a_i \in V$;

2. $a \in V \implies (\exists b \ll a)\, b \in V$;

3. $a \ll b \implies a \notin V \vee b \in V$,

where $\ll$ is the *way-below* relation:

$$ a \ll b \overset{\text{def}}{\iff} (\exists \{a_0, \ldots, a_{n-1}\} \subseteq \mathsf{wb}(b))\, a \lhd \{a_0, \ldots, a_{n-1}\}. $$

Classically the third condition is superfluous; constructively however, it is non-trivial and of significant importance as the following examples of located subsets suggest; decidable subsets of any set, spreads on the binary tree, Dedekind reals, semi-located subsets of a locally compact metric spaces. See Troelstra and van Dalen [6] and Bishop [1] for details about those examples.

In the setting of continuous cover, one can define an analogue of the notion of Dedekind cut. A pair (L, U) of subsets of S is a *cut* if

1. $a \lhd \{a_0, \ldots, a_{n-1}\} \ \& \ a \in U \implies (\exists i < n)\, a_i \in U$;

2. $a \in U \implies (\exists b \ll a)\, b \in U$;

3. $a \lhd \{a_0, \ldots, a_{n-1}\} \subseteq L \implies a \in L$;

4. $a \in L \implies (\exists \{a_0, \ldots, a_{n-1}\} \gg a)\, \{a_0, \ldots, a_{n-1}\} \subseteq L$;

5. $a \ll b \implies a \in L \vee b \in U$;

6. $L \cap U = \emptyset$.

Proposition 2. *There is a bijection between located subsets and cuts.*

The notion of cut is geometric in the sense of propositional geometric theory [7, Chapter 2]. This leads us to the following definition of Lawson topology as a formal space [5].

Definition 3. The *Lawson topology* of a continuous cover $\mathcal{S}$ is the formal space $\mathcal{L}(\mathcal{S})$ presented by the geometric theory $T_{\mathcal{L}}$ whose models are cuts of $\mathcal{S}$.

A *perfect map* between continuous covers $(S, \lhd, \mathsf{wb})$ and $(S', \lhd', \mathsf{wb}')$ is a relation $r \subseteq S \times S'$ such that

1. $a \lhd' U \implies r^{-}\{a\} \lhd r^{-}U$;

2. $a \ll' b \implies r^{-}\{a\} \ll r^{-}\{b\}$.

Let **CCov** be the category of continuous covers and perfect maps, and let **KReg** be the category of compact regular formal spaces. Then, the universal property of Lawson topology is recovered in the setting of continuous covers.

Theorem 4. *The construction $\mathcal{L}(\mathcal{S})$ extends to a functor $\mathcal{L}\colon \mathbf{CCov} \to \mathbf{KReg}$ which is right adjoint to the forgetful functor $U\colon \mathbf{KReg} \to \mathbf{CCov}$.*

The above adjunction induces a monad $K_{\mathcal{L}}$ on **KReg**.

Theorem 5. *The monad $K_{\mathcal{L}}$ induced by the adjunction is naturally isomorphic to the Vietoris monad on **KReg**.*

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Weak Fraïssé categories

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We develop category-theoretic framework for the theory of limits of weak Fraïssé classes. Fraïssé theory now belongs to the folklore of model theory, however it actually can easily be formulated in pure category theory, see [3]. The crucial point is the notion of *amalgamation*, saying that two embeddings of a fixed object can be joined by further embeddings into a single one. More precisely, for every two arrows f, g with the same domain there should exist compatible arrows f', g' with the same codomain, such that $f' \circ f = g' \circ g$. A significant relaxing of the amalgamation property, called the *weak amalgamation property* has been discovered by Ivanov [1] and independently by Kechris and Rosendal [2] during their study of generic automorphisms in model theory. It turns out that the weak amalgamation property is sufficient for constructing special objects satisfying certain variant of homogeneity.

Let $\mathfrak{K}$ be a fixed category. We say that $\mathfrak{K}$ has the *weak amalgamation property* (briefly: WAP) if for every $z \in \mathrm{Obj}(\mathfrak{K})$ there exists a $\mathfrak{K}$-arrow $e : z \to z'$ such that for every $\mathfrak{K}$-arrows $f : z' \to x$, $g : z' \to y$ there are $\mathfrak{K}$-arrows $f' : x \to w$, $g' : y \to w$ satisfying $f' \circ f \circ e = g' \circ g \circ e$. In other words, the square in the diagram

$$
\begin{array}{ccc}
y & \xrightarrow{\ g'\ } & w \\
{\scriptstyle g}\big\uparrow & & \big\uparrow{\scriptstyle f'} \\
z' & \xrightarrow{\ f\ } & x
\end{array}
$$

with $e : z \nearrow z'$

may not be commutative.

We work within the following setup. Namely, $\mathfrak{K}$ is a fixed category, $\mathfrak{L} \supseteq \mathfrak{K}$ is a bigger category such that $\mathfrak{K}$ is full in $\mathfrak{L}$ and the following conditions are satisfied:

(L0) All $\mathfrak{L}$-arrows are monic.

(L1) Every $\mathfrak{L}$-object is the co-limit of a sequence in $\mathfrak{K}$.

(L2) Every $\mathfrak{K}$-object is ω-small in $\mathfrak{L}$.

We say that $\mathfrak{K}$ is *directed* if for every $x, y \in \mathrm{Obj}(\mathfrak{K})$ there are $v \in \mathrm{Obj}(\mathfrak{K})$ and $\mathfrak{K}$-arrows $i : x \to v$, $j : y \to v$. In model theory, this is usually called the *joint embedding property*. We say that $\mathfrak{K}$ is *weakly dominated* by a subcategory $\mathfrak{S}$ if the following conditions are satisfied.

(D1) For every $x \in \mathrm{Obj}(\mathfrak{K})$ there is $f \in \mathfrak{K}$ such that $\mathrm{dom}(f) = x$ and $\mathrm{cod}(f) \in \mathrm{Obj}(\mathfrak{S})$.

(D2) For every $y \in \mathrm{Obj}(\mathfrak{S})$ there exists $j : y \to y'$ in $\mathfrak{S}$ such that for every $\mathfrak{K}$-arrow $f : y' \to z$ there is a $\mathfrak{K}$-arrow $g : z \to u$ satisfying $g \circ f \circ j \in \mathfrak{S}$.

We say that $V \in \mathrm{Obj}(\mathfrak{L})$ is $\mathfrak{K}$-*universal* if for every $\mathfrak{K}$-object x there is an $\mathfrak{L}$-arrow from x to V. Finally, we say that $\mathfrak{K}$ is a *weak Fraïssé category* if it is directed, has the WAP, and is weakly dominated by a countable subcategory. An $\mathfrak{L}$-object V is *weakly $\mathfrak{K}$-homogeneous* if for every $\mathfrak{K}$-object a there is a $\mathfrak{K}$-arrow $e : a \to b$ such that for every $\mathfrak{L}$-arrows $i : b \to U$, $j : b \to U$ there exists an automorphism $h : U \to U$ satisfying $h \circ i \circ e = j \circ e$. This is illustrated in the following diagram

in which the triangle is not necessarily commutative.

$$
\begin{array}{ccc}
 & & U \\
 & \nearrow^{\,j} & \uparrow \\
a \xrightarrow{\;e\;} b & & h \\
 & \searrow_{\,i} & \downarrow \\
 & & U
\end{array}
$$
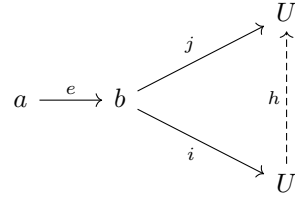
Below is the main result.

Theorem 1. *Let $\mathfrak{K} \subseteq \mathfrak{L}$ be as above. The following conditions are equivalent.*

(a) *$\mathfrak{K}$ is a weak Fraïssé category.*

(b) *There exists a $\mathfrak{K}$-universal weakly $\mathfrak{K}$-homogeneous object in $\mathfrak{L}$.*

Furthermore, a weakly $\mathfrak{K}$-homogeneous object is unique up to isomorphisms, as long as it exists.

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Tensor products of Cuntz semigroups

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Abstract Cuntz semigroups, also called Cu-semigroups, are domains with a compatible semigroup structure. They naturally arise as invariants of C^*-algebras. (A C^*-algebra is a norm-closed $*$-algebra of operators on a Hilbert space - it can be thought of as a noncommutative topological space.) Given a C^*-algebra A, its (concrete) Cuntz semigroup $\mathrm{Cu}(A)$ is constructed from positive elements in matrix algebras over A (and in fact, from the stabilization $A \otimes \mathcal{K}$) in a similar way that the K-theory group $K_0(A)$ is constructed from projections in matrix algebras over A.

The Cuntz semigroup plays an important role in the ongoing program to classify simple, amenable C^*-algebras. This classification program was initiated by George Elliott in the 80s in order to parallel the successful classification of amenable von Neumann algebras. (A von Neumann algebra, or W^*-algebra, is weak*-closed $*$-algebra of operators on a Hilbert space - it can be thought of as a noncommutative measure space.) The classification of amenable von Neumann algebras, accomplished by Connes, Haagerup and others in the 70s and 80s, is considered one of the greatest accomplishments in operator algebra theory.

The category of Cu-semigroups was introduced in 2008 by Coward, Elliott and Ivancescu, [CEI08]. The morphisms in this category, called Cu-morphisms, are Scott continuous semigroup maps that preserve the way-below relation. From the perspective of domain theory it might seem unusual to insist that morphisms preserve the way-below relation. However, one can show that every $*$-homomorphism between C^*-algebras, $A \to B$, naturally induces a map between their Cuntz semigroups, $\mathrm{Cu}(A) \to \mathrm{Cu}(B)$, which preserves the way-below relation. Moreover, by considering such morphisms, the category Cu has many desirable properties. For instance, it admits inductive limits (even arbitrary colimits) - a result that is no longer true without requiring that morphisms preserve the way-below relation. Similarly, the category of domains (with Scott continuous maps) does not admit inductive limits, unless one requires the involved maps to preserve the way-below relation.

In [APT14], together with Ramon Antoine and Francesc Perera, we initiated a systematic study of the category Cu. We showed that Cu admits tensor products. The concept of a tensor product is based on the notion of bimorphisms. Given Cu-semigroups S, T and P, a Cu-bimorphism $\varphi \colon S \times T \to P$ is a map that is additive and Scott continuous in each variable, and that preserves the joint way-below relation: If $s' \ll s$ and $t' \ll t$, then $\varphi(s', t') \ll \varphi(s, t)$. The tensor product of Cu-semigroups S and T is a Cu-semigroup $S \otimes T$ together with a universal Cu-bimorphism $S \times T \to S \otimes T$ that linearizes all Cu-bimorphisms from $S \times T$.

It follows that Cu has the structure of a symmetric, monoidal category: The tensorial unit is $\overline{\mathbb{N}} := \{0, 1, 2, \ldots, \infty\}$, with the obious addition and partial order. Note that $\overline{\mathbb{N}}$ is the Cuntz semigroup of the complex numbers $\mathbb{C}$. Moreover, we have natural isomorphisms

$$(S \otimes T) \otimes P \cong S \otimes (T \otimes P), \quad \text{and} \quad S \otimes T \cong T \otimes S.$$

Given C^*-algebras A and B, there is a natural Cu-morphism

$$\mathrm{Cu}(A) \otimes \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes B).$$

In some cases (but not always) this map is an isomorphism.

Recently, together with Antoine and Perera, [APT17], we showed that Cu is even a *closed monoidal category*. This means that for Cu-semigroups S and T, there is a Cu-semigroup $[\![S, T]\!]$ playing the role of morphisms from S to T, such that for any other Cu-semigroup P there is a natural bijection

$$\mathrm{Hom}\big(S \otimes T, P\big) \cong \mathrm{Hom}\big(S, [\![T, P]\!]\big).$$

The Cu-semigroup $[\![S, T]\!]$ is a *bivariant* Cu-*semigroup*. These bivariant Cu-semigroups behave very reasonable and provide many new examples of Cu-semigroups. For instance, compact elements in $[\![S, T]\!]$ naturally correspond to Cu-morphisms $S \to T$. (An element a in a Cu-semigroup is called compact if $a \ll a$.)

The step from Cu-semigroups to bivariant Cu-semigroups is similar to the step from K-theory to Kasparov's KK-theory (which is a bivariant version of K-theory). Since KK-theory plays an important role in topology, in index theory, and in the structure theory of C^*-algebras, we expect that bivariant Cuntz semigroups will turn out as a powerful tool in analysis and related areas as well.

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Properties of the annhilator of fuzzy subgroups

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1 Introduction

Duality theory for locally compact abelian groups was initially studied by Pontryagin [3]. This duality lies at the core of Fourier transform techniques in abstract harmonic analysis. We can describe Pontryagin's approach in the following way: Take first the circle group of the complex plane $\mathbb{T}$ endowed with its natural topology, as dualizing object. Then assign to a group X in the class of locally compact abelian groups the group $X^\wedge := \mathrm{CHom}(X; \mathbb{T})$ of continuous homomorphisms, and endow it with the compact open topology. This is precisely the dual group of X. After observing that the dual of a discrete group is compact and conversely, Pontryagin proved that the dual of a locally compact abelian group X is again a locally compact abelian group. This operation can be done a second time and then we obtain the second dual group $X^{\wedge\wedge}$ which, as it is known today, is topologically isomorphic to the initial locally compact group X.

In this context the notion of annihilator of a subgroup plays an important role. If $G \subset X$ is a subgroup of X, its *annihilator* is defined as the subgroup $G^\perp := \{\varphi \in X^\wedge \colon \varphi(G) = \{1\}\}$. If L is a subgroup of $X^\wedge$, the *inverse annihilator* is defined by $^\perp L := \{g \in X \colon \varphi(g) = 1, \forall \varphi \in L\}$. One important reason to study annihilators is the following: If G is a closed subgroup of a locally compact abelian group then $^\perp(G^\perp) = G$ [2].

2 Results

We introduce an extension of the notion of annihilator of a subgroup to the more general framework of fuzzy subgroups. Observe that constant functions $\underline{\lambda}$ are elementary examples of fuzzy subgroups. This shows that the class of fuzzy subgroups of a group is much larger than the class of subgroups.

As it can be noted, the definition of annihilator and inverse annihilator depends only of the group of continuous characters of the group X. In case of discrete subgroups, since $\mathrm{CHom}(X; \mathbb{T})$ coincides exactly with $\mathrm{Hom}(X; \mathbb{T})$ it is a purely algebraic notion. We develop a notion of annihilator of a fuzzy subgroup in this context. Our definition relies on the use of the α-levels of the fuzzy subgroup [1].

We will show that the formula $^\perp(G^\perp) = G$ is also true in the fuzzy framework. Then we will present another properties of the annihilator related with unions and intersections and we will conclude with some non trivial examples [4] were our definition can be applied .

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Compact Enriched Categories

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The interplay between order and topology has attracted a great deal of attention of researchers working in these fields. Of particular inspiration to us is the work [7] of Nachbin about topological spaces equipped with an additional partial order relation, subject to certain compatibility conditions. A particular class of such spaces, the compact ones, can be equivalently described in purely topological terms: the category of partially ordered compact spaces and monotone continuous maps is equivalent to the category of stably compact spaces and spectral maps (see [2] for details). As a consequence, the scope of various important notions and results in topology can be substantially extended, we mention here the concept of order-normality and the Urysohn Lemma. Turning the emphasis "up-side down", one might also ask what properties of the partial order are guaranteed by the existence of a compatible compact topology? One quick answer to this question is implied by [5, Lemma II.1.9]: since every partially ordered compact space corresponds to a stably compact space which is in particular sober, every partially ordered compact space has directed suprema.

Another important source of inspiration for our research over the past years has been Lawvere's ground-breaking paper [6] presenting generalised metric spaces as "order relations enriched in the quantale $[0, \infty]$", or better: as enriched categories. Undoubtedly, topology is omnipresent in the study of metric spaces; however, there does not seem to exist a systematic account in the literature connecting both lines of research.

The principal aim of this talk is to investigate compact quantale-enriched categories, encompassing this way ordered, metric, and probabilistic metric compact spaces. We place this study in the general framework of topological theories [3] and monad-quantale-enriched categories (see [4]).

Accordingly, in this talk we consider an (almost) strict topological theory $\mathcal{U} = (\mathbb{U}, \mathcal{V}, \xi)$ in the sense of [3] based on the ultrafilter monad $\mathbb{U}$, on a quantale $\mathcal{V}$ and on a convergence relation $\xi : U\mathcal{V} \to \mathcal{V}$ that makes $\mathcal{V}$ a compact Hausdorff topological space. The term "almost strict" refers to the fact that we do not require continuity of $\otimes$ but only lax continuity. Based on this data, one obtains a natural extension of the ultrafilter monad on Set to a monad (U, m, e) on $\mathcal{V}\text{-}\mathsf{Cat}$ [8] such that $e_X : X \to UX$ and $m_X : UUX \to UX$ become $\mathcal{V}$-functors, for each $\mathcal{V}$-category (X, a_0). The objects of the Eilenberg-Moore category for this monad, $\mathcal{V}\text{-}\mathsf{Cat}^{\mathbb{U}}$, can be described as triples (X, a_0, α) where (X, a_0) is a $\mathcal{V}$-category and $\alpha : UX \to X$ is the convergence of a compact Hausdorff topology on X. For $\mathcal{V}$ being the two-element lattice with the discrete topology, compact $\mathcal{V}$-categories coincide with Nachbin's ordered compact Hausdorff spaces; correspondingly, for $\mathcal{V}$ being Lawvere's quantale $[0, \infty]$ with the canonical compact Hausdorff topology, we call compact $\mathcal{V}$-categories *metric compact Hausdorff spaces*.

We recall that ordered compact Hausdorff spaces can be considered as special topological spaces, via a comparison functor

$$\mathsf{Ord}^{\mathbb{U}} \longrightarrow \mathsf{Top}$$
$$\searrow \quad \downarrow$$
$$\mathsf{Ord}$$

commuting with the forgetful functors to the category Ord. To bring this construction into our setting, we consider the category $\mathcal{U}$-Cat of $\mathcal{U}$-categories and $\mathcal{U}$-functors. A $\mathcal{U}$-category is a pair (X, a) were a is a $\mathcal{U}$-relation, meaning that it is a V-relation of the type $UX \nrightarrow X$, subject to reflexivity and transitivity. Furthermore, $\mathcal{U}$-relations compose through Kleisli convolution, $\circ$; unfortunately, associativity of this operation depends on the continuity of $\otimes$. Due to this fact, extreme care is needed when handling notions and results transferred from the framework of $\mathcal{V}$-categories. In this talk we revise and expand results regarding "Lawvere completeness in topology" (see [1]). For instance, since in general $\mathcal{U}$-distributors do not compose, the notion of Lawvere-complete $\mathcal{U}$-category is defined with respect to the set of those adjunctions $\varphi \dashv \psi$ where the composites $\varphi \circ \psi$ and $\psi \circ \varphi$ are $\mathcal{U}$-distributors.

Finally, there are suitable functors $(-)_0 : \mathcal{U}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$ and $K : (\mathcal{V}\text{-}\mathsf{Cat})^{\mathbb{U}} \to \mathcal{U}\text{-}\mathsf{Cat}$ that make the diagram

$$
\begin{array}{ccc}
(\mathcal{V}\text{-}\mathsf{Cat})^{\mathbb{U}} & \xrightarrow{\ K\ } & \mathcal{U}\text{-}\mathsf{Cat} \\
& {\scriptstyle G^{\mathbb{U}}} \searrow & \downarrow {\scriptstyle (-)_0} \\
& & \mathcal{V}\text{-}\mathsf{Cat}
\end{array}
$$

commutative. In particular, under some conditions that also include the continuity of the tensor $\otimes$, we have proven that compact Hausdorff $\mathcal{V}$-categories are Lawvere-complete since compact Hausdorff $\mathcal{V}$-categories correspond to Lawvere-complete $\mathcal{U}$-categories and the functor $(-)_0$ preserves Lawvere-completeness.

This is joint work with Dirk Hofmann.

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Theories of relational lattices

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The natural join and inner union of tables in relational databases can be algebraically modeled as the meet and the join operations in a class of lattices, the class of relational lattices. The connection between these lattices and databases is well illustrated in previous work on the subject, see [8, 3]. We recall here only the mathematical definition of these lattices and discuss some recent advances on their quasiequational and equational theories.

The set of functions from A to D—noted here D^A—can be endowed with the structure of a generalized ultrametric space where the distance takes values in the powerset algebra $P(A)$, see [5, 1]. Namely, define the distance between $f, g \in D^A$ by $\delta(f, g) := \{\, a \in A \mid f(a) \neq f(g) \,\}$. A subset $X \subseteq D^A$ is α-closed if $\delta(f, g) \subseteq \alpha$ and $g \in X$ implies $f \in X$; a pair $(\alpha, X) \in P(A) \times D^A$ is closed if X is α-closed; the closed pairs form a Moore family on $P(A) \times P(D^A)$. The relational lattice $\mathsf{R}(D, A)$ is, up to isomorphism, the lattice of closed pairs of $P(A) \times P(D^A)$.

It was proved in [3] that the quasiequantional theory of relational lattices, over the signature which contains the lattices operations $\wedge, \vee$ as well as an additional constant H (the header constant), is undecidable. We recently refined this result and proved that the quasiequational theory of relational lattices, over the pure lattice signature, is undecidable, [6, 7]. We actually proved there a stronger statement:

Theorem 1. *It is undecidable whether a finite subdirectly irreducible lattice can be embedded into a relational lattice.*

The proof is a reduction from the coverability problem for $S5$ universal product frames, see [2]. It also allows us to find a quasiequation that holds in all the finite $\mathsf{R}(E, A)$, but failing in some infinite $\mathsf{R}(D, A)$, with A finite. A universal product frame is a special dependent product, thus of the form $\prod_{a \in A} D_a$; with the same definition as above, we can give to dependent products the structure of a generalized ultrametric space. The reduction crucially relies on the following statement, whose proof appears in [6].

Theorem 2. *The spaces $(\prod_{a \in A} D_a, \delta)$ are, up to isomorphism, the pairwise-complete and spherically complete generalized ultrametric spaces.*

Using a result from [1] these spaces are, up to isomorphism, the injective objects in the category of generalized ultrametric spaces over $P(A)$.

Coming back to the theories of relational lattices, a natural aim is to move from quasiequations to equations and to relate equational theories of infinite relational lattices to the equational theories of the finite ones. Many informations can be deduced by analysing the functorial properties of the construction $\mathsf{R}(\_, \_)$. For $\psi : E \to D$, $\pi : A \to B$, and $(\alpha, X) \in \mathsf{R}(D, A)$, put

$$\mathsf{R}(D, \pi)(\alpha, X) := (\forall_\pi(\alpha), \pi^{*-1}(X)), \qquad \mathsf{R}(\psi, A)(\alpha, X) := (\alpha, \psi_*^{-1}(X)).$$

Here, for $f \in D^A$, we have $\pi^*(f) = f \circ \psi$, $\psi_*(f) = \psi \circ f$, and $\forall_\pi$ is right adjoint to π^{-1}. Notice that $\mathsf{R}(D, \pi) \circ \mathsf{R}(\psi, A) = \mathsf{R}(\psi, A) \circ \mathsf{R}(D, \pi)$, so we can define $\mathsf{R}(\psi, \pi) := \mathsf{R}(D, \pi) \circ \mathsf{R}(\psi, A)$.

Proposition 3. *The construction* $\mathsf{R}(\_,\_)$ *is a functor from* $\mathsf{Set}^{op} \times \mathsf{Set}$ *to* $\mathsf{cSL}_\wedge$, *the category of complete meet-semilattices and maps preserving all meets.*

For ψ and π as above, let $\ell_\psi : \mathsf{R}(E, A) \to \mathsf{R}(D, A)$ be left adjoint to $\mathsf{R}(\psi, A) : \mathsf{R}(D, A) \to \mathsf{R}(E, A)$; let $\ell_\pi : \mathsf{R}(D, B) \to \mathsf{R}(D, A)$ be left adjoint to $\mathsf{R}(D, \pi) : \mathsf{R}(D, A) \to \mathsf{R}(D, B)$. The following two observations are crucial.

Proposition 4. *If $E \neq \emptyset$ and $\psi : E \to D$ is injective, then ℓ_ψ is injective and preserves all the meets. If $\psi : A \to B$ is surjective, then ℓ_π is injective and preserves all the meets.*

In particular, $\mathsf{R}(E, B)$ belongs to the variety generated by $\mathsf{R}(D, A)$ whenever $E \subseteq D$ and $A \subseteq B$. When all these sets are finite, it is possible to look at the combinatorial proprieties the OD-graphs to assert that the two varieties are not equal, see [4]. Using Proposition 2, we derive the following theorem, showing that, for equations, the situation is quite different from the one of quasiequations.

Theorem 5. *If A is finite, then $\mathsf{R}(D, A)$ belongs to the variety generated by all the finite $\mathsf{R}(E, A)$.*

Indeed, $\mathsf{R}(D, A)$ is an algebraic lattice, thus it is isomorphic to the ideal completion of the join-semilattice of its compact elements. Yet, this join-semilattice is the colimit of the diagram $\ell_{\psi_{E_0,E_1}}$ where $E_0 \subseteq E_1 \subseteq D$, E_0, E_1 are non-empty and finite, and ψ_{E_0,E_1} is the inclusion of E_0 into E_1. In particular this colimit is a lattice in the variety generated by the finite $\mathsf{R}(E, A)$. It is well known that the ideal completion of a lattice and the lattice satisfy same the same identities.

If A is infinite, then $\mathsf{R}(D, A)$ is not an algebraic lattice, yet something can be said when D is finite.

Theorem 6. *If D is finite, then $\mathsf{R}(D, A)$ lies in the variety generated by all the finite $\mathsf{R}(D, B)$.*

Let $\mathrm{Part}_\mathfrak{f}(A)$ be the set of finite partitions of A, and consider the canonical maps $\pi_Q : A \to Q$ with $Q \in \mathrm{Part}_\mathfrak{f}(A)$, as well as the maps $\pi_{Q,P} : Q \to P$, for $Q, P \in \mathrm{Part}_\mathfrak{f}(A)$ such that Q refines P, sending a block of Q to the block of P that contains it. The maps $\mathsf{R}(D, \pi_Q)$ induce a canonical map $\pi : \mathsf{R}(D, A) \to \lim_{Q \in \mathrm{Part}_\mathfrak{f}(A)} \mathsf{R}(D, Q)$ in the category $\mathsf{cSL}_\wedge$, where $\lim_{Q \in \mathrm{Part}_\mathfrak{f}(A)} \mathsf{R}(D, Q)$ is the inverse limit of the maps $\mathsf{R}(D, \pi_{Q,P})$. We argue that if D is finite, then π is injective and preserves finite joins. Now, $\lim_{Q \in \mathrm{Part}_\mathfrak{f}(A)} \mathsf{R}(D, Q)$ is an algebraic lattice, and the poset of its compact element can be identified with the colimit (in the category of join-semilattices) of the diagram $\ell_{\pi_{Q,P}} : \mathsf{R}(D, P) \to \mathsf{R}(D, Q)$, for $Q, P \in \mathrm{Part}_\mathfrak{f}(A)$ and Q refines P. As before, $\lim_{Q \in \mathrm{Part}_\mathfrak{f}(A)} \mathsf{R}(D, Q)$ belongs to the variety generated by the $\mathsf{R}(D, Q)$, that are finite. As $\mathsf{R}(D, A)$ embeds into $\lim_{Q \in \mathrm{Part}_\mathfrak{f}(A)} \mathsf{R}(D, Q)$, then the same holds of $\mathsf{R}(D, A)$.

If both D and A are infinite, then the canonical map π is not an embedding. The tools used to prove Theorems 3 and 4 allow us to identify a complete lattice R_ω—the limit $\lim_{Q \in \mathrm{Part}_\mathfrak{f}(A)} \mathsf{R}(D, Q)$—which is a unique generator for the variety generated by the finite $\mathsf{R}(E, B)$. The quest for a characterization of the equational theory of relational lattices might involve recognizing R_ω as a sublattice of $\mathsf{R}(D, A)$ and how equational properties extend from the smaller lattice to its envelope.

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Deepening the link between logic and functional analysis via Riesz MV-algebras

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Riesz Spaces are lattice-ordered linear spaces over the field of real numbers $\mathbb{R}$. They have had a predominant rôle in the development of functional analysis over ordered structures, due to the simple remark that most of the spaces of functions one can think of are indeed Riesz Spaces.

Not very known is the rôle that vector lattices play in logic. Given any positive element u of a Riesz Space V, the interval $[0, u]$ can be endowed with a stucture of Riesz MV-algebra. These structures have been defined in the setting of Łukasiewicz logic, as expansion of MV-algebras – the *standard* semantics of the infinite valued Łukasiewicz logic – and in [1] is proved that Riesz MV-algebras are categorical equivalent to Riesz Spaces with a strong unit. Henceforth, vector lattices and logic are closely related.

Our aim is to exploit the connection between Riesz Spaces and MV-algebras to deepen the link between functional analysis and Łukasiewicz logic.

The first step is to introduce a notion of *limit of formulas* and use it to characterize the (uniform) norm convergence in Riesz Spaces. Consider the logic $\mathbb{R}\mathcal{L}$ that has Riesz MV-algebras as models (and it is a conservative expansion of Łukasiewicz logic) and let η_r denote the formula $\Delta_r \top$ of $\mathbb{R}\mathcal{L}$, where $\{\Delta_r\}$ is the family of connectives that models the scalar operation and $\top$ is defined as usual. Thus, $\Delta_r \top$ is evaluated into r by any $[0, 1]$-evaluation.

Definition 1. *We say that φ is the limit of the sequence $\{\varphi_n\}_{n \in \mathbb{N}}$, and we write $\varphi = \lim_n \varphi_n$ if for any $r \in [0, 1)$ there exists k such that $\vdash \eta_r \to (\varphi \leftrightarrow \varphi_n)$ for any $n \geq k$.*

This notion of *logical limit* is strictly connected to the one of convergence. Indeed, it is well known that the Lindenbaum-Tarski algebra of Łukasiewicz logic is isomorphic with the algebra of piecewise linear functions with integer coefficients. The same holds for the Lindenbaum-Tarski algebra of $\mathbb{R}\mathcal{L}$, which is isomorphic with the algebra of piecewise linear functions with real coefficients. If we denote by f_φ the function that correspond to the equivalence class of φ in the Lindenbaum-Tarski algebra of $\mathbb{R}\mathcal{L}$, we have the following result.

Theorem 1. *The following are equivalent*
(1) $\lim_n \varphi_n = \varphi$,
(2) $\lim_n f_{\varphi_n} = f_\varphi$ (uniform convergence in the Lindenbaum-Tarski algebra of $\mathbb{R}\mathcal{L}$),
(3) $f_{\varphi_n} \to f_\varphi$ (order convergence in the Lindenbaum-Tarski algebra of $\mathbb{R}\mathcal{L}$).

The above-mentioned result allows us to explore the possibility of studying the norm-completion of the Lindenbaum-Tarski algebra of $\mathbb{R}\mathcal{L}$ by its Dedekind σ-completion.

If RL_n denotes the Lindenbaum-Tarski algebra of $\mathbb{R}\mathcal{L}$ (where formulas have at most n variables), we characterize two different norm-completions of it. To do so, we consider the following notations:
(i) $\|[\varphi]\|_u = \sup\{f_\varphi(\mathbf{x}) | \mathbf{x} \in [0, 1]^n\}$, where $[\varphi] \in RL_n$,
(ii) $\|f\|_\infty = \sup\{f(\mathbf{x}) | \mathbf{x} \in [0, 1]^n\}$, where $f \in C([0, 1]^n)$,

(iii) $I([\varphi]) = \int f_\varphi(\mathbf{x})d\mathbf{x}$, where $[\varphi] \in RL_n$.

All of the above defined operator are norms in the corresponding spaces and the following theorem holds.

Theorem 2. *(1) The norm-completion of the normed space $(RL_n, \|\cdot\|_u)$ is isometrically isomorphic with $(C([0,1]^n), \|\cdot\|_\infty)$,*

(2) The norm-completion of the normed space (RL_n, I) is isometrically isomorphic with $(L^1(\mu)_u, s_\mu)$, where

(i) μ be the Lebesgue measure associated to I,

(ii) $L^1(\mu)_u$ is the algebra of $[0,1]$-valued integrable functions on $[0,1]^n$,

(iii) $s_\mu(\hat{f}) = I(f)$ and $\hat{f}$ is the class of f, provided we identify two functions that are equal μ-almost everywhere.

With the goal of capturing the unit norm $\|\cdot\|_u$ in a purely syntactic way, we now define the logical system $\mathcal{IRL}$, which stands for *Infinitary Riesz Logic*, by adding an infinitary disjunction to the systems $\mathbb{RL}$ (as well as appropriate axioms and a deduction rule). The system has Dedekind σ-complete Riesz MV-algebras as models and the following results hold.

Theorem 3. *(1) IRL, the Lindenbaum-Tarski algebra of $\mathcal{IRL}$ is a Dedekind σ-complete Riesz MV-algebra;*

(2) $\mathcal{IRL}$ is complete with respect to to all algebras in $\mathbf{RMV_{dc\sigma}}$, the class of Dedekind σ-complete Riesz MV-algebras;

Moreover, we can characterize the models of $\mathcal{IRL}$ by means of particular compact Hausdorff spaces.

Theorem 4. *(1) All Dedekind σ-complete Riesz MV-algebras are norm-complete w.r.t. $\|\cdot\|_u$.*

(2) Let A be a Dedekind σ-complete Riesz MV-algebra. There exists a quasi-Stonean compact Hausdorff space (i.e. it has a base of open F_σ sets) X such that $A \simeq C(X)_u$, the unit interval of $C(X)$.

We conclude this abstract by recalling how all the different completions of RL_n we have defined are linked to each other.

(1) If one consider the sup-norm, the norm completion of RL_n is $C([0,1]^n, \|\cdot\|_\infty)$, which is not Dedekind complete and it is contained in IRL_n.

(2) IRL is a norm-complete Riesz MV-algebra and IRL_n contains $C([0,1]^n)$, as the latter is the norm-completion of RL_n.

(3) If one consider the integral norm, the norm completion of RL_n is $L^1(\mu)_u$, i.e. the unit interval of the space of μ-integrable functions in n variables. This space is Dedekind complete as it is an abstract L-space and hence contains IRL_n.

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On the Complexity of the Equational Theory of Generalized Residuated Boolean Algebras

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1 Introduction

A residuated Boolean algebra is an algebra $(A, \wedge, \vee, ', \top, \bot, \bullet, \backslash, /)$ where $(A, \wedge, \vee, ', \top, \bot)$ is a Boolean algebra, and $\bullet, \backslash$ and $/$ are binary operators on A satisfying the following residuation property: for any $a, b, c \in A$,

$$a \bullet b \leq c \text{ iff } b \leq a \backslash c \text{ iff } a \leq c/b$$

The operators $\backslash$ and $/$ are called right and left residuals of the fusion $\bullet$ respectively.

Residuated boolean algebras are introduced by Jónsson and Tsinakis [3] as generalizations of relation algebras. Jispen [2] proved that the equational theory of residuated boolean algebras with unit, and that of many relative classes of algebras are decidable. Buszkowski [1] showed the finite embeddability property for residuated boolean algebras, which yields the decidability of the universal theory of residuated boolean algebras. The complexity of the equational theory of residuated boolean algebras is solved in [4], where the main result is that the equational theory of residuated boolean algebras is PSPACE-complete.

Generalized residuated Boolean algebras are introduced in [1]. The generalization is from binary to arbitrary $n \geq 2$ ary residuals. Instead of a single binary operator $\bullet$, generalized residuated algebras admit a finite number of finitary operations o. With each n-ary operation o_i $(1 \leq i \leq m)$ there are associated n residual operations o/j $(1 \leq j \leq n)$ which satisfy the following generalized residuation law:

$$o_i(a_1, \ldots, a_n) \leq b \quad \text{iff} \quad a_j \leq (o_i/j)(a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_n)$$

A *generalized residuated Boolean algebra* is a Boolean algebra with generalized residual operations. The aim of this paper is to show that the complexity of the equational theory of such algebras is still PSAPCE-complete. Our proof is by reducing the decidability of the equational theory into the decidability of a sequent calculus for generalized Boolean residuated algebra.

2 Generalized BFNL

The sequent calculus for Boolean residuated algebras, namely Boolean full nonassociative Lambek calculus (BFNL), is introduced in [1]. Here we shall introduce the sequent calculus GBFNL for generalized residuated Boolean algebras. The formulae are defined as usual (cf. [4]). Structures are defined inductively as follows:

(1) All formulae are structures.

(2) For n-ary operator o_i $(n \geq 2)$ and structures $\Gamma_1, \ldots, \Gamma_n$, $(\Gamma_1, \ldots, \Gamma_n)_{o_i}$ is a structure.

By $\Gamma[\ ]$ we mean a structure with a single position which can be filled with a structure.

Definition 2.1. *The sequent calculus GBFNL for generalized residuated Boolean algebras consists of the following axioms and rules:*

(1) *Axioms:*

$$(Id) \ A \Rightarrow A \quad (D) \ A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$$

$$(\top) \ \Gamma \Rightarrow \top \quad (\bot) \ \Gamma[\bot] \Rightarrow A \quad (\neg 1) \ A \wedge \neg A \Rightarrow \bot \quad (\neg 2) \ \top \Rightarrow A \vee \neg A$$

(2) *Rules:*

$$\frac{\Gamma[(A_1,\ldots,A_n)_{o_i}] \Rightarrow A}{\Gamma[o_i(A_1,\ldots,A_n)] \Rightarrow A} \ (\mathrm{o_i L}) \qquad \frac{\Gamma_1 \Rightarrow A_1; \ldots; \Gamma_n \Rightarrow A_n}{(\Gamma_1,\ldots,\Gamma_n)_{o_i} \Rightarrow o_i(A_1,\ldots,A_n)} \ (\mathrm{o_i R})$$

$$\frac{\Delta[A_j] \Rightarrow B; \Gamma_1 \Rightarrow A_1; \ldots; \Gamma_n \Rightarrow A_n}{\Delta[(\Gamma_1,\ldots,(o_i/j)(A_1,\ldots,A_n),\ldots,\Gamma_n)_{o_i}] \Rightarrow B} \ (\mathrm{o_i/jL})$$

$$\frac{(A_1,\ldots,\Gamma,\ldots,A_n)_{o_i} \Rightarrow A_j}{\Gamma \Rightarrow o_i/j(A_1,\ldots,A_n)} \ (\mathrm{o_i/jR})$$

$$\frac{\Gamma[A_i] \Rightarrow B}{\Gamma[A_1 \wedge A_2]} (\wedge \mathrm{L}) \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} (\wedge \mathrm{R})$$

$$\frac{\Gamma[A_1] \Rightarrow B \quad \Gamma[A_2] \Rightarrow B}{\Gamma[A_1 \vee A_2] \Rightarrow B} (\vee \mathrm{L}) \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2} (\vee \mathrm{R}) \qquad \frac{\Delta \Rightarrow A \quad \Gamma[A] \Rightarrow B}{\Gamma[\Delta] \Rightarrow B} (\mathrm{Cut})$$

The sequent calculus GBFNL can be simulated by a multi-sorted Boolean nonassociative Lambek calculus which is denoted by MBFNL. This means that n-ary residuals can be translated into binary ones. A translation $\ddagger$ from GBFNL to MBFNL can be defined inductively as usual. In particular, we have the following translation of residuals:

(1) $((A_1,\ldots,A_n)_{o_i})^{\ddagger} = (\cdots((A_1 \bullet_i A_2)\cdots) \bullet_i A_n).$

(2) $(o_i/j)(A_1,\ldots,A_n)^{\ddagger} = (\cdots(A_1 \bullet_i A_2)\cdots) \bullet_i A_{j-1})\backslash_i(\cdots(A_j/_i A_n)\cdots)/_i A_{j+1}).$

(3) $(\Gamma_1,\ldots,\Gamma_n)_{o_i}^{\ddagger} = (\cdots((\Gamma_1 \circ_i \Gamma_2)\cdots) \circ_i \Gamma_n).$

This translation is faithful. We may easily obtain the following theorem of simulation:

Theorem 2.2. $\vdash_{\mathrm{GBNFL}} \Gamma \Rightarrow A$ *iff* $\vdash_{\mathrm{MBFNL}} \Gamma^{\dagger} \Rightarrow A^{\dagger}.$

3 Complexity of GBFNL

The second step to solve the complexity problem is to simulate MBFNL by a multi-sorted tense logic MKt which is the multi-modal version of basic tense logic Kt. The translation # defined in [4], which embeds BFNL into two-sorted tense logic $\mathrm{K}^{\mathrm{t}}_{1,2}$, can be extended to simulate MBFNL. Each n-ary product operator o_i is translated via n pairs of tense operators. The following results can be obtained as in [4].

Theorem 3.1. $\vdash_{\mathrm{MBFNL}} \Gamma \Rightarrow A$ *iff* $\vdash_{\mathrm{MKt}} (f(\Gamma))^{\#} \supset A^{\#}.$

Moreover, using the technique in [4], one can simulate MKt by the basic tense logic Kt via a similar translation $*$ as in [4].

Theorem 3.2. $\vdash_{\mathrm{MKt}} A$ *iff* $\vdash_{\mathrm{Kt}} A^{*}.$

Since the complexity of Kt is PSPACE-complete, it follows that GBFNL is in PSPACE. On the other hand, we may define a translation † from the modal logic K to GBNFL as in [4] such that $(\Diamond A)^{\dagger} = o(m_1, \ldots, m_{n-1}, A)$. Then we obtain the following simulation result:

Theorem 3.3. *For any modal formula A, $\vdash_{\mathrm{K}} A$ iff $\vdash_{\mathrm{GBNFL}} \top \Rightarrow A^{\dagger}$.*

Since the modal logic K is PSAPCE-complete, it follows that GBFNL is PSPACE-hard. Therefore we get the following theorem:

Theorem 3.4. *GBFNL is PSPACE-complete.*

As a consequence, the equational theory of generalized Boolean residuated algebras is PSPACE-complete.

4   More complexity results

If we change the Boolean basis of a generalized Boolean residuated algebra into distributive lattices, we get generalized distributive residuated lattices. We also obtain the generalized distributive full nonassociative Lambek calculus (GDFNL) for such algebras.

Theorem 4.1. *GBFNL is a conservative extension of GDFNL.*

It follows that GDFNL is in PSPACE. We reduce the satisfiability of a QBF to the validity of consequence relation of distributive lattice with bi-tense operators. The equational theory of distributive lattices with bi-tense operators is PSPACE-hard. Then GDFNL is PSPACE-hard.

Theorem 4.2. *GDFNL is PSPACE-complete. Hence DFNL is PSPACE-complete.*

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A Constructive Four-Valued Logic

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1 Introduction

The Belnap-Dunn four-valued logic is the logic of De Morgan lattices. A De Morgan lattice is an algebra $(A, \wedge, \vee, \neg, 0, 1)$ where $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $\neg$ is the De Morgan negation, namely $\neg$ is an unary operation on A satisfying the following conditions:

(1) $\neg(a \wedge b) = \neg a \vee \neg b$;

(2) $\neg(a \vee b) = \neg a \wedge \neg b$;

(3) $\neg\neg a = a$;

(4) $\neg 0 = 1$ and $\neg 1 = 0$.

It is well-known that every De Morgan lattice can be embedded in a (subdirect) product of the lattice **4**. Belnap's four-valued logic [1] is the logic of the following lattice **4**:
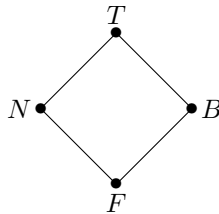
Belnap's Lattice **4**

Dunn's logic of De Morgan lattices [4, 5] is the same as Belnap's four-valued logic due to the representation theorem. Dunn [2] developed a theory of negation which is adapted with the theory of information. For a comprehensive survey on negation, see Dunn [4].

In the present paper, we shall present a constructive four-valued logic **C4L**. Dunn's four-valued semantics for De Morgan logic introduces two semantics consequence relations $\varphi \models_1 \psi$ and $\varphi \models_0 \psi$ which can be interpreted via Belnap's concepts of acceptance and rejection. A formula φ is accepted if 1 belongs to the value of φ, and it is rejected if 0 belongs to the value of φ. Our idea is to generalize Belnap-Dunn four-valued logic to a weak logic which is constructive in the following sense: if a formula is accepted, then it is accepted at any future state, and if it is rejected, it is rejected at any future state. The underling temporal structure is assumed to be a linear order. We call this logic as a constructive four-valued logic because it is a sublogic of Belnap-Dunn four-valued logic in which the law of double negation elimination is refuted. The logic **C4L** can be viewed the weakening of Belnap-Dunn logic in the way that intuitionistic logic is the weakening of classical propositional logic. We shall introduce the Kripke semantics for **C4L**. Consequently, Belnap-Dunn four-valued logic can be represented as the logic of a single reflexive point which is an extension of **C4L**.

The negation in **C4L** is a new one to Dunn's kite of negations [4]. Intuitively it is a modal negation which is interpreted on linearly ordered sets.

2 Language and Semantics

The language $\mathcal{L}$ of the constructive four-valued logic **C4L** consists of a denumerable set of propositional variables $\mathbf{Var} = \{p_i \mid i \in \mathbb{N}\}$, and propositional connectives $\top, \bot, \neg, \wedge$ and $\vee$. The set of all formulae is defined by the following inductive rule:

$$\varphi ::= p \mid \top \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi, \text{ where } p \in \mathbf{Var}.$$

A *sequent* is an expression of the form $\varphi \vdash \psi$ where φ and ψ are formulae.

Definition 2.1. A *frame* is a pair $\mathcal{F} = (W, \leq)$ where W is a non-empty set of states, and $\leq$ is a linear order on W. A subset $A \subseteq W$ is called an *upset* in $\mathcal{F}$ if A is closed under $\leq$, namely, if $w \in A$ and $w \leq w'$, then $w' \in A$. Let $Up(W)$ be the set of all upsets in $\mathcal{F}$.

A *model* is a tuple $\mathcal{M} = (W, \leq, V)$, where $(W, \leq)$ is a frame, and $V : \mathbf{Var} \to Up(W) \times Up(W)$ is valuation. When $V(p) = (A^+, A^-)$, we say that A^+ is the set of states which accept p, and A^- is the set of states which reject p. We also write $V^+(p) = A^+$ and $V^-(p) = A^-$.

Obviously for any frame $\mathcal{F} = (W, \leq)$, we have $\emptyset, W \in Up(W)$. Moreover, in any model $\mathcal{M} = (W, \leq, V)$, each propositional variable p is assigned with a pair $(V^+(p), V^-(p))$ of upsets. There is no need to require that $V^+(p) \cup V^-(p) = W$ because there may be states which neither accept nor reject a propositional variable.

Definition 2.2. For any formula φ, model $\mathcal{M} = (W, \leq, V)$ and $w \in W$, the acceptance and rejection relations $\mathcal{M}, w \models^+ \varphi$ and $\mathcal{M}, w \models^- \varphi$ are defined inductively as follows:

(1) $\mathcal{M}, w \models^+ \top$ and $\mathcal{M}, w \not\models^- \top$; $\mathcal{M}, w \not\models^+ \bot$ and $\mathcal{M}, w \models^- \bot$.

(2) $\mathcal{M}, w \models^+ p$ iff $w \in V^+(p)$; $\mathcal{M}, w \models^- p$ iff $w \in V^-(p)$.

(3) $\mathcal{M}, w \models^+ \varphi \wedge \psi$ iff $\mathcal{M}, w \models^+ \varphi$ and $\mathcal{M}, w \models^+ \varphi$.

(4) $\mathcal{M}, w \models^- \varphi \wedge \psi$ iff $\mathcal{M}, w \models^- \varphi$ or $\mathcal{M}, w \models^- \psi$.

(5) $\mathcal{M}, w \models^+ \varphi \vee \psi$ iff $\mathcal{M}, w \models^+ \varphi$ or $\mathcal{M}, w \models^+ \psi$.

(6) $\mathcal{M}, w \models^- \varphi \vee \psi$ iff $\mathcal{M}, w \models^- \varphi$ and $\mathcal{M}, w \models^- \psi$.

(7) $\mathcal{M}, w \models^+ \neg\varphi$ iff for any $w' \in W$, if $w \leq w'$, then $\mathcal{M}, w' \models^- \varphi$.

(8) $\mathcal{M}, w \models^- \neg\varphi$ iff there is $w' \in W$ such that $w \leq w'$ and $\mathcal{M}, w' \models^+ \varphi$.

A formula φ is *accepted in* $\mathcal{M}$, notation $\mathcal{M} \models^+ \varphi$, if for any $w \in W$, $\mathcal{M}, w \models^+ \varphi$. A formula ψ is a *positive consequence* of φ, notation $\varphi \models^+ \psi$, if for any model $\mathcal{M}$ and w in $\mathcal{M}$, if $\mathcal{M}, w \models^+ \varphi$, then $\mathcal{M}, w \models^+ \psi$. We say that ψ is a *negative consequence* of φ, notation $\varphi \models^- \psi$, if for any model $\mathcal{M}$ and w in $\mathcal{M}$, if $\mathcal{M}, w \models^- \varphi$, then $\mathcal{M}, w \models^- \psi$.

Lemma 2.3. For any model $\mathcal{M} = (W, \leq, V)$ and $w, w' \in W$, the following hold:

(1) If $w \leq w'$ and $\mathcal{M}, w \models^+ \varphi$ then $\mathcal{M}, w' \models^+ \varphi$.

(2) If $w \leq w'$ and $\mathcal{M}, w \models^- \varphi$ then $\mathcal{M}, w' \models^- \varphi$.

(3) $\mathcal{M}, w \models^- \varphi$ iff $\mathcal{M}, w \models^+ \neg\varphi$.

(4) $\mathcal{M} \models^- \varphi$ iff $\mathcal{M} \models^+ \neg\varphi$.

Corollary 2.4. *For any formulae φ and ψ, $\varphi \models^- \psi$ iff $\neg\varphi \models^+ \neg\psi$.*

We say that a sequent $\varphi \vdash \psi$ is *valid* if $\varphi \models^+ \psi$ and $\psi \models^- \varphi$. A sequent rule of the form

$$\frac{\varphi_1 \vdash \psi_1, \ldots, \varphi_n \vdash \psi_n}{\varphi_0 \vdash \psi_0}(\rho)$$

is *valid*, if the premisses $\varphi_i \vdash \psi_i$ for $1 \le i \le n$ are valid, then $\varphi_0 \vdash \psi_0$ is valid.

Proposition 2.5. *The following sequents and rule are valid:*

(1) $\neg\neg\neg\neg\varphi \vdash \neg\neg\varphi$.

(2) $\neg(\varphi \wedge \psi) \vdash \neg\varphi \vee \neg\psi$ *and* $\neg\varphi \vee \neg\psi \vdash \neg(\varphi \wedge \psi)$.

(3) $\neg(\varphi \vee \psi) \vdash \neg\varphi \wedge \neg\psi$ *and* $\neg\varphi \wedge \neg\psi \vdash \neg(\varphi \vee \psi)$.

(4) $\varphi \wedge \neg\neg\psi \vdash \neg\neg\varphi \vee \psi$.

(5) *The contraposition rule:*

$$\frac{\varphi \vdash \psi}{\neg\psi \vdash \neg\varphi}(CP)$$

One can easily show that the sequents $\neg\neg\varphi \vdash \varphi$ and $\neg\neg\neg\varphi \vdash \neg\varphi$ are not valid.

3 The Logic C4L

The set of all valid sequents under the Kripke semantics can be axiomatized as a sequent system. We have the following definition of the constructive logic **C4L**:

Definition 3.1. *The logic* **C4L** *consists of the following axioms and rules:*

(1) *Axioms:*

$$\text{(Id) } \varphi \vdash \varphi \quad (\bot) \; \bot \vdash \varphi \quad (\top) \; \varphi \vdash \top \quad \text{(D) } \varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$$

$$(\neg\bot) \; \varphi \vdash \neg\bot \quad (\neg\top) \; \neg\top \vdash \varphi \quad (\text{N1}) \; \neg\varphi \vdash \neg\neg\neg\varphi \quad (\text{N2}) \; \varphi \wedge \neg\neg\psi \vdash \neg\neg\varphi \vee \psi$$

$$(\text{DM1}) \; \neg(\varphi \vee \psi) \vdash \neg\varphi \wedge \neg\psi \quad (\text{DM2}) \; \neg\varphi \wedge \neg\psi \vdash \neg(\varphi \vee \psi)$$

(2) *Rules for lattice operations:*

$$\frac{\varphi_i \vdash \psi}{\varphi_1 \wedge \varphi_2 \vdash \psi}(\wedge\text{L})(i = 1, 2) \quad \frac{\varphi \vdash \psi \quad \varphi \vdash \chi}{\varphi \vdash \psi \wedge \chi}(\wedge\text{R})$$

$$\frac{\varphi \vdash \chi \quad \psi \vdash \chi}{\varphi \vee \psi \vdash \chi}(\vee\text{L}) \quad \frac{\varphi \vdash \psi_i}{\varphi \vdash \psi_1 \vee \psi_2}(\vee\text{R})(i = 1, 2)$$

(3) *Cut rule and contraposition rule:*

$$\frac{\varphi \vdash \psi \quad \psi \vdash \chi}{\varphi \vdash \chi}(\text{Cut}) \quad \frac{\varphi \vdash \psi}{\neg\psi \vdash \neg\varphi}(\text{CP})$$

A sequent $\varphi \vdash \psi$ is derivable in **C4L** *if there is a derivation of the sequent.*

Theorem 3.2. *A sequent $\varphi \vdash \psi$ is derivable in* **C4L** *if and only if $\varphi \vdash \psi$ is valid.*

The logic **C4L** is a sublogic of Belnap-Dunn four-valued logic. From the semantic perspective, if one considers the special model which has only one single reflexive state $\circ$, then the set of all valid sequents in this frame is exactly the Belnap-Dunn four-valued logic.

4   Weak De Morgan Algebras

From algebraic perspective, there is a class of algebras for **C4L**. We call them *weak De Morgan algebras*. We shall prove the algebraic completeness of **C4L**, and develop a cut-free sequent calculus for **C4L** by which we get the decidability of the derivation of a sequent in the system.

Definition 4.1. A *weak De Morgan algebra* is an algebra $(A, \wedge, \vee, \neg, 0, 1)$ where $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $\neg$ is an operation on A satisfying the following conditions:

(1) $\neg(a \wedge b) = \neg a \vee \neg b$.

(2) $\neg(a \vee b) = \neg a \wedge \neg b$.

(3) $\neg a \leq \neg\neg\neg a$.

(4) $a \wedge \neg\neg\neg b \leq \neg\neg a \vee \neg b$.

(5) $\neg 0 = 1$ and $\neg 1 = 0$.

The class of all weak De Morgan algebras is denoted by **WDM**.

Theorem 4.2. *A sequent $\varphi \vdash \psi$ is derivable in* **C4L** *iff $\varphi \vdash \psi$ is valid in* **WDM**.

Obviously the class of all De Morgan algebras is a subvariety of **WDM**. Note that the difference between De Morgan algebras and weak De Morgan algebras is about number of negations. Semi-De Morgan algebras proposed in [6] are weakened from De Morgan algebras by dropping the double negation law and one De Morgan law. However, the logic of semi-De Morgan algebras (**SDM**) is incomparable with **C4L**. The following map of logics show their relationships with classical and intuitionistic logic:
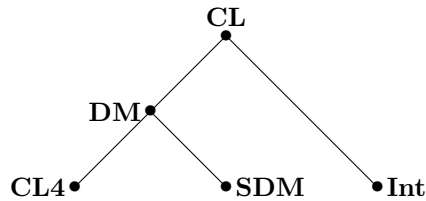
Gentzen sequent calculus for **C4L** shall be presented in the full version of this abstract.

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Undecidability of $\{\cdot, 1, \vee\}$-equations in subvarieties of commutative residuated lattices

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The decidability of the equational and quasi-equational theories for commutative residuated lattices ($\mathcal{CRL}$) axiomatized by $\{\cdot, 1, \leq\}$-inequalities have been fully classified. It has been shown that quasi-equational theories axiomatized by *knotted inequations* (k_n^m), i.e. universally quantified inequations of the form $x^n \leq x^m$ for $n \neq m$, are not only decidable, but also have the finite embedability property (FEP) [5]. In fact, $\mathcal{CRL} + (\mathrm{k}_n^m) + \Gamma$ has the FEP for any set Γ of $\{\cdot, 1, \leq\}$-equations [2]. Viewed proof-theoretically, these results show that the Full Lambek calculus with exchange ($\mathbf{FL}_e$) axiomatized by knotted inference rules have decidable consequence relations.

In [1], it is shown that $\mathbf{FL_c}$ is undecidable, which algebraically corresponds to $\mathcal{RL} + (\mathrm{k}_1^2)$ having undecidable quasi-equational and equational theories. In fact, for $1 \leq n \leq m$, [1] shows that there exists a residuated lattice $\mathbf{R}$ in the variety $\mathcal{RL} + (\mathrm{k}_n^m)$ such that, for any variety $\mathcal{V}$,

$$\mathbf{R} \in \mathcal{V} \implies \mathcal{V} \text{ has undecidable quasi-equational and equational theories.}$$

As a consequence of this, certain non-commutative varieties satisfying equations in the signature $\{\cdot, 1, \vee\}$ are also shown to be undecidable.

However, in the commutative case, little is known about the decidability of $\mathcal{CRL}$'s axiomatized by equations in the signature $\{\cdot, 1, \vee\}$, e.g. the effect of inequations such as $x \leq x^2 \vee 1$ or $xy \leq x^2 y \vee x^3 y^2$ on decidabilty in $\mathcal{CRL}$ is unknown.

The present work defines a class D of $\{\cdot, 1, \vee\}$-equations such that the following theorem is obtained:

Theorem 1. *If* (d) $\in D$*, then there exists* $\mathbf{R}_\mathrm{d}$ *in* $\mathcal{CRL} + (\mathrm{d})$ *such that for every variety* $\mathcal{V}$*,*

$$\mathbf{R}_\mathrm{d} \in \mathcal{V} \implies \mathcal{V} \text{ has an undecidable quasi-equational theory.}$$

Furthermore, as a consequence of the above theorem, there is a subclass $D' \subset D$ such that

Corollary 2. *If* (d) $\in D'$*, then there exists* $\mathbf{R}_\mathrm{d}$ *in* $\mathcal{CRL} + (\mathrm{d})$ *such that for every variety* $\mathcal{V}$*,*

$$\mathbf{R}_\mathrm{d} \in \mathcal{V} \implies \mathcal{V} \text{ has an undecidable equational theory.}$$

As in [1], [3], and [4], we use *counter machines* (CM), a variant of Turing Machines, for our undecidable problem. From a given a CM M and rule (d) $\in D$, we construct a new machine M_d and a commutative idempotent semi-ring $\mathbf{A}_{M_d}$. We interpret machine instructions of M_d as relations on $\mathbf{A}_{M_d}$, and define a new relation $<_M$ on $\mathbf{A}_N$ such that

$$M \text{ halts on input } C \iff \theta(C) <_M q_f,$$

where q_f is a designated element A_{M_d} and θ is a certain function on the configurations of M into the set A_{M_d}.

We then simulate the rule (d) in $\mathbf{A}_{M_d}$ by a new relation $<_{\mathbf{d}}$ extending $<_M$ that, on the one hand, satisfies certain restricted consequences of the rule (d), and on the other hand, maintains the property that

$$M \text{ halts on input } C \iff \theta(C) <_{\mathrm{d}} q_f.$$

Lastly, following the methods utilized in [1], we use the theory of residuated frames [2] to construct a residuated lattice $\mathbf{W}^+$ in $\mathcal{CRL} + (\mathrm{d})$ from $\langle A_{M_d}, <_d, \cdot \rangle$ that has the halting problem from the machine M encoded into the order of $\mathbf{W}^+$, effectively interpreting a halting problem into any variety that contains $\mathbf{W}^+$. Membership in D is equivalent to whether certain systems of linear equations admit positive solutions. Let

$$\forall(x_1, ..., x_n) \; x_1 x_2 \cdots x_n \leq \bigvee_{(c_1, ..., c_n) \in C} x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n}$$

be the linearization of some $\{\cdot, 1, \vee\}$-equation (r), where $C \subset \mathbb{N}^n$ is finite. Then (r) $\notin D$ if and only if there exists a positive solution to the system of linear equations

$$\left\{ \sum_{i=1}^n c_i x_i = \sum_{i=1}^n d_i x_i : (c_1, ..., c_n), (d_1, ..., d_n) \in C_X \right\},$$

where

$$C_X := \{(c_1, ..., c_n) \in C : (\exists i \in X) \; c_i > 0\},$$

for some $X \subseteq \{1, ..., n\}$.

The members of D' are those equations (r) in D such that (r) has, as a consequence, an inequation of the form:

$$(\forall x) \; x^n \leq \bigvee_{i=1}^m x^{n+c_i},$$

where $n, c_1, ..., c_m > 0$, and as a consequence of membership in D, $m \geq 2$.

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Duality for Relations on Ordered Algebras

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In algebraic logic, one is accustomed to considering, for example, the inherent order on a distributive lattice as capturing entailment between propositions within a particular logic. Generalizing this to morphisms between algebras, one thinks about binary relations that capture a notion of entailment between logics. At a mimimum, these relations should respect the algebraic structure under consideration, and should in some sense still capture a notion of entailment (that is, order). Respect for the algebraic structure means, essentially, that the relations ought to be relations in the category of the algebras. Capturing entailment means that the relations should be closed under strengthening of premises and weakening of conclusions. Putting these ideas together leads to a natural relational setting for algebraic logic.

Natural duality has its most familiar instances in categories of algebras and spaces that are relevant to (positive) algebraic logic by virtue of being concrete over posets. The objects come equipped with partial order with respect to which the morphisms and operations are monotonic. For example, Priestley duality, Stone duality, Banaschewski duality (between partially ordered sets and Stone distributive lattices), and Hofmann-Mislove-Stralka duality (between semilattices and Stone semilattices) all are concrete over posets. Note that while a Stone space has a trivial order, that fact is precisely the feature that distinguishes a Stone space from a Priestley space. So even Stone duality fits the general ordered scheme, when one takes the duals to be complemented distributive lattices.

We study how one extends a duality between ordered algebras and ordered spaces to relations. The motivation is to understand the general setting in which relation lifting carries over to these dualities.

For this abstract, we restrict our attention only to DL, the category of bounded distributive lattices, and Pri, the category of Priestley spaces. In the full paper we consider a more general setting to include other varieties of algebras and their dual spaces.

In a category $\mathcal{A}$ with pullbacks, one defines $\mathsf{Span}(\mathcal{A})$ as the category of isomorphism classes of spans $A \leftarrow R \rightarrow B$ with composition being defined by pullbacks. So in particular $\mathsf{Span}(\mathsf{DL})$ and $\mathsf{Span}(\mathsf{Pri})$ make sense because both categories have pullbacks.

The categories DL and Pri are both equipped with suitable factorization systems $(\mathcal{E}, \mathcal{M})$ for spans (factoring a span into an epimorphism e followed by a jointly monic span m), so that categories $\mathsf{Rel}(\mathsf{DL})$ and $\mathsf{Rel}(\mathsf{Pri})$ arise by taking morphisms to be the monomorphic spans. In DL, these are essentially sublattices of $A \times B$. In Pri, they are merelty compact subspaces (with the induced order) of the $X \times Y$. Composition is defined by pullback and renormalizing via the factorization system. Again in both $\mathsf{Rel}(\mathsf{DL})$ and $\mathsf{Rel}(\mathsf{Pri})$, this means that composition is concretely the usual relational composition.

Looking toward duality, we are faced immediately with a problem. The dual of a span $A \leftarrow R \rightarrow B$ in distributive lattices is a *cospan* $2^A \rightarrow 2^R \leftarrow 2^B$ in Priestley spaces, and vice versa. Nevertheless, $\mathsf{Rel}(\mathsf{DL})$ provides precisely those relations that respect the algebraic structure of the objects. And $\mathsf{Rel}(\mathsf{Pri})$ provides a similar service for topological structure of Priestley spaces.

To obtain relations that also respect entailment (closure under strengthening of premises and weakening of conclusions), we consider *weakening* relations, i.e., those binary relations between posets that are closed under the following rule: $a \leq a'$, $a'Rb'$ and $b' \leq b$ implies aRb. Because

DL and Pri are both concrete over Pos, we can define weakening relations between objects to be morphisms in Rel(DL) or Rel(Pri) that are closed under the weakening rule.

Putting things together, DL and Pri have suitable structure for defining relations generally, and have forgetful functors into Pos so that weakening relations make (forgetful) sense. Moreover, the composition of relations in DL and Pri coincides concretely with composition of weakening relations in Pos. So we define categories $\overline{\mathsf{DL}}$ and $\overline{\mathsf{Pri}}$ as the subcategories of Rel(DL) and Rel(Pri) consisting of relations which forgetfully are weakening relations.

Specifically, in $\overline{\mathsf{DL}}$, a morphism corresponds exactly to a relation closed under the familar proof calculus rules for positive logic. In $\overline{\mathsf{Pri}}$, a morphism is simply a compact upper set in $X^{\mathrm{op}} \times Y$. Notice that these categories are both order-enriched, by taking relations orered by inclusion.

The main problem now is to understand how the natural duality of DL and Pri lifts to $\overline{\mathsf{DL}}$ and $\overline{\mathsf{Pri}}$. Our main additional tool is the weighted limits of cospans and weighted colimits of spans. Call a cospan $P \xleftarrow{j} C \xrightarrow{k} Q$ in Pos *bipartite* if k and j are embeddings and for every $p \in P$, every $q \in Q$, $kq \not\leq jp$. We show that the duals of weakening relations in DL are exactly the bipartite cospans in Pri, and that commas of bipartite cospans in Pri are exact and determine weakening spans in Pri. Thus we have the main theorem.

Theorem 1. *The order enriched categories $\overline{\mathsf{DL}}$ and $\overline{\mathsf{Pri}}$ are dually equivalent on 1-cells and equivalent on 2-cells.*

Now from this duality, we recover the original duality of DL and Pri by noting that is both settings, adjoint pairs of weakening relations determine and are determined by functions. That is, im $\overline{\mathsf{DL}}$, define $\mathsf{Map}(\overline{\mathsf{DL}})$ to consist of pairs of relations (R, S) so that $1_A \leq S \circ R$ and $R \circ S \leq 1_B$. Define $\mathsf{Map}(\overline{\mathsf{Pri}})$ likewise.

Lemma 1. *The category $\mathsf{Map}(\overline{DL})$ is equivalent (actually isomorphic) to DL.*

Now since the duality for relations preserves order on hom-sets, it also follows that $\mathsf{Map}(\overline{Pri})$ is dually equivalent to $\mathsf{Map}(\overline{\mathsf{DL}})$.

Although we have paid attention to Priestley duality here, many of the technical results depend only more general structure of DL and Pri. In the full paper, we discuss sufficient conditions for a natural duality between categories $\mathcal{A}$ and $\mathcal{X}$ that are conrete over Pos to lift to $\overline{\mathcal{A}}$ and $\overline{\mathcal{X}}$.

Semi-Constructive Versions of the Rasiowa-Sikorski Lemma and Possibility Semantics for Intuitionistic Logic

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The celebrated Rasiowa-Sikorski Lemma [6] states that for any Boolean algebra B, any countable set Q of subsets of B, and any non-zero $a \in B$, there exists an ultrafilter U over B such that $a \in U$ and U preserves all existing meets in Q. Rasiowa and Sikorski [5] famously applied the lemma to the Lindenbaum-Tarski algebra of Classical Predicate Logic in order to prove its completeness with respect to Tarskian semantics. However, their proof of the lemma was an application of the Stone representation theorem [7], which relies on the non-constructive Boolean Prime Ideal Theorem (BPI). In fact, Goldblatt [1] observes that the Rasiowa-Sikorski Lemma is equivalent over ZF to the conjunction of BPI and Tarski's Lemma, a weaker proposition that states that for any Boolean algebra B, any countable set Q of subsets of B, and any $a \in B$, there exists an filter F over B such that $a \in F$ and for any $X \in Q$ such that $\bigwedge X$ exists in B, $\bigwedge X \in F$ or there is $x \in X$ such that $\neg x \in F$. Goldblatt also proves that Tarski's Lemma is *semi-constructive*, in the sense that it is equivalent over ZF to the Axiom of Dependent Choices (DC).

In this talk, we provide a generalization of Tarski's Lemma to the variety of distributive lattices (DL), the Q-Lemma, that states that for any distributive lattice L, any countable sets Q_M and Q_J of subsets of L, and any two elements $a, b \in L$, if $a \nleq b$, then there exists a pair (F, I) such that:

- F and I are a filter and an ideal over L respectively;

- $F \cap I = \emptyset$, $a \in F$ and $b \in I$;

- for any $X \in Q_M$, if $\bigwedge X$ exists in L and distributes over all joins, then $\bigwedge X \in F$ or there is $x \in X \cap I$;

- for any $Y \in Q_J$, if $\bigvee Y$ exists in L and distributes over all meets, then $\bigvee Y \in I$ or there is $y \in Y \cap F$.

We show that the Q-Lemma is a semi-constructive version of the Rasiowa-Sikorski Lemma for DL as stated in [2], in the sense that it is equivalent over ZF to Tarski's Lemma, and that the Rasiowa-Sikorski Lemma for DL is equivalent to the conjunction of the Prime Filter Theorem and the Q-Lemma.

Moreover, we generalize some of the ideas behind possibility semantics for classical logic, developed in Holliday [3] and Humberstone [4], to intuitionistic logic. We show in particular how to provide a choice-free bitopological representation of distributive lattices and Heyting algebras based on pairs of filters and ideals, and how this framework combined with the Q-Lemma yields an alternative semantics for Intuitionistic Predicate Logic with the Axiom of Constant Domains.

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Rooted frames for fusions of multimodal logics

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Most of monomodal logics are characterized by classes of frames (see e.g. [1],[2]). It is even possible to use single connected frames for some logics. The additional modalities make the problem of seeking one connected frame more demanding.

Consider a propositional n-modal language $\mathcal{L}_1$ with modal operators $\square_1, \ldots, \square_n$ and a propositional m-modal language $\mathcal{L}_2$ with modal operators $\square_{n+1}, \ldots, \square_{n+m}$. Let us denote by $\mathcal{L}_{1,2}$ the propositional $n+m$-modal language with operators $\square_1, \ldots, \square_n, \square_{n+1}, \ldots, \square_{n+m}$. The smallest $n+m$-modal logic in the language $\mathcal{L}_{1,2}$ containing $L_1 \cup L_2$ is called a *fusion* of $L_1 \subset \mathcal{L}_1$ and $L_2 \subset \mathcal{L}_2$. We write $L_1 \oplus L_2$ for the fusion of L_1 and L_2.

A Kripke n-frame $\mathfrak{B} = \langle V, H_1, \ldots, H_n \rangle$ is called a *subframe* of a frame $\mathfrak{F} = \langle W, R_1, \ldots, R_n \rangle$ if $V \subseteq W$ and H_i is the restriction of R_i to V (i.e. $H_i = R_i \cap (V \times V)$), for all $i \in \{1, \ldots, n\}$. A subframe $\mathfrak{B}$ of $\mathfrak{F}$ is called a *generated subframe* of $\mathfrak{F}$ if for each $y \in W$, $y \in V$ if $xR_i y$ for some $x \in V$ and some $i \in \{1, \ldots, n\}$. The subframe of the frame $\mathfrak{F}$ generated by the set $U \subseteq W$ will be denoted by $[U]_{\mathfrak{F}}$. If $U = \{x\}$, we write $[x]_{\mathfrak{F}}$ instead of $[\{x\}]_{\mathfrak{F}}$. For a given class $\mathcal{C}$ of n-frames, let $PGS(\mathcal{C})$ be the class of all subframes of the frames from the class $\mathcal{C}$ generated by a single point. In symbols

$$PGS(\mathcal{C}) = \{[x]_{\mathfrak{F}} \colon \mathfrak{F} = \langle W, R_1, \ldots, R_n \rangle \in \mathcal{C}, x \in W\}.$$

A Kripke n-frame $\mathfrak{F} = \langle W, R_1, \ldots, R_n \rangle$ is *rooted* if $\mathfrak{F} = [x]_{\mathfrak{F}}$ for some $x \in W$ i.e. if there exists $x \in W$ such that for each $y \in W \setminus \{x\}$ there exists a sequence $(x_1, \ldots, x_{k-1})$ of elements from W such that

$$xR_{i_1}x_1, x_1R_{i_2}x_2, \ldots, x_{k-2}R_{i_{k-2}}x_{k-1}, x_{k-1}R_{i_{k-1}}y,$$

where $i_j \in \{1, \ldots, n\}$. The point x is called a *root* of the frame $\mathfrak{F}$.

Let L_1 be an n-modal logic and L_2 be an m-modal logic. Assume that L_1 and L_2 are characterized by classes of rooted frames $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. It is already known that there exists a class of $n+m$-frames that characterizes $n+m$-modal logic $L_1 \oplus L_2$ (see e.g. [3],[4]).

Consider a class $\mathcal{C}$ of rooted frames. Let $\mathfrak{F}$ be a frame with a root x. We say that the point x is a $\mathcal{C}$-*root* if for each $\mathfrak{G} \in \mathcal{C}$ and a root y of $\mathfrak{G}$ there exists a p-morphism from $\mathfrak{F}$ to $\mathfrak{G}$ sending x to y.

Let us consider the class $\mathcal{C}_{Grz.3} = \{\mathfrak{F}_{Grz.3}^n = \langle \{1, \ldots, n\}, \geq \rangle \colon n \in \mathbb{N}\}$ of all finite chains. A frame with a $\mathcal{C}_{Grz.3}$-root is $\mathfrak{F}_{Grz.3}^r = \langle W', \leq \rangle$, where

$$W' = \left\{ \frac{1}{n} \colon n \in \mathbb{N} \right\} \cup \{0\}.$$

Let us consider the chain $\mathfrak{F}_{Grz.3}^6$. Point 6 is a root of the frame $\mathfrak{F}_{Grz.3}^6$, therefore $f(0) = 6$. It is necessary to preserve order. In next steps $f(1) = 1$, $f(\frac{1}{2}) = 2$, $f(\frac{1}{3}) = 3$, $f(\frac{1}{5}) = 5$, $f(\frac{1}{k}) = 6$ for $k \geq 6$.

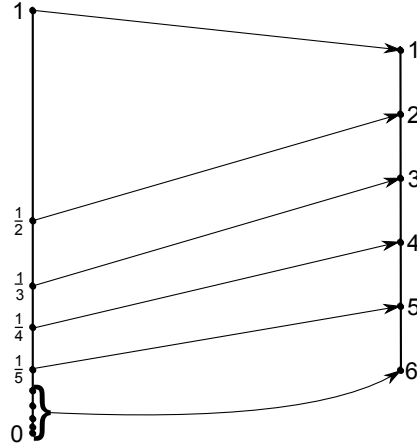Figure 1: $\mathfrak{F}^r_{Grz.3} \to \mathfrak{F}^6_{Grz.3}$

Let L_1 be an n-modal logic and L_2 be an m-modal logic. Assume that L_1 and L_2 are characterized by classes of rooted frames $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Classes $\mathcal{C}'_1$ and $\mathcal{C}'_2$ are closures of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, under the formation of disjoint unions and isomorphic copies. Moreover, let $\mathfrak{F}^1 = \langle W_1, R_1, \ldots, R_n \rangle$ be an L_1-frame with $PGS(\mathcal{C}_1)$-root and $\mathfrak{F}^2 = \langle W_2, R_{n+1}, \ldots, R_{n+m} \rangle$ be an L_2-frame with $PGS(\mathcal{C}_2)$-root.

In the talk we will show how to construct a rooted frame $\mathfrak{F}^r = \langle W^r, S_1, \ldots, S_{n+m} \rangle$ which characterizes the $n + m$-modal logic $L_1 \oplus L_2$ and has the following properties

(a) $\mathfrak{F}^r$ is countable if $\mathfrak{F}^1$ and $\mathfrak{F}^2$ are countable;

(b) each $S_1, \ldots, S_n$-connected component of the frame $\mathfrak{F}^r$ is isomorphic to the frame $\mathfrak{F}^1$;

(c) each $S_{n+1}, \ldots, S_{n+m}$-connected component of the frame $\mathfrak{F}^r$ is isomorphic to the frame $\mathfrak{F}^2$;

(d) $\mathfrak{F}^r$ is a frame with a $PGS(\mathcal{C}'_1 \oplus \mathcal{C}'_2)$-root;

(e) for each $n + m$-formula φ, $\mathfrak{F}^r \models \varphi$ if and only if φ is valid in a $PGS(\mathcal{C}'_1 \oplus \mathcal{C}'_2)$-root of the frame $\mathfrak{F}^r$.

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The computational complexity of the Leibniz hierarchy

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Abstract algebraic logic is a field that studies uniformly propositional logics [2, 3, 4]. One of its main achievements is the development of the so-called *Leibniz hierarchy* (see Figure 1), which provides a taxonomy that classifies propositional systems accordingly to the way their notions of *logical equivalence* and of *truth* can be defined.

A fundamental question, that arose in the study of the Leibniz hierarchy, is whether there is an algorithm that allows to classify logics in the Leibniz hierarchy. The answer to this question depends on the way in which these logics are presented. More precisely, in [7] it is shown that the problem of classifying logics presented *syntactically*, i.e. by means of finite Hilbert calculi, in the Leibniz hierarchy is in general undecidable. On the other hand, it is not difficult to see that logics presented *semantically*, i.e. by means of finite sets of finite (logical) matrices of finite type, can be classified mechanically in the Leibniz hierarchy. It is therefore natural to ask which is the computational complexity of the problem of classifying semantically presented logics in the Leibniz hierarchy. More precisely, in this contribution we will present a solution to the following problems:

- Let K be a level of the Leibniz hierarchy. Which is the computational complexity of the problem $\mathsf{Class\text{-}K}$ of determining whether a semantically presented logic belongs to K?

Elementary considerations show that the naive algorithms, that solve $\mathsf{Class\text{-}K}$, run in exponential time. The interesting part of our proof consists in establishing a hardness result, according to which these algorithms cannot be substantially improved. In [1] it was established that the following problem, which we denote by $\mathsf{Gen\text{-}Clo}_2^1$, is complete for **EXPTIME**:

- Let $\boldsymbol{A}$ be a finite algebra of finite type, whose basic operations are at most binary, and a h be a unary function on A. Does h belong to the clone of $\boldsymbol{A}$?

We will construct a polynomial-time reduction of $\mathsf{Gen\text{-}Clo}_2^1$ to $\mathsf{Class\text{-}K}$.

To this end, consider a non-trivial algebra $\boldsymbol{A}$ whose basic operations $\mathcal{F}$ are at most binary, and a unary function h on A. For sake of simplicity, we assume that $\mathcal{F}$ contains no constant symbols. Our goal is to define a new algebra $\boldsymbol{A}^\natural$, related to $\boldsymbol{A}$ and h. The construction of the $\boldsymbol{A}^\natural$ is partially reniniscent of ideas exploited in [6] and [5] to prove some hardness results related to type sets and Maltsev conditions. The universe of $\boldsymbol{A}^\natural$ is given by eight disjoint copies $A_1, \ldots, A_8$ of A. Given an element $a \in A$, we will denote by a^i its copy in A_i. The basic operation of $\boldsymbol{A}^\natural$ are the ones in $\mathcal{F}$ plus a new ternary operation $\heartsuit$ and a new unary operation $\square$. Their interpretation is defined as follows. Given an n-ary operation $f \in \mathcal{F}$ and $a_1^{m_1} \ldots, a_n^{m_n} \in A^\natural$, we set

$$f(a_1^{m_1} \ldots, a_n^{m_n}) \coloneqq f^{\boldsymbol{A}}(a_1, \ldots, a_n)^5.$$

Observe that all the operations $f^{\boldsymbol{A}^\natural}$ with $f \in \mathcal{F}$ give values in A_7. Given $a^m, b^n, c^k \in A^\natural$, we set

$$\heartsuit(a^m, b^n, c^k) \coloneqq \begin{cases} a^1 & \text{if } a^m = c^k \text{ and } h(a)^5 = b^n \text{ and } m \in \{1, 3, 4\} \\ a^2 & \text{if } a^m = c^k \text{ and } h(a)^5 = b^n \text{ and } m \in \{2, 5, 6, 7, 8\} \\ a^4 & \text{if } m, k \in \{1, 3, 4\} \text{ and (either } a^m \neq c^k \text{ or } h(a)^5 \neq b^n) \\ a^7 & \text{if } \{m, k\} \cap \{2, 5, 6, 7, 8\} \neq \emptyset \text{ and} \\ & \text{(either } a^m \neq c^k \text{ or } h(a)^5 \neq b^n). \end{cases}$$
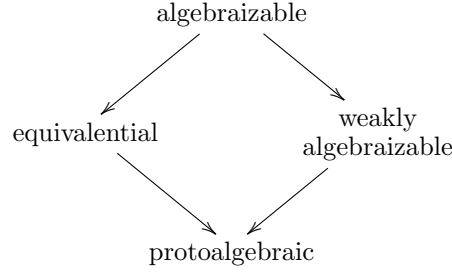
Figure 1: The main classes in the Leibniz hierarchy.

Given $a^m \in A^\natural$, we set

$$\Box(a^m) := \left\{ \begin{array}{ll} a^m & \text{if } m = 1 \text{ or } m = 2 \\ a^{m-1} & \text{if } m \text{ is even and } m \geq 3 \\ a^{m+1} & \text{if } m \text{ is odd and } m \geq 3. \end{array} \right.$$

Now, consider the matrix $\langle \boldsymbol{A}^\natural, F^\natural \rangle$, where $F^\natural := A_1 \cup A_2$. Observe that the matrix $\langle \boldsymbol{A}^\natural, F^\natural \rangle$ can be constructed out of $\boldsymbol{A}$ in polynomial time, since the arity of the basic operations of $\boldsymbol{A}$ is bounded by 2. The hearth of our proof consists in showing that if $\vdash$ is the logic determined by the matrix $\langle \boldsymbol{A}^\natural, F^\natural \rangle$, then the following conditions are equivalent:

1. $\vdash$ is algebraizable.

2. $\vdash$ is protoalgebraic.

3. h belongs to the clone of $\boldsymbol{A}$.

As a consequence, there is a polynomial time reduction of the $\mathsf{Gen\text{-}Clo}_2^1$ to the problem $\mathsf{Class\text{-}K}$ for every level K of the Leibniz hierarchy. Hence we obtain the following:

Theorem 1. *Let* K *be a level of the Leibniz hierarchy.* $\mathsf{Class\text{-}K}$ *is complete for* **EXPTIME**.

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Unifying the Leibniz and Maltsev hierarchies

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Universal algebra and abstract algebraic logics are two theories that study, respectively, arbitrary algebraic structures and arbitrary substitution-invariant consequence relations (sometimes called deductive systems). The interplay between the two theories can be hardly overestimated. On the one hand, techniques from universal algebra have been fruitfully applied to the study of propositional logics in the framework of abstract algebraic logic. On the other hand, any class of algebras K is naturally associated with a substitution-invariant equational consequence $\vDash_{\mathsf{K}}$ (representing the validity of generalized quasi-equations in K), which is amenable to the techniques of abstract algebraic logic. The fact that universal algebra and abstract algebraic logic pursue two tightly connected paths is nicely reflected in the fact that one of the main achievements of both theories is a taxonomy in which, respectively, varieties and deductive systems are classified. In universal algebra, this taxonomy is called *Maltsev hierarchy*, while in abstract algebraic logic it is known as *Leibniz hierarchy*.

The goal of this contribution is to show that this analogy between the Maltsev and Leibniz hierarchies can be made mathematically precise, in a such way that the traditional Maltsev hierarchy coincides with the restriction of a suitable *finite companion* of the Leibniz hierarchy formulated for two-deductive systems. To this end, we need to solve a fundamental asymmetry between the theories of the Maltsev and Leibniz hierarchy: while there is a precise definition of what the Maltsev hierarchy is [3, 4, 5], no such agreement exists for the case of the Leibniz hierarchy.

For the sake of simplicity, we will introduce the main new definitions for *logics*, i.e. substitution-invariant consequence relations formulated over the set of formulas (built up with an arbitrarily large infinite set of variables) of an algebraic language. Recall that each logic $\vdash$ is naturally associated with a class of matrices $\mathsf{Mod}^{\mathrm{Su}}(\vdash)$, called the *Suszko models* of $\vdash$ [1]. An *interpretation* of a logic $\vdash$ into a logic $\vdash'$ is a map $\boldsymbol{\tau}$ assigning an n-ary term $\boldsymbol{\tau}(f)$ of $\vdash'$ to every n-ary connective f of $\vdash$ in such a way that

$$\text{if } \langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash'), \text{ then } \langle \boldsymbol{A}^{\boldsymbol{\tau}}, F \rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash)$$

where $\boldsymbol{A}^{\boldsymbol{\tau}}$ is the algebra in the language of $\vdash$, whose universe is A and in which the connective f is interpreted as the term-function $\boldsymbol{\tau}(f)^{\boldsymbol{A}}$ of $\boldsymbol{A}$. We write $\vdash \, \leq \, \vdash'$ to denote the fact that $\vdash$ is interpretable into $\vdash'$. The interpretability relation $\leq$ is a preorder on the class of all logics. We denote by Log the poset obtained identifying equi-interpretable logics.

Theorem 1. Log *is a complete meet-semilattice, meaning that infima of all its subsets exist. Moreover,* Log *is not a join-semilattice. Finally,* Log *has no minimum element, it has a maximum and a coatom (that under Vopěnka's Principle is unique).*

A *Leibniz condition* is a sequence $\Phi := \{\vdash_{\alpha} : \alpha \in \mathsf{Ord}\}$ of logics indexed by all ordinals Ord, satisfying the following additional condition: if $\alpha \leq \beta$, then $\vdash_{\beta} \, \leq \, \vdash_{\alpha}$. The class of *models* of Φ is $\mathsf{Mod}(\Phi) := \{\vdash : \vdash_{\alpha} \, \leq \, \vdash \text{ for some } \alpha \in \mathsf{Ord}\}$. A *Leibniz class* is a class of logics M for which there is a Leibniz condition Φ such that $\mathsf{M} = \mathsf{Mod}(\Phi)$. It is not difficult to see that all classes of

logics traditionally included into the Leibniz hierarchy are in fact Leibniz classes in this general sense. For this reason, we propose to identify the Leibniz hierarchy with the poset of all Leibniz classes. Leibniz classes can be characterized in terms of closure under certain constructions, that we call *Taylorian products* and *compatible expansions*, as follows (cf. [4, 5]):

Theorem 2. *Let* M *be a class of logics. The following conditions are equivalent:*

1. M *is a Leibniz class.*

2. M *is closed under term-equivalence, compatible expansions and Taylorian products.*

3. M *is a complete filter of* Log.

The fact that Leibniz classes can be identified with complete filters of Log rises the question of understanding which of the classical Leibniz classes determine a meet-irreducible or prime filter (cf. [2]). This is a completely new direction of research. Nevertheless, we were able to obtain some promising results: for example, it turns out that, in the setting of logics with theorems, the class of equivalential logics is meet-reducible, while (under the assumption of Vopěnka's Principle) the classes of truth-equational and assertional logics are prime.

As we mentioned, it is possible to associate a *finite companion* to the Leibniz hierarchy, understood as the poset of all Leibniz classes. Roughly speaking, this is the collection of Leibniz classes determined by Leibniz conditions of the form $\Phi = \{\vdash_n : \alpha \in \omega\}$, where $\vdash_n$ is a finitely presentable and finitely equivalential logic. We call *finitely presentable Leibniz classes* the classes of logics in the finite companion of the Leibniz hierarchy. The Maltsev hierarchy is then the restriction of the finite companion of the Leibniz hierarchy of two-deductive system to equational consequences. More precisely, we have the following:

Theorem 3. *Let* K *be a class of varieties.* K *is a Maltsev class iff there is a finitely presentable Leibniz class* M *of two-deductive systems such that* $\mathsf{K} = \{\mathsf{V} : \mathsf{V}$ *is a variety and* $\vDash_\mathsf{V} \in \mathsf{M}\}$.

The above result shows that the logical theory of the Leibniz hierarchy may be seen as a generalization of the algebraic theory of Maltsev classes. Moreover, in our opinion, this perspective shows that the conceptual taxonomies, which lie at the heart of modern abstract algebraic logic and universal algebra, have a common root.

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Admissibility for universal classes and multi-conclusion consequence relations

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The notions of admissibility and structural completeness for logics and consequence relations has received considerable attention for many years. Recently a study of these concepts has been undertaken by Rosalie Iemhoff [1]. It appears that admissibility may be considered in various nonequivalent ways. This leads to variants of structural completeness. Here we investigate this topic from an algebraic perspective. We provide algebraic characterizations of variants of structural completeness for universal classes (which are algebraic counterparts of multi-conclusion consequence relations). Then we study the preservation of these properties by the Blok-Esakia isomorphism.

A (multi-conclusion) *rule* is an ordered pair, written as Γ/Δ, of finite sets of formulas in a given propositional language. When $|\Delta| = 1$ we talk about a *single-conclusion rule*. A set of rules, written as a relation $\vdash$, is a *multi-conclusion consequence relation* (mcr) if for all finite sets $\Gamma, \Gamma', \Delta, \Delta'$ of formulas, for every formula φ and for every substitution s the following holds

- $\varphi \vdash \varphi$;
- if $\Gamma \vdash \Delta, \varphi$ and $\Gamma, \varphi \vdash \Delta$, then $\Gamma \vdash \Delta$;
- if $\Gamma \vdash \Delta$, then $\Gamma, \Gamma' \vdash \Delta, \Delta'$;
- if $\Gamma \vdash \Delta$, then $s(\Gamma) \vdash s(\Delta)$.

(We omit the curly brackets for sets, write commas for unions and omit the empty set.)

Informally speaking, admissible rules are rules that may be added to a mcr in order to improve the search of (multi)theorems. A formal definition of admissible rules for the basic and narrow variants was given by Rosalie Iemhoff in [1] (called there *full* and *strict*). The weak variant is taken from [3]. For simplicity, we consider the admissibility for single-conclusion rules.

Definition 1. For a rule $r = \Gamma/\delta$ and a mcr $\vdash$ let $\vdash_r$ be a least mcr extending $\vdash$ and containing r. Then r is

- *admissible for* $\vdash$ provided $\vdash \Delta$ iff $\vdash_r \Delta$ for every finite set Δ of formulas;

- *weakly admissible for* $\vdash$ provided $\vdash \varphi$ iff $\vdash_r \varphi$ for formula φ;

- *narrowly admissible for* $\vdash$ provided for every substitution s ($\forall \gamma \in \Gamma \ \vdash s(\gamma)$) yields $\vdash s(\delta)$.

And $\vdash$ is (*strongly, widely*) *structurally complete* if every (weakly, narrowly) admissible for $\vdash$ single-conclusion rule belongs to $\vdash$.

Fact 2. *For a single-conclusion rule r and a mcr $\vdash$ we have the implications:*

r is admissible for $\vdash$ $\Rightarrow$ r is weakly admissible for $\vdash$ $\Rightarrow$ r is narrowly admissible for $\vdash$,
$\vdash$ is widely struct. complete $\Rightarrow$ $\vdash$ is strongly struct. complete $\Rightarrow$ $\vdash$ is struct. complete.

Algebraic counterparts of single conclusion consequence relations are quasivarieties of algebras. A main tool to deal with the admissibility is then the notion of free algebras for quasivarieties: the admissibility corresponds exactly to the validity on free algebras.

Algebraic counterparts of mcrs are universal classes of algebras[1]. Clearly, for every universal clas $\mathcal{U}$ free algebras exist. But they do not have to belong to $\mathcal{U}$. We overcome this obstacle by introducing the notion of a free family. It consists of quotients of a term algebra chosen in a certain minimal way [4]. Thus this object is indeed similar to a free algebra.

Recall that a counterpart of a single-conclusion rule is a quasi-identity. We skip definitions of the variants of the admissibility for quasi-identities. Let us just note that they are direct translations of Definition 1 [4]. In order to formulate our theorem let us introduce a bit of notation: For a universal class $\mathcal{U}$ let $\mathbf{F}$ be its free algebra and $\mathcal{F}$ be its free family, both of denumerable rank. For a class $\mathcal{K}$ of algebras let $\mathsf{Q}(\mathcal{K})$ be a least quasivariety containing $\mathcal{K}$.

Theorem 3. *Let q be a quasi-identity and $\mathcal{U}$ be a universal class. Then*

- *q is admissible for $\mathcal{U}$ iff $\mathcal{F} \models q$;*

- *q is weakly admissible for $\mathcal{U}$ iff $\mathbf{F} \in \mathsf{Q}(\{\mathbf{A} \in \mathcal{U} \mid \mathbf{A} \models q\})$;*

- *q is narrowly admissible for $\mathcal{U}$ iff $\mathbf{F} \models q$.*

Consequently,

- *$\mathcal{U}$ is structurally complete iff $\mathsf{Q}(\mathcal{F}) = \mathsf{Q}(\mathcal{U})$;*

- *$\mathcal{U}$ is strongly structurally complete iff $\mathbf{F} \in \mathsf{Q}(\mathcal{U} \cap \mathcal{Q})$ yields $\mathcal{U} \subseteq \mathcal{Q}$ for every quasivariety $\mathcal{Q}$;*

- *$\mathcal{U}$ is widely structurally complete iff $\mathsf{Q}(\mathbf{F}) = \mathsf{Q}(\mathcal{U})$.*

The classical Blok-Esakia theorem states that there is one to one correspondence between extensions of intuitionistic logic and normal extensions of modal Grzegorczyk logic. In [2] Emil Jeřábek extended this fact to mcrs. Algebraically it says that there is an isomorphism σ from the lattice of universal classes of Heyting algebras onto the lattice of universal classes of modal Grzegorczyk algebras (see [3] for an algebraic treatment of this topic). In [3, 4] we obtained the following preservation and reflection facts.

Theorem 4. *Let $\mathcal{U}$ be a universal class of Heyting algeras. Then*

- *$\mathcal{U}$ is structurally complete iff $\sigma(\mathcal{U})$ is structurally complete;*

- *$\mathcal{U}$ is strongly structurally complete iff $\sigma(\mathcal{U})$ is strongly structurally complete;*

- *$\mathcal{U}$ is widely structurally complete iff $\sigma(\mathcal{U})$ is widely structurally complete.*

Let us finish with the remark that the admissibility and the weak admissibility properties may be directly defined also for multi-conclusion rules. Then the analogs of the presented results may be proved. However there is a problem with the narrow variant of the admissibility. It may be defined in at least two different ways. The connection to the strong and the basic variants is more complex.

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Disjunction and Existence Property in Inquisitive Logic

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The system `InqBQ` ([1], [4], [3]) generalizes `FOL` (first order classical logic) to study *dependencies* between `FOL` structures in a similar fashion to Dependence Logic ([5]) and other logics based on team semantics ([2], [6]). In this paper we introduce several model theoretic constructions useful to study the entailment relation of `InqBQ`, and prove the disjunction and existence for the classical fragment of the logic (presented in [3]).

GENERALIZING `FOL` SEMANTICS

In the rest of the paper with $\Sigma = \{f, \dots; R, \dots\}$ we indicate a fixed `FOL` signature.

Definition (Skeleton of a model). Given $M = \langle D^M; f^M, \dots, R^M, \dots; \sim^M \rangle$ a `FOL` structure (note that we introduce here an extensional equality $\sim^M$, i.e. a congruence wrt $f^{\mathcal{M}}$ and $R^{\mathcal{M}}$) we define its **skeleton** as the tuple $\mathrm{Sk}(M) = \langle D^M; f^M, \dots \rangle$ consisting of the domain and the interpretation of the function symbols.

Definition (Information model). An **information model** is a tuple $\mathcal{M} = \langle M_w | w \in W^{\mathcal{M}} \rangle$ where the M_w are `FOL` structures sharing the same skeleton: $\forall w, w'. \mathrm{Sk}(M_w) = \mathrm{Sk}(M_{w'})$. Conceptually, an information model represents a *collection of possible states of affairs* and we can represent a body of information by selecting the structures compatible with it.

Definition (Info state). Given a model $\mathcal{M}$ we call a subset of the structures that compose it an **info state**: $s \subseteq W$ (modulo a natural identification). We call a model $\mathcal{M}_s = \langle M_w | w \in s \rangle$ for $s \subseteq W^{\mathcal{M}}$ a **submodel of** $\mathcal{M}$.

Definition (Support semantics). Let $\mathcal{M}$ be a model, s an info state of $\mathcal{M}$ and $g : \mathrm{Var} \to D^{\mathcal{M}}$ a valuation. Let α be a `FOL` formula. We define the support relation by the following inductive clauses. As a notational convention, we will omit s if $s = W^{\mathcal{M}}$.

$$\mathcal{M}, s \vDash_g \bot \iff s = \emptyset$$
$$\mathcal{M}, s \vDash_g [t_1 = t_2] \iff \forall w \in s. M_w \vDash_g^{\texttt{FOL}} [t_1 = t_2]$$
$$\mathcal{M}, s \vDash_g R(\bar{t}) \iff \forall w \in s. M_w \vDash_g^{\texttt{FOL}} R(\bar{t})$$
$$\mathcal{M}, s \vDash_g \phi \wedge \psi \iff \mathcal{M}, s \vDash_g \phi \text{ and } \mathcal{M}, s \vDash_g \psi$$
$$\mathcal{M}, s \vDash_g \phi \to \psi \iff \forall t \subseteq s. \text{ if } \mathcal{M}, t \vDash_g \phi$$
$$\text{then } \mathcal{M}, t \vDash_g \psi$$
$$\mathcal{M}, s \vDash_g \forall x.\phi \iff \forall d \in D^{\mathcal{M}}. \mathcal{M}, s \vDash_{g[x \mapsto d]} \phi$$

We say that a theory Γ entails a formula α (notation $\Gamma \vDash \alpha$) if and only if for every tuple $\langle \mathcal{M}, s, g \rangle$ that supports Γ, this supports also α.

Lemma (Properties of the support semantics for `FOL` formulas).

Flatness: $\mathcal{M}, s \vDash_g \alpha \iff \forall w \in s. \mathcal{M}, \{w\} \vDash_g \alpha$.

Classical World Support: $\mathcal{M}, \{w\} \vDash_g \alpha \iff M_w \vDash_g^{\texttt{FOL}} \alpha$.

Classical Validity Preservation: $\Gamma \vDash \alpha \iff \Gamma \vDash^{\texttt{FOL}} \alpha$.

ADDING NEW OPERATORS TO THE LOGIC

Defined this generalized semantics, we can now introduce new logical operators to describe *connections and relations between models sharing the same skeleton*. We consider here **the logic** `InqBQ` obtained by adding the **operator** $\vee\!\!\!\vee$ and the **quantifier** $\exists$, and their associated semantical clauses.

$$\mathcal{M}, s \vDash_g \phi \vee\!\!\!\vee \psi \iff \mathcal{M}, s \vDash_g \phi \text{ or } \mathcal{M}, s \vDash_g \psi$$
$$\text{(a disjunct holds at the *whole state*)}$$
$$\mathcal{M}, s \vDash_g \exists x.\phi \iff \exists d \in D^{\mathcal{M}}. \mathcal{M}, s \vDash_{g[x \mapsto d]} \phi$$
$$\text{(an element is a *uniform* witness of } \phi \text{ at } s)$$

Lemma (**Downward Closure**). $\mathcal{M}, s \vDash_g \phi$ and $t \subseteq s$, then $\mathcal{M}, t \vDash_g \phi$.

Note that *flatness holds exactly for those formulas ϕ which are semantically equivalent to a* `FOL` *formula* ([3]).

DISJUNCTION AND EXISTENCE PROPERTIES

Theorem. Let Γ be a `FOL` theory and α a `FOL` formula. Then:

Disjunction Property: If $\Gamma \vDash \phi \lor\!\!\!\lor \psi$, then $\Gamma \vDash \phi$ or $\Gamma \vDash \psi$.

Existence Property: If $\Gamma \vDash \overline{\exists} x.\phi(x)$, then $\Gamma \vDash \phi(t)$ for some term t.

The proof of this theorem is based on the introduction of some relevant model-theoretic constructions.

The $\oplus$ operator: We can define an operator $\oplus$ such that, given a set of models $\{\mathcal{M}_i | i \in I\}$, it produces a model $\oplus_{i \in I} \mathcal{M}_i$ with the following properties

$$\mathcal{M}_i \nvDash \phi \implies \oplus_{i \in I} \mathcal{M}_i \nvDash \phi \qquad\qquad \forall i \in I. \ \mathcal{M}_i \vDash \alpha \iff \oplus_{i \in I} \mathcal{M}_i \vDash \alpha \text{ for } \alpha \text{ classical}$$

This construction strongly relies on *downward closure* and *flatness* for classical formulas.

Characteristic model of Γ: Consider a classical theory Γ. For every non entailment $\Gamma \nvDash \phi$ we can select a model $\mathcal{M}_\phi$ that is a *witness* of it, meaning $\mathcal{M}_\phi \vDash_{g_\phi} \Gamma$ and $\mathcal{M}_\phi \nvDash_{g_\phi} \phi$. If we define now $\mathcal{M}_\Gamma = \oplus_{\Gamma \nvDash \phi} \mathcal{M}_\phi$, by the property of $\oplus$ we obtain $\mathcal{M}_\Gamma \vDash \psi \iff \Gamma \vDash \psi$.

Note that this model can be used to easily prove the disjunction property:

$$\Gamma \vDash \phi \lor\!\!\!\lor \psi \iff \mathcal{M}_\Gamma \vDash \phi \lor\!\!\!\lor \psi \iff \mathcal{M}_\Gamma \vDash \phi \text{ or } \mathcal{M}_\Gamma \vDash \psi \iff \Gamma \vDash \phi \text{ or } \Gamma \vDash \psi$$

Blow-up model: given a model $\mathcal{M}$ we can define an elementarily equivalent model $\mathcal{B}\mathcal{M}$ (the **blow-up of $\mathcal{M}$**) whose domain is $\mathcal{T}\Sigma(D^\mathcal{M})$, the free algebra of terms in the extended signature $\Sigma(D^\mathcal{M})$ obtained by adding to Σ a fresh constant symbol for every element of $D^\mathcal{M}$. In this step, the intensional equality plays a fundamental role.

Permutation models: given a model $\mathcal{M}$ and a permutation of its elements $\sigma \in \mathfrak{S}\left(D^\mathcal{M}\right)$, we can naturally extend such permutation to $\mathcal{T}\Sigma\left(D^\mathcal{M}\right)$. Using this, we can define a model $\mathcal{B}^\sigma \mathcal{M}$ by *permuting the names of the elements of $\mathcal{B}\mathcal{M}$ according to σ*.

Note that this operation *preserves skeletons* $(\mathrm{Sk}(\mathcal{B}\mathcal{M}) = \mathrm{Sk}(\mathcal{B}^\sigma \mathcal{M}))$ and that *a closed term t of $\Sigma\left(D^\mathcal{M}\right)$ is fixed under every permutation σ iff t is a closed term of Σ*. These two properties (of great importance for the proof of the existence property) wouldn't hold if the permutation was applied directly to $\mathcal{M}$, thus the necessity of defining the model $\mathcal{B}\mathcal{M}$.

Lemma. $\mathcal{B}^\sigma \mathcal{M} \vDash \phi(\sigma(d_1), \ldots, \sigma(d_n)) \iff \mathcal{B}\mathcal{M} \vDash \phi(d_1, \ldots, d_n) \iff \mathcal{M} \vDash \phi(d_1, \ldots, d_n)$

The full permutation model: Consider now the model $\mathcal{M}_\Gamma$. As the action of a permutation $\sigma \in \mathfrak{S}\left(D^{\mathcal{M}_\Gamma}\right)$ preserves the skeleton of $\mathcal{B}(\mathcal{M}_\Gamma)$, we can consider the new model $\mathfrak{S}\left(\mathcal{M}_\Gamma\right) = \{M | \exists \sigma. \ M \in \mathcal{B}^\sigma\left(\mathcal{M}_\Gamma\right)\}$. By building $\mathcal{M}_\Gamma$ in a suitable way, we can obtain the following two properties:

$\underline{\mathfrak{S}\,\mathcal{M}_\Gamma \vDash \phi \iff \Gamma \vDash \phi}$: since $\mathfrak{S}\left(\mathcal{M}_\Gamma\right) \vDash \Gamma$ can be tested on single worlds by *flatness*, and $\mathcal{B}\left(\mathcal{M}_\Gamma\right)$ is a submodel of $\mathfrak{S}\left(\mathcal{M}_\Gamma\right)$.

$\underline{\mathfrak{S}\,\mathcal{M}_\Gamma \vDash \overline{\exists} x.\phi(x) \Rightarrow \mathfrak{S}\,\mathcal{M}_\Gamma \vDash \phi(t)}$ for some closed t: the intuitive reason being that the role of two elements can be *swapped* in the model $\mathfrak{S}\left(\mathcal{M}_\Gamma\right)$ as long as they are not fixed by every permutation σ. From this we obtain that, if there exists an element without the property ϕ, then every element that is not the interpretation of a closed term of Σ does not have the property.

Note that this model can be used to easily prove the existence property:

$$\Gamma \vDash \overline{\exists} x.\phi(x) \iff \mathfrak{S}\left(\mathcal{M}_\Gamma\right) \vDash \overline{\exists} x.\phi(x) \iff \exists t \text{ closed.} \ \mathfrak{S}\left(\mathcal{M}_\Gamma\right) \vDash \phi(t) \iff \exists t \text{ closed.} \ \Gamma \vDash \phi(t)$$

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The Free Algebra in a Two-sorted Variety of State Algebras

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States of MV-algebras [6] are $[0, 1]$-valued functions, which generalise finitely-additive probability measures on boolean algebras, and whose domains are MV-algebras [2]. Flaminio and Montagna [3] introduced an internal state as an additional unary operation $\sigma\colon M \to M$ satisfying certain equational laws on an MV-algebra M. Internal states capture the basic properties of states in a setting amenable to universal-algebraic techniques.

In our note [5] we made first steps towards a general two-sorted algebraic model for expressing the notion of state between two MV-algebras M and N, making thus a fundamental distinction between events (captured by elements of the domain M) and probability degrees (represented by the co-domain N). A *generalised state* of M with values in N is a mapping $s\colon M \to N$ such that for every $a, b \in M$ the following hold: $s(a \oplus b) = s(a) \oplus s(b \wedge \neg a)$, $s(\neg a) = \neg s(a)$, and $s(\top) = \top$. A *state algebra* is a two-sorted algebra (M, N, s), where the operations of M and N are in the single sorts given by M and N, respectively, and the only operation between the two sorts is the generalised state s. The class of all state algebras constitutes a two-sorted algebraic variety. Most universal-algebraic constructions and results have analogous correspondents in the multi-sorted setting [1].

In this contribution we will characterise the free state algebra $\mathbf{F}(S_1, S_2)$ generated by a two-sorted set of generators (S_1, S_2). The free state algebra can be expressed as

$$\mathbf{F}(S_1, S_2) = \mathbf{F}(S_1, \emptyset) \amalg \mathbf{F}(\emptyset, S_2),$$

where $\mathbf{F}(S_1, \emptyset)$ and $\mathbf{F}(\emptyset, S_2)$ are the free state algebras over $(S_1, \emptyset)$ and $(\emptyset, S_2)$, respectively, and $\amalg$ denotes the coproduct operation in the multi-sorted algebraic category of state algebras. First, the algebra $\mathbf{F}(S_1, \emptyset)$ is isomorphic to the state algebra $(F(S_1), \langle \widehat{F(S_1)} \rangle, \alpha)$, where $F(S_1)$ is the free MV-algebra over S_1, $\langle \widehat{F(S_1)} \rangle$ is the affine representation of $F(S_1)$ (see [4]), and α is the evaluation map $F(S_1) \to \langle \widehat{F(S_1)} \rangle$ sending elements of $F(S_1)$ to $[0, 1]$-valued affine functions over the state space of $F(S_1)$. Second, the free state algebra over $(\emptyset, S_2)$ is $\mathbf{F}(\emptyset, S_2) = (2, F(S_2), s_0)$, where 2 is the two-element MV-algebra, $F(S_2)$ is the free MV-algebra generated by S_2, and s_0 is the only possible generalised state $2 \to F(S_2)$. We will show that

$$\mathbf{F}(S_1, \emptyset) \amalg \mathbf{F}(\emptyset, S_2) = (F(S_1), \langle \widehat{F(S_1)} \rangle \amalg_{MV} F(S_2), \beta_1 \circ \alpha),$$

where

$$\langle \widehat{F(S_1)} \rangle \amalg_{MV} F(S_2)$$

is the coproduct (free product [7]) of MV-algebras $\langle \widehat{F(S_1)} \rangle$ and $F(S_2)$, and the map $\beta_1\colon \langle \widehat{F(S_1)} \rangle \to \langle \widehat{F(S_1)} \rangle \amalg_{MV} F(S_2)$ is the coproduct injection.

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Sublocales and a Boolean extension of a frame [*]

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This talk is about sublocales, the natural subobjects in the category of locales (which one may think about as generalized topological spaces), that is, in the dual category of the category of frames ([3]).

Sublocales of a frame L are well defined subsets of L, and constitute, in the natural inclusion order, a coframe $\mathsf{S}(L)$. Hence sublocale lattices are more complicated than their topological counterparts (complete and atomic Boolean algebras). One of the main differences is that only complemented sublocales (and most sublocales are not complemented) distribute over all joins of sublocales. But, as J. Isbell emphasized, a locale has enough complemented sublocales to compensate for this shortcoming: one has open and closed sublocales (precisely corresponding to classical open and closed subspaces), complementing each other.

A separation axiom called subfitness (making sense for classical spaces as well, slightly weaker than T_1) is characterized by the property that *every open sublocale is a join of closed ones*, and another, stronger, called fitness (akin to regularity) is characterized by the fact that *every closed sublocale is an intersection of open ones*. These properties sound dual to each other, but is not quite so: in fact *in a fit frame every sublocale whatsoever is an intersection of open ones* which has no counterpart in the subfit case. Now what does the property that *every sublocale whatsoever is a join of closed ones* mean? In [1] it was shown that it characterizes the so called scattered frames (quite analogous to scattered topological spaces), formally the L with Boolean $\mathsf{S}(L)$. The main goal of our talk will be to discuss the system $\mathsf{S}_{\mathfrak{c}}(L)$ of all the sublocales of a general L that are joins of closed ones.

We will start the talk by presenting the basics about $\mathsf{S}_{\mathfrak{c}}(L)$ ([5]). First, $\mathsf{S}_{\mathfrak{c}}(L)$ is always a frame. Since it is a join-sublattice, is it not also a coframe, or even a subcolocale of $\mathsf{S}(L)$? We give a complete answer for subfit frames L. There, indeed, $\mathsf{S}_{\mathfrak{c}}(L)$ is a subcolocale (and in fact this is another characterization of subfitness). Moreover, it is a Boolean algebra and in fact precisely the Booleanization of $\mathsf{S}(L)$. Further, we have here a Boolean extension $L \to \mathsf{S}_{\mathfrak{c}}(L)$ by *open* sublocales; this is compared with the well known frame extension $L \to \mathsf{S}(L)^{\mathrm{op}}$ by *closed* sublocales (the embedding into the frame of congruences), and the relation is analysed.

Subspaces of a space can be viewed as sublocales (more precisely, sublocales of the associated frame of open sets $\Omega(X)$). But in general there are more sublocales than subspaces (a space has typically generalized subspaces that are not classical induced ones). In case of a T_1-space X, it turns out that the classical ones constitute precisely the $\mathsf{S}_{\mathfrak{c}}(\Omega(X))$, and hence the Booleanization of $\mathsf{S}(\Omega(X))$.

Point-free modeling of real-valued functions on a frame L that are not necessarily continuous has been so far based on the extension of L to its frame of sublocales $\mathsf{S}(L)^{\mathrm{op}}$, mimicking the replacement of a topological space by its discretization ([2]). If time permits, we will explain why the smaller $\mathsf{S}_{\mathfrak{c}}(L)$ can replace $\mathsf{S}(L)^{\mathrm{op}}$ with advantages in the case of a subfit L ([4]).

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[*]A part of the talk is joint work with Anna Tozzi [5].

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Canonical extensions of archimedean vector lattices with strong order unit

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Canonical extensions of Boolean algebras with operators were introduced in the seminal paper of Jónsson and Tarski [7]. They were generalized to distributive lattices with operators [4, 5], lattices with operators [2], and further to posets [6, 3].

Stone duality provides motivation for the definition of the canonical extension. For example, the canonical extension B of a Boolean algebra A is isomorphic to the powerset of the Stone space X of A, and the embedding $e : A \to B$ is realized as the inclusion of the Boolean algebra $\mathsf{Clop}(X)$ of clopen subsets of X into the powerset $\wp(X)$. The inclusion $\mathsf{Clop}(X) \hookrightarrow \wp(X)$ is dense and compact, and these are the defining properties of the canonical extension:

Definition 1. The *canonical extension* of a Boolean algebra A is a pair $A^\sigma = (B, e)$, where B is a complete Boolean algebra and $e : A \to B$ is a Boolean monomorphism satisfying:

1. (Density) Each $x \in B$ is a join of meets and a meet of joins of elements of $e[A]$.

2. (Compactness) For $S, T \subseteq A$, from $\bigwedge e[S] \leq \bigvee e[T]$ it follows that $\bigwedge e[S'] \leq \bigvee e[T']$ for some finite $S' \subseteq S$ and $T' \subseteq T$.

A similar situation arises for archimedean vector lattices with strong order unit. Let A be an archimedean vector lattice with strong order unit. By Yosida representation [8], A is represented as a uniformly dense vector sublattice of the vector lattice $C(Y)$ of all continuous real-valued functions on the Yosida space Y of A. Moreover, if A is uniformly complete, then A is isomorphic to $C(Y)$. Since Y is compact, every continuous real-valued function on Y is bounded. Therefore, $C(Y)$ is a vector sublattice of the vector lattice $B(Y)$ of all bounded real-valued functions on Y.

The inclusion $C(Y) \hookrightarrow B(Y)$ has many similarities with the inclusion $\mathsf{Clop}(X) \hookrightarrow \wp(X)$. In particular, $C(Y)$ is dense in $B(Y)$. However, it is never compact in the sense of Definition 1. Indeed, if Y is a singleton, then both $C(Y)$ and $B(Y)$ are isomorphic to $\mathbb{R}$. Now, if $S = \{\beta \in \mathbb{R} : 1/2 < \beta \leq 1\}$ and $T = \{\alpha \in \mathbb{R} : 0 \leq \alpha < 1/2\}$, then $\bigwedge S \leq \bigvee T$ as both are $1/2$, but there are not finite subsets $S' \subseteq S$ and $T' \subseteq T$ with $\bigwedge S' \leq \bigvee T'$.

Our goal is to tweak the definition of compactness appropriately, so that coupled with density, it captures algebraically the behavior of the inclusion $C(Y) \hookrightarrow B(Y)$.

Let A be an archimedean vector lattice and let $u \in A$ be the strong order unit of A. We identify $\mathbb{R}$ with a subalgebra of A by identifying $\alpha \in \mathbb{R}$ with $\alpha u \in A$.

Definition 2. The *canonical extension* of an archimedean vector lattice with strong order unit A is a pair $A^\sigma = (B, e)$, where B is a Dedekind complete (archimedean) vector lattice with strong order unit and $e : A \to B$ is a unital vector lattice monomorphism satisfying:

1. (Density) Each $x \in B$ is a join of meets and a meet of joins of elements of $e[A]$.

2. (Compactness) For $S, T \subseteq A$ and $0 < \varepsilon \in \mathbb{R}$, from $\bigwedge e[S] + \varepsilon \leq \bigvee e[T]$ it follows that $\bigwedge e[S'] \leq \bigvee e[T']$ for some finite $S' \subseteq S$ and $T' \subseteq T$.

Theorem 3. *Let X be a completely regular space, and let $C^*(X)$ be the vector lattice of bounded continuous real-valued functions on X. Then $B(X)$ is the canonical extension of $C^*(X)$ if and only if X is compact.*

Regardless of whether X is compact, the vector lattice $C^*(X)$ is dense in $B(X)$ in the sense of Definition 2. Thus, the theorem shows that the compactness axiom of Definition 2 when applied to $C^*(X)$ and $B(X)$ gives an algebraic formulation of topological compactness.

Theorem 4. *Let A be an archimedean vector lattice with strong order unit, Y the Yosida space of A, and $e : A \to C(Y)$ the Yosida embedding. Then the pair $(B(Y), e)$ is up to isomorphism the canonical extension of A. Thus, canonical extensions of archimedean vector lattices with strong order unit always exist and are unique up to isomorphism.*

In fact, the correspondence $A \mapsto A^\sigma$ is functorial. This functoriality of canonical extensions contrasts with the lack of it for Dedekind completions [1].

It is well known that a Boolean algebra can be realized as the canonical extension of some other Boolean algebra if and only if it is complete and atomic. We give a similar characterization in our setting. Suppose that B is an archimedean vector lattice with strong order unit. If B is Dedekind complete, then it has a unique multiplication which makes it a lattice-ordered ring (see, e.g., [1, Sec. 8]). Viewing B as a ring, since B is Dedekind complete, the idempotents $\mathrm{Id}(B)$ of B form a complete Boolean algebra. Then B is a canonical extension of some vector lattice A with strong order unit if and only $\mathrm{Id}(B)$ is atomic. We also give a purely ring-theoretic characterization of B as a Baer ring with essential socle.

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Antistructural completeness in propositional logics

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In this contribution, we shall investigate the notion of an antistructural completion $\alpha\mathcal{L}$ of a propositional logic $\mathcal{L}$, which is in a natural sense dual to the well-known notion of a structural completion of a logic, and provide several equivalent characterizations of such completions under some mild conditions on the logic in question.

Recall that the *structural completion* of a logic $\mathcal{L}$ is the largest logic $\sigma\mathcal{L}$ which has the same theorems as $\mathcal{L}$ (see [2]). A logic $\mathcal{L}$ is then called *structurally complete* if $\sigma\mathcal{L} = \mathcal{L}$. The logic $\sigma\mathcal{L}$ exists for each $\mathcal{L}$ and it has a simple description: $\Gamma \vdash_{\sigma\mathcal{L}} \varphi$ if and only if the rule $\Gamma \vdash \varphi$ is *admissible*, that is, for each substitution σ we have $\emptyset \vdash_{\mathcal{L}} \sigma\varphi$ whenever $\emptyset \vdash_{\mathcal{L}} \sigma\gamma$ for each $\gamma \in \Gamma$.

Antistructural completions involve the same notions, but with respect to antitheorems rather than theorems. Here some clarification is in order: an *antitheorem* of $\mathcal{L}$ is a set of formulas Γ such that no valuation into a model of $\mathcal{L}$ designates each $\gamma \in \Gamma$. Equivalently, Γ is an antitheorem of $\mathcal{L}$ (symbolically, $\Gamma \vdash_{\mathcal{L}} \emptyset$) if $\sigma\Gamma \vdash_{\mathcal{L}} \mathrm{Fm}_{\mathcal{L}}$ for each substitution σ, where $\mathrm{Fm}_{\mathcal{L}}$ is the set of all formulas of $\mathcal{L}$. A set of formulas Γ is an antitheorem of $\mathcal{L}$ if $\Gamma \vdash_{\mathcal{L}} \mathrm{Fm}$ *provided that $\mathcal{L}$ has an antitheorem* (or provided that Γ is finite). It may happen, however, that a logic has no antitheorems, e.g. the positive fragment of classical or intuitionistic logic.

The *antistructural completion* of a logic $\mathcal{L}$ is defined as the largest logic $\alpha\mathcal{L}$ (whenever it exists) which has the same antitheorems as $\mathcal{L}$. Naturally, a logic $\mathcal{L}$ is then *antistructurally complete* if $\alpha\mathcal{L} = \mathcal{L}$. As a first example, consider intuitionistic logic $\mathcal{IL}$. Its antistructural completion may be computed using Glivenko's theorem. We have:

$$\Gamma \vdash_{\mathcal{IL}} \emptyset \;\Leftrightarrow\; \emptyset \vdash_{\mathcal{IL}} \sim\bigwedge\Gamma \;\Leftrightarrow\; \emptyset \vdash_{\mathcal{CL}} \sim\bigwedge\Gamma \;\Leftrightarrow\; \Gamma \vdash_{\mathcal{CL}} \emptyset$$

for finite Γ, hence $\mathcal{IL}$ and $\mathcal{CL}$ have the same antitheorems. Therefore classical logic $\mathcal{CL}$ is the antistructural completion of $\mathcal{IL}$ by virtue of being its largest non-trivial extension. Our aim will be to generalize this Glivenko-like connection between $\mathcal{IL}$ and $\mathcal{CL}$ to a wider setting.

For this purpose, the following notion is the natural counterpart of admissibility. A rule $\Gamma \vdash \varphi$ will be called *antiadmissible* in $\mathcal{L}$ if for each substitution σ and each Δ we have:

$$\sigma\Gamma, \Delta \vdash_{\mathcal{L}} \emptyset \text{ whenever } \sigma\varphi, \Delta \vdash_{\mathcal{L}} \emptyset$$

Lemma. *The antiadmissible rules of each logic form a reflexive monotone structural relation which is closed under finitary cuts (but not necessarily under arbitrary cuts).*

However, unlike the admissible rules, the antiadmissible rules in general need not define a logic and the antistructural completion of a logic need not exist.

Example. *Consider the standard Gödel chain $[0,1]_G$ expanded by a constant c_q for each rational $q \in \mathbb{Q} \cap [0,1]$. The logic defined semantically by all the* principal *filters on this chain does not have an antistructural completion.*

The existence of antistructural completions is therefore a rather more delicate matter than in the case of structural completions. Our main result now provides a widely applicable sufficient condition for the existence of $\alpha\mathcal{L}$ and several equivalent descriptions of this logic.

It involves a technical property which we call the *maximal consistency property (MCP)* which states that each consistent theory, i.e. a theory Γ such that $\Gamma \nvdash \emptyset$, may be extended to a maximal consistent theory. In particular, each finitary logic enjoys this property.

Theorem. *Let $\mathcal{L}$ be a logic with a finite antitheorem which enjoys the MCP. (For example, let $\mathcal{L}$ be a finitary logic with an antitheorem.) Then $\alpha\mathcal{L}$ exists and the following are equivalent:*

(i) $\Gamma \vdash_{\alpha\mathcal{L}} \varphi$.

(ii) $\Gamma \vdash \varphi$ is antiadmissible in $\mathcal{L}$.

(iii) $\Gamma \vdash \varphi$ is valid in all $\mathcal{L}$-models $\langle \boldsymbol{Fm}, \Gamma \rangle$ where Γ is a maximal consistent theory.

If $\mathcal{L}$ is moreover protoalgebraic, then these are equivalent to:

(iv) $\sigma\varphi, \Delta \vdash_{\mathcal{L}} \emptyset$ implies $\sigma\Gamma, \Delta \vdash_{\mathcal{L}} \emptyset$ for each Δ and each invertible substitution σ.

(v) $\Gamma \vdash \varphi$ is valid in all (reduced) κ-generated $\mathcal{L}$-simple matrices for $\kappa = |\operatorname{Var}_{\mathcal{L}}|$.

If $\mathcal{L}$ enjoys the local deduction theorem (LDDT) and finitarity, then these are equivalent to:

(vi) $\varphi, \Delta \vdash_{\mathcal{L}} \emptyset$ implies $\Gamma, \Delta \vdash_{\mathcal{L}} \emptyset$ for each Δ.

(vii) $\Gamma \vdash \varphi$ is valid in all (reduced) $\mathcal{L}$-simple matrices.

Proposition. *A finitary logic with an antitheorem $\mathcal{L}$ which enjoys the LDDT is antistructurally complete if and only if $\operatorname{Mod} \mathcal{L}$ is semisimple (each subdirectly irreducible $\mathcal{L}$-model is $\mathcal{L}$-simple).*

Item (iv) above may in fact be replaced by item (vi) whenever Γ is finite, or more generally whenever there are at least κ variables which do not occur in Γ for $\kappa = |\operatorname{Var}_{\mathcal{L}}|$.

Let us now provide some examples of antistructural completions of known logics to illustrate this notion. The following two claims are essentially reformulations of the results of [1] and [3].

Example. *The antistructural completion of Hájek's Basic Fuzzy Logic is the (infinitary) Łukasiewicz logic. Each axiomatic extension of the Full Lambek calculus with exchange and weakening which validates the axiom $p \vee \neg(p^n)$ for some $n \in \omega$ is antistructurally complete.*

Our main result has some use even outside the realm of protoalgebraic logics.

Example. *The antistructural completion of the four-valued Belnap–Dunn logic $\mathcal{B}$ is Priest's three-valued Logic of Paradox. The antistructural completion of the extension of $\mathcal{B}$ by the rule $p, \neg p \vdash q$ is the Exactly True Logic, i.e. the extension of $\mathcal{B}$ by the rule $p, \neg p \vee q \vdash q$.*

Observe also that the same notions can be considered for algebras rather than logics. For example, the variety of De Morgan algebras is antistructurally complete, while the antistructural completion of the variety of De Morgan lattices is the variety of Kleene lattices.

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Extended Contact Logic

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1 Introduction

Region Connection Calculus is a formalism for reasoning about the relation of contact between regions in topological spaces [5]. Its role in artificial intelligence and computer science stems from the importance of spatial informations in systems for natural language understanding, robotic navigation, etc [2]. After the introduction of Region Connection Calculus, Contact Logic and its different variants have been proposed [1, 4, 6]. Most of them are based on the binary predicate of ordinary contact which holds between regular closed subsets A and B iff $A \cap B \neq \emptyset$ ("regions A and B are in contact"). Recently, a ternary predicate of extended contact has been introduced which holds between regular closed subsets A, B and C iff $A \cap B \subseteq C$ ("regions A and B are jointly bounded by region C"). Remark that two regions are in ordinary contact iff they are not jointly bounded by the empty region. Moreover, the interest to consider the new ternary relation of extended contact lies in the possibility it gives to define the unary predicate of internal connectedness. See [3, Chapter 2] for details. In this note, we introduce the syntax and the semantics of Extended Contact Logic. Then, we give an axiomatization of the set of all valid formulas this semantics gives rise to. Finally, we prove the decidability of the set of all theorems this axiomatization gives rise to.

2 Syntax and semantics

Let VAR be a countable set of variables (p, q, etc). The set TER of all terms (α, β, etc) is defined by

- $\alpha ::= p \mid 0 \mid \alpha^\star \mid (\alpha + \beta)$.

Reading terms as regions, the constructs 0, * and $+$ should be regarded as the empty region, the complement operation and the union operation. The set FOR of all formulas (φ, ψ, etc) is defined by

- $\varphi ::= \alpha \leq \beta \mid (\alpha, \beta) \triangleright \gamma \mid \perp \mid \neg\varphi \mid (\varphi \vee \psi)$.

For $\leq$ and $\triangleright$, we propose the following readings: $\alpha \leq \beta$ can be read "region α is contained in region β", $(\alpha, \beta) \triangleright \gamma$ can be read "regions α and β are jointly bounded by region γ". We will write $\alpha \equiv \beta$ for $(\alpha \leq \beta \wedge \beta \leq \alpha)$. Terms and formulas are interpreted in topological models, i.e. structures of the form (X, τ, V) where (X, τ) is a topological space and V is a valuation on (X, τ), i.e. a map associating with every term α a regular closed subset $V(\alpha)$ of (X, τ) such that

- $V(0) = \emptyset$,

- $V(\alpha^\star) = Cl_\tau(X \setminus V(\alpha))$ where Cl_τ denotes the closure operator in (X, τ),

- $V(\alpha + \beta) = V(\alpha) \cup V(\beta)$.

The connectives $\perp$, $\neg$ and $\vee$ being classically interpreted, the satisfiability of a formula φ in (X, τ, V) (in symbols $(X, \tau, V) \models \varphi$) is defined as follows:

- $(X, \tau, V) \models \alpha \leq \beta$ iff $V(\alpha) \subseteq V(\beta)$,

- $(X, \tau, V) \models (\alpha, \beta) \rhd \gamma$ iff $V(\alpha) \cap V(\beta) \subseteq V(\gamma)$.

We will say that the formula φ is valid (in symbols $\models \varphi$) iff for all topological models (X, τ, V), $(X, \tau, V) \models \varphi$.

3   Axiomatization and decidability

Let $\mathbb{L}_{min}$ be the Hilbert-style axiomatic system consisting of the inference rule of modus ponens and the following axioms:

sentential axioms: instances of tautologies of propositional classical logic,

identity axioms: $\alpha \equiv \alpha$, $\alpha \equiv \beta \to \beta \equiv \alpha$, $\alpha \equiv \beta \wedge \beta \equiv \gamma \to \alpha \equiv \gamma$,

congruence axioms: $\alpha \equiv \beta \to \alpha^\star \equiv \beta^\star$, $\alpha \equiv \beta \wedge \gamma \equiv \delta \to \alpha + \gamma \equiv \beta + \delta$,

Boolean axioms: $(\alpha + \beta) + \gamma \equiv \alpha + (\beta + \gamma)$, $\alpha + \beta = \beta + \alpha$, etc,

nondegenerate axiom: $0 \not\equiv 1$,

extended contact axioms: **(i)** $(\alpha, \beta) \rhd \gamma \to (\beta, \alpha) \rhd \gamma$, **(ii)** $\alpha \leq \gamma \to (\alpha, \beta) \rhd \gamma$, **(iii)** $(\alpha, \beta) \rhd \gamma \wedge (\alpha, \beta) \rhd \delta \wedge (\gamma, \delta) \rhd \epsilon \to (\alpha, \beta) \rhd \epsilon$, **(iv)** $(\alpha, \beta) \rhd \gamma \to \alpha \cdot \beta \leq \gamma$, **(v)** $(\alpha, \gamma) \rhd \delta \wedge (\beta, \gamma) \rhd \delta \to (\alpha + \beta, \gamma) \rhd \delta$.

The notion of proof in $\mathbb{L}_{min}$ is the standard one. All provable formulas will be called theorems of $\mathbb{L}_{min}$.

Proposition 1. *For all formulas φ, $\models \varphi$ iff φ is a theorem of $\mathbb{L}_{min}$.*

Proof. For the soundness, it suffices to check that the inference rule of modus ponens preserve validity and that all axioms of $\mathbb{L}_{min}$ are valid. For the completeness, a detour through a semantical interpretation of terms and formulas in extended contact algebras as the ones studied in [3, Chapter 2] can be done. $\square$

Proposition 2. *The set of all theorems of $\mathbb{L}_{min}$ is decidable.*

Proof. A detour through a semantical interpretation of terms and formulas in relational structures as the ones studied in [1] and an associated finite model property can be done. $\square$

Nevertheless, the exact complexity of the set of all theorems of $\mathbb{L}_{min}$ is not known.

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Prime injective S-acts

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In this article, we define and study prime injectivity which is a generalization of $\mathcal{M}$-injectivity and investigate Skornjakov criterion respect to prime injectivity of acts. We charecterize the behaviour of the property considered under well-known constructions such as product, coproduct and direct sum. Ultimately, among the following results it is proved that an S-act is prime injective if and only if it is a prime-absolute retract if and only if it has no prime-essential extension.

1 Prime injective acts

In this section we define a generalization of injectivity of S-acts and we are going to study some behaviour of it.

Definition 1.1. (1) An act A is said to be *prime injective, if for any prime monomorphism* $g : B \to C$, *any homomorphism* $f : B \to A$ *can be lifted to a homomorphism* $\bar{f} : C \to A$, *such that* $\bar{f}g = f$.
(2) *An act A is said to be weakly prime injective, if it is injective relative to embeddings of all prime ideals into S.*
(3) *An S-act A is called to be f-g prime injective (cyclic prime injective), whenever for each prime homomorhism $g : F \to C$ from finitely generated (cyclic) act F to an act C, and for any homomorphism $f : F \to A$ there exists a homomorphism $h : C \to A$ such that $hg = f$.*

It is clear every injective act is prime injective act and all prime injective acts are weakly prime injective and each $\mathcal{M}$-injective act is a (weakly) prime injective act.

Proposition 1.2. *An act A is weakly prime injective if and only if for any homomorphism $f : I_S \to A$, where $I \subseteq S$ is a prime right ideal. there exists an element $a \in A$ such that $f(s) = as$ for every $s \in I$.*

Lemma 1.3. *The following statements are equivalent for monoid S.*

(1) *Every prime ideal of S is a retract of S.*

(2) *Every prime ideal of S is weakly prime injective.*

Recall that every cofree act is injective. Now we can say every cofree act is prime injective with the similary proof of theorem 3.1.5 of [4]. It is implies that every act can be embedded into a prime injective act. It means that the category of **S-act** has enough prime injective acts.

Lemma 1.4. *Every prime injective act contains a zero.*

Note that the category **S-act** is complete and cocomplete and has all products, coproducts, pushouts, pullbacks.

Proposition 1.5. *Let $\{A_i : i \in I\}$ be a family of S-acts. Then*

(1) *$\prod_{i \in I} A_i$ is prime injective (f-g prime injective, cyclic prime injective) if and only if A_i's are prime injective (f-g prime injective, cyclic prime injective) for all $i \in I$.*

(2) *If the coproduct $\coprod_{i \in I} A_i$ is prime injective (f-g prime injective, cyclic prime injective), then each A_i is prime injective (f-g prime injective, cyclic prime injective) act.*

Proposition 1.6. *Each direct sum of f-g prime injective (cyclic prime injective) acts is f-g prime injective (cyclic prime injective) .*

The converse of part (2) of Proposition 1.5, is not necessarily true in general. But we will show in Proposition 1.9, its converse is true for special S.

Theorem 1.7. *Assume an act A contain a zero θ. A is prime injective if and only if it is injective relative to all inclusions prime subact of cyclic acts.*

Definition 1.8. A monoid S is called *left prime reversible if $I \cap J \neq \emptyset$ for any prime right ideals I and J of S.*

Proposition 1.9. *The following statements are equivalent for any monoid S*

(1) *All coproducts of prime injective right acts are prime injective.*

(2) *$\{x, y\}$ is prime injective where x, y are fixed elements.*

(3) *S is left prime reversible.*

Theorem 1.10. *Pushouts transfer prime monomorphisms.*

Next theorem is one of the most interesting theorems about injectivity of S-acts with respect to any subclass of prime monomorphisms. This was proved by P. Berthiaume in [3], for injective acts and B. Banaschewski [2] has proved it for $\mathcal{M}$-injective acts when $\mathcal{M}$ is subclass of monomorhisms.

Theorem 1.11. *Let S be a semigroup. The following are equivalent for an S-act A:*

(1) *A is prime injective.*

(2) *A is a prime-absolute retract.*

(3) *A has no prime-essential extension.*

Theorem 1.11 immediately implies

Corollary 1.12. *For every right S-act there exists an prime injective hull.*

We are going to show that absolutely pure acts are absolutely prime-retract and then by Theorem 1.11, they are prime injective acts.

Theorem 1.13. *Every absolutely pure acts are prime injective.*

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First-order interpolation may be derived from propositional interpolation

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Following the ground breaking results of Maksimova [6] many families of propositional logics have been classified w.r.t. the interpolation property. However, on first-order level, the knowledge about interpolation is restricted. Moreover, it is not known which of the seven interpolating intermediary propositional logics [5] admit first-order interpolation (first-order infinitely-valued Gödel logic $G_{[0,1]}$ is the most notable example).

This lecture develops a general methodology to connect propositional and first-order interpolation. The construction of the first-order interpolant follows this procedure:

$$\left.\begin{array}{c} \text{existence of suitable Skolemizations } + \\ \text{existence of Herbrand expansions } + \\ \text{propositional interpolant} \end{array}\right\} \rightarrow \begin{array}{c} \text{first-order} \\ \text{interpolation.} \end{array}$$

This methodology is realized for lattice-based finitely-valued logics, the top element representing true and can be extended to (fragments of) infinitely-valued logics.

The construction of the first-order interpolant from the propositional interpolant follows this procedure:

1. Develop a validity equivalent Skolemization replacing all strong quantifiers (negative existential or positive universal quantifiers) in the valid formula $A \supset B$ to obtain the valid formula $A_1 \supset B_1$.

2. Construct a valid Herbrand expansion $A_2 \supset B_2$ for $A_1 \supset B_1$. Occurrences of $\exists x B(x)$ and $\forall x A(x)$ are replaced by suitable finite disjunctions $\bigvee B(t_i)$ and conjunctions $\bigwedge B(t_i)$, respectively.

3. Interpolate the propositionally valid formula $A_2 \supset B_2$ with the propositional interpolant I^*: $A_2 \supset I^*$ and $I^* \supset B_2$ are propositionally valid.

4. Reintroduce weak quantifiers to obtain valid formulas $A_1 \supset I^*$ and $I^* \supset B_1$.

5. Eliminate all function symbols and constants not in the common language of A_1 and B_1 by introducing suitable quantifiers in I^* (note that no Skolem functions are in the common language, therefore they are eliminated). Let I be the result.

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[*]The first author discussed the problem of deciding the admissibility of interpolation in first-order logics on the basis of the admissibility interpolation in propositional logics with Petr Hájek who suggested that proof-theoretic approaches might help to overcome the lack of algebraization of first-order logics.

6. I is an interpolant for $A_1 \supset B_1$. $A_1 \supset I$ and $I \supset B_1$ are Skolemizations of $A \supset I$ and $I \supset B$. Therefore I is an interpolant of $A \supset B$.

This methodology is realized for lattice-based finitely-valued logics and can be extended to (fragments of) infinitely-valued logics (more precisely to fragments of first-order infinitely-valued Gödel logic).

Consider Gödel logic $G_{[0,1]}$, the logic of all linearly ordered Kripke frames with constant domains. Its connectives can be interpreted as functions over the real interval $[0,1]$ as follows: $\perp$ is the logical constant for 0, $\vee, \wedge, \exists, \forall$ are defined as *maximum, minimum, supremum, infimum*, respectively. $\neg A$ is an abbreviation for $A \rightarrow \perp$ and $\rightarrow$ is defined as

$$u \rightarrow v = \begin{cases} 1 & u \leq v \\ v & \text{else} \end{cases}$$

The weak quantifier fragment of $G_{[0,1]}$ admits Herbrand expansions. This follows from cut-free proofs in hypersequent calculi [1, 2, 3]. This can be easily shown by proof transformation steps in the hypersequent calculus. Indeed, we can transform proofs by eliminating weak quantifier inferences:

i If there is an occurrence of an $\exists$ introduction, we select all formulas A_i that correspond to this inference and eliminate the $\exists$ introduction by the use of $\bigvee_i A_i$.

ii If there is an occurrence of a $\forall$ introduction, we select all formulas B_i that correspond to this inference and eliminate the $\forall$ introduction by the use of $\bigwedge_i B_i$.

With this procedure we do not infer weak quantifiers and combine the disjunctions/conjunctions to accommodate contractions. Propositional Gödel logic interpolates and therefore the weak quantifier fragment of $G_{[0,1]}$ interpolates, too.

The fragment $A \supset B$, A, B prenex also interpolates: Skolemize as in classical logic, construct a Herbrand expansion, interpolate, go back to the Skolem form and use an immediate analogy of the 2nd ε-theorem [4] to go back to the original formulas.

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On exponentiability in quantale-enriched categories

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1 Introduction

The purpose of this work is to present results on the existence of "function spaces" in categories of quantale-enriched categories, with particular emphasis on *generalized metric spaces* and *generalized probabilistic metric spaces* and their corresponding *non-expansive maps* (see [7] and [6]), that is, *V-categories* and *V-functors* when V is respectively the quantale of the *complete half-real line equipped with addition* and the *quantale of distribution functions*, but focusing also on categories enriched in the unit interval equipped with a *continuous t-norm*.

Most of the material presented is part of joint work with Dirk Hofmann that is published in [2] and [3] (see also [4]).

2 Exponentiable quantale-enriched categories

Given a quantale $(V, \otimes, k)$, it is well-known that the category V-Cat of enriched V-categories and V-functors is *closed*, that is, for each V-category X the functor $- \otimes X : V\text{-Cat} \to V\text{-Cat}$ induced in V-Cat by the tensor $\otimes$ in V has a right adjoint (see [7]). Here we concentrate on *cartesian closedness* of V-Cat, that is, on the existence of a right adjoint $(\ )^X : V\text{-Cat} \to V\text{-Cat}$ to the functor $- \times X : V\text{-Cat} \to V\text{-Cat}$ for each V-category X. This is not always the case, and in fact this ends up on the existence of a convenient V-category structure on the set Y^X of V-functors from X to Y (= exponential of Y with exponent X), for every V-category Y.

We will also pose this problem more generally, studying instead the existence of exponentials in the comma categories $(V\text{-Cat}) \downarrow Y$. Inspired by known results on exponentiability for continuous maps between topological spaces (see [5] and [1]), we prove in particular that *every proper and every étale V-functor is exponentiable in V-Cat.*

3 Final remarks

This study leads also to some interesting open problems on the properties of the quantales involved, namely on the quantale of distribution functions.

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A Vietoris functor for bispaces and d-frames

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Vietoris topology [12], hyperspace, powerlocale or powerdomain are many names for the same phenomenon. The scope of its applications ranges from semantics of programming languages [1, 10], coalgebraic logic [7, 8] to modal logic [2, 7, 11, 9]. Also, in Abramsky's *"Domain theory in logical form"* [1], a Vietoris construction has been an important tool for establishing a connection between syntax and semantics. An example of another such situation is the Jónsson-Tarski duality [5]. In modern parlance, we have an endofunctor $\mathbb{V}$ on Stone spaces and an endofunctor on Boolean algebras $\mathbb{M}$:

$$\mathbb{V} \left(\quad \textbf{Stone} \underset{\text{spec}}{\overset{\text{Clp}}{\rightleftarrows}} \textbf{Bool} \quad \right) \mathbb{M}$$
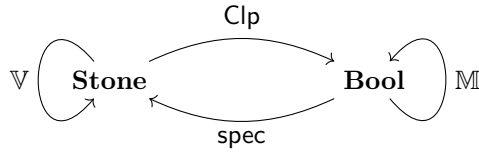
Moreover, the duality of Stone spaces and Boolean algebras extends to a duality of $\mathbb{V}$-coalgebras and $\mathbb{M}$-algebras. The category of $\mathbb{V}$-coalgebras is isomorphic to the category of descriptive general Kripke frames and the category of $\mathbb{M}$-algebras is isomorphic to the category of modal Boolean algebras.

Jónnson-Tarski duality is an instance of a more general picture where we can substitute **Stone** by a suitable category of spaces (modelling semantics) and **Bool** by a suitable category of algebras (modelling syntax) such that those categories are dually equivalent and some interconnected power-constructions $\mathbb{V}$ and $\mathbb{M}$ still exist. Other examples, where we can replace the base categories, include Priestley spaces and distributive lattices, or compact Hausdorff spaces and compact regular frames.

It was a beautiful insight by Jung and Moshier that all the dualities mentioned in the previous paragraph, and many more, sit in the duality between compact regular bitopological spaces **biKReg** and compact regular d-frames **d-KReg** [6]. Here d-frames are algebraic duals of bitopological spaces in the same way as frames[1] are algebraic duals of (ordinary) spaces. We give Vietoris endofunctors $\mathbb{W}$ and $\mathbb{M}^d$ which are generalisations of the corresponding Vietoris constructions for Stone, Priestley and frame dualities mentioned above.

$$\mathbb{W} \left(\quad \textbf{biKReg} \underset{\Sigma}{\overset{\Omega}{\rightleftarrows}} \textbf{d-KReg} \quad \right) \mathbb{M}^d$$
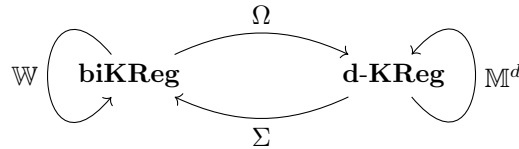
The construction

On the semantic side, we have the endofunctor $\mathbb{W} \colon \textbf{biKReg} \to \textbf{biKReg}$. Similarly to the Vietoris endofunctor for spaces or domains, the points of $\mathbb{W}(X; \tau_+, \tau_-)$ are compact *convex* subsets of X

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[1] *Frames* are complete lattices satisfying the equation: $a \wedge (\bigvee_i b_i) = \bigvee_i (a \wedge b_i)$.

and subbases of the topologies of $\mathbb{W}(X)$ are sets $\{\boxtimes U_+, \oplus U_+ : U_+ \in \tau_+\}$ and $\{\boxtimes U_-, \oplus U_- : U_- \in \tau_-\}$ where

$$K \in \boxtimes U \text{ iff } K \subseteq U \quad \text{and} \quad K \in \oplus U \text{ iff } K \cap U \neq \emptyset.$$

On the algebraic side we have d-frames, i.e. structures of the form $(L_+, L_-; \mathsf{con}, \mathsf{tot})$ where L_+ and L_- are frames corresponding to the two topologies and $\mathsf{con} \subseteq L_+ \times L_-$ is a relation which captures when two abstract opens are disjoint from each other and, similarly, $\mathsf{tot} \subseteq L_+ \times L_-$ representing when two abstract opens cover the whole space. The endofunctor $\mathbb{M}^d \colon \mathbf{d\text{-}Frm} \to \mathbf{d\text{-}Frm}$ is computed as follows

$$\mathbb{M}^d \colon \ (L_+, L_-; \mathsf{con}, \mathsf{tot}) \quad \longmapsto \quad (\mathbb{M}^{\mathbf{Frm}} L_+, \ \mathbb{M}^{\mathbf{Frm}} L_-; \ \mathsf{con}^{\mathbb{M}}, \mathsf{tot}^{\mathbb{M}}).$$

Here, $\mathbb{M}^{\mathbf{Frm}}$ is the Johnstone's powerlocale construction for frames [4]. To describe the consistency and totality relations we need to develop a free construction of a d-frame and then $\mathsf{con}^{\mathbb{M}}$ and $\mathsf{tot}^{\mathbb{M}}$ can be given by a set of generators. Similarly as in frames, $\mathbb{M}^d$ is comonadic and we also have the following familiar result:

Theorem. *Let $\mathcal{L}$ be a d-frame. If $\mathcal{L}$ is regular, zero-dimensional or compact regular then also $\mathbb{M}^d \mathcal{L}$ is.*

Thanks to this, we can restrict $\mathbb{M}^d$ to an endofunctor $\mathbb{M}^d \colon \mathbf{d\text{-}KReg} \to \mathbf{d\text{-}KReg}$ and formalise the connection between $\mathbb{W}$ and $\mathbb{M}^d$. The first step is to investigate the spectrum bispace $\Sigma\mathbb{M}^d(\mathcal{L})$. It turns out that its points have a very natural description. Namely, they are in bijection with the set of $\alpha \in L_+ \times L_-$ such that

$$(\mathrm{A}+) \qquad \forall u_+ \in L_+ \colon \text{ if } (\alpha_+ \vee u_+, \alpha_-) \in \mathsf{tot} \text{ then } (u_+, \alpha_-) \in \mathsf{tot}$$

$$(\mathrm{A}-) \qquad \forall u_- \in L_- \colon \text{ if } (\alpha_+, \alpha_- \vee u_-) \in \mathsf{tot} \text{ then } (\alpha_+, u_-) \in \mathsf{tot}$$

We can now prove the main result:

Theorem. *Let $\mathcal{L}$ be a compact regular d-frame. Then, $\mathbb{W}\Sigma(\mathcal{L}) \cong \Sigma\mathbb{M}^d(\mathcal{L})$. Moreover, this bi-homeomorphism is natural in $\mathcal{L}$.*

Thanks to this, we can lift the dual equivalence of categories $\mathbf{biKReg}$ and $\mathbf{d\text{-}KReg}$ to a dual equivalence of the category of $\mathbb{W}$-coalgebras and the category of $\mathbb{M}^d$-algebras.

As mentioned above, $\mathbb{W}$ is a generalisation of the corresponding Vietoris constructions for Stone spaces, Priestley spaces or compact regular spaces. Similarly, $\mathbb{M}^d$ is a generalisation of the constructions for Boolean algebras, distributive lattices and compact regular frames.

Another duality that embeds into the duality $\mathbf{biKReg}^{\mathrm{op}} \cong \mathbf{d\text{-}KReg}$ is the duality between the category of stably compact spaces (i.e. compact Hausdorff ordered spaces) and the category of strong proximity lattices. In fact, those categories are equivalent to $\mathbf{biKReg}$ and $\mathbf{d\text{-}KReg}$, respectively. Although a Vietoris construction is known for stably compact spaces [3], $\mathbb{M}^d$ is (to our knowledge) the first algebraic counterpart for it. Moreover, the free construction we developed for d-frame Vietoris functor is the first free construction of a d-frame.

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Modal extensions of $Ł_n$-valued logics, coalgebraically

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We show how previous work on modal extensions of $Ł_n$-valued logics fits naturally into the coalgebraic framework and indicate some of the ensuing generalisations.

Modal extensions of $Ł_n$-valued logics. We study logics with a modal operator $\Box$ and built from a countable set of propositional variables Prop using the connectors $\neg, \rightarrow, \Box, 1$ in the usual way. To interpret formulas on structures, we use a (crisp) many-valued generalization of the KRIPKE models. We fix a positive integer n and we denote by $Ł_n$ the subalgebra $Ł_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ of the standard MV-algebra $\langle [0,1], \neg, \rightarrow, 1 \rangle$. A *frame* is a couple $\langle W, R \rangle$ where W is a nonempty set and R is an binary relation. We denote by FR the class of frames.

Definition 0.1 ([2, 4, 5, 9]). An *$Ł_n$-valued model*, or a *model* for short, is a couple $\mathcal{M} = \langle \mathfrak{F}, \mathrm{Val} \rangle$ where $\mathfrak{F} = \langle W, R \rangle$ is a frame and $\mathrm{Val} \colon W \times \mathsf{Prop} \rightarrow Ł_n$. The valuation map Val is extended inductively to $W \times \mathsf{Form}$ using ŁUKASIEWICZ' interpretation of the connectors $0, \neg$ and $\rightarrow$ in $[0,1]$ and the rule

$$\mathrm{Val}(u, \Box\phi) = \min\{\mathrm{Val}(w, \phi) \mid w \in Ru\}. \tag{1}$$

A formula ϕ is *true* in an $Ł_n$-valued model $\mathcal{M} = \langle \mathfrak{F}, \mathrm{Val} \rangle$, in notation $\mathcal{M} \models \phi$, if $\mathrm{Val}(u, \phi) = 1$ for every world u of $\mathfrak{F}$. If Φ is a set of formulas that are true in every $Ł_n$-valued model based on an frame $\mathfrak{F}$, we write

$$\mathfrak{F} \models_n \Phi$$

and say that Φ is *$Ł_n$-valid* in $\mathfrak{F}$.

Apart from the signature of frames, there is another first-order signature that can be used to interpret formulas. We denote by $\preceq$ the dual order of divisibility on $\mathbb{N}$, that is, for every $\ell, k \in \mathbb{N}$ we write $\ell \preceq k$ if ℓ is a divisor of k, and $\ell \prec k$ if ℓ is a proper divisor of k.

Definition 0.2 (*n*-frames, [5, 9]). An *n-frame* is a tuple $\langle W, (r_m)_{m \preceq n}, R \rangle$ where $\langle W, R \rangle$ is a frame, $r_m \subseteq W$ for every $m \preceq n$, and

1. $r_n = W$ and $r_m \cap r_q = r_{\gcd(m,q)}$ for any $m, q \preceq n$,

2. $Ru \subseteq r_m$ for any $m \preceq n$ and $u \in r_m$.

FR^n is the class of *n*-frames. For $\mathfrak{F} \in \mathsf{FR}^n$, a model $\mathcal{M} = \langle \mathfrak{F}, \mathrm{Val} \rangle$ is based on $\mathfrak{F}$ if $\mathrm{Val}(u, \mathsf{Prop}) \subseteq Ł_m$ for every $m \preceq n$ and $u \in r_m$. We write

$$\mathfrak{F} \models \Phi$$

if Φ holds in all models based on $\mathfrak{F}$.

It is apparent from [4, 8, 5, 9] that $\models$ is better behaved then $\models_n$ because there is a nice duality between *n*-frames and modal $\mathcal{MV}_n$-algebras, very much analogous to the classical duality between Kripke frames and Boolean algebras with operators. For example, the Goldblatt-Thomason theorem for modal $Ł_n$-valued logic in [9] is first proved for *n*-frames and $\models$. The Goldblatt-Thomason theorem for frames and $\models_n$ then appears as a corollary. Morevoer, the canonical extension of a modal $\mathcal{MV}_n$-algebra $\mathbf{A}$ can be obtained as

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the complex algebra of a canonical n-frame associated with $\mathbf{A}$. This construction leads to completeness-through-canonicity results [5] with regards to classes of n-frames.

Modal extensions of Ł$_n$-valued logics, coalgebraically. We account for $\models_n$ by following well-established coalgebraic methodology, summarised in

$$T \, \rotatebox{90}{$\subset$} \ \mathsf{Set} \ \underset{S}{\overset{P}{\rightleftarrows}} \ \mathsf{MV_n} \ \rotatebox{-90}{$\subset$} \, L \tag{2}$$

where $T = \mathcal{P}$ is the powerset functor and LA is the free $\mathcal{MV}_n$ algebra generated by $\{\Box a \mid a \in A\}$ modulo the axioms of modal $\mathcal{MV}_n$-algebras. P and S are the contravariant functors given by homming into Ł$_n$. (1) allows us to extend P to a functor $\widetilde{P}$ from T-coalgebras to L-algebras, assigning to a T-coalgebra its 'complex algebra'. Similarly, the functor S can be extended to a functor $\widetilde{S}$ from L-algebras to T-coalgebras assigning to an L-algebra its 'canonical structure'.

A Kripke frame $\mathfrak{F} = \langle W, R \rangle$ is exactly a T-coalgebra (for $T = \mathcal{P}$). The Lindenbaum algebras (over a set of atomic propositions) are free L-algebras. We have $\mathfrak{F} \models \phi$ iff all morphisms from the free L-algebra (over the atomic propositions of ϕ) to $\widetilde{P}\mathfrak{F}$ map ϕ to W.

To account for $\models$, we replace, in (2), Set by the category $\mathsf{Set}_{\mathcal{V}_n}$ defined as follows. Let $\mathcal{V}_n = \{1, \ldots, n\}$ be the lattice of all divisors of n ordered by $n \leq m$ if m divides n (so that n is bottom and 1 is top). Then $\mathsf{Set}_{\mathcal{V}_n}$ has as objects pairs (X, v) with $v : X \to \mathcal{V}$ and arrows are maps $f : (X, v) \to (X', v')$ such that $v'fx \geq vx$. Note that this definition makes sense for any complete lattice $\mathcal{V}$ and that $\mathsf{Set}_{\mathcal{V}}$ coincides with Goguen's category of fuzzy sets [3].[1]

In order to extend functors $T : \mathsf{Set} \to \mathsf{Set}$ as in (2) to functors $\mathsf{Set}_{\mathcal{V}_n} \to \mathsf{Set}_{\mathcal{V}_n}$ we notice that $\mathsf{Set}_{\mathcal{V}}$ can be described equivalently as a category of 'continuous presheaves'. A continuous presheaf is a collection of sets $(X, (X_i)_{i \in \mathcal{V}})$ such that (i) $i \leq j$ only if $X_j \subseteq X_i$ (ii) $X_{\bigvee I} = \bigcap_{i \in I} X_i$ (iii) $X_0 = X$. Under mild conditions, this allows us to extend T pointwise by mapping $(X, (X_i)_{i \in \mathcal{V}})$ to $(TX, (TX_i)_{i \in \mathcal{V}})$.

In case of $\mathcal{V} = \mathcal{V}_n$ and $T = \mathcal{P}$, a T-coalgebra is precisely an n-frame, and capture the situation for $\models$:

$$T \, \rotatebox{90}{$\subset$} \ \mathsf{Set}_{\mathcal{V}_n} \ \underset{S}{\overset{P}{\rightleftarrows}} \ \mathsf{MV_n} \ \rotatebox{-90}{$\subset$} \, L \tag{3}$$

The adjunction (3) has better properties than (2). In particular, (3) restricts to a dual equivalence on finite structures. This shows that (3) falls into the framework of [7] and allows us to obtain the Goldblatt-Thomason theorems of [9] from the coalgebraic Goldblatt-Thomason theorem of [6]. In particular, this generalises the theorems of [9] to other functors T.

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Topology from enrichment:
the curious case of partial metrics

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Following Fréchet [1], a metric space (X, d) is a set X together with a real-valued function d on $X \times X$ such that the following axioms hold:

[M0] $d(x, y) \geq 0$,
[M1] $d(x, y) + d(y, z) \geq d(x, z)$,
[M2] $d(x, x) = 0$,
[M3] if $d(x, y) = 0 = d(y, x)$ then $x = y$,
[M4] $d(x, y) = d(y, x)$,
[M5] $d(x, y) \neq +\infty$.

The categorical content of this definition, as first observed by Lawvere [6], is that the extended real interval $[0, \infty]$ underlies a commutative quantale $([0, \infty], \bigwedge, +, 0)$, so that a "generalised metric space" (i.e. a structure as above, minus the axioms M3-M4-M5) is exactly a category enriched in that quantale. It was furthermore shown in [4] that to any category enriched in a commutative quantale one can associate a closure operator on its collection of objects. For a metric space (X, d), viewed as an $[0, \infty]$-enriched category, that "categorical closure" on X coincides precisely with the metric (topological) closure defined by d. And Lawvere [6] famously reformulated the Cauchy completeness of a metric space in terms of adjoint distributors.

More recently, see e.g. [7], the notion of a partial metric space (X, p) has been proposed to mean a set X together with a real-valued function p on $X \times X$ satisfying the following axioms:

[P0] $p(x, y) \geq 0$,
[P1] $p(x, y) + p(y, z) - p(y, y) \geq p(x, z)$,
[P2] $p(x, y) \geq p(x, x)$,
[P3] if $p(x, y) = p(x, x) = p(y, y) = p(y, x)$ then $x = y$,
[P4] $p(x, y) = p(y, x)$,
[P5] $p(x, y) \neq +\infty$.

The categorical content of *this* definition was discovered in two steps: first, Höhle and Kubiak [5] showed that there is a particular quantaloid of positive real numbers, such that categories enriched in that quantaloid correspond to ("generalised") partial metric spaces; and second, we realised in [8] that Höhle and Kubiak's quantaloid of real numbers is actually a universal construction on Lawvere's quantale of real numbers: namely, the quantaloid $\mathcal{D}[0, \infty]$ of diagonals in $[0, \infty]$.

In this talk we shall show how every small quantaloid-enriched category has a canonical closure operator on its set of objects: this makes for a functor from quantaloid-enriched categories to closure spaces. Under mild necessary-and-sufficient conditions on the base quantaloid, this functor lands in the category of topological spaces; and an involutive quantaloid is Cauchy-bilateral (a property discovered earlier in the context of distributive laws [2]) if and only if the closure on any enriched category is identical to the closure on its symmetrisation. As this now applies to metric spaces and partial metric spaces alike, we demonstrate how these general categorical constructions produce the "correct" definitions of convergence and Cauchyness of sequences in generalised partial metric spaces. Finally we describe the Cauchy-completion (and,

if time premits, also the Hausdorff contruction and exponentiability) of a partial metric space, again by application of general quantaloid-enriched category theory.

This talk is based on a joint paper with Dirk Hofmann [3].

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Modal logics over finite residuated lattices

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Several publications in the literature address the study of modal expansions of many-valued logics (see eg. [3, 2] [1], [4], [5]), and it is the aim of the current work to contribute to the better understanding of this topic. In particular, the framework developed in [1], which focused on modal logics defined from classes of Kripke models evaluated over finite residuated lattices, proposed several interesting open problems, and along the following lines we solve some of them.

In [1] we can find an standard definition of the Kripke models of a finite integral bounded commutative residuated lattice $\mathbf{A}$ (simply called finite residuated lattices in the rest of the abstract). There, the authors provide an axiomatic system for the local consequence relation arising from the models of $\mathbf{A}^c$ (the algebra $\mathbf{A}$ expanded with canonical constants[1]), while it is left as open problem the formulation of the corresponding system for the analogous global consequence relation.[2] On the other hand, the modal logics studied in [1] consider only the $\square$ modal operator, interpreted in a world of the model, as usual, by $e(v, \square\varphi) = \bigwedge_{w \in W} Rvw \rightarrow e(w, \varphi)$. It is well known that, over residuated lattices the dual $\diamond$ operator, for $e(v, \diamond\varphi) = \bigwedge_{w \in W} Rvw \odot e(w, \varphi)$, is in general no longer definable from $\square$, since the negation needs not to be involutive.

We wish to address the previous topics towards the full characterization, and subsequent understanding, applicability and possible generalisation of modal many-valued logics.

For $\mathbf{B}$ a finite residuated lattice (with or without canonical constants) consider the following logics, all of them defined from classes of Kripke models as usual:

- $\Vdash^l_{\square \mathbf{B}}$ and $\Vdash^l_{\diamond \square \mathbf{B}}$: the local consequence relation over the Kripke models of $\mathbf{B}$, with only $\square$ and with both $\square$ and $\diamond$ operators,

- $\Vdash^g_{\square \mathbf{B}}$ and $\Vdash^g_{\diamond \square \mathbf{B}}$: the global consequence relation over the Kripke models of $\mathbf{B}$, with only $\square$ and with both $\square$ and $\diamond$ operators.

Let $\vdash^l_{\square \mathbf{A}^c}$ denote the axiomatic system for $\Vdash^l_{\square \mathbf{A}^c}$ provided in [1, Th. 4.11] (called there $\Lambda(l, \mathbf{Fr}, \mathbf{A}^c)$). As far as we know, all the other logics from the previous list have not been axiomatized in the literature, for arbitrary $\mathbf{B}$. We can first exhibit recursively enumerable axiomatic systems for $\Vdash^l_{\diamond \square \mathbf{A}^c}$ and for both global logics arising from algebras with constants.

1. Let $\vdash^l_{\diamond \square \mathbf{A}^c}$ be the axiomatic system $\vdash^l_{\square \mathbf{A}^c}$ extended with $\square(\varphi \rightarrow \bar{c}) \leftrightarrow (\diamond\varphi \rightarrow \bar{c})$ for each $c \in A$. Then for any set of formulas Γ, φ it holds that

$$\Gamma \vdash^l_{\diamond \square \mathbf{A}^c} \varphi \iff \Gamma \Vdash^l_{\diamond \square \mathbf{A}^c} \varphi$$

2. Let $\vdash^g_{\square \mathbf{A}^c}$ be the axiomatic system $\vdash^l_{\square \mathbf{A}^c}$ expanded with the rule $(\mathtt{Mon}): \varphi \rightarrow \psi \rhd \square\varphi \rightarrow \square\psi$. Similarly, let $\vdash^g_{\diamond \square \mathbf{A}^c}$ be the axiomatic system $\vdash^l_{\diamond \square \mathbf{A}^c}$ expanded with the rule $(\mathtt{Mon})$. Then for any set of formulas Γ, φ

$$\Gamma \vdash^g_{\square \mathbf{A}^c} \varphi \iff \Gamma \Vdash^g_{\square \mathbf{A}^c} \varphi \quad and \quad \Gamma \vdash^g_{\diamond \square \mathbf{A}^c} \varphi \iff \Gamma \Vdash^g_{\diamond \square \mathbf{A}^c} \varphi$$

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[1]That is, with one additional constant symbol for each one of its elements, interpreted in the natural way.

[2]We are interested here in the models where the accessibility relation is evaluated in A. In [1] it is presented an axiomatization of the global consequence of the smaller class of models with $\{0, 1\}$-valued accessibility relation. It is worth to remark that the usual $\mathtt{K}$ axiom holds in this restricted class of models, while it does not in the general framework.

In order to prove the previous completeness results, finitarity of the propositional logic (due to finiteness of the algebra) and presence of constants are crucial. Nevertheless, resorting to those properties it is not hard to prove the Truth Lemma for the usual Canonical Models (both for the local and global logics). We remark here for the interested reader a cornerstone property from which the Truth Lemma follows easily, provable for all the previous canonical models:[3]

For any formula ψ and any $h \in W^c$, if $R^c hg \le g(\psi)$ for all $g \in W^c$ then $h(\Box\psi) = 1$.

The relation between the local and global modal logics of the class of frames over an algebra $\mathbf{A}$ (and over its corresponding $\mathbf{A}^c$) is now easy to understand. First, from the previous completeness result is immediate that the logic $\Vdash^g_{\Box\mathbf{A}^c}$ ($\Vdash^g_{\Diamond\Box\mathbf{A}^c}$) is the smallest consequence relation containing $\Vdash^l_{\Box\mathbf{A}^c}$ (respectively, $\Vdash^l_{\Diamond\Box\mathbf{A}^c}$) and closed under (Mon). On the other hand, by the definition of the modal logics arising from a class of models, is clear that for any set of formulas Γ, φ *without (non-trivial) constant symbols* it holds that

$$\Gamma \Vdash^*_{\mathtt{M}\mathbf{A}} \varphi \quad \Longleftrightarrow \quad \Gamma \Vdash^*_{\mathtt{M}\mathbf{A}^c} \varphi$$

for $* \in \{l, g\}$ and $\mathtt{M} \in \{\Box, \Diamond\Box\}$.

Even if we do not have a syntactical characterization of the modal logics arising from Kripke models over residuated lattices without canonical constants, the previous relation provides interesting information about these logics. A first immediate observation is that the logics $\Vdash^*_{\mathtt{M}\mathbf{A}}$ are finitary, since so are their corresponding versions over $\mathbf{A}^c$. Finitarity of both, the logic and the rule (Mon), provide (*via* eg. Zorn's Lemma) a very simple characterization of $[\Vdash^l_{\mathtt{M}\mathbf{A}}]^{(\mathtt{Mon})}$, the minimum consequence relation expanding $\Vdash^l_{\mathtt{M}\mathbf{A}}$ closed under (Mon). From there, it is easy to prove that for any Γ, φ without non-trivial constant symbols

$$\Gamma[\Vdash^l_{\mathtt{M}\mathbf{A}}]^{(\mathtt{Mon})}\varphi \quad \Longleftrightarrow \quad \Gamma[\Vdash^l_{\mathtt{M}\mathbf{A}^c}]^{(\mathtt{Mon})}\varphi$$

After the previous observations and using the completeness results stated above for logics over lattices with canonical constants, we can prove the following chain of equivalences:

$$\Gamma \Vdash^g_{\mathtt{M}\mathbf{A}} \varphi \quad \Longleftrightarrow \quad \Gamma \Vdash^g_{\mathtt{M}\mathbf{A}^c} \varphi \quad \Longleftrightarrow \quad \Gamma[\Vdash^l_{\mathtt{M}\mathbf{A}^c}]^{(\mathtt{Mon})}\varphi \quad \Longleftrightarrow \quad \Gamma[\Vdash^l_{\mathtt{M}\mathbf{A}}]^{(\mathtt{Mon})}\varphi.$$

This answers positively the open question (4) from [1], on whether the global deduction is the minimum closure operator containing the local deduction one and closed under the (Mon), both for finite residuated lattices with and without canonical constants.

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[3]This property is applicable also in some cases based on infinite residuated lattices, see eg. [6].

Varieties of De Morgan Monoids I:
Minimality and Irreducible Algebras

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A *De Morgan monoid* $\boldsymbol{A} = \langle A; \cdot, \wedge, \vee, \neg, e \rangle$ comprises a distributive lattice $\langle A; \wedge, \vee \rangle$, a commutative monoid $\langle A; \cdot, e \rangle$ satisfying $x \leqslant x^2 := x \cdot x$, and a function $\neg \colon A \longrightarrow A$, called an *involution*, such that $\boldsymbol{A}$ satisfies $\neg\neg x = x$ and $x \cdot y \leqslant z \iff x \cdot \neg z \leqslant \neg y$. (The derived operations $x \to y := \neg(x \cdot \neg y)$ and $f := \neg e$ turn $\boldsymbol{A}$ into an involutive residuated lattice in the sense of [3].)

The class DMM of all De Morgan monoids is a variety that algebraizes the relevance logic $\mathbf{R^t}$ of [1]. Its lattice of subvarieties Λ_{DMM} is dually isomorphic to the lattice of axiomatic extensions of $\mathbf{R^t}$. A *Sugihara monoid* is a De Morgan monoid that is idempotent, i.e., it satisfies $x^2 = x$. Sugihara monoids are subdirect products of chains. They are locally finite and well-understood (see Dunn's contributions to [1]).

In contrast, relatively little is known about the structure of (i) arbitrary De Morgan monoids and (ii) the lattice Λ_{DMM}. This situation is lamented in [8, p. 263] and [2, Sec. 3.5], which pre-date many recent papers on residuated lattices. But the latter have concentrated mainly on varieties incomparable with DMM (e.g., Heyting and MV-algebras), larger than DMM (e.g., full Lambek algebras) or smaller (e.g., Sugihara monoids). On the positive side, Slaney [5, 6] showed that the free 0–generated De Morgan monoid is finite, and that there are only seven non-isomorphic subdirectly irreducible 0–generated De Morgan monoids. No finiteness result of this kind holds in the 1–generated case, however. This talk and its sequel report on an attempt to enlarge our knowledge of DMM and its subvariety lattice.

Like any commutative residuated lattice, a De Morgan monoid $\boldsymbol{A}$ is finitely subdirectly irreducible iff its neutral element e is join-irreducible. In this case, however, the extra features of De Morgan monoids imply additional properties, e.g., $\boldsymbol{A}$ consists only of upper bounds of e and lower bounds of f, i.e., $A = [e) \cup (f]$. To this description, we can add a new result:

Theorem 1. *Every finitely subdirectly irreducible De Morgan monoid $\boldsymbol{A}$ consists of an interval subalgebra $[\neg a, a]$ and two chains of idempotent elements, $(\neg a]$ and $[a)$, where a is e or f^2.*

In the former case, $[\neg a, a]$ has at most two elements, and $\boldsymbol{A}$ is a Sugihara monoid. The case $a = f^2 \neq e$ is more challenging, as it involves non-idempotent elements and an order that need not be linear. In both cases, e and f belong to the interval $[\neg a, a]$.

To describe the atoms of Λ_{DMM}, we need to refer to the De Morgan monoids depicted below. (If b is the least element of a De Morgan monoid, then $a \cdot b = b$ for all elements a.) Note that $\mathbf{2}$ is a Boolean algebra, and $\boldsymbol{S}_3$ is a Sugihara monoid. In what follows, $\mathbb{V}(\boldsymbol{A})$ [resp. $\mathbb{Q}(\boldsymbol{A})$] denotes the smallest variety [resp. quasivariety] containing an algebra $\boldsymbol{A}$.

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$$\mathbf{2}: \quad \mathbf{S}_3: \quad \mathbf{C}_4: \quad \mathbf{D}_4:$$
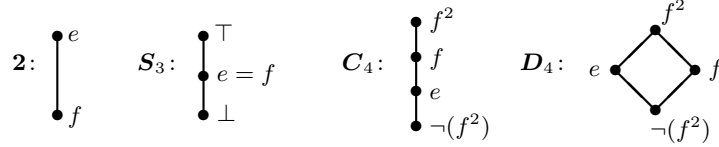
Lemma 2. *Up to isomorphism,* $\mathbf{2}$, $\mathbf{C}_4$ *and* $\mathbf{D}_4$ *are the only simple 0–generated De Morgan monoids.*

Theorem 3. *The distinct classes* $\mathbb{V}(\mathbf{2})$, $\mathbb{V}(\mathbf{S}_3)$, $\mathbb{V}(\mathbf{C}_4)$ *and* $\mathbb{V}(\mathbf{D}_4)$ *are precisely the minimal varieties of De Morgan monoids.*

Lemma 2 is implicit in Slaney's identification of the 0–generated subdirectly irreducible De Morgan monoids, but it is easier to prove it directly. Theorem 3 (which uses Lemma 2) does not seem to have been stated explicitly in the relevance logic literature.

It can also be shown that a subquasivariety of DMM is minimal (i.e., it contains no nontrivial proper subquasivariety) iff it is $\mathbb{V}(\mathbf{S}_3)$ or $\mathbb{Q}(\mathbf{A})$ for some nontrivial 0–generated De Morgan monoid $\mathbf{A}$. Combining this observation with Slaney's description of the free 0–generated De Morgan monoid in [5], we obtain:

Theorem 4. *The variety of De Morgan monoids has just 68 minimal subquasivarieties.*

For philosophical reasons, the relevance logic literature also emphasizes a system called $\mathbf{R}$, which differs from $\mathbf{R^t}$ in that it lacks the so-called Ackermann truth constant $\mathbf{t}$ (corresponding to the neutral element e of a De Morgan monoid). The logic $\mathbf{R}$ is algebraized by the variety RA of *relevant algebras*. Świrydowicz [7] showed that the subvariety lattice of RA has a unique atom, with just three covers. We remark that this result can be derived more easily from Theorem 3 and the following finding of Slaney [6]: if $h\colon \mathbf{A} \to \mathbf{B}$ is a homomorphism from a finitely subdirectly irreducible De Morgan monoid into a nontrivial 0–generated De Morgan monoid, then h is an isomorphism or $\mathbf{B} \cong \mathbf{C}_4$.

Świrydowicz's theorem has been applied recently to show that no consistent axiomatic extension of $\mathbf{R}$ is structurally complete, except for classical propositional logic [4]. The situation for $\mathbf{R^t}$ is very different and is the subject of ongoing algebraic investigation by the present authors.

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Varieties of De Morgan Monoids II: Covers of Atoms

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This is the second half of a two-part talk on the lattice Λ_{DMM} of subvarieties of the variety DMM of all De Morgan monoids. The investigation is motivated by an anti-isomorphism between Λ_{DMM} and the lattice of axiomatic extensions of the relevance logic $\mathbf{R^t}$ of [1]. Recall that a De Morgan monoid $\boldsymbol{A} = \langle A; \cdot, \wedge, \vee, \neg, e \rangle$ is the expansion of a commutative monoid $\langle A; \cdot, e \rangle$ by a residuated distributive lattice order and a compatible antitone involution $\neg$, where $a \leqslant a^2$ for all elements a, and that $f := \neg e$.

The first talk established that the atoms of Λ_{DMM} (i.e., the minimal varieties of De Morgan monoids) are just the four varieties generated, respectively, by the De Morgan monoids depicted below. They include the two-element Boolean algebra $\mathbf{2}$, and the three-element Sugihara monoid $\boldsymbol{S}_3$. In the present talk, we aim to say as much as possible about the covers of these four atoms in Λ_{DMM}, since these define the 'pre-maximal' consistent axiomatic extensions of $\mathbf{R^t}$.
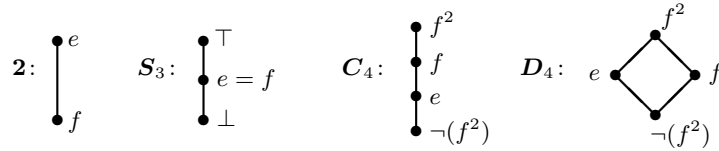
In Λ_{DMM}, a cover K of one of the atoms ($\mathbb{V}(\boldsymbol{A})$, say) will be called *interesting* if K is not the varietal join of $\mathbb{V}(\boldsymbol{A})$ and one of the other three minimal varieties. We can show:

Theorem 1. (i) $\mathbb{V}(\mathbf{2})$ *has no interesting cover within* DMM.

  (ii) *The only interesting cover of* $\mathbb{V}(\boldsymbol{S}_3)$ *within* DMM *is the variety* $\mathbb{V}(\boldsymbol{S}_5)$ *generated by the five-element (totally ordered) Sugihara monoid.*

  (iii) *Every interesting cover of* $\mathbb{V}(\boldsymbol{D}_4)$ *within* DMM *has the form* $\mathbb{V}(\boldsymbol{A})$ *for some simple* 1– *generated De Morgan monoid* $\boldsymbol{A}$, *where* $\boldsymbol{D}_4$ *embeds into* $\boldsymbol{A}$ *but is not isomorphic to* $\boldsymbol{A}$.

The situation with $\mathbb{V}(\boldsymbol{C}_4)$ is more complex, as can be guessed from the following result of Slaney [2]: if $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism from a finitely subdirectly irreducible De Morgan monoid into a nontrivial 0–generated De Morgan monoid, then h is an isomorphism or $\boldsymbol{B} \cong \boldsymbol{C}_4$. This motivates study of the class W of all De Morgan monoids that map homomorphically onto $\boldsymbol{C}_4$ or are trivial, as well as its subclass N, consisting of De Morgan monoids that have $\boldsymbol{C}_4$ as a retract or are trivial. It can be shown that W and N are quasivarieties, but neither is a variety.

Theorem 2. W *has a largest subvariety, denoted here by* U. *Also,* N *has a largest subvariety, denoted here by* M. *The varieties* U *and* M *are finitely axiomatized, and* M *consists of the De Morgan monoids in* U *that satisfy* $e \leqslant f$.

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Theorem 3. *If* K *is an interesting cover of* $\mathbb{V}(\boldsymbol{C}_4)$ *within* DMM, *then exactly one of the following holds.*

(i) K $\subseteq$ M.

(ii) K $= \mathbb{V}(\boldsymbol{A})$ *for some finite* 0*–generated subdirectly irreducible De Morgan monoid* $\boldsymbol{A} \in$ U $\setminus$ M.

(iii) K $= \mathbb{V}(\boldsymbol{A})$ *for some simple* 1*–generated De Morgan monoid* $\boldsymbol{A}$, *such that* $\boldsymbol{C}_4$ *embeds into* $\boldsymbol{A}$ *but is not isomorphic to* $\boldsymbol{A}$.

As $\mathbb{V}(\boldsymbol{C}_4)$ is the only minimal subvariety of U, all covers of $\mathbb{V}(\boldsymbol{C}_4)$ within U are interesting (i.e., they are not joins of atoms in Λ_{DMM}). Only four De Morgan monoids $\boldsymbol{A}$ satisfy the demand in Theorem 3(ii); they are depicted in Slaney [2], where they are labeled $\boldsymbol{C}_5, \boldsymbol{C}_6, \boldsymbol{C}_7, \boldsymbol{C}_8$. Infinitely many covers of $\mathbb{V}(\boldsymbol{C}_4)$ exemplify Theorem 3(iii). Not all of them are finitely generated varieties, and it appears to be difficult to classify them structurally.

Here, however, we are able to describe completely the covers of $\mathbb{V}(\boldsymbol{C}_4)$ within M, i.e., the witnesses of Theorem 3(i). In particular:

Theorem 4. *There are exactly six covers of* $\mathbb{V}(\boldsymbol{C}_4)$ *within* M. *Consequently, there are just ten covers of* $\mathbb{V}(\boldsymbol{C}_4)$ *within* U. *All ten of these covers are finitely generated varieties.*

A *Dunn monoid* is a distributive commutative residuated lattice, satisfying $x \leqslant x^2$, so De Morgan monoids are just Dunn monoids with a compatible involution. Slaney [3] discusses ways of constructing De Morgan monoids $\mathrm{S}^{\leqslant}(\boldsymbol{B})$ from Dunn monoids $\boldsymbol{B}$, where $\boldsymbol{B}$ is a subalgebra of the Dunn monoid reduct of $\mathrm{S}^{\leqslant}(\boldsymbol{B})$. We refer to these methods as *skew reflection* constructions. Each construction first creates a copy b' of every element b of $\boldsymbol{B}$ and orders the new elements so that $b' \leqslant c'$ iff $c \leqslant b$. A new upper bound 1 and lower bound 0 for all of these elements is introduced, and $b' \cdot c'$ is defined to be 1 for all $b, c \in B$. No element of the form b' is a lower bound of an element of $\boldsymbol{B}$, but certain elements of $\boldsymbol{B}$ may be lower bounds of new elements b' (thus expanding the order relation $\leqslant$ on the superstructure $\mathrm{S}^{\leqslant}(\boldsymbol{B})$ of $\boldsymbol{B}$), subject to certain axioms. The axioms ensure that $\mathrm{S}^{\leqslant}(\boldsymbol{B})$ really is a De Morgan monoid.

We can prove that a De Morgan monoid belongs to U iff it is a subdirect product of skew reflections of Dunn monoids, where the bottom element 0 is meet-irreducible in every subdirect factor. This limits the choices of algebras $\boldsymbol{A}$ such that $\mathbb{V}(\boldsymbol{A})$ generates a cover of $\mathbb{V}(\boldsymbol{C}_4)$ within M. It forces $\boldsymbol{A}$ to be finite, and leads eventually to the proof of Theorem 4.

There are additional motivations for study of M, which come from considerations of structural completeness. The minimal varieties of De Morgan monoids are structurally complete, as are the well-understood varieties of odd Sugihara monoids. We have shown that all remaining structurally complete subvarieties of DMM lie within M, though not all subvarieties of M are structurally complete. The following result is therefore of interest:

Theorem 5. *Every cover of* $\mathbb{V}(\boldsymbol{C}_4)$ *within* M *has no proper subquasivariety other than* $\mathbb{V}(\boldsymbol{C}_4)$, *and is thus (hereditarily) structurally complete.*

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$\aleph_1$, ω_1, and the modal μ-calculus

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The modal μ-calculus $\mathbf{L}_\mu$, see [4], enriches the syntax of (poly)modal logic $\mathbf{K}$ with least and greatest fixed-point constructors μ and ν. In a Kripke model $\mathcal{M}$, the formula $\mu_x.\phi$ (resp., $\nu_x.\phi$) denotes the least (resp., the greatest) fixed-point of the function $\phi_\mathcal{M}$ (of the variable x) obtained by evaluating ϕ in $\mathcal{M}$ under the additional condition that x is interpreted as a given subset of worlds. It is required that every occurrence of x is positive in ϕ, so $\phi_\mathcal{M}$ is monotone and the least fixed-point exists by the Tarski-Knaster theorem.

A formula $\phi(x)$ is said to be continuous if, for every model $\mathcal{M}$, the function $\phi_\mathcal{M}$ is continuous, in the usual sense. The continuous fragment $\mathcal{C}_0(X)$ of the modal μ-calculus is the set of formulas generated by the following syntax:

$$\phi := x \mid \psi \mid \top \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \langle a \rangle \phi \mid \mu_z.\chi \,,$$

where $x \in X$, $\psi \in \mathbf{L}_\mu$ is a μ-calculus formula not containing any variable $x \in X$, and $\chi \in \mathcal{C}_0(X \cup \{ z \})$. Fontaine [3] proved that a formula $\phi \in \mathbf{L}_\mu$ is continuous in x if and only if it is equivalent to a formula in $\mathcal{C}_0(x)$; she also proved that it is decidable whether a formula of the modal μ-calculus is continuous. We add to the above grammar one more production and study the fragment $\mathcal{C}_1(X)$ of $\mathbf{L}_\mu$ defined as follows:

$$\phi := x \mid \psi \mid \top \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \langle a \rangle \phi \mid \mu_z.\chi \mid \nu_z.\chi \,,$$

with the same constraints as above but w.r.t $\mathcal{C}_1(X \cup \{ z \})$.

Definition 1. Let κ be a regular cardinal. A set $\mathcal{I} \subseteq P(X)$ is κ-*directed* if every subset of $\mathcal{I}$ of cardinality smaller than κ has an upper bound in $\mathcal{I}$. A function $f : P(X) \to P(X)$ is κ-*continuous* if it preserves unions of κ-directed sets.

Notice that, if $\kappa = \aleph_0$, then κ-continuity is the standard notion of continuity. The following proposition is an immediate consequence of the fact that $\aleph_1$-continuous functions are closed under parametrized least and greatest fixed-points, see [5, 6].

Proposition 2. *Every formula in $\phi(x) \in \mathcal{C}_1(x)$ is $\aleph_1$-continuous.*

The folllowing theorem is a sort of converse to the previous statement.

Theorem 3. *For each formula $\phi(x) \in \mathbf{L}_\mu$ we can construct a formula $\psi(x) \in \mathcal{C}_1(x)$ such that $\phi(x)$ is κ-continuous for some regular cardinal κ if and only if $\phi(x)$ is equivalent to $\psi(x)$.*

The consequences of this theorem are twofold.

Corollary 4. *It is decidable whether a formula $\phi(x)$ is κ-continuous for some regular cardinal κ.*

Corollary 5. *If a formula is κ-continuous for some regular cardinal κ, then it is κ-continuous for some $\kappa \in \{\aleph_0, \aleph_1\}$.*

That is, there are no other relevant fragments of the modal μ-calculus, apart from $\mathcal{C}_0$ and $\mathcal{C}_1$, that are determined from some continuity condition.

Let us recall that, for a monotone function $f : P(X) \to P(X)$, we can define the *approximants to the least fixed-point of f* as follows: $f^{\alpha+1}(\emptyset) = f(f^\alpha(\emptyset))$ and $f^\beta(\emptyset) = \bigcup_{\alpha < \beta} f^\alpha(\emptyset)$ (so $f^0(\emptyset) = \emptyset$). If $f^{\alpha+1}(\emptyset) = f^\alpha(\emptyset)$, then $f^\alpha(\emptyset)$ is the least fixed-point of f.

Definition 6. We say that and ordinal α is the *closure ordinal* of $\phi(x) \in \mathbf{L}_\mu$ if, for every model $\mathcal{M}$, $\phi_\mathcal{M}^\alpha(\emptyset)$ is the least fixed-point of $\phi_\mathcal{M}$, and moreover there exists a model $\mathcal{M}$ for which $\phi^\beta(\emptyset)$ is not the least fixed-point of $\phi_\mathcal{M}$, for every $\beta < \alpha$.

Of course, not every formula $\phi(x) \in \mathbf{L}_\mu$ has a closure ordinal. For example $[\;]x$ has no closure ordinal, while ω_0 is the closure ordinal of $[\;]\bot \vee \langle\;\rangle x$. Czarnecki [2] proved that every ordinal $\alpha < \omega_0^2$ is the closure ordinal of a formula $\phi \in \mathbf{L}_\mu$. Afshari and Leigh [1] proved that if a formula $\phi(x) \in \mathbf{L}_\mu$ does not contain greatest fixed-points and has a closure ordinal α, then $\alpha < \omega_0^2$. Considering that every ordinal below ω_0^2 can be written as a polynomial in the inderterminates $1, \omega_0$, our next theorem can be used to recover Czarnecki's result:

Theorem 7. *Closure ordinals of formulas of the modal μ-calculus are closed under ordinal sum.*

Since a formula $\phi(x)$ in the syntactic fragment $\mathcal{C}_1(x)$ is $\aleph_1$-continuous, the maps $\phi_\mathcal{M}$ converge to their least fixed-point in at most ω_1 steps, where ω_1 is the least uncountable ordinal (considering cardinals as specific ordinals, we have $\omega_1 = \aleph_1$). In particular, every formula in this fragment has a closure ordinal with ω_1 as an upper bound. We prove that ω_1 is indeed a closure ordinal:

Theorem 8. ω_1 *is the closure ordinal of the formula* $\phi(x) := \nu_z.(\langle v \rangle x \wedge \langle h \rangle z) \vee [v]\bot$.

Extending Thomason's coding to the full modal μ-calculus, it is also possible to construct a monomodal formula in $\mathbf{L}_\mu$ whose only free variable is x, with ω_1 as closure ordinal. Consequently, we extend Czarnecki's result by showing that polynomials in the inderterminates $1, \omega_0, \omega_1$ denote closure ordinals.

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ALBA-style Sahlqvist preservation for modal compact Hausdorff spaces

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Canonicity, i.e. the preservation of validity of formulas from descriptive general frames to their underlying Kripke frames, is an important notion in modal logic, since it provides a uniform strategy for proving the strong completeness of axiomatic extensions of a basic (normal modal) logic. Thanks to its importance, the notion of canonicity has been explored also for other non-classical logics. In [19], Jónsson gave a purely algebraic reformulation of the frame-theoretic notion of canonicity, which he defined as the preservation of validity under taking canonical extensions, and proved the canonicity of Sahlqvist identities in a purely algebraic way. The construction of canonical extension was introduced by Jónsson and Tarski [20] as a purely algebraic encoding of the Stone spaces dual to Boolean algebras. In particular, the denseness requirement directly relates to the zero-dimensionality of Stone spaces. A natural question is then for which classes of formulas do canonicity-type preservation results hold in topological settings in which compactness is maintained and zero-dimensionality is generalized to the Hausdorff separation condition. This question has been addressed in [1, 2]. Specifically, in [1], Bezhanishvili, Bezhanishvili and Harding gave a canonicity-type preservation result for Sahlqvist formulas from modal compact Hausdorff spaces to their underlying Kripke frames, and in [2], Bezhanishvili and Sourabh generalized this result to modal fixed point formulas.

The proposed talk reports on an ongoing work in which the canonicity-type preservation results in [1, 2] are reformulated in a purely algebraic way in the spirit of Bjarni Jónsson. This work pertains to the wider theory of *unified correspondence*, which aims at identifying the underlying principles of Sahlqvist-type canonicity and correspondence for non-classical logics. As explained in [11, 7], this theory is grounded on the Stone-type dualities between the algebraic and the relational semantics of non-classical logics, and explains the "Sahlqvist phenomenon" in terms of the order-theoretic properties of the algebraic interpretations of the connectives of a non-classical logic. The focus on these properties has been crucial to the possibility of generalizing the Sahlqvist-type results from modal logic to a wide array of non-classical logics, including intuitionistic and distributive and general (non-distributive) lattice-based (modal) logics [8, 10, 6], non-normal (regular) modal logics [25], monotone modal logic [15], hybrid logics [14], many valued logics [22] and bi-intuitionistic and lattice-based modal mu-calculus [3, 4, 5]. In addition, unified correspondence has effectively provided overarching techniques unifying different methods for proving both canonicity and correspondence: in [24], the methodology pioneered by Jónsson [19] and the one pioneered by Sambin-Vaccaro [26] were unified; in [9, 12], constructive canonicity proposed by Ghilardi and Meloni [17] was unified with the Sambin-Vaccaro methodology; in [13], the Sambin-Vaccaro correspondence has been unified with the methodology of correspondence via translation introduced by Gehrke, Nagahashi and Venema in [16]. Recently, a very surprising connection has been established between the notions and techniques developed in unified correspondence and structural proof theory, which made it possible to solve a problem, opened by Kracht [21], concerning the characterization of the axioms which can be transformed into analytic structural rules [18, 23].

The main tools of unified correspondence are a purely order-theoretic definition of *inductive formulas/inequalities*, and the algorithm ALBA, which computes the first-order correspondent

of input formulas/inequalities and is guaranteed to succeed on the inductive class.

In this talk, we illustrate how the preservation result of [1, 2] can be encompassed into unified correspondence theory. Intermediate steps toward this goal are: the identification of the order-theoretic properties which guarantee the Esakia lemma to hold, the proof of a suitably adapted version of the topological Ackermann lemma, and the introduction of the version of the algorithm ALBA appropriate for the "compact Hausdorff" setting.

Together, these results show that the same proof techniques introduced by Jónsson to prove the canonicity of Sahlqvist identities directly fuel the canonicity-type preservation of Sahlqvist formulas in the setting of modal compact Hausdorff spaces. I will also discuss further generalizations, and more in general a systematic approach to canonicity-type preservation results to which these preliminary results pave the way.

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On interpolation in $NEXT(\mathbf{KTB})$ and $NEXT(\mathbf{KB})$

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We study two modal logics: the Brouwer logic $\mathbf{KTB} := \mathbf{K} \oplus T \oplus B$ and its interesting sub-logic – the logic $\mathbf{KB} := \mathbf{K} \oplus B$, where:

$$T := \Box p \to p$$
$$B := p \to \Box \Diamond p$$

The logic $\mathbf{KTB}$ (logic $\mathbf{KB}$) is complete with respect to the class of reflexive and symmetric Kripke frames (symmetric Kripke frames).

We shall study n-branching Brouwerian modal logics $\mathbf{KTB}.\mathbf{Alt}(\mathbf{n}) := \mathbf{KTB} \oplus alt_n$ as well as $\mathbf{KB}.\mathbf{Alt}(\mathbf{n}) := \mathbf{KB} \oplus alt_n$ where

$$alt_n := \Box p_1 \vee \Box(p_1 \to p_2) \vee ... \vee \Box((p_1 \wedge ... \wedge p_n) \to p_{n+1}).$$

For $n = 3$ the above axiom involves linearity of the appropriate reflexive frames – they are chains of reflexive points. Chains of (possibly) irreflexive points characterize logic $\mathbf{KB}.\mathbf{Alt}(\mathbf{2})$.

Definition 1. *A logic L has the Craig interpolation property (CIP) if for every implication $\alpha \to \beta$ in L, there exists a formula γ (interpolant for $\alpha \to \beta$ in L) such that*

$$\alpha \to \gamma \in L \text{ and } \gamma \to \beta \in L \quad and \quad Var(\gamma) \subseteq Var(\alpha) \cap Var(\beta).$$

The symbol $Var(\alpha)$ means the set of all propositional variables of the formula α.
The weaker notion of interpolation for deducibility is defined as follows:

Definition 2. *A logic L has interpolation for deducibility (IPD) if for any α and β the condition $\alpha \vdash_L \beta$ implies that there exists a formula γ such that*

$$\alpha \vdash_L \gamma \text{ and } \gamma \vdash_L \beta, \text{ and } Var(\gamma) \subseteq Var(\alpha) \cap Var(\beta).$$

It is a logical folklore that (CIP) together with (MP) and deduction theorem implies (IPD).

It is known that $\mathbf{K}$, $\mathbf{T}$, $\mathbf{K4}$ and $\mathbf{S4}$ have (CIP), see Gabbay [3]. Also the logics from $NEXT(\mathbf{S4})$ are well characterized as regards interpolation (see [5], also [1], p.462-463). It is also known that $\mathbf{S5}$ has (CIP). The last fact can be proven by applying a very general method of construction of inseparable tableaux (see i.e. [1], p. 446-449). The same method can be applied in the case of $\mathbf{KTB}$ and $\mathbf{KB}$. Then we get that the logics $\mathbf{KTB}$ and $\mathbf{KB}$ have (CIP).

The following facts were proven in [4]:

Theorem 1. *The logic $\mathbf{KTB}.\mathbf{Alt}(\mathbf{3})$ does not have (CIP).*

Theorem 2. *There are only two tabular logics from $NEXT(\mathbf{KTB}.\mathbf{Alt}(\mathbf{3}))$ having (IPD). They are the trivial logic $L(\circ)$ and the logic determined by two element cluster $L(\circ - -\circ)$.*

In [4] the following conjectures are placed:

Conjecture 1. *The logic determined by a reflexive and symmetric Kripke frame having the structure of a Boolean cube has (IDP).*

Conjecture 2. *The logic determined by a reflexive and symmetric Kripke frame having the structure of 2^n-element Boolean cube, $n \geq 3$, has (IDP).*

In our talk we disprove these conjectures and prove others negative results on interpolation in $NEXT(\mathbf{KTB}.\mathbf{Alt}(\mathbf{n}))$ for $n \geq 3$. We also provide a similar research for the logics from $NEXT(\mathbf{KB}.\mathbf{Alt}(\mathbf{n}))$. First result, a similar to Theorem 1 is the following:

Theorem 3. *The logic $\mathbf{KB}.\mathbf{Alt}(\mathbf{2})$ does not have (CIP).*

Second, in contrast to logics from $NEXT(\mathbf{KTB}.\mathbf{Alt}(\mathbf{3}))$ we prove:

Theorem 4. *There are infinitely many tabular logics from $NEXT(\mathbf{KB}.\mathbf{Alt}(\mathbf{2}))$ having (IPD).*

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A new characterization of a class $\mathrm{HSP}_{\mathrm{U}}(\mathcal{K})$

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We recall that a class of algebras $\mathcal{K}$ has *f*inite embeddability property, briefly (FEP), if any finite partial sublagebra of any member is embeddable into some finite one from $\mathcal{K}$. This property was generalized in [1] by the following way.

Definition 1. *Let* $\mathbf{A} = (A, \mathrm{F})$ *be an algebra and* $X \subseteq A$. *A* partial subalgebra *is a pair* $\mathbf{A}|_X = (X, \mathrm{F})$, *where for any* $f \in F_n$ *and all* $x_1, \ldots, x_n \in X$, $f^{\mathbf{A}|_X}(x_1, \ldots, x_n)$ *is defined if and only if* $f^{\mathbf{A}}(x_1, \ldots, x_n) \in X$. *We put then*

$$f^{\mathbf{A}|_X}(x_1, \ldots, x_n) := f^{\mathbf{A}}(x_1, \ldots, x_n).$$

Definition 2. *An algebra* $\mathbf{A} = (A; \mathrm{F})$ *satisfies* the general finite embeddability *(*finite embeddability property*) property for the class* $\mathcal{K}$ *of algebras of the same type* F *if for any finite subset* $X \subseteq A$, *there exist a (finite) algebra* $\mathbf{B} \in \mathcal{K}$ *and an embedding* $\rho : \mathbf{A}|_X \hookrightarrow \mathbf{B}$, *i.e., an injective mapping* $\rho : X \to B$ *satisfying the property* $\rho(f^{\mathbf{A}|_X}(x_1, \ldots, x_n)) = f^{\mathbf{B}}(\rho(x_1), \ldots, \rho(x_n))$ *if* $x_1, \ldots, x_n \in X$, $f \in \mathrm{F}_n$ *and* $f^{\mathbf{A}|_X}(x_1, \ldots, x_n)$ *is defined.*

The most important idea of this generalization is described in the following theorems. Both theorems are direct consequences of well known model theory facts (see [4]) or alternatively the direct proofs are published in [1].

Theorem 1. *Let* $\mathbf{A} = (A; \mathrm{F})$ *be an algebra and let* $\mathcal{K}$ *be a class of algebras of the type* F. *If* $\mathbf{A}$ *satisfies the general finite embeddability property for* $\mathcal{K}$ *then* $\mathbf{A} \in \mathrm{ISP}_{\mathrm{U}}(\mathcal{K})$.

Theorem 2. *Let* $\mathbf{A} = (A; \mathrm{F})$ *be an algebra such that* F *is finite and let* $\mathcal{K}$ *be a class of algebras of the type* F. *If* $\mathbf{A} \in \mathrm{ISP}_{\mathrm{U}}(\mathcal{K})$ *then* $\mathbf{A}$ *satisfies the general finite embeddability property for* $\mathcal{K}$.

The great applicability of these concepts in logic theory is obvious. Examples of results obtained using by some type of partial embeddability are contained for example in [1, 2, 5, 6, 7] etc.

In our talk we present analogous property (a finite covering property of an algebra $\mathbf{A}$ by a class $\mathcal{K}$) and then we show that this property is equivalent to $\mathbf{A} \in \mathrm{HSP}_{\mathrm{U}}(\mathcal{K})$.

In next we denote the set of all terms of type F over the set A by $\mathbf{T}_{\mathrm{F}}(A)$. Let $t \in \mathbf{T}_{\mathrm{F}}(A)$ be a term then we denote by $|t|$ the set of all variables (members of A) used in the term t. For any set $T \subseteq \mathbf{T}_{\mathrm{F}}(A)$ we denote

$$|T| = \bigcup_{t \in t} |t|.$$

Now we are ready to define the finite covering property.

Definition 3. *Let* $\mathbf{A} = (A, \mathrm{F})$ *be an algebra and* $\mathcal{K}$ *be a class of algebras of the type* F. *We say that the class* $\mathcal{K}$ finitely partially covers the algebra $\mathbf{A}$ *(or* $\mathbf{A}$ *satisfies* the finite covering property for the class $\mathcal{K}$) *if for every finite set of terms* $T \subseteq \mathbf{T}_{\mathrm{F}}(A)$ *there exist an algebra* $\mathbf{B} \in \mathcal{K}$, *a mapping* $f : B \to A$ *and a set* $Y \subseteq B$ *such that*

i) $f|_Y : Y \to |T|$ *is a bijection,*

ii) *if* $t(a_1, \ldots, a_n) \in T$ *and* $y_1, \ldots, y_n \in Y$ *are such that* $fy_i = a_i$ *then*

$$ft^{\mathcal{B}}(y_1, \ldots, y_n) = t^{\mathcal{A}}(a_1, \ldots, a_n).$$

The following theorem is the most important result of our talk.

Theorem 3. *Let* **A** *satisfy the finite covering property for the class* $\mathcal{K}$ *if and only if* **A** $\in$ HSP$_U(\mathcal{K})$.

This theorem has several natural consequences, for example, using well known Bjarni Jónsson Lemma (see [3]) we obtain:

Theorem 4. *Let* $\mathcal{K}$ *be a generating class of congruence distributive variety* $\mathcal{V}$ *then any subdirectly irreducible member of* $\mathcal{V}$ *is finitely coverable by* $\mathcal{K}$.

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Algebras from a Quasitopos of Rough Sets

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Rough set theory was defined by Pawlak [3] to deal with incomplete information. Since then it has been studied from many directions including algebra and category theory. A summary of previous work on categories of rough sets can be found in [2]. Our work is an amalgamation of the algebraic and category-theoretic approaches. In this work, we introduce the class of *contrapositionally complemented pseudo-Boolean algebras* and the corresponding logic, emerging from the study of algebras of strong subobjects in a generalized category of rough sets.

Elementary topoi were defined to capture properties of the category of sets. With a similar goal in mind, in [2] we proposed the following natural generalization $RSC(\mathscr{C})$ of the category RSC of rough sets. RSC has the pairs (X_1, X_2) as objects, where X_1, X_2 are sets and $X_1 \subseteq X_2$, and the set functions $f : X_2 \to Y_2$ as arrows with domain (X_1, X_2) and codomain (Y_1, Y_2) such that $f(X_1) \subseteq Y_1$. By replacing sets with objects of an arbitrary topos $\mathscr{C}$, we obtain

Definition 1. *[2] The category $RSC(\mathscr{C})$ has the pairs (A, B) as objects, where A and B are $\mathscr{C}$-objects such that there exists a monic arrow $m : A \to B$ in $\mathscr{C}$. m is said to be a* monic *corresponding to the object (A, B). The pairs (f', f) are the arrows with domain (X_1, X_2) and codomain (Y_1, Y_2), where $f' : X_1 \to Y_1$ and $f : X_2 \to Y_2$ are $\mathscr{C}$-arrows such that $m'f' = fm$, and m and m' are monics corresponding to the objects (X_1, X_2) and (Y_1, Y_2) in $RSC(\mathscr{C})$ respectively.*

The category $RSC(\mathscr{C})$ forms a quasitopos [2]. Any quasitopos, just like a topos, has an internal (intuitionistic) logic associated with the strong subobjects of its objects [6]. Let $\mathcal{M}((U_1, U_2))$ be the set of strong monics of an $RSC(\mathscr{C})$-object (U_1, U_2). $\mathcal{M}((U_1, U_2))$ thus forms a pseudo-Boolean algebra. Moreover, the operations on $\mathcal{M}((U_1, U_2))$ are characterized as follows.

Proposition 1. *The operations on $\mathcal{M}((U_1, U_2))$ obtained by taking the pullbacks of specific characteristic arrows along the $RSC(\mathscr{C})$-subobject classifier $(\top, \top) : (1, 1) \to (\Omega, \Omega)$ are:*

$$\cap : \ (f', f) \cap (g', g) = (f' \cap g', f \cap g), \quad \cup : \ (f', f) \cup (g', g) = (f' \cup g', f \cup g),$$
$$\neg : \ \neg(f', f) = (\neg f', \neg f), \qquad\qquad \to : \ (f', f) \to (g', g) = (f' \to g', f \to g),$$

where (f', f) and (g', g) are strong monics with codomain (U_1, U_2), and $\top : 1 \to \Omega$ is the subobject classifier of the topos $\mathscr{C}$. The operations on f', g' (f, g) used above are those of the algebra of subobjects of U_1 (U_2) in the topos $\mathscr{C}$.

In the context of the algebra of strong subobjects of an RSC-object (U_1, U_2), we had noted in [2] that, since the complementation $\neg$ is with respect to the object (U_1, U_2), we actually require the concept of *relative* rough complementation. Iwiński's *rough difference* operator [1] is what we use, and we define a new negation $\sim$ on $\mathcal{M}((U_1, U_2))$ as:

$$\sim (f', f) := (\neg f', \neg(m \circ f')),$$

where (f', f) is a strong monic with codomain (U_1, U_2) and $m : U_1 \to U_2$ is a monic arrow corresponding to (U_1, U_2). We observe that $\mathcal{A} := (\mathcal{M}((U_1, U_2)), (Id_{U_1}, Id_{U_2}), \cap, \cup, \to, \sim)$ forms a contrapositionally complemented (*c.c.*) lattice [5], with $1 := (Id_{U_1}, Id_{U_2})$. In fact, $\mathcal{A}$ satisfies the property $\sim a = a \to \neg\neg \sim 1$, which is not true in general for an arbitrary *c.c.* lattice. Moreover, $\sim$ is neither a semi-negation nor involutive. These observations indicate that $\mathcal{A}$ is an instance of a new algebraic structure, involving two negations $\sim$ and $\neg$, and defined as follows.

Definition 2. *An abstract algebra* $\mathcal{A} := (A, 1, 0, \rightarrow, \cup, \cap, \neg, \sim)$ *is said to be a* contrapositionally complemented pseudo-Boolean algebra *(c.c.-pseudo-Boolean algebra) if* $(A, 1, 0, \rightarrow, \cup, \cap, \neg)$ *forms a pseudo-Boolean algebra and* $\sim a = a \rightarrow (\neg\neg \sim 1)$*, for all* $a \in A$.

An entire class of *c.c.*-pseudo-Boolean algebras can be obtained as follows, starting from any pseudo-Boolean algebra $\mathcal{H} := (H, 1, 0, \rightarrow, \cup, \cap, \neg)$.

Theorem 2. *Let* $\mathcal{H}^{[2]} := \{(a, b) : a \leq b, a, b \in H\}$*, and* $u := (u_1, u_2) \in \mathcal{H}^{[2]}$*. Consider the set* $A_u := \{(a_1, a_2) \in \mathcal{H}^{[2]} : a_2 \leq u_2 \text{ and } a_1 = a_2 \wedge u_1\}$*. Define the following operators on* A_u*:*

$$\sqcup: \ (a_1, a_2) \sqcup (b_1, b_2) := (a_1 \vee b_1, a_2 \vee b_2), \qquad \sqcap: \ (a_1, a_2) \sqcap (b_1, b_2) := (a_1 \wedge b_1, a_2 \wedge b_2),$$
$$\neg: \ \neg(a_1, a_2) := (u_1 \wedge \neg a_1, u_2 \wedge \neg a_2), \qquad \sim: \ \sim(a_1, a_2) := (u_1 \wedge \neg a_1, u_2 \wedge \neg a_1),$$
$$\rightarrow: \ (a_1, a_2) \rightarrow (b_1, b_2) := ((a_1 \rightarrow b_1) \wedge u_1, (a_2 \rightarrow b_2) \wedge u_2).$$

Then $\mathcal{A}_u := (A_u, u, (0, 0), \rightarrow, \sqcup, \sqcap, \neg, \sim)$ *is a c.c.-pseudo-Boolean algebra.*

We define in the standard way, a *c.c.*-pseudo-Boolean set lattice. Using the representation theorem for pseudo-Boolean algebras [5], one obtains the following.

Theorem 3 (**Representation Theorem**). *Let* $\mathcal{A} := (A, 1, 0, \rightarrow, \cup, \cap, \neg, \sim)$ *be a c.c.-pseudo-Boolean algebra. There exists a monomorphism h from* $\mathcal{A}$ *into a c.c.-pseudo-Boolean set lattice.*

Note that, as the class of all pseudo-Boolean algebras is equationally definable, the class of all *c.c.*-pseudo-Boolean algebras is also so. Thus we define the logic corresponding to *c.c.*-pseudo-Boolean algebras, and call it *Intuitionistic logic with minimal negation* (ILM).

Various definitions of mappings from one formal system to another can be found in literature. A detailed study of connections between Classical logic (CL), Intuitionistic logic (IL) and Minimal logic (ML) can be found in [4], which has first formally defined the notion of 'interpretability' of formulas of one logic into another. In our work, we generalize the notion as follows. The mapping $r : L_1 \rightarrow L_2$ from formulas in logic L_1 to formulas in logic L_2 is called an *interpretation*, if for any formula $\alpha \in L_1$, we have $\vdash_{L_1} \alpha$ if and only if $\Delta_\alpha \vdash_{L_2} r(\alpha)$, where Δ_α is a finite set of formulas in L_2 corresponding to α. r is an embedding, if it is the inclusion map and $\Delta_\alpha = \emptyset$ for any α in L_1. IL can clearly be embedded into ILM. Furthermore, we have

Theorem 4. *There exists an interpretation from* ILM *into* IL.

The proof is similar to the one used to show connections between constructive logic with strong negation [5, Chapter XII] and IL.

We may also compare ILM and ML. Since ML corresponds to the class of *c.c* lattices [5] and any *c.c.*-pseudo-Boolean algebra is a *c.c.* lattice, ML can be embedded inside ILM. Using Theorem 4 and an interpretation of IL into ML [4, Theorem B], we have

Corollary 5. *There exists an interpretation from* ILM *into* ML.

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The Variety of Nuclear Implicative Semilattices is Locally Finite

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We study the variety of *nuclear algebras* — algebras of the form $(A, \wedge, 1, \rightarrow, \mathbf{j})$ where $(A, \wedge, 1, \rightarrow)$ is a meet-implicative semilattice and $\mathbf{j} : A \rightarrow A$ is a *nucleus*.

The latter means that $\mathbf{j}$ is an idempotent inflationary multiplicative unary operator, that is, the identities

$$x \leqslant \mathbf{j}x$$
$$\mathbf{jj}x = \mathbf{j}x$$
$$\mathbf{j}(x \wedge y) = \mathbf{j}x \wedge \mathbf{j}y$$

hold in A.

It has been proved by Diego in [1] that the variety of meet-implicative semilattices is locally finite. Our main result is that the same remains true after extending the signature with a nucleus as above.

Archetypal example of an implicative semilattice: let $(X, \leqslant)$ be a poset (partially ordered set), and let $A = \mathbf{D}(X, \leqslant)$ be the set of *downsets* of $(X, \leqslant)$ (subsets $D \subseteq X$ satisfying $x \in D$, $y \leqslant x \Rightarrow y \in D$ for all $x, y \in X$). Let us equip A with the semilattice structure via $D_1 \wedge D_2 := D_1 \cup D_2$; it has unit $1 := \varnothing$ and the implication given by $D_1 \rightarrow D_2 := \downarrow(D_2 - D_1)$, where for a subset $S \subseteq X$, we denote by $\downarrow(S)$ the smallest downset containing S, i. e. $\downarrow(S) = \{x \in X : \exists\, s \in S\ x \leqslant s\}$.

NB. The partial order $\leqslant$ on A resulting from this structure is the *opposite* of the subset inclusion, i. e. $D \leqslant D'$ iff $D \supseteq D'$.

In fact it follows from the work of Köhler [3] that every *finite* implicative semilattice is isomorphic to one of the above form. Moreover, Köhler obtained a dual description of homomorphisms between finite implicative semilattices in terms of certain partial maps between posets.

We extend this finite duality of Köhler to nuclear algebras. Every subset $S \subseteq X$ of a poset $(X, \leqslant)$ gives rise to a nucleus $\mathbf{j}_S : \mathbf{D}(X, \leqslant) \rightarrow \mathbf{D}(X, \leqslant)$ defined by $\mathbf{j}_S(D) = \downarrow(S \cap D)$. Moreover for finite X, every nucleus $\mathbf{j}$ on $\mathbf{D}(X, \leqslant)$ is equal to some such $\mathbf{j}_S$, for a unique subset $S \subseteq X$. We also obtain description of homomorphisms of nuclear algebras in terms of partial maps between the corresponding posets, as in [3].

This in turn makes it possible to give a dual description of nuclear subalgebras, and a dual characterization of situations when a nuclear algebra A is generated by its elements $a_1, ..., a_n \in A$. In particular, we have

Theorem 1. *Given downsets $D_1, ..., D_n$ of a finite poset $(X, \leqslant)$ and a subset $S \subseteq X$, if the nuclear algebra $(\mathbf{D}(X, \leqslant), \mathbf{j}_S)$ is generated by its elements $D_1, ..., D_n$ then for any $x \in X$, either $x \in \max(D_k)$ for some $k \in \{1, ..., n\}$ or $x \in \max(S \cap \downarrow y)$ for some $y \neq x$.*

This then enables us to apply the general construction of the universal model from [2] to our case.

Given a natural number n, we construct a poset $L(n)$, a subset $S(n) \subseteq L(n)$ and downsets $D(n, 1), ..., D(n, n) \in \mathbf{D}(L(n))$ with the following universal property: for any finite poset X and any subset $S \subseteq X$, if the nuclear algebra $(\mathbf{D}(X), \mathbf{j}_S)$ is generated by the downsets $D_1, ..., D_n \in \mathbf{D}(X)$, then there is a unique isomorphism $\varphi : X \to X'$ to a downset $X' \subseteq L(n)$ with $\varphi(S) = X' \cap S(n)$ and $\varphi(D_k) = X' \cap D(n, k)$, $k = 1, ..., n$.

The construction is inductive: we start with $L(n)_0$ empty; having constructed $S(n)_i \subseteq L(n)_i$ and $D(n, 1)_i, ..., D(n, n)_i \in \mathbf{D}(L(n)_i)$, we define $L(n)_{i+1} \supseteq L(n)_i$, $S(n)_{i+1} \supseteq S(n)_i$, $D(n, k)_{i+1} \supseteq D(n, k)_i$, $k = 1, ..., n$, as follows.

$L(n)_{i+1} \setminus L(n)_i$ consists of elements $r_{\alpha, \sigma} \notin S(n)_{i+1}$, one for each antichain α in $L(n)_i$, with $\alpha \not\subseteq L(n)_{i-1}$ if $i > 0$, and each $\sigma \subsetneqq \{k \in \{1, ..., n\} : \alpha \subseteq D(n, k)_i\}$, as well as elements $s_{\alpha, \sigma} \in S(n)_{i+1}$, one for each such pair α, σ that $\sigma \subseteq \{k \in \{1, ..., n\} : \alpha \subseteq D(n, k)_i\}$ and either $\sigma \neq \{k \in \{1, ..., n\} : \alpha \subseteq D(n, k)_i\}$ or $\alpha \not\subseteq S(n)_i$.

We then define $D(n, k)_{i+1} = D(n, k)_i \cup \{r_{\alpha, \sigma} : k \in \sigma\} \cup \{s_{\alpha, \sigma} : k \in \sigma\}$, $k = 1, ..., n$.

Extension of the partial order to $L(n)_{i+1}$ is uniquely determined by the requirements $\max(\downarrow(r_{\alpha, \sigma}) \setminus \{r_{\alpha, \sigma}\}) = \alpha$ and $\max(\downarrow(s_{\alpha, \sigma}) \setminus \{s_{\alpha, \sigma}\}) = \alpha$.

We then have

Theorem 2. *The above construction stops after finite number of steps, i. e. there is an i such that $L(n) = L(n)_i$ with $S(n) = S(n)_i$, $D(n, k) = D(n, k)_i$ has the above universal property.*

On the other hand we have

Theorem 3. *The variety of nuclear algebras has the finite model property.*

These facts enable us to conclude

Corollary. *For each n, the finite nuclear algebra $(\mathbf{D}(L(n)), \mathbf{j}_{S(n)})$ is the free n-generated nuclear algebra. In particular, the variety of nuclear algebras is locally finite.*

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A proof-theoretic approach to abstract interpretation

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Abstract interpretation is a theory of formal program verification which generates sound approximations of the semantics of programs, and has been used as the basis of methods and effective algorithms to approximate undecidable or computationally intractable problems such as the verification of safety-critical software (e.g. medical, nuclear, aviation software).

Typically, a complex concrete model (such as the powerset $\mathcal{P}(\Sigma)$ of a possibly infinite set modelling program executions) is related to a model that can be efficiently represented and manipulated (such as a finite lattice A, encoding the relevant – logically interconnected – properties about these executions) by means of an adjoint pair of maps. Specifically, the right adjoint (the *concretization* map $\gamma : A \to \mathcal{P}(\Sigma)$) provides the intended interpretation of the symbolic properties (that is, $S \models a$ iff $S \subseteq \gamma(a)$ for any $S \in \mathcal{P}(\Sigma)$ and $a \in A$); the left adjoint (the *abstraction* map $\alpha : \mathcal{P}(\Sigma) \to A$) classifies the executions of the given program according to their satisfying the relevant properties.

Although this theory was connected to logic since its inception [2, 1, 4], it is only in the last decade that the connection was made systematic. In particular, the notion of an (internal) logic of an abstraction was introduced in [5] and systematically related to the order-theoretic properties of the concretization map. In [3], this line of research is further developed. Namely, the logics underlying specific abstractions are identified, together with explicit specification of proof-theoretic presentations for each of them.

The present talk reports on the preliminary results of an ongoing work in which, using duality theory and algebraic logic, we generalise the results of [3] and introduce a general procedure for generating the (internal) logic of an abstraction together with the specification of a proof system for it. The main idea is to generate a logic whose Lindenbaum-Tarski algebra is isomorphic to the abstract algebra A. In particular, we highlight the connection between properties of the logic, such as its expressiveness and its completeness, and the preservation properties of the concretization map. Ongoing research directions concern the extension of these results to richer abstract algebras A endowed with modal (dynamic) operators.

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Generalized bunched implication algebras

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1 Introduction

A *residuated lattice* is an algebra $(A, \wedge, \vee, \cdot, \backslash, /, 1)$, where $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid and $x \cdot y \leq z$ iff $x \leq y/z$ iff $y \leq x \backslash z$, for all $x, y, z \in A$. If $\cdot$ is equal to $\wedge$, then $\mathbf{A}$ is called a Brouwerian algebra (these are the bottom-free subreducts of Heyting algebras) and in this case we write $x \to y$ for $x \backslash y$; it also follows that $y/x = x \backslash y$ so we suppress this operation. A *generalized bunch implication algebra*, or *GBI-algebra*, is an algebra $\mathbf{A} = (A \wedge, \vee, \cdot, \backslash, /, 1, \to, \top)$, where $(A, \wedge, \vee, \cdot, \backslash, /, 1)$ is a residuated lattice and $(A, \wedge, \vee, \to, \top)$ is a Brouwerian algebra.

Commutative and bounded GBI-algebras are known as BI-algebras and they form algebraic semantics for bunched implication logic. The later is of interest in computer science and it is used in proving correctness of concurrent programs.

2 Decidability and FMP

We present a Gentzen calculus for GBI, which enjoys cut elimination; the proof proceeds by considering distributive residuated frames, two-sorted structures that form relational semantics for GBI-algebras. This allows to prove cut elimination for any extension of GBI with equations over the signature $\{\vee, \wedge, \cdot, 1\}$. In particular we recover the known cut elimination for the system of bunched implication (BI) logic as a special case.

We further prove decidability of GBI. The decidability of the $\to$-free fragment can be shown by defining an appropriate complexity measure on sequents. We demonstrate that this complexity measure fails to be decreasing for the $\to$ rules of the calculus and also discuss the difficulties in finding any complexity measure that is decreasing. Nevertheless, we prove the decidability by defining a binary graph on the sequent tree of each sequent and showing that certain aspects of these graphs are reduced as we trace a proof upwards. This can be combined with the fact that we can restrict our attention to special types of sequents in a proof (3-reduced) to put a bound on the overall search space, thus yielding decidability.

We further prove that from the termination of the proof search we not only obtain decidability but also the finite model property. We do that by creating a distributive residuated frame whose dual GBI-algebra is finite.

3 Congruences

Congruences in residuated lattices are determined by certain subsets (in a way similar to the fact that congruences in groups are determined by normal subgroups). Given $a, x \in A$ we define $\rho'_a x = ax/a$ and $\lambda'_a(x) = a \backslash xa$ (which are akin to conjugates in group theory). A subset is called *normal* if it is closed under ρ'_a and λ'_a for all $a \in A$. A *(RL)-deductive filter* of a residuated lattice $\mathbf{A}$ is defined to be a normal upward closed subset of A that is closed under

multiplication and meet and contains the element 1. It is known that if θ is a congruence on $\mathbf{A}$ then $\uparrow[1]_\theta$, the upset of the equivalence class of 1, is a deductive filter. Conversely, if F is a deductive filter of a residuated lattice $\mathbf{A}$, then the relation θ_F is a congruence on $\mathbf{A}$, where $a \; \theta_F \; b$ iff $a\backslash b \wedge b\backslash a \in F$.

Note that if A is a Brouwerian or a Heyting algebra, then deductive filters are usual lattice filters.

We prove that the GBI-deductive filters are exactly the RL-deductive filters that are further closed under $r_{a,b}(x) = (a \to b)/(xa \to b)$ and $s_{a,b}(x) = (a \to bx)/(a \to b)$, for all a, b.

Alternatively, congruences are characterized by their equivalence classes of $\top$. These are usual lattice filters that are closed under the $t_{a,b}(x) = a/b \to (a \wedge x)/b$, $t'_{a,b}(x) = b/a \to b/(a \wedge x)$, $u_{a,b}(x) = a/(b \wedge x) \to a/b$, $u'_{a,b}(x) = (b \wedge x)\backslash a \to b\backslash a$, $v_{a,b}(x) = ab \to (a \wedge x)b$ and $v'_{a,b}(x) = ab \to a(b \wedge x)$ for all a, b.

4 Examples

A *weak conucleus* on a residuated lattice $\mathbf{A}$ is an interior operator σ on $\mathbf{A}$ such that $\sigma(x)\sigma(y) \leq \sigma(xy)$, for all $x, y \in \mathbf{A}$. Then $\sigma[\mathbf{A}] = (\sigma[A], \wedge_\sigma, \vee, \cdot, \backslash_\sigma, /_\sigma)$ is a residuated lattice-ordered semigroup, where $x \bullet_\sigma y = \sigma(x \bullet y)$, where $\bullet \in \{\wedge, \backslash, /\}$; we are interested in the cases where this algebra also has an identity element e and hence $(\sigma[A], e)$ is a residuated lattice. A *topological weak conucleus* on a GBI-algebra $\mathbf{A}$ is a conucleus on both the residuated lattice and the Brouwerian algebra reducts of $\mathbf{A}$.

Given a residuated lattice $\mathbf{A}$ and a positive idempotent element p we define the map σ_p by $\sigma_p(p) = p\backslash x/p$. Then σ_p is a topological weak conucleus (which we call the *double division conucleus by p*), and p is the identity element $\sigma_p(\mathbf{A})$; we denote the resulting residuated lattice $(\sigma_p(\mathbf{A}), p)$ by $p\backslash\mathbf{A}/p$. If $\mathbf{A}$ is involutive then so is $p\backslash\mathbf{A}/p$ and the latter is a subalgebra of $\mathbf{A}$ with respect to the operations $\wedge, \vee, \cdot, +, \sim, -$. Recall that an *involutive* residuated lattice is an expansion of a residuated lattice with an extra constant 0 such that $\sim(-x) = x = -(\sim x)$, where $\sim x = x\backslash 0$ and $-x = 0/x$; we also define $x + y = \sim(-y \cdot -x)$.

Given a poset $\mathbf{P} = (P, \leq)$, we define the set $Wk(\mathbf{P})$ of all binary relations R on P such that $a \leq b \; R \; c \leq d$ implies $a \; R \; d$, for all $a, b, c, d \in P$; these are called $\leq$-*weakening relations*. In other words $Wk(\mathbf{P}) = \mathcal{O}(\mathbf{P} \times \mathbf{P}^\partial)$, where $\mathcal{O}$ denotes the downset operator, and it supports a structure of a GBI-algebra, under union and intersection, and composition of relations.

We note that we also have that $Wk(\mathbf{P}) \cong Res(\mathcal{O}(\mathbf{P}))$, where for a complete join semilattice $\mathbf{L}$, $Res(\mathbf{L})$ denotes the residuated lattice of all residuated maps on $\mathbf{L}$; recall that a map on f on a poset $\mathbf{P}$ is called *residuated* if there exists a map f^* on P such that $f(x) \leq y$ iff $x \leq f^*(y)$, for all $x, y \in P$.

Given a poset $\mathbf{P} = (P, \leq)$, we set $\mathbf{A} = Rel(P)$, to be the involutive GBI algebra of all binary relations on the set P. Note that $p = \leq$ is a positive idempotent element of $\mathbf{A}$. It is easy to see that $p\backslash\mathbf{A}/p$ is exactly $Wk(\mathbf{P})$. Since $\mathbf{A}$ is an involutive GBI-algebra, so is $Wk(\mathbf{P})$.

Exponentiability in Stone Spaces and Priestley Spaces[*]

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Introduction. Exponentiability of objects and morphisms is one of the important good properties for a category. The problem of exponentiability is studied in many contexts and starts its history since 1940's. Among the past works on the subject the most useful for the author were: the great note [4]; the articles [1],[7], [8]; and the books [3], [5],[6].

Due to the existence of important dualities between Stone spaces and Boolean algebras, as well as between Priestley spaces and distributive lattices, our aim is to characterize exponentiable objects and exponentiable morphisms in the categories of Stone spaces and Priestley spaces. The presented work is a part of the more extensive program, which aims to study local homeomorphisms of the so called logical spaces e.g. Stone spaces, Priestley spaces, Spectral spaces, Esakia spaces. This is motivated by the importance of local homeomorhphisms not only in topology, but in algebraic geometry and other areas of mathematics due to their attractive properties.

Given objects X, Y in the small category $\mathbf{C}$ with finite limits, the object Y^X (if it exists in $\mathbf{C}$) is said to be an *exponential* of Y by X, if for any object A in $\mathbf{C}$ there is a natural bijection between the set of all morphisms from $A \times X$ to Y and the set of all morphisms from A to Y^X, i.e. $\mathbf{C}(A \times X, Y) \cong \mathbf{C}(A, Y^X)$. An object X of a category $\mathbf{C}$ is said to be *exponentiable* if the exponent Y^X exists in $\mathbf{C}$ for any object Y. Given object X in $\mathbf{C}$, consider the family $\mathbf{C}/X$ of morphisms $f : Y \to X$ with codomain X in $\mathbf{C}$. Let morphisms between members of the mentioned family be obvious commutative triangles. It is easy to check that the family together with the defined morphisms between them is a category. It is the case that if $\mathbf{C}$ has all finite limits, then $\mathbf{C}/X$ also does so. Let us note that the product of two objects of $\mathbf{C}/X$ is a pullback in $\mathbf{C}$ with the obvious projection to X. As in the case of $\mathbf{C}$, given two morphisms $f : Y \to X$ and $g : Z \to X$ the object g^f (if it exists in $\mathbf{C}/X$) is said to be an exponential of g by f, if for any object $h : W \to X$ in $\mathbf{C}/X$ there is a natural bijection between the set of morphisms from $h \times_X f$ to g and the set of morphisms from h to g^f, i.e. $\mathbf{C}/X(h \times_X f, g) \cong \mathbf{C}/X(h, g^f)$. A morphism f of a category $\mathbf{C}$ is said to be *exponentiable morphism* if the exponent g^f exists in $\mathbf{C}/X$ for any morphism g of $\mathbf{C}$.

Note that for categories of structured sets, the problem of exponentiability reduces to finding appropriate corresponding structure on the set of structure-preserving maps between structured sets. In the following subsections we state the main result already obtained regarding exponentiable objects and morphisms in the categories of Stone spaces and Priestley spaces. For brevity, the supporting lemmas and propositions are omitted.

Exponentiability in Stone spaces. A compact, Hausdorff, and zero-dimensional topological space is called a *Stone space*. The first category we are interested in is the category of Stone spaces and continuous maps. Let us denote the mentioned category by **Stone**. Investigation of exponentiability of object in **Stone** showed that only the finite spaces are exponentiable (unlike the case of all topological spaces where only so-called core-compacts are exponentiable, which can be infinite [2],[4]). Thus we obtain the following result:

Proposition 1. *A Stone space X is exponentiable in* **Stone** *if and only if X is finite.*

After that we are able to prove the full characterization of exponentiable maps of Stone spaces. That is the following result holds:

Proposition 2. *The map $f : X \to B$ between Stone spaces is exponentiable in* **Stone**$/B$ *if and only if f is a local homeomorphism.*

Exponentiability in Priestley spaces. A partially ordered topological space $(X, \leq)$ is called a Priestley space, if X is compact Hausdorff space and for any pair $x, y \in X$ with $x \not\leq y$, there exists a clopen up-set U of X such that $x \in U$ and $y \notin U$. It turns out that the topology on a Priestley space is compact Hausdorff and zero-dimensional, i.e. is a Stone topology. The second category we are interested in is the category of Priestley spaces and continuous order-preserving maps. Let us denote this category by **PS** (**P**riestley **S**paces). Investigation of exponentiability of objects in **PS** showed that, similarly to the case of Stone spaces, only finite spaces are exponentiable in **PS**. Hence the following:

Proposition 3. *A Priestley space X is exponentiable in* **PS** *if and only if X is finite.*

Due to this fact, given a Priestley space B we get the following corollary about exponentiability of $\pi_2 : X \times B \to B$ in **PS**$/B$:

Corollary 3.1. $\pi_2 : X \times B \to B$ *is exponentiable in* **PS**$/B$ *if and only if X is finite.*

Moreover, we were able to prove a necessary condition for exponentiability of a map between Priestley spaces. An order preserving map $f : X \to B$ is called an interpolation-lifting map if given $x \leq y$ in X and $f(x) \leq b \leq f(y)$, there exists $x \leq z \leq y$ such that $f(z) = b$.

Proposition 4. *If $f : X \to B$ is exponentiable in* **PS**$/B$ *then f is interpolation-lifting.*

We are still unable to find a necessary and sufficient condition for exponentiability of Priestley maps. Already obtained results draw quite interesting picture of considered categories. Only the smallest part of the considered categories (only finite objects) have such strong property as exponentiability. Further work is in progress, namely we are investigating whether exponentiable morphisms in **PS** are precisely the local homeomorphisms that are also interpolation-lifting maps.

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Filter pairs: A new way of presenting logics

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In the work we present we introduce the notion of filter pair as a tool for creating and analyzing logics. We sketch the basic idea of this notion:

By *logic* we mean a pair $(\Sigma, \vdash)$ where Σ is a signature, i.e. a collection of connectives with finite arities, and $\vdash$ is a Tarskian consequence relation, i.e. an idempotent, increasing, monotone, finitary and structural relation between subsets and elements of the set of formulas $Fm_\Sigma(X)$ built from Σ and a set X of variables.

It is well-known that every logic gives rise to an algebraic lattice contained in the powerset $\wp(Fm_\Sigma(X))$, namely the lattice of theories. This lattice is closed under arbitrary intersections (since intersections of theories are theories) and suprema of directed subsets.

Conversely an algebraic lattice $L \subseteq \wp(Fm_\Sigma(X))$ that is closed under arbitrary intersections and unions of increasing chains gives rise to a finitary closure operator (assigning to a subset $A \subseteq Fm_\Sigma(X)$ the intersection of all members of L containing A). This closure operator need not be structural — this is an extra requirement.

We observe that the structurality of the logic just defined is equivalent to the *naturality* (in the sense of category theory) of the inclusion of the algebraic lattice into the power set of formulas with respect to endomorphisms of the formula algebra: Structurality means that the preimage under a substitution of a theory is a theory again or, equivalently, that the following diagram commutes for any substitution σ:

$$
\begin{array}{ccc}
Fm_\Sigma(X) & L \overset{i}{\hookrightarrow} \wp(Fm_\Sigma(X)) \\
\sigma \downarrow \quad \sigma^{-1}|_L \uparrow \quad & \quad \uparrow \sigma^{-1} \\
Fm_\Sigma(X) & L \overset{i}{\hookrightarrow} \wp(Fm_\Sigma(X))
\end{array}
$$

Further, it is equivalent to demand this naturality for all Σ-algebras and homomorphisms instead of just the formula algebra. We thus arrive at the definition of *filter pair*:

Definition. (i) *A* filter pair *for the signature Σ is a contravariant functor G from Σ-algebras to algebraic lattices together with a natural transformation $i\colon G \to \wp(-)$ from G to the functor that takes an algebra to the power set of its underlying set, which preserves arbitrary infima and suprema of directed subsets.*

(ii) *The* logic associated to a filter pair (G, i) *is the logic associated (in the above fashion) to the algebraic lattice given by the image $i(G(Fm_\Sigma(X))) \subseteq \wp(Fm_\Sigma(X))$.*

Thus a filter pair can be seen as a *presentation* of a logic, different from the usual style of presentation by axioms and derivation rules. For a given logic L one can take $G := Fi_L$, the functor which associates to a Σ-algebra the lattice of L-filters on it; this shows that every logic admits a presentation by a filter pair.

A more interesting case is when G is the functor associating to a Σ-structure the lattice of congruences relative to some quasivariety K, that is, $G\colon A \mapsto \{\theta \mid A/\theta \in K\}$ – we call these

filter pairs *congruence filter pairs.* There is a huge supply of congruence filter pairs by the following result:

Theorem. *Let K be a quasivariety, and $\tau = \langle \epsilon, \delta \rangle$ a set of equations (i.e. pairs of unary formulas in the signature of K). For every Σ-algebra A denote by $Con_K(A) := \{\theta \mid A/\theta \in K\}$ the set of congruences relative to K. Then*

$$(G \colon A \mapsto Con_K(A), \quad i \colon \theta \mapsto \{a \in A \mid \epsilon(a) = \delta(a) \text{ in } A/\theta\})$$

defines a filter pair

It follows from [**BP**, Thm 5.1(ii)] that every algebraizable logic admits a presentation by such a congruence filter pair. But strictly more logics arise in this way, even non-protoalgebraic logics. A presentation by a congruence filter pair can give means of determining the position of a logic in the Leibniz hierarchy; e.g. the logic is algebraizable if the natural transformation i is injective. Similar criteria can be given for being truth-equational or Lindenbaum algebraizable.

Further, we have algebraic tools available for dealing with logics which admit a presentation by a congruence filter pair. For example, (under an additional technical assumption) the logic presented by a congruence filter pair has the Craig interpolation property if the quasivariety has the amalgamation property. Further correspondence principles between algebraic and logical properties are under investigation.

Thus on the one hand filter pairs give a way of analyzing logics, on the other hand they give a new way of attacking the problem of associating a logic to a given quasivariety.

In the talk we will motivate and introduce the notions of filter pair and congruence filter pair, and make the above statements concrete in examples.

We will further offer a point of view on congruence filter pairs as being an approach to algebraizing logic which is dual to the one via the Leibniz operator. The Leibniz operator is a map from the lattice of filters to the lattice of congruences which is "trying to be" a *right* adjoint (and actually is a right adjoint for protoalgebraic logics). The natural transformation i of a congruence filter pair, in contrast, has a *left* adjoint going from the lattice of filters to the lattice of congruences. Thus the congruence filter pair approach to defining and analyzing logics, while being equivalent to the Leibniz operator approach in the algebraizable case, diverges into a different direction in the non-algebraizable case. We will also make this point of view concrete with sample calculations.

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The logic of resources and capabilities

Apostolos Tzimoulis

joint work with M. Bílková, G. Greco, A. Palmigiano, and N. Wijnberg

Organizations are social units of agents who are structured and managed to meet a need, or to pursue collective goals. The study of organizations in economics and social science has led to substantial literature, which explains the various forms of organization structure and the relations they bear to the generation of competitive advantage in terms of agency, knowledge, goals, capabilities and inter-agent coordination. As such, organization theory is very amenable to be studied with the logical tools developed in the context of the study of information flow.

However, presently there are not many instances of logical systems specifically designed to describe the internal dynamics of organizations. Furthermore, existing logics aimed at capturing notions of agency and information flow typically lack a comparable proof-theoretic development. More often than not, the hurdles preventing their standard proof-theoretic development are due to the very features which make them capture essential aspects of the real world, such as their not being closed under uniform substitution, or the presence of certain extralinguistic labels and devices encoding key interactions between logical connectives.

In [2], a framework similar in spirit to STIT logics is presented, that aims at achieving an emergent notion of dynamics which is based on a hierarchy of more primitive notions, the most basic of which are agency and agents' capabilities.

With [2] as a starting point, we develop a logic aimed to describe dynamics of organizations, the logic of resources and capabilities [1]. The key feature of this logic lies on the idea that a better grasp on the notion of capabilities can be achieved if we simultaneously can talk about resources. For instance we can compare the capabilities of different agents in terms on their being able to perform a certain task with less resources or we can extend the reasoning over a planning problem in terms of the order of which resources needs to be used to perform a certain task.

The core aspect of this logic is based on *multi-type display-type calculi*, a methodology introduced in [4, 3] motivated by considerations discussed in [8, 6] to provide DEL and PDL with analytic calculi, and further developed in [5, 7, 11, 12, 13, 10], in synergy with algebraic techniques [9]. In the present framework, resources and formulas are represented as terms of different types, each with an independently defined logic, which interact thanks to operators, such as the capability operators in this specific setting, which take arguments of different types (resources and formulas in this specific setting).

The present contribution reports on the technical aspects involved in the logic of resources and capabilities, that is, the analyticity of its axioms, the rules of the proof system, the corresponding semantics, the canonicity of the rules and cut elimination. Finally, some examples will be presented that illustrate what this logic can be used for.

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Fixed-point elimination in the
Intuitionistic Propositional Calculus

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We approach Intuitionistic Logic and Heyting algebras from fixed-point theory and μ-calculi [1]. A μ-calculus is a prototypical computational logic, obtained from a base logic or a base algebraic system by addition of distinct forms of iteration, least and greatest fixed-points, so to increase expressivity. We consider therefore $\mathbf{IPC}_\mu$, the Intuitionistic Propositional μ-Calculus, whose formula-terms are generated by the grammar

$$\phi = x \mid \top \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \rightarrow \phi \mid \mu_x.\phi \mid \nu_x.\phi \,,$$

where it is required in the last two productions that the variable x occurs positively in ϕ. Formulas are interpreted over complete Heyting algebras, with $\mu_x.\phi$ (resp. $\nu_x.\phi$) denoting the least fixed-point (resp. the greatest fixed-point) of the intepretation of $\phi(x)$, as a monotone function of the variable x. These extremal fixed-points exist, by the Knaster-Tarski theorem.

Ruitenburg [3] proved that for each formula $\phi(x)$ of the $\mathbf{IPC}$ there exists a number $\rho(\phi)$ such that $\phi^{\rho(\phi)}(x)$—the formula obtained from ϕ by iterating $\rho(\phi)$ times substitution of ϕ for the variable x—and $\phi^{\rho(\phi)+2}(x)$ are equivalent in Intuitionistic Logic. An immediate consequence of this result is that a syntactically monotone intuitionisitc formula $\phi(x)$ converges both to its least fixed-point and to its greatest fixed-point in at most $\rho(\phi)$ steps. In the language of μ-calculi, we have $\mu_x.\phi(x) = \phi^{\rho(\phi)}(\bot)$ and $\nu_x.\phi(x) = \phi^{\rho(\phi)}(\top)$. These identities witness that the $\mathbf{IPC}_\mu$ is degenerated, meaning that every formula from the above grammar is equivalent to a fixed-point free formula. They also witness that nor completeness neither the Knaster-Tarski theorem are needed to interpret the above formulas over Heyting algebras.

Ruitenburg's result is not the end of the story. We aim at computing explicit representations of fixed-point expressions by means of fixed-point free formulas. Such an algorithm would provide an axiomatization of fixed-points in the $\mathbf{IPC}$ and also a decision procedure for the $\mathbf{IPC}_\mu$. We also aim at computing closure ordinals of intuitionisitc formulas $\phi(x)$, that is, the least number n such that $\mu_x.\phi(x) = \phi^n(\bot)$ and the least number m for which $\nu_x.\phi(x) = \phi^m(\top)$. Notice that bounds on Ruitenberg's numbers $\rho(\phi)$ might be over-approximation of closure ordinals of ϕ, for example, for an arbitrary intuitionistic formula ϕ, $\nu_x.\phi(x) = \phi^k(\top)$ for $k = 1$, while $\rho(\phi)$ might be arbitrarily large. We tackled these problems in a recent work [2]. We achieve there an effective transformation of intuitionisitc μ-formulas into equivalent fixed-point free intuitionisitc formulas. Such a transformation allows to estimate upper bounds of closure ordinals, which are tight in many cases.

We sketch in what follows the ideas by which we devise our effective transformation.

Lemma. *Every polynomial* $\mathfrak{f} : H \longrightarrow H$ *over a Heyting algebra* H *is* compatible, *meaning that the equation* $\mathfrak{f}(x) \wedge y = \mathfrak{f}(x \wedge y) \wedge y$ *holds.*

A first consequence of the above statement is that, for such a polynomial, $\mathfrak{f}^2(\top) = \mathfrak{f}(\top)$, so $\mathfrak{f}(\top)$ is the greatest fixed-point of $\mathfrak{f}$ when $\mathfrak{f}$ is monotone. This observation is generalized to systems of equations as follows.

Lemma. *If H is an Heyting algebra and, for $i = 1, \ldots, n$, $\mathfrak{f}_i : H^n \longrightarrow H$ is a monotone polynomial, then $\langle \mathfrak{f}_1, \ldots, \mathfrak{f}_n \rangle^n(\top)$ is the greatest fixed-point of $\langle \mathfrak{f}_1, \ldots, \mathfrak{f}_n \rangle : H^n \longrightarrow H^n$.*

Fact. If $f : P \longrightarrow Q$ and $g : Q \longrightarrow P$ are monotone functions such that the least fixed-point $\mu.(g \circ f)$ of $g \circ f$ exists, then $f(\mu.(g \circ f))$ is the least fixed-point of $f \circ g$.

These statements allow us to give an explicit representation of $\mu_x.\phi(x)$ when all the occurrences of the variable x are under the left side of an implication. Namely, if we write $\phi(x) = \psi_0[\psi_1(x)/y_1, \ldots, \psi_n(x)/y_n]$ with y_i under the left side of just one implication, then

$$\mu_x.\phi(x) = \psi_0(\, \nu_{y_1,\ldots,y_n}.\langle \psi_1(\psi_0(y_1, \ldots, y_n)), \ldots, \psi_n(\psi_0(y_1, \ldots, y_n)) \rangle \,)$$
$$= \psi_0(\, \langle \psi_1(\psi_0(y_1, \ldots, y_n)), \ldots, \psi_n(\psi_0(y_1, \ldots, y_n)) \rangle^n(\top) \,).$$

Other two important consequences of compatibility of polynomials are the following distribution laws of least fixed-points w.r.t. the residuated structure:

$$\mu.(\bigwedge_{i \in I} \mathfrak{f}_j) = \bigwedge_{i \in I} \mu.\mathfrak{f}_i , \qquad\qquad \mu.(\alpha \to \mathfrak{f}) = \alpha \to \mu.\mathfrak{f} , \qquad (1)$$

which holds when $\mathfrak{f}$ and $\mathfrak{f}_i$ are monotone polynomials and α is a constant.

Fact. The least fixed-point of a monotone function $f(x, x)$ can be computed by firstly computing the least fixed-point of $f(x, y)$ in the variable y, parametrizing in the variable x, and then by computing the least fixed-point of the resulting monotone function in the variable x.

This observation allows us to split the search of an explicit representation of the least fixed-point of a formula into two steps: first we can assume that every occurrence of the variable x is under the left side of an implication; then we can assume that there are no occurrences of the variable x under the left side of an implication. A formula with the latter property is then equivalent to a conjunction of disjunctive formulas, that is, formulas generated by the grammar below on the left:

$$\phi = x \mid \beta \vee \phi \mid \phi \vee \beta \mid \alpha \to \phi \mid \phi \vee \phi , \qquad \mu_x.\phi = (\bigwedge_{\alpha \in \mathsf{Head}(\phi)} \alpha) \to (\bigvee_{\beta \in \mathsf{Side}(\phi)} \beta) , \qquad (2)$$

where α and β are formulas with no occurrence of the variable x. The first of the relations (1) reduces the computation of the least fixed-point of a formula to the computation of the least fixed-point of a disjunctive formula ϕ. For such a formula, call α a head formula and β a side formula; let $\mathsf{Head}(\phi)$ denote the set of head formulas in a parse tree of ϕ and, similarly, let $\mathsf{Side}(\phi)$ be the set of side formulas in the same parse tree of ϕ. Using the second of the relations (1) and the fact that disjunctive formulas give rise to monotone inflationary functions, an expression for the least fixed-point of a disjunctive formula appears on the right of (2).

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First-order logic properly displayed

G. Greco, M. A. Moshier, A. Palmigiano, and A. Tzimoulis

The existing sequent calculi for first-order logic [18] contain special rules for the introduction of quantification and for substitution. The application of these rules depends on the unbounded and bounded variables occurring in formulas. For example, in the standard Gentzen calculus for first-order logic the rules

$$\frac{\Gamma \vdash \Delta, A[x]}{\Gamma \vdash \Delta, \forall x A} \qquad \frac{\Gamma, A[x] \vdash \Delta}{\Gamma, \exists x A \vdash \Delta}$$

are sound only when x does not appear free in the conclusions of the rules.

A proposal for a display calculus for fragments of first-order logic was first presented in [21, 20]. The key idea of this approach is that existential quantification can be viewed as a diamond-like operator of modal logic, and universal quantification can be seen as a box-like operator as discussed in [14, 19]. The underlying reason for these similarities which have been observed and exploited in [14, 19, 21, 20] is order-theoretic and pertains to the phenomenon of adjunction: indeed the set theoretic semantic interpretation of the existential and universal quantification are the left and right adjoint respectively of the inverse projection map and more generally, in categorical semantics, the left and right adjoint of the pullbacks along projections [15],[7, Chapter 15]. However, the display calculus of [21] contains rules with side conditions on the free and bounded variables of formulas similar to the ones presented above. This implies that the rules are not closed under uniform substitution, that is, the display calculus is not *proper* [20, Section 4.1].

We present results based on ongoing work in [10] on a proper display calculus for first-order logic. The design of our calculus is based on the multi-type methodology first presented in [5, 2], motivated by considerations discussed in [8, 4], for DEL and PDL and further developed in [3, 6, 11, 12, 13, 1], in synergy with algebraic techniques [9]. The multi-type approach allows for the co-existence of terms of different types bridged by heterogeneous connectives. The requirement for the calculus is that in a derivable sequent $x \vdash y$ the structures x and y must be of the same type. In this framework properness means uniform substitution within each type.

Using insights from [15, 16, 17] we introduce a proper display calculus for first-order logic. The conditions on rules are internalised in the calculus by the use of appropriate types. The language of first-order logic is expanded with a unary heterogeneous connective that serves as the right adjoint of the existential quantifier and the left adjoint of the universal quantifier. In the context of the calculus this connective signifies the introduction of a fresh variable to a formula.

In my talk I will present the proper display calculus for first-order logic and discuss its completeness, soundness, cut-elimination and conservativity.

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2

An approach to parts of d-frames
and an Isbell-type density theorem *

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With the purpose of finding a Stone duality for bitopological spaces, A. Jung and A. Moshier introduced in [2] the category dFrm of *d-frames* in which objects are structures that comprise two frames, thought of as lattices of open sets, and two relations that connect both frames, as abstractions of the covering and disjointness relation. Morphisms in this category, named d-frame homomorphims, are pairs of frame homomorphisms preserving those relations. The aim of this talk is to explore an approach to the notion of parts of a space in this pointfree bitopological setting.

In the category Loc of locales, subobjects, namely sublocales, can be thought of as *generalized subspaces* and they form a far more complex and richer structure than their classical counterpart [1]. They can be represented in several different ways: frame congruences, nuclei, sublocale sets and sublocale maps (onto frame homomorphisms). The categorical interpretation of the last one provides a candidate for the bitopological case, as onto frame homomorphisms are precisely extremal epimorphisms in the category Frm = Loc$^{\mathrm{op}}$ of frames. Motivated by this fact, in this talk, we will present a characterization of extremal epimorphisms in the category of d-frames. They are given by certain pairs of onto frame homomorphisms and, consequently, they can be represented by pairs of sublocale sets endowed with appropriate covering and disjointness relations. However, non-trivial examples are not easily found and even though one can easily show that they form a complete lattice, this structure does not seem to be as rich as in the localic case.

Furthermore, we will define dense d-frame homomorphisms and show that, given a pair of sublocale sets containing all the regular elements of a d-frame, their associated sublocale maps form a dense extremal epimorphism in dFrm. Conversely, an extremal epimorphism in dFrm is dense if and only if its associated pair of sublocale sets contains all the regular elements. Accordingly, we will show that there is a least dense extremal epimorphism for each d-frame, obtaining in this way a Isbell-type density theorem for the category of d-frames.

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Wild Algebras in Cartesian Categorical Logic

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Let k denote an algebraically closed field. A "k-algebra" is a ring with compatible k-vector space structure. A "module" over a k-algebra A is a k-vector space M with an action of A that is compatible with the group and vector space structures on M. It is known that the classical first-order theory of modules over the free algebra $k\langle X, Y \rangle$ is undecidable. More precisely, the classical first-order theory is undecidable in that there is no Turing machine algorithm that will establish whether a given sentence of the theory is a theorem [4], [1]. A k-algebra S is "wild" if its category of modules admits a "representation embedding." This is a finitely generated $(S, k\langle X, Y \rangle)$-bimodule M, free over $k\langle X, Y \rangle$, such that an induced functor $M \otimes_{k\langle X, Y \rangle} -\colon k\langle X, Y \rangle\text{-mod} \to S\text{-mod}$, between categories of finite-dimensional modules, preserves and reflects indecomposability and isomorphism (p. 272 of [4]). The conjecture of M. Prest is that any finite-dimensional wild algebra has an undecidable theory of modules (p. 350 of [4]).

There are two goals of our recent work [3]. First is to reformulate the theory of modules over a fixed k-algebra within the cartesian fragment of first-order categorical logic as described in D1.2 of [2]; and then to prove that the cartesian theory of $k\langle X, Y \rangle$-modules in particular is undecidable. The second is to reformulate Prest's conjecture in a manner appropriate for cartesian logic and then to prove it. Our hope is that this rephrasing in cartesian categorical logic will shed light on the original problem in classical model theory. However, we make no claim to resolve the original conjecture.

That the cartesian theory of $k\langle X, Y \rangle$-modules is undecidable can be seen by adapting the proof idea of the original source [1]. That is, it can be seen that each element of a distinguished class of sequents of the cartesian theory is provable in the theory if, and only if, the two words of a corresponding pair of words of a fixed monoid with an undecidable word problem are equivalent. Thus, the module theory can be seen to interpret the undecidable word problem of a finitely presented monoid, making it undecidable as well. The affirmative result here meets the first goal stated above.

To attain the second goal, undecidability must somehow be "transmitted" between cartesian theories of modules over $k\langle X, Y \rangle$ and S of the first paragraph. This is accommplished by giving what amounts to a translation between the theories. This is by the device of "syntactic categories." The syntactic category $\mathscr{C}_\mathbb{T}$ of a cartesian theory $\mathbb{T}$ is a categorification of the syntax of $\mathbb{T}$, described for example in D1.4 of [2]. A translation of theories $\mathbb{T} \to \mathbb{T}'$ is essentially the same thing as a functor of syntactic categories $\mathscr{C}_\mathbb{T} \to \mathscr{C}_{\mathbb{T}'}$. Our main result is that what we call a "representation embedding" of cartesian theories $\mathbb{T}$ and $\mathbb{T}'$ induces such a functor of syntactic categories. The functor amount to a conservative translation of theories, so that if $\mathbb{T}$ is undecidable, then so is $\mathbb{T}'$.

In a bit more detail, a representation embedding between categories of models of cartesian theories can be defined to be a functor between the categories of **Set**-models that preserves indecomposability and projectivity and that reflects epics when restricted to the full subcategory of indecomposable projective models. The main result, then, is that if there is such a representation embedding between cartesian theories say $\mathbb{T}$ and $\mathbb{T}'$, then if $\mathbb{T}$ is undecidable,

so is $\mathbb{T}'$. Our reformulation of Prest's conjecture in cartesian logic can be seen to provide a representation embedding in this sense between the respective cartesian theories of $k\langle X, Y \rangle$- and S-modules. Thus, the main result is applied to obtain an affirmative resolution of the reformulation in cartesian logic of Prest's conjecture as a corollary.

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On two concepts of ultrafilter extensions of first-order models and their generalizations

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There exist two known concepts of ultrafilter extensions of first-order models, both in a certain sense canonical. One of them [1] comes from universal algebra where it goes back to a seminal paper by Jónsson and Tarski [2] and also modal logic [3, 4]. Another one [5, 6] has its sources in iterated ultrapowers in model theory [7, 8, 9] and especially algebra of ultrafilters, with ultrafilter extensions of semigroups [10] as its main precursor. By a classical fact of general topology, the space of ultrafilters over a discrete space is its largest compactification. The main result of [5, 6], which confirms a canonicity of the extension introduced there, generalizes this fact to discrete spaces endowed with arbitrary first-order structure. An analogous result for the former type of ultrafilter extensions was obtained in [11].

Here we offer a uniform approach to both types of extensions. It is based on the idea to extend the extension procedure itself. We propose a generalization of the standard concept of first-order models in which functional and relational symbols are interpreted rather by ultrafilters over sets of functions and relations than by functions and relations themselves. We provide two specific operations which turn generalized models into ordinary ones, characterize the resulting ordinary models in topological terms, and establish necessary and sufficient conditions under which the latter are the two canonical ultrafilter extensions of some models. For details, we refer the reader to the forthcoming [12].

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Unification in first order logics: superintuitionistic and modal.

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1. Introduction. We introduce and apply unification in predicate logics that extend intuitionistic predicate logic Q-INT and modal predicate logic Q-S4 (or Q-K4). S. Ghilardi succesfully applied unification in propositional logic [5], [6], [7]. We show that unification in $L \supseteq$ Q-INT is projective iff $L \supseteq$ P.Q–LC, Gödel-Dummett's predicate logic plus Plato's Law (in modal case: $L \supseteq$ mP.Q-S4.3); hence, such L is almost structurally complete: each admissible rule is either derivable or passive and unification in L is unitary. We provide an explicit basis for all passive rules in Q-INT (Q-S4). We show that every unifiable Harrop's formula is projective and we extend the classical results of Kleene (on disjunction and existence quantifier under implication) to projective formulas and to all extensions of $Q - INT$. Rules that are admissible in all extensions of Q-INT are given. We prove that L has filtering unification iff L extends Q-KC: $=$ Q-INT $+ (\neg A \vee \neg\neg A)$ (Q-K4.2$^+$), and that unification in Q-LC, Q-KC (Q-S4.3, Q-S4.2) is nullary and in Q-INT (Q-S4) it is not finitary, contrary to the propositional cases.

Q-L denotes the least predicate logic extending a propositional logic L, e.g. Q-CL, Q-INT, Q-S4. We follow the axioms and notation of [2], [3]. We consider a standard *first-order* (or *predicate*) language $\{\rightarrow, \wedge, \vee, \bot, \forall, \exists\}$ (plus modal $\Box, \Diamond$) with *free individual variables*: $\{a_1, a_2, a_3, \ldots\}$, *bound individual variables*: $\{x_1, x_2, x_3, \ldots\}$, *predicate variables*: $Pr= \{P_1, P_2, P_3, \ldots\}$; no function symbols or $=$. *Formulas* (Fm) are q-formulas (q-Fm) in which no bound variable occurs free. A 2nd-order *substitution for predicate variables* is used.

2. Unifiability. A basis for passive rules. A *unifier* for A in a logic L is a substitution (for predicate variables) τ making A a theorem of L, i.e. $\tau(A) \in L$. A formula A is *unifiable* in L (L-unifiable) if it has a unifier in L. A unifier $v\colon Pr \rightarrow \{\bot, \top\}$ is called *ground*. Note: (i) A is L-unifiable iff (ii) there is a ground unifier for A in L iff (iii) A is valid in a classical model with 1-element universe. Hence unifiability is *absolute*. Note: Unifiable $\neq$ Consistent. A rule A/B is *passive*, if A is not unifiable. Consider the following (schematic) rules:

$$(P\forall): \quad \frac{\neg\forall_{\bar{z}}C(\bar{z}) \wedge \neg\forall_{\bar{z}}\neg C(\bar{z})}{\bot} \qquad \left((\Box P\forall): \quad \frac{\Diamond\exists_{\bar{x}}A(\bar{x}) \wedge \Diamond\exists_{\bar{x}}\neg A(\bar{x})}{\bot}\right)$$

Theorem 1. $P\forall$ ($\Box P\forall$) form a basis for all passive rules over Q–INT (Q–K4D)

3. Projective unification and Harrop formulas. A unifier ε for a formula A in a logic L is *projective* if $\vdash_L (\Box)A \rightarrow \forall_{x_1,\ldots,x_n}(\varepsilon(P_i(x_1,\ldots,x_n)) \leftrightarrow P_i(x_1,\ldots,x_n))$, for each predicate variable P_i. A logic L enjoys *projective unification* if each L-unifiable formula has a projective unifier. P.Q-LC (mP.Q-S4.3) denotes the Gödel-Dummett (S4.3 modal) predicate logic extended with the following formula called *(modal) Plato's Law*
(P): $\quad \exists_x(\exists_x B(x) \rightarrow B(x))$, $\qquad$ (mP): $\quad \exists_x \Box(\exists_x \Box B(x) \rightarrow B(x))$.

Theorem 2. *A superintuitionistic predicate logic L enjoys projective unification if and only if* P.Q-LC $\subseteq L$. *If a modal logic L enjoys projective unification, then* mP.Q-S4.3 $\subseteq L$.

Corollary 3. *Every logic containing* P.Q-LC *is almost structurally complete i.e. every admissible rule is either derivable or passive.*

Corollary 4. P.Q-LC *is the least logic* $L \supseteq$ Q-INT *in which* $\vee$ *and* $\exists$ *is definable by* $\wedge, \rightarrow, \forall$.

Theorem 5. *For an infinite rooted Kripke frame* $\mathcal{F} = <W, \leqslant, \mathcal{D}>$, (m)P *is valid in* $\mathcal{F}$ *iff* $\mathcal{F}$ *has constant domain* $\mathcal{D}$ *and* W *is well (quasi-)ordered.* IP.Q-LC *(*mP.Q-S4.3*) is Kripke incomplete.*

Harrop q-formulas $q\text{-}Fm_H$ (or Harrop formulas Fm_H) are defined by the clauses:
1. all elementary q-formulas (including $\perp$) are Harrop; 2. if $A, B \in q\text{-}Fm_H$, then $A \wedge B \in q\text{-}Fm_H$; 3. if $B \in q\text{-}Fm_H$, then $A \rightarrow B \in q\text{-}Fm_H$;   4. if $B \in q\text{-}Fm_H$, then $\forall_{x_j} B \in q\text{-}Fm_H$.

Theorem 6. *Any unifiable Harrop's formula is projective in* Q–INT.

Theorem 7. *For any L-projective sentence A and any formulas $B_1, B_2, \exists_x C(x)$, we have*
(i)   *if* $\vdash_L A \rightarrow B_1 \vee B_2$, *then* $\vdash_L (A \rightarrow B_1) \vee (A \rightarrow B_2)$,
(i)'   *if* $\vdash_L A \rightarrow \Box B_1 \vee \Box B_2$, *then* $\vdash_L \Box(\Box A \rightarrow B_1) \vee \Box(\Box A \rightarrow B_2)$, *(in the modal case)*,
(ii)   *if* $\vdash_L A \rightarrow \exists_x C(x)$, *then* $\vdash_L \exists_x (A \rightarrow C(x))$,
(ii)' *if* $\vdash_L A \rightarrow \exists_x \Box C(x)$, *then* $\vdash_L \exists_x \Box(\Box A \rightarrow C(x))$, *(in the modal case)*.

Example: The following non-passive rule is admissible in every predicate logic $L \supseteq$ Q-INT: $\neg(\exists_x P(x) \wedge \exists_x \neg P(x)) \rightarrow \exists_y Q(y) \;/\; \exists_y[\neg(\exists_x P(x) \wedge \exists_x \neg P(x)) \rightarrow Q(y)]$.

4. Filtering unification and unification types. Recall: σ is *more general than* τ, if $\vdash_L \tau(x) \leftrightarrow \theta(\sigma(x))$, for some substitution θ (σ, τ are defined on finite sets of variables). A *most general unifier, mgu,* for a formula A is a unifier that is more general than any unifier for A. Unification in L is *unitary*, 1, if every L-unifiable formula has a mgu. The other unification types: *finitary, infinitary* and *nullary*, 0, depend on the number of maximal unifiers see [1]. [7] characterized modal logics in which unification is *filtering*, that is, for every two unifiers for a formula there is another unifier that is more general than both of them, (type 1 or 0).

Theorem 8. *Let L be a superintuitionistic predicate logic (modal logic extending* Q-K4*). Unification in L is filtering iff the Stone law* $\neg\neg A \vee \neg A$ *(*$2^+ : \Diamond^+\Box^+ A \rightarrow \Box^+\Diamond^+ A$*) is in L.*

Corollary 9. *For every superintuitionistic (modal) predicate logic L (containing* Q-S4*)*
(i) if Q-KC $\subseteq L$ *(*Q-S4.2 $\subseteq L$*), then unification in L is either unitary or nullary;*
(ii) if L enjoys unitary unification, then Q-KC $\subseteq L$ *(*Q-S4.2$\subseteq L$*).*

Corollary 10. *Unification in* Q-LC, Q-KC *(*Q-S4.3, Q-S4.2*) is nullary and in* Q-INT *(*Q-S4*) it is either infinitary or nullary, contrary to the corresponding propositional cases.*

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Filtrations for many-valued modal logic with applications

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Introduction. The methods of filtration and selective filtration are among the oldest and best known techniques for obtaining finite models in modal logic. In the present work we define the filtration construction in the context of many-valued modal logics with arbitrary residuated lattices as truth spaces. We prove an accompanying filtration theorem and show that many-valued filtrations exist by exhibiting the smallest and largest filtrations satisfying the definition. Next, we apply filtrations to show that certain natural many-valued analogues of **T**, **K4** and **S4** have the strong finite model property. A more challenging example perhaps is the many valued analogue of Gödel-Löb logic. We show that this logic is characterized by the class of all finite MV-frames which satisfy a certain many-valued version of transitivity and which contain no infinite non-0 paths. As in the classical case, this latter result requires the use of a selective filtration construction.

Many valued modal logic. In [2, 3] Fitting introduced a family of many-valued modal logics over Heyting algebras where both the valuation and the accessibility relations of the associated Kripke models are many-valued. This can be generalized by replacing Heyting algebras with residuated lattices, as is done in e.g. [1]. We follow this framework. Formulas are given by the following recursion: $\varphi := \bot \mid p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \rightarrow \varphi \mid \Diamond\varphi \mid \Box\varphi$, with p from a denumerably infinite set PROP of proposition letters. Let $\mathbf{A} = (A, \wedge, \vee, \circ, \rightarrow, 1, 0)$ be a residuated lattice. In other words, the reduct $(A, \wedge, \vee, 1, 0)$ is a bounded lattice while the reduct $(A, \circ, 1)$ is a commutative monoid, and moreover $\rightarrow$ is the right residual of $\circ$, i.e. $a \circ b \leq c$ iff $a \leq b \rightarrow c$ for all $a, b, c \in A$. An $\mathbf{A}$-frame is a triple $\mathfrak{F} = (W, D, B)$ with a non-empty universe W and $\mathbf{A}$-valued accessibility relations $D : W \times W \rightarrow \mathbf{A}$ and $B : W \times W \rightarrow \mathbf{A}$. An $\mathbf{A}$-model is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ where $\mathfrak{F}$ is an $\mathbf{A}$-frame and $V : \mathsf{PROP} \times W \rightarrow \mathbf{A}$ is an $\mathbf{A}$-valued valuation. The valuation can be extended to all formulas. In particular,

$$V(\Diamond\varphi, w) = \bigvee\{D(w, v) \circ V(\varphi, v) \mid v \in W\} \text{ and}$$
$$V(\Box\varphi, w) = \bigwedge\{B(w, v) \rightarrow V(\varphi, v) \mid v \in W\}.$$

Let $a \in \mathbf{A}$, then a formula φ is said to be *a-true* in a model at $w \in W$, denoted by $\mathfrak{M}, w \Vdash_a \varphi$, if $V(\varphi, w) \geq a$.

Filtrations. Given a subformula-closed set of formulas Σ, define an equivalence relation $\leftrightsquigarrow_\Sigma$ on an $\mathbf{A}$-model $\mathfrak{M} = (W, D, B, V)$ such that $w \leftrightsquigarrow_\Sigma^{\mathbf{A}} v$ iff $V(\varphi, w) = V(\varphi, v)$ for all $\varphi \in \Sigma$. Let $[w]_\Sigma^{\mathbf{A}}$ denote the equivalence class of $w \in W$ under $\leftrightsquigarrow_\Sigma^{\mathbf{A}}$.

Definition 1. *Let $W_\Sigma = \{[w]_\Sigma \mid w \in W\}$. Let $\mathfrak{M}_f = (W_\Sigma, D_f, B_f, V_\Sigma)$ be any model such that:*

(R1) *Let $a \in \mathbf{A}$. If $D\,wv \geq a$, then $D_f[w][v] \geq a$.*
(R2) *Let $a, a' \in \mathbf{A}$. If $D_f[w][v] \geq a$, then for every $\Diamond\varphi \in \Sigma$, if $\mathfrak{M}, v \Vdash_{a'} \varphi$, then*

$\mathfrak{M}, w \Vdash_{a \circ a'} \Diamond \varphi$.

(R3) *Let $a \in \mathbf{A}$. If $\mathrm{B}\, wv \geq a$, then $\mathrm{B}_f[w][v] \geq a$.*

(R4) *Let $a, a' \in \mathbf{A}$. If $\mathrm{B}_f[w][v] \geq a$, then for every $\Box \varphi \in \Sigma$, if $\mathfrak{M}, w \Vdash_{a'} \Box \varphi$, then $\mathfrak{M}, v \Vdash_{a \circ a'} \varphi$.*

(V) $V_\Sigma([w], p) = V(w, p)$ *for all $p \in \Sigma$.*

*Then $\mathfrak{M}_f$ is called an **A**-valued filtration of $\mathfrak{M}$ through Σ.*

Theorem 2 (**A**-valued Filtration Theorem). *Let $\mathfrak{M}_f = (W_\Sigma, \mathrm{D}_f, \mathrm{B}_f, V_\Sigma)$ be a filtration of $\mathfrak{M}$ through a subformula closed set Σ over **A**. Then, for all formulas $\varphi \in \Sigma$, all states w in $\mathfrak{M}$ and any truth value $a \neq 0$ in **A**, we have that $\mathfrak{M}, w \Vdash_a \varphi \iff \mathfrak{M}_f, [w]_\Sigma^{\mathbf{A}} \Vdash_a \varphi$. Moreover, if **A** and Σ are both finite, then so is $\mathfrak{M}_f$.*

We show that **A**-valued filtrations exist by exhibiting the smallest and largest filtrations (in the sense of producing, respectively, the smallest and largest relations D_f and B_f in terms of the order of **A**) satisfying the definition. We also define a filtration which preserves a-transitivity of models (see below).

Applications: Many-valued modal logics with the FMP.   In this section, for simplicity, we restrict to **A**-frames $\mathfrak{F} = (W, D, B)$ where $D = B$ which we notate as $\mathfrak{F} = (W, R)$. Moreover, we will assume that **A** is a finite Heyting algebra. In [1] Bou et. al. axiomatize the logic $\Lambda(\mathsf{Fr}, \mathbf{A}^c)$ of the class of all **A**-frames. In analogue to the classical case, let $\mathbf{T}(\mathbf{A})$, $\mathbf{K4}(\mathbf{A})$ and $\mathbf{S4}(\mathbf{A})$ be the logics obtained by adding to to the system of Bou et. al. the axioms $\Box p \to p$ and $\Box p \to \Box \Box p$ individually and in combination. Let **gl** be the Löb formula $\Box(\Box p \to p) \to \Box p$ and let $\mathbf{GL}(\mathbf{A})$ be the logic obtained by adding **gl** and **4** to an axiomatization of $\Lambda(\mathsf{Fr}, \mathbf{A}^c)$.

An **A**-frame $\mathfrak{F} = (W, R)$ is a-reflexive if $Rwv \geq a$ for all $w, v \in W$; it is a-transitive if $a \leq (Rwv \wedge Rvu \to Rwu)$ for all $w, v, u \in W$. It follows that $\mathfrak{F}$ is 1 transitive iff $Rwv \wedge Rvu \leq Rwu$ for all $w, v, u \in W$. A non-0 path in $\mathfrak{F}$ is a finite or infinite sequence $w_0, w_1, \ldots$ such that $Rw_i w_{i+1} > 0$ for all $i \geq 0$.

Lemma 3. *The axioms $\Box p \to p$ and $\Box p \to \Box \Box p$ are canonical for 1-reflexivity and 1-transitivity, i.e., the canonical models (see [1]) of $\mathbf{T}(\mathbf{A})$, $\mathbf{K4}(\mathbf{A})$ are 1-reflexive and 1-transitive, respectively.*

Now we can obtain the following theorem by judicious application of filtrations.

Theorem 4. $\mathbf{T}(\mathbf{A})$ *and* $\mathbf{K4}(\mathbf{A})$ *are characterized by the classes of all finite 1-reflexive and 1-transitive **A**-frames, respectively.* $\mathbf{S4}(\mathbf{A})$ *is by the class of all finite 1-reflexive and 1-transitive **A**-frames.*

A more intricate argument, combining transitive and selective filtration, establishes the following analogue of the well-known classical result.

Theorem 5. *Let **A** be a finite Heyting chain. Then $\mathbf{GL}(\mathbf{A})$ is determined by the class of finite, 1-transitive **A**-frames with no infinite non-0 paths.*

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Bimodal Bilattice Logic

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Many-valued modal logics provide a natural formalisation of reasoning with modal notions such as knowledge or action in contexts where the two-valued classical picture is not sufficient. Such contexts typically involve reasoning with incomplete, inconsistent or graded information.

A prominent example of a (non-modal) many-valued logic designed to deal with incomplete and incosistent information is is the Dunn–Belnap four-valued logic [4, 2, 3]. Ginsberg [7] generalized the Dunn–Belnap four-valued matrix *FOUR* by introducing the notion of a *bilattice* and shows that bilattices emerge naturally in many computer science applications; see also [5, 6].

Formally, bilattices are sets equipped with two partial orders $\leq_t$ (the "truth order") and $\leq_i$ (the "information order") that both satisfy the lattice properties (plus other assumptions that need not be discussed now). Intuitively, $\leq_t$ orders members of a bilattice with respect to how *truthful* they are; $\leq_i$ orders them with respect to *how much information* they represent. For instance, in Belnap's four-valued matrix the value "true" is above the value "both" with respect to $\leq_t$ but below it with respect to $\leq_i$.

Arieli and Avron [1] study a (non-modal) logic based on bilattices using the full language $\{\wedge, \vee, t, f, \otimes, \oplus, \bot, \top, \neg, -, \supset\}$ containing constants for maximal $(\top, t)$ / minimal $(\bot, f)$ elements and suprema $(\vee, \oplus)$ / infima $(\wedge, \otimes)$ operators for both of the orderings, with two negations $(\neg, -)$ and an implication connective $(\supset)$.

Several modal extensions of Dunn–Belnap and Arieli–Avron have been studied recently [9, 8, 10]. These modal extensions add a modal operator $\Box$ to either the full Arieli–Avron language [8, 10] or to its fragment $\{\wedge, \vee, \neg, f, \supset\}$ [9]. The operator $\Box$ is interpreted in terms of the truth-order infimum (simplifying a bit, the value of $\Box\phi$ in world w of a Kripke model is the truth-order infimum of the values of ϕ in worlds w' accessible from w.)

However, a modal operator $\Box_i$ corresponding to the information-order infimum is a natural addition to consider. If worlds in a Kripke model are seen as "sources" of information, then the value of $\Box_i\phi$ at w is the *minimal information* about ϕ on which all the sources agree. If accessible worlds are seen as possible outcomes of some information-modifying operation (such as adding or removing information), then the value of $\Box_i\phi$ at w is the minimal information about ϕ that is guaranteed to be preserved by the operation. (This extension is briefly considered but not pursued in [8, 10]).

The present paper studies the bimodal bilattice logic arising from such an extension. It is well known that $\Box_i$ is expressible in any language extending $\{\wedge, \vee, \neg, \bot, \Box\}$; define $\Box_i\phi :=$ $(\bot \wedge \neg\Box\neg\phi) \vee \Box\phi$. We focus here on the case where $\bot$ is not available and extend the modal language used in [9] with $\Box_i$. For the sake of simplicity, we use Belnap's *FOUR* as our bilattice of truth values (the non-modal logic of arbitrary bilattices is identical to the the non-modal logic of *FOUR*, [1]).

Our main technical result is a sound and complete axiomatization. The axiomatization reflects the fact that $\Box_i\phi$ has a designated value (i.e. one of $\top, t$) iff $\Box\phi$ has a designated value; but $\Box_i$ is distinctive in the context of negation. More specifically, we add the following axioms to the non-modal base: $\Box\phi \equiv \Box_i\phi$, $\Box\neg\phi \equiv \neg\Box_i\phi$, $(\neg\Box\phi \supset f) \equiv \Box(\neg\phi \supset f)$, $\Box t$,

$(\Box\phi \wedge \Box\psi) \supset \Box(\phi \wedge \psi)$, together with the inference rule $\dfrac{\phi \supset \psi}{\Box\phi \supset \Box\psi}$.

Potential applications of the logic in knowledge representation and expressiveness of the language are discussed as well. The work done in this paper is preliminary – a version of the framework with many-valued accessibility is a topic for future research.

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Modular proof theory for axiomatic extensions and expansions of lattice logic

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Lattice logic is the $\{\wedge, \vee, \top, \bot\}$-fragment of classical propositional logic without distributivity. Lattice logic is captured by a basic Gentzen-style sequent calculus (cf. e.g. [18]), which we refer to as L0. Such a calculus has the usual rules of Identity (restricted to atomic formulas with empty contexts on both sides of the sequent), Cut (with empty contexts on both sides of the sequents) and the standard introduction rules for the logical connectives in *additive* form.[1] L0 is perfectly adequate as a proof calculus for lattice logic, when this logic is regarded in isolation. However, the main interest of lattice logic lays in it serving as base for a variety of logics, which are either its *axiomatic extensions* (e.g. the logics of modular and distributive bounded lattices and their variations [16]), or its proper *language-expansions* (e.g. the full Lambek calculus [17, 8], bilattice logic [2], orthologic [9], linear logic [15]). Hence, it is sensible to require of an adequate proof theory of lattice logic to be able to account in a modular way for these logics as well. A source of nonmodularity arises from the fact that L0 lacks structural rules. Indeed, the additive formulation of the introduction rules of L0 encodes the information which is stored in standard structural rules such as weakening, contraction, associativity, and exchange. Hence, one cannot use L0 as a base to capture logics aimed at 'negotiating' these rules, such as the Lambek calculus [17] and other substructural logics [8].

To remedy this, in [10] the first and the fourth author introduced two sequent calculi, which we refer here as L1 and L2. L1 is a sequent calculus that adopts the *visibility*[2] principle isolated by Sambin, Battilotti and Faggian in [19] to formulate a general strategy for cut elimination. L2 is a sequent calculus which enjoys the *display*[3] principle isolated by Belnap in [1]. *Properness* (i.e. closure under uniform substitution of all parametric parts in rules, see [20]) is the main interest and added value of L2 and allows for the smoothest Belnap-style proof of cut-elimination. The second attempt is motivated by and embeds in a more general theory—that of the so-called *proper multi-type calculi*, introduced in [13, 5, 6, 4] and further developed in [7, 3, 14, 11]—which creates a proof-theoretic environment designed on the basis of algebraic and order-theoretic insights (see [12]), which aims at encompassing in a uniform and modular way a very wide range of non-classical logics, spanning from dynamic epistemic logic, PDL, and inquisitive logic to lattice-based substructural (modal) logics. Proper multi-type calculi are a natural generalization of Belnap's display calculi [1] (later refined by Wansing's notion of proper display calculi [20]), the salient features of which they inherit. L1 and L2 have a structural language and the introduction rules for the logical connectives are formulated in *multiplicative* form.[4] This more general formulation of the introduction rules implies that

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[1]A logical rule is in additive form if each occurrence of non-active formulas in the conclusion occurs in each premise and conversely (in the literature such rules are also called context-sharing rules). Moreover, in the unary introduction rules for conjunction and disjunction only one immediate subformula of the principal formula appears as active formula in the premise. An introduction rule for the logical connectives is in multiplicative form if each occurrence of non-active formulas in the conclusion occurs in exactly one premise and conversely (in the literature such rules are also called context-splitting rules). Moreover, in the unary introduction rules for conjunction and disjunction both immediate subformulas of the principal formula appear as active formulas in the premise.

[2]A sequent calculus verifies the *visibility* property if both the auxiliary formulas and the principal formula of the introduction rules for the logical connectives occur in an *empty* context.

[3]A sequent calculus verifies the *display* property if each substructure can be isolated on exactly one side of the turnstile by means of structural rules. Notice that display property implies visibility, but not vice versa.

[4]The multiplicative form of the introduction rules is the most important aspect in which L1 departs from the calculus of [19]. Indeed, the introduction rules for conjunction and disjunction in [19] are in additive form.

the structural rules of weakening, exchange, associativity, and contraction are not anymore subsumed by the introduction rules. L1 and L2 are more uniform and modular compared to L0 in a precise sense. All these calculi block the derivation of the distributivity axiom, as well as of any other weaker form of distributivity. However, in the literature there are no instances of analytic sequent calculi in which axiomatic extension of lattice logic which are weaker than distributive lattice logic are captured using structural rules.

In this talk I will expand on an ongoing work on modular proof theory for axiomatic extensions and expansions of lattice logic. In particular, I will present a sequent calculus enjoying a weaker form of visibility that derives the modularity axiom but still blocks distributivity, thanks to a generalized form of the binary logical rules for conjunction and disjunction.

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Quantifiers on languages and codensity monads [*]

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The main content of this talk concerns recent joint work (see [4]) with Mai Gehrke and Daniela Petrişan on the understanding, at the level of recognisers, of the effect of applying a layer of various kinds of quantifiers in the context of logic on words.

Two approaches have been remarkably effective in the study of languages: the algebraic one, and the logical one. Whereas the former relies on the notions of recognition by a monoid and of syntactic monoid of a language, the latter is based on a semantic on finite words. Let us briefly recall these two approaches.

Consider a finite set A (the *alphabet*) and an A-language, i.e. a subset L of the monoid A^* free on A. We say that a monoid M *recognises* the language L provided there is a monoid morphism $\phi\colon A^* \to M$ and $P \subseteq M$ such that $\phi^{-1}(P) = L$. This condition is equivalent to the existence of a homomorphism $A^* \to M$ whose kernel saturates L. The maximal congruence $\sim_L$ on A^* saturating L is defined by $(x, y) \in \sim_L$ if $uxv \in L \Leftrightarrow uyv \in L$ for all $u, v \in A^*$. The quotient $A^*/\sim_L$ is called the *syntactic monoid of L*, and one can define a *regular* language to be one whose syntactic monoid is finite.

It turns out that, beyond the regular case, monoids do not provide a notion of recognition that is fine-grained enough to be useful. This led us to introduce in [3] the notion of a *Boolean space with an internal monoid* (*BM*, for short), which behaves well with respect to recognition in the non-regular setting. A *BM* is a pair (X, M) given by a Boolean space X (i.e, a compact and Hausdorff space that is zero-dimensional) along with a dense subset M carrying a monoid structure, such that $\forall m \in M$ the maps $\lambda_m, \rho_m \colon M \to M$ given by left and right multiplication by m, respectively, can be extended to continuous functions on X. An example is provided by the pair $(\beta(A^*), A^*)$, where $\beta(A^*)$ is the Stone-Čech compactification of the discrete set A^*. Now, define a morphism $(X, M) \to (Y, N)$ to be a continuous function $X \to Y$ whose restriction is a monoid morphism from M to N. Recalling the bijection $L \mapsto \widehat{L}$ between $\mathcal{P}(A^*)$ and the clopens of $\beta(A^*)$, we say that a *BM* (X, M) *recognises* the language L if there is a morphism $\phi\colon (\beta(A^*), A^*) \to (X, M)$ and a clopen subset $C \subseteq X$ such that $\phi^{-1}(C) = \widehat{L}$. This extends the classical definition of recognition in the regular case.

The second approach stems from the interpretation of a word $w \in A^*$, say of length n, as a relational structure on the set $\{1, \ldots, n\}$. These structures are equipped with (interpretations of) unary relations P_a, one for each $a \in A$, selecting the positions in the word w in which the letter a appears. Additional relations, such as the natural order on $\{1, \ldots, n\}$, are sometimes considered in specific situations. Every (first-order, or higher-order) sentence ψ in a language interpretable over words determines a language $L_\psi \subseteq A^*$ consisting of all those words satisfying ψ. However, if $\psi(x)$ is a formula containing a free first-order variable x, in order to be able to interpret the free variable we extend the alphabet to $(A \times \{0, 1\})^*$ and use the more compact notation $a_1 a_2' a_3' \cdots a_n$ for the word $(a_1, 0)(a_2, 1)(a_3, 1) \cdots (a_n, 0) \in (A \times \{0, 1\})^*$. The language

$L_{\psi(x)} \subseteq (A \times \{0, 1\})^*$ is then the collection of all the words in the extended alphabet, with *only one* marked position, in which the formula $\psi(x)$ is satisfied when the variable x points at that position. Finally, one can consider the quantified formula $\exists x.\psi(x)$ which yields the language over the alphabet A^* of all those words $a_1 \cdots a_n$ such that there exists $1 \leq i \leq n$ with $a_1 \cdots a_i' \cdots a_n \in L_{\psi(x)}$. There are other quantifiers of interest in language theory. An example is provided by *modular quantifiers*: a word w satisfies the sentence $\exists_{p \bmod q} x.\psi(x)$ if there are $p \bmod q$ positions in the word w in which the formula $\psi(x)$ is satisfied.

The question we pose, and answer, is the following: Suppose a language, defined by a formula $\psi(x)$, is recognised by a *BM* (X, M). If Q is some quantifier (e.g. a modular quantifier), how can we construct a *BM* recognising the language associated to the sentence $Qx.\psi(x)$? The question is motivated by open problems on the separation of Boolean circuit complexity classes, where classes of languages are characterised in terms of logic fragments.

The answer employs duality-theoretic and categorical tools. Several quantifiers of interest can be modelled using commutative semirings S (e.g. $S = \mathbb{Z}/q\mathbb{Z}$ for the modular quantifiers) or, from a categorical viewpoint, the free S-semimodule monad on **Set** (=the category of sets and functions). On the way to our answer, we prove that whenever an operation on languages — quantification being a particular case — can be modelled by a finitary *commutative* monad (in the sense of [6]) T on **Set**, then a recogniser for the languages obtained by applying the operation represented by T can be built by means of the *profinite monad* $\widehat{T}$ on the category of Boolean spaces and continuous functions. The profinite monad $\widehat{T}$ associated to T was first defined in [1], building on the ideas introduced in [2], and it is based on the notion of *codensity monad* of a functor which has its origins in the work of Kock in the 60's (see also [5]).

In the case of quantifiers modelled by a finite and commutative semiring S, that is when T is the free S-semimodule monad, we provide a concrete description of the Boolean space $\widehat{T}X$, for X any Boolean space, in terms of certain S-valued *measures* on X. If in addition the semiring S is idempotent (hence a semilattice), $\widehat{T}X$ can be equivalently described as the space of all continuous functions $X \to S$, where S is equipped with the topology of all downsets with respect to its semilattice order. We remark that, in the case $S = \mathbb{2}$ is the two-element Boolean algebra, $\widehat{T}$ is the Vietoris monad on Boolean spaces (already related to the existential quantifier in [3]) and we essentially recover the classical description of the Vietoris space in terms of functions into the Sierpiński space.

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Classical and intuitionistic relation algebras

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In their seminal 1951-52 papers [1] on Boolean algebras with operators (BAOs), Jónsson and Tarski showed that many varieties of BAOs, including the variety of relation algebras, are closed under canonical extensions, and that a relation algebra is complete and atomic with all atoms as functional elements if and only if it is the complex algebra of a generalized Brandt groupoid. The results about canonical extensions were extended to distributive lattices with operators by Gehrke and Jónsson in 1994, and to lattices with operators by Gehrke and Harding in 2001. Here we show results about relation algebras can also be generalized to certain distributive residuated lattices and involutive distributive residuated lattices, in some cases expanded by a Heyting implication. These varieties include (generalized) bunched implication algebras and weakening relation algebras, which have applications in computer science and algebraic logic.

A *relation algebra* $(A, \wedge, \vee, ', \top, \bot, \cdot, \breve{}, 1)$ is a Boolean algebra $(A, \wedge, \vee, ', \top, \bot)$ and a monoid $(A, \cdot, 1)$ such that $xy \leq z \iff x^\breve{} \cdot z' \leq y'$. An excellent introduction to relation algebras is in [2] and several results about them were extended to residuated Boolean monoids in [3].

A *cyclic involutive generalized bunched implication algebra* (or CyGBI-algebra for short) $(B, \wedge, \vee, \rightarrow, \top, \bot, \cdot, 1, \sim)$ is a Heyting algebra $(B, \wedge, \vee, \rightarrow, \top, \bot)$ and a monoid $(B, \cdot, 1)$ with a linear cyclic negation $\sim$ that satisfies $\sim\sim x = x$ and $x \leq \sim y \iff xy \leq \sim 1$. So they are involutive residuated lattices expanded with a Heyting implication, and both relation algebras and CyGBI-algebras can be defined by identities. A relation algebra is a CyGBI-algebra if we define $x \rightarrow y = x' \vee y$ and $\sim x = x'^\breve{}$. In a CyGBI-algebra define $x^\breve{} = \sim\neg x$ where $\neg x = x \rightarrow \bot$, then it is a relation algebra if it satisfies the identities $\neg\neg x = x$ and $(xy)^\breve{} = y^\breve{} x^\breve{}$.

We define algebras of binary relations that are cyclic involutive GBI-algebras and generalize representable relation algebras: Let $\mathbf{P} = (P, \sqsubseteq)$ be a partially ordered set, $Q \subseteq P^2$ an equivalence relation that contains $\sqsubseteq$, and define the set of *weakening relations* on $\mathbf{P}$ by $\mathrm{Wk}(\mathbf{P}, Q) = \{\sqsubseteq \circ R \circ \sqsubseteq : R \subseteq Q\}$. Note that this set is closed under intersection $\cap$, union $\cup$ and composition $\circ$, but not under complementation $R' = Q - R$ or converse $R^\breve{}$.

Weakening relations are the natural analogue of binary relations when the category **Set** of sets and functions is replaced by the category **Pos** of partially ordered sets and order-preserving functions. Since sets can be considered as discrete posets (i.e. ordered by the identity relation), **Pos** contains **Set** as a full subcategory, which implies that weakening relations are a substantial generalization of binary relations. They have applications in sequent calculi, proximity lattices/spaces, order-enriched categories, cartesian bicategories, bi-intuitionistic modal logic, mathematical morphology and program semantics, e.g. via separation logic.

Theorem 1. *Let $\mathbf{P} = (P, \sqsubseteq)$ be a poset, Q an equivalence relation that contains $\sqsubseteq$, and for $R, S \in \mathrm{Wk}(\mathbf{P}, Q)$ define $\top = Q$, $\bot = \emptyset$, $1 = \sqsubseteq$, $\sim R = R^{\breve{}'}$ and $R \rightarrow S = (\sqsupseteq \circ (R \cap S') \circ \sqsupseteq)'$ where $S' = Q - S$. Then $\mathbf{Wk}(\mathbf{P}, Q) = (\mathrm{Wk}(\mathbf{P}, Q), \cap, \cup, \rightarrow, \top, \bot, \circ, 1, \sim)$ is a CyGBI-algebra.*

Algebras of the form $\mathbf{Wk}(\mathbf{P}, Q)$ are called representable weakening relation algebras, and if $Q = P \times P$, then we write $\mathbf{Wk}(\mathbf{P})$ and call this algebra the *full weakening relation algebra on* $\mathbf{P}$. If $\mathbf{P}$ is a discrete poset then $\mathbf{Wk}(\mathbf{P})$ is the full representable set relation algebra on the set P, so algebras of weakening relations play a role similar to representable relation algebras. We define

the class RwRA of *representable weakening relation algebras* as all algebras that are embedded in a weakening relation algebra $\mathbf{Wk}(\mathbf{P}, Q)$ for some poset $\mathbf{P}$ and equivalence relation Q that contains $\sqsubseteq$. In fact the variety RRA of representable relation algebras is finitely axiomatized over RwRA.

Theorem 2.   *1.* RwRA *is a discriminator variety closed under canonical extensions.*

     2. RRA *is the subvariety of* RwRA *defined by* $\neg\neg x = x$.

     3. The class RwRA *is not finitely axiomatizable relative to the variety of all CyGBI-algebras.*

A *groupoid* is defined as a partial algebra $\mathbf{G} = (G, \circ, ^{-1})$ such that $\circ$ is a partial binary operation and $^{-1}$ is a (total) unary operation on G that satisfy the following axioms:

1. $(x \circ y) \circ z \in G$ or $x \circ (y \circ z) \in G \implies (x \circ y) \circ z = x \circ (y \circ z)$,
2. $x \circ y \in G \iff x^{-1} \circ x = y \circ y^{-1}$,
3. $x \circ x^{-1} \circ x = x$ and $(x^{-1})^{-1} = x$.

Typical examples of groupoids are disjoint unions of groups and the *pair-groupoid* $(X \times X, \circ, ^{\smile})$, where $(x, y)^{\smile} = (y, x)$ and $(x, y) \circ (z, w) = (x, w)$ if $y = z$ (undefined otherwise). A *partially-ordered groupoid* $(G, \leq, \circ, ^{-1})$, or *po-groupoid* for short, is a groupoid $(G, \circ, ^{-1})$ such that $(G, \leq)$ is a poset and

4. $x \leq y$ and $x \circ z, y \circ z \in G \implies x \circ z \leq y \circ z$,
5. $x \leq y \implies y^{-1} \leq x^{-1}$,
6. $x \circ y \leq z \circ z^{-1} \implies x \leq y^{-1}$.

If $\mathbf{P} = (P, \sqsubseteq)$ is a poset with dual poset $\mathbf{P}^{\partial} = (P, \sqsupseteq)$ then $\mathbf{P} \times \mathbf{P}^{\partial} = (P \times P, \leq, \circ, ^{\smile})$ is a po-groupoid, called a *po-pair-groupoid*, with $(a, b) \leq (c, d) \iff a \sqsubseteq c$ and $b \sqsupseteq d$. The set of order-ideals of $\mathbf{P}$ is denoted by $\mathcal{O}(\mathbf{P})$.

Theorem 3. *Let* $\mathbf{G} = (G, \leq, \circ, ^{-1})$ *be a po-groupoid. Then* $(\mathcal{O}(\mathbf{G}), \cap, \cup, \rightarrow, \top, \emptyset, \cdot, 1, \sim)$ *is a CyGBI-algebra, where* $X \rightarrow Y = \mathcal{O}(\mathbf{G}) - \uparrow(X - Y)$, $X \cdot Y = \downarrow\{x \cdot y : x \in X, y \in Y\}$, $1 = \downarrow\{x \circ x^{-1} : x \in G\}$ *and* $\sim X = \mathcal{O}(\mathbf{G}) - \{x^{-1} : x \in X\}$.

For example, for a poset $\mathbf{P} = (P, \sqsubseteq)$ the weakening relation algebra $\mathbf{Wk}(\mathbf{P})$ is obtained from the po-pair-groupoid $\mathbf{G} = \mathbf{P} \times \mathbf{P}^{\partial}$, and for an equivalence relation $Q \subseteq P^2$, $\mathbf{Wk}(\mathbf{P}, Q)$ is obtained from the sub-po-groupoid $(Q, \leq, \circ, ^{\smile})$. If one takes the 2-element chain $\mathbf{P} = \mathbf{C}_2 = (\{0, 1\}, \sqsubseteq)$ with the usual order $0 \sqsubseteq 1$, then $P^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ and

$$\mathrm{Wk}(\mathbf{C}_2) = \{\emptyset, \{(0, 1)\}, \{(0, 0), (0, 1)\}, \{(0, 1), (1, 1)\}, \{(0, 0), (0, 1), (1, 1)\}, P^2\}.$$

Theorem 4. *For an n-element chain* $\mathbf{C}_n$ *the algebra* $\mathbf{Wk}(\mathbf{C}_n)$ *has cardinality* $\binom{2n}{n}$.

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Proper multi-type display calculi for classical and intuitionistic inquisitive logic[*]

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Introduction. In this work, we define proper multi-type display calculi for both classical and intuitionsitic inquisitive logic, which enjoy Belnap-style cut-elimination and subformula property.

Inquisitive logic is the logic of inquisitive semantics, a semantic framework developed by Ciardelli, Groenendijk and Roelofsen [5, 1] which captures both assertions and questions in natural language. A distinguishing feature of inquisitive logic is that formulas are evaluated on *information states*, i.e., sets of possible worlds, rather than on single possible worlds. Inquisitive logic defines a *support* relation between states and formulas, the intended understanding of which is that in uttering a sentence, a speaker proposes to enhance the current common ground to one that supports the sentence. This semantics is also known as *team semantics*, which was introduced by Hodges [6, 7] in the context of *dependence logic* [8]. Recent work [2] generalised the original classical logic-based framework of inquisitive logic [1], and introduced inquisitive logic on the basis of intuitionistic propositional logic.

The Hilbert-style presentations of both classical and intuitionistic inquisitive logic are not closed under uniform substitution, and some axioms are sound only for a certain subclass of formulas, called *standard formulas*. This and other features make the quest for analytic calculi for the logics not straightforward. A first step in this direction was taken in [3], where a multi-type sequent calculus was developed for classical inquisitive logic. However, this calculus does not enjoy display property. In this work, we generalise the methodology of [3] and propose a proper multi-type display calculi for both classical and intuitionistic inquisitive logic. We develop a certain algebraic and order-theoretic analysis of the support semantics, which provides the guidelines for the design of a multi-type environment accounting for two domains of interpretation, for standard and for general formulas, as well as for their interaction. This multi-type environment in its turn provides the semantic environment for the multi-type calculi for both classical and intuitionistic inquisitive logic we propose in this work.

Classical and intuitionistic inquisitive logic. The following grammar defines the language of both classical (Clnq) and intuitionistic inquisitive logic (Ilnq) presented as a language of two types:

$$\text{Standard} \ni \alpha ::= p \mid 0 \mid \alpha \sqcap \alpha \mid \alpha \rightarrowtail \alpha \qquad \text{General} \ni A ::= \,\downarrow\!\alpha \mid A \wedge A \mid A \vee A \mid A \rightarrow A$$

Standard formulas of Clnq and Ilnq adopt the standard semantics for classical and intuitionistic propositional logic, respectively. General type formulas are evaluated on *information states*, which *sets* of classical valuations for Clnq, or *sets* of possible worlds in intuitionistic Kripke models $M = (W, R, V)$ for Ilnq. The *support* relation $S \models \phi$ of a general type formula ϕ in either logic on a state S is defined as:

$$
\begin{array}{llll}
S \models \,\downarrow\!\alpha & \text{iff} & v \models \alpha \text{ for all } v \in S & S \models \phi \vee \psi \quad \text{iff} \quad S \models \phi \text{ or } S \models \psi \\
S \models \phi \wedge \psi & \text{iff} & S \models \phi \text{ and } S \models \psi & S \models \phi \rightarrow \psi \quad \text{iff} \quad \text{for any } T \leq S, \text{ if } T \models \phi, \text{ then } T \models \psi
\end{array}
$$

where the extension relation $\leq$ between information states is defined as $T \leq S$ iff $T \subseteq S$ in the Clnq case, and as $T \leq S$ iff $T \subseteq R[S]$ in the Ilnq case. Ilnq and Clnq are complete with respect to the systems below:

System of Ilnq: Rule: Modus Ponens for both types

Axioms:
- Axiom schemata of (disjunction-free) intuitionistic logic (**IPC**) for Standard-formulas
- Axiom schemata of **IPC** for General-formulas
- $(\downarrow\!\alpha \rightarrow (A \vee B)) \rightarrow (\downarrow\!\alpha \rightarrow A) \vee (\downarrow\!\alpha \rightarrow B)$ (Split axiom)

System of Clnq: The system of Ilnq extended with two extra axioms: • $\sim\!\sim \alpha \rightarrowtail \alpha$ • $\neg\neg\!\downarrow\!\alpha \rightarrow \,\downarrow\!\alpha$

Order-theoretic analysis. In the setting of Clnq, the base logic, namely classical propositional logic, gives rise to a Boolean algebra $\mathbb{B} = (\mathcal{P}(2^V), \cap, \cup, (\cdot)^c, \varnothing, 2^V)$. The set $\mathcal{P}^{\downarrow}(\mathcal{P}(2^V))$ of downward closed collections of states forms a perfect Heyting algebra $\mathbb{A} := (\mathcal{P}^{\downarrow}(\mathcal{P}(2^V)), \cap, \cup, \Rightarrow, \varnothing, \mathcal{P}(2^V))$ as the complex algebra of the relational structure $(\mathcal{P}(2^V), \subseteq)$. The following mappings between the two algebras

$$\mathrm{f}^* : \mathbb{B} \to \mathbb{A}\,;\, S \mapsto \{\{v\} \mid v \in S\} \cup \{\varnothing\} \qquad \mathrm{f} : \mathbb{A} \to \mathbb{B}\,;\, S \mapsto \{T \mid T \subseteq S\} \qquad \downarrow : \mathbb{B} \to \mathbb{A}\,;\, S \mapsto \{T \mid T \subseteq S\},$$

turn out to be adjoints to one another: $\mathrm{f}^* \dashv \mathrm{f} \dashv \downarrow$, since $\mathrm{f}\mathcal{S} \subseteq S$ iff $\mathcal{S} \subseteq \downarrow S$ and $\mathrm{f}^* S \subseteq \mathcal{S}$ iff $S \subseteq \mathrm{f}\mathcal{S}$. Similar observations can be made for Ilnq, and similar mappings can be found between a Heyting algebra for the base logic, intuitionistic logic with single-world semantics, and a Heyting algebra on the higher level.

Proper multi-type display calculi for Clnq and Ilnq. Building on the order-theoretic analysis, we introduce the corresponding structural operators F^*, F and $\Downarrow$ for the mappings f^*, f and $\downarrow$. The structural languages for the standard type and general type and their interpretations are presented as follows:

Standard $\Gamma ::= \alpha \mid \Phi \mid \Gamma\,,\,\Gamma \mid \Gamma \sqsupset \Gamma \mid \mathrm{F}X$ General $X ::= A \mid \Downarrow\Gamma \mid \mathrm{F}^*\Gamma \mid X\,;\,X \mid X > X$

Structural symbols	Φ		,		$\sqsupset$		;	>	F^*		F		$\Downarrow$
Operational symbols	(1)	0	$\sqcap$	$(\sqcup)$	$(\mapsto)$	$\to$	$\wedge$	$\vee$	$(\rightarrowtail)$	$\to$	(f^*)		(f)	(f)		$\downarrow$

Our calculi for Clnq and Ilnq are built on the basis of the one introduced [3], but there are major differences in the following structural rules that characterise the interaction between the two types:

$$\frac{\mathrm{F}^*\Gamma \vdash \Delta}{\Gamma \vdash \mathrm{F}\Delta}\ \mathrm{f\ adj} \qquad \frac{\mathrm{F}X \vdash \Gamma}{X \vdash \Downarrow\Gamma}\ \mathrm{d\ adj} \qquad \frac{X \vdash \Downarrow\mathrm{F}Y}{X \vdash Y}\ \mathrm{d\text{-}f\ elim} \qquad \frac{\Gamma \vdash \Delta}{\mathrm{F}^*\Gamma \vdash \Downarrow\Delta}\ \mathrm{bal} \qquad \frac{X \vdash Y}{\mathrm{F}X \vdash \mathrm{F}Y}\ \mathrm{f\ mon}$$

$$\frac{X \vdash \Downarrow(\Gamma \sqsupset \Delta)}{X \vdash \mathrm{F}^*\Gamma > \Downarrow\Delta}\ \mathrm{d\ dis} \qquad \frac{\mathrm{F}X\,,\,\mathrm{F}Y \vdash Z}{\mathrm{F}(X\,;\,Y) \vdash Z}\ \mathrm{f\ dis} \qquad \frac{X \vdash \mathrm{F}^*\Gamma > (Y\,;\,Z) \qquad X \vdash \mathrm{F}^*\Gamma > (Y\,;\,Z)}{X \vdash (\mathrm{F}^*\Gamma > Y)\,;\,(\mathrm{F}^*\Gamma > Z)}\ \mathrm{Split}$$

We adopt a standard display calculus for standard formulas of Ilnq, and we add the following classical Grishin rule for standard formulas of Clnq:

$$\frac{\Pi \vdash \Gamma \sqsupset (\Delta\,,\,\Sigma)}{\Pi \vdash (\Gamma \sqsupset \Delta)\,,\,\Sigma}\ \mathrm{CG}$$

The completeness of the calculi is proved by deriving the axioms and rules of the Hilbert systems. In particular, the split axiom in both logics is derived by applying the Split rule, and the double negation law for Clnq is derived by applying the Grishin rule for classical standard formulas. The proposed calculi are *proper* multi-type display calculi, a strict and particularly well-behaved subclass of multi-type sequent calculi, therefore cut-elimination and subformula property follow from the general result in [4].

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Arithmetic interpretation of the monadic fragment of intuitionistic predicate logic and Casari's formula

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By a celebrated theorem of Solovay, the Gödel-Löb logic GL is the modal logic of the provability predicate of Peano Arithmetic PA. This entails arithmetic interpretation of the intuitionistic propositional calculus IPC as was shown by Goldblatt, by Boolos, and by Kuznetsov and Muravitsky in the late 1970's and early 1980's. To see this, first use the Gödel translation t to embed IPC into the modal logic Grz, and then use the splitting translation sp—that maps $\Box\varphi$ to $\varphi \wedge \Box\varphi$—to embed Grz into GL as in the following diagram.

$$\text{IPC} \xrightarrow{\quad t \quad} \text{Grz} \xrightarrow{\quad sp \quad} \text{GL}$$

$$\text{IPC} \vdash \varphi \quad \text{iff} \quad \text{Grz} \vdash t(\varphi) \quad \text{iff} \quad \text{GL} \vdash sp(t(\varphi))$$

Finally, use Solovay's theorem to interpret GL into PA. The aim of this talk is to lift the above correspondences to monadic extensions of the logics in question completing the work of Esakia [2, 3]. To motivate the exact statement, we recollect some obstacles one encounters when trying to extend the above correspondences to the predicate setting. Let QIPC, QGrz, and QGL be the full predicate extensions of IPC, Grz, and GL, respectively. As was shown by Montagna [9], the analogue of Solovay's theorem is no longer true for QGL. Regarding the remaining correspondences, the situation seems at least severely more complicated than in the propositional case. While it is a well-known result of Kripke [8] that QIPC is complete with respect to Kripke frames, neither QGL nor QGrz is complete with respect to Kripke frames (see [9] and [5]). So the standard proofs for the propositional case do not extend to the predicate setting since they make use of Kripke semantics for IPC, Grz, and GL, respectively.

Unlike the full predicate logics, their one-variable fragments often behave much nicer. We will refer to them as monadic fragments. Bull [1] showed that the intuitionistic bi-modal logic MIPC axiomatizes the monadic fragment of QIPC (by interpreting $\Box$ and $\Diamond$ as the universal and existential quantifiers, respectively). Esakia [2] introduced the monadic fragments MGL and MGrz of QGL and QGrz, respectively, and conjectured that—in contrast to the full predicate case—Solovay's theorem extends to MGL. This conjecture was verified by Japaridze [6, 7].

The (extended) Gödel translation embeds MIPC into MGrz. However, the (extended) splitting translation fails to embed MGrz into MGL. To remedy this, Esakia adopted Casari's formula Cas—a modified version of the rule of universal quantification—to the monadic setting.

$$(\text{Cas}) \qquad \forall x[(p(x) \to \forall x p(x)) \to \forall x p(x)] \to \forall x p(x)$$

Let MCas be the monadic version of Casari's formula and let

$$\text{M}^+\text{IPC} = \text{MIPC} + \text{MCas} \quad \text{and} \quad \text{M}^+\text{Grz} = \text{MGrz} + t(\text{MCas}).$$

Esakia anticipated that the desired correspondence can be lifted to M$^+$IPC, M$^+$Grz and, MGL (note that MGL $\vdash$ sp(t(MCas)), so M$^+$GL = MGL):

$$\mathsf{M^+IPC} \xrightarrow{\quad \mathsf{t} \quad} \mathsf{M^+Grz} \xrightarrow{\quad \mathsf{sp} \quad} \mathsf{MGL}$$

$$\mathsf{M^+IPC} \vdash \varphi \qquad \text{iff} \qquad \mathsf{M^+Grz} \vdash \mathsf{t}(\varphi) \qquad \text{iff} \qquad \mathsf{MGL} \vdash \mathsf{sp}(\mathsf{t}(\varphi))$$

The goal of this talk is to verify this. Our main technical contribution consists in proving the finite model property (fmp) for the logics $\mathsf{M^+IPC}$ and $\mathsf{M^+Grz}$. We prove this by carefully modifying the selective filtration method for MIPC as presented in [4, Section 10.3].

Theorem 1. *The logics* $\mathsf{M^+IPC}$ *and* $\mathsf{M^+Grz}$ *have the fmp.*

Using that finite $\mathsf{M^+IPC}$-frames coincide with finite $\mathsf{M^+Grz}$-frames, we can now show:

Corollary 2. *The Gödel translation embeds* $\mathsf{M^+IPC}$ *into* $\mathsf{Q^+Grz}$, *and the splitting translation embeds* $\mathsf{M^+Grz}$ *into* MGL.

Using Theorem 1, we can also draw the connection to the full predicate case. Let

$$\mathsf{Q^+IPC} = \mathsf{QIPC} + \mathsf{Cas} \quad \text{and} \quad \mathsf{Q^+Grz} = \mathsf{QGrz} + \mathsf{t}(\mathsf{Cas}).$$

Using a semantic criterion from [10], we derive:

Corollary 3. $\mathsf{M^+IPC}$ *is the monadic fragment of* $\mathsf{Q^+IPC}$ *and* $\mathsf{M^+Grz}$ *is the monadic fragment of* $\mathsf{Q^+Grz}$.

Recall that by Japaridze's results, MGL is arithmetically complete. We therefore obtain arithmetic interpretation of the one-variable fragment of $\mathsf{Q^+IPC}$.

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Algebraic generalizations of completeness and canonicity

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Dedicated to the memory of Bjarni Jónsson

In their landmark 1951 work, Jónsson and Tarski identified defining properties of boolean algebras with operators (BAOs) dual to relational structures (which we know today as *Kripke frames*) and showed that every BAO $\mathfrak{A}$ can be embedded into such a dual suitably constructed from $\mathfrak{A}$, which they called the *perfect* extension and which we know today as the *canonical* extension of $\mathfrak{A}$. Furthermore, they isolated a class of equations preserved by this process, thus pioneering a line of work which much later came to include results such as the Sahlqvist theorem and its generalizations to, e.g., *inductive (in-)equalities*.

Let us briefly recall these defining properties of duals of relational structures, restricting attention to modal algebras (MAs, i.e., BAOs with a single unary $\diamond$) for simplicity: they are lattice-$\mathcal{C}$omplete, $\mathcal{A}$tomic (thus also atomistic, being boolean algebras) and completely additi$\mathcal{V}$e, i.e., for any set X of elements, if $\bigvee X$ exists, then $\bigvee\{\diamond x \mid x \in X\}$ exists and

$$\diamond \bigvee X = \bigvee \{\diamond x \mid x \in X\}.$$

Hence, it is natural to call such algebras $\mathcal{CAV}$-BAOs and write $\mathcal{CAV}$ to denote this class. It is also natural to use similar conventions for classes of algebras obtained by dropping some of these conditions, e.g., $\mathcal{CA}$ or $\mathcal{CV}$. Some of these classes are dual disguises of more general semantics of modal logic, e.g., $\mathcal{CA}$-BAOs are dually equivalent to neighbourhood frames (Došen 1989), thus also providing an algebraic framework for *coalgebraic* semantics; $\mathcal{CV}$-BAOs, as shown recently by Holliday, allow a dual representation in terms of *possibility semantics*; and $\mathcal{AV}$-BAOs are dual incarnations of *discrete general frames*.

For every variety of BAOs V defined by equations satisfying the conditions of Jónsson and Tarski, or perhaps by in-equalities studied in Jónsson's 1994 work, or by Sahlqvist/inductive (in-)equalities, the following meta-level "equation" holds:

$$V = \mathbb{S}(V \cap \mathcal{CAV}), \tag{1}$$

i.e., every $\mathfrak{A} \in V$ is (an isomorphic copy of) a subalgebra of a $\mathcal{CAV}$-BAO from the same variety. Several authors, like Goldblatt, call this property being *complex*; in our setting, to be more precise, we should speak of being $\mathcal{CAV}$-*complex*. As shown by Wolter in the 1990's, this is a proper generalization of canonicity. In other words, there is a variety whose defining equations are not preserved in general by canonical (perfect) extensions, yet satisfying (1); furthermore, this variety happens to correspond to a very natural tense logic. Wolter has also shown that $\mathcal{CAV}$-complexity is the algebraic counterpart of two distinct notions of modal completeness: strong *global* completeness and strong *local* completeness, corresponding to the two natural notions of modal consequence.

What happens when $\mathcal{CAV}$ in (1) is replaced by a broader class of algebras? First of all, note that there is a natural generalization of canonicity, proposed by Chellas 1980. This notion allowed Surendonk (2001) to prove that some flagship examples of varieties failing (1) are, e.g., $\mathcal{CA}$-complex. But, in general, for many non-canonical varieties even $\mathcal{C}$-complexity (i.e., closure under completions) is too much too ask. Furthermore, while the Wolterian correspondence between $\mathcal{X}$-complexity and strong *global* $\mathcal{X}$-completeness is quite robust (to wit, it survives

whenever $\mathcal{X}$ is closed under products), strong *local* $\mathcal{X}$-completeness can be a weaker property when $\mathcal{X} \not\subseteq \mathcal{AV}$. Results of Shehtman imply strong local $\mathcal{CA}$-completeness of many logics which are not closed under completions.

Finally, an obvious corollary of canonicity or $\mathcal{CAV}$-complexity of V is *weak* Kripke completeness, i.e., completeness for *theoremhood*, i.e., the satisfaction of the following meta-level "equation":

$$V = \mathbb{HSP}(V \cap \mathcal{CAV}). \tag{2}$$

Given that, on the one hand, a) even (1) itself is a weaker property than canonicity and (2) is still a *much* weaker property than (1) and that on the other hand, b) weak completeness can be proved by more constructive methods, which do not involve the Axiom of Choice and yet establish a strong property (fmp) for possibly non-canonical logics (cf., e.g., Fine 1975, Moss 2007 or Bezhanishvili and Ghilardi 2014), there is some irony in the fact that canonicity appears in many presentations of modal logic mostly *en route* to weak completeness. This state of affairs does not seem to do full justice to either notion. Still, weak completeness is quite often *the* notion of completeness of interest from modal logicians' point of view.[1] An obvious question is thus, again, what happens when $\mathcal{CAV}$ is replaced in (2) by other classes of BAOs? Note, for example, that weak $\mathcal{AV}$-completeness, strong $\mathcal{AV}$-completeness and $\mathcal{AV}$-complexity coincide, so while we can expect numerous negative results, there are some unexpected positive ones too.

More than a decade ago, I attempted to clarify the picture during my PhD studies (Litak 2004, 2005, 2008), unifying, expanding, and building on earlier results by Thomason, Fine, Gerson, van Benthem, Blok, Chagrova, Chagrov, Wolter, Zakharyaschev, Venema and other researchers. As it turns out, every possible combination of $\mathcal{C}$, $\mathcal{A}$, $\mathcal{V}$ and related properties allows to produce examples of logics/varieties for which completeness fails in a different way. Moreover, negative results concerning Kripke completeness, such as the Blok Dichotomy (sometimes also called the *Blok Alternative*), generalize to these weaker completeness notions. The only major piece of the puzzle missing was the status of $\mathcal{V}$-completeness—and I only managed to solve this in a recent collaboration with Holliday, using a first-order formulation of complete additivity inspired by his work on possibility semantics (some additional insights on this issue have been obtained by Andréka, Gyenis and Németi and more recently by van Benthem). We were surprised how natural some of our counterexamples turned out to be.

Where do we go from here? Even as far as weak completeness of modal logics is concerned, there are numerous unanswered questions like availability of broader completeness results in smaller lattices of logics (are all extensions of K4 $\mathcal{AV}$-complete, for example?) or the status of the Blok Dichotomy for $\mathcal{A}$-completeness. Our understanding of the hierarchy of notions refining strong completeness and canonicity seems even more sketchy—and further study could yield dividends for coalgebraic semantics and possibility semantics (which, as observed by Holliday, can be used to present a constructive perspective on canonical extensions). But our ignorance in these matters as far as other non-classical logics are concerned is most striking. We have some isolated results: we know, for example, that MV-algebras are not only non-canonical (Gehrke and Priestley 2002), but fail to be closed under completions (Gehrke and Jónsson 2004) and the same applies to many other varieties of GBL-algebras (Kowalski and Litak 2008). Thanks to Shehtman 1977, we also know that there are Kripke-incomplete si-logics, even uncountably many ones (Litak 2002), but this is the border of *hic sunt leones* area: Kuznetsov's earlier question about the existence of topologically incomplete si-logics remains unanswered until today. And for substructural logics in general, not much more seems to be known. Where will the door opened by Jónsson and Tarski in 1951 finally lead us?

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[1] Surely enough, things looks different when a logic is taken to be a *consequence relation* rather than a *set of theorems*.

The periodic sequence property

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In 1984, Wim Ruitenburg [19] published a surprising result[1] about the intuitionistic propositional calculus (IPC). It does not seem well-known: one of the few researchers making extensive use of it was the late Sergey Mardaev [9–13]; apart from it, some recent references quoting Ruitenburg's paper include Ghilardi et al. [6] or Humberstone's monograph [7]. Moreover, most of these references use it in the context of definability (eliminability) of fixpoints, where it is just one of possible lines of attack (the other being via uniform interpolation [16]; see [6] for a discussion). The property established by Ruitenburg deserves more attention though: to begin with, it turns out to be a natural generalization of *local finiteness*.

Consider a propositional formula A. Fix a propositional variable p, which can be thought of as representing the context hole or the argument of A taken as a polynomial (other propositional variables being additional constants). Given any other formula B, write $A(B)$ for the result of substituting B for p. Also, write $A \equiv_L B$ for $\vdash_L A \leftrightarrow B$. Now define the obvious iterated substitution operation $A^0(p) := p, A^{n+1}(p) := A(A^n(p))$. Such a sequence turns almost immediately into a cycle modulo $\equiv_{\mathsf{CPC}}$:

Lemma 1 ([19], Lemma 1.1). *For any A, $\quad A(p) \equiv_{\mathsf{CPC}} A^3(p)$.*

The above observation can be reformulated as asserting that CPC has *uniformly globally periodic sequences* (ugps). A logic L has this property if there exist b, c s.t. for any formula A, $A^b(p) \equiv_L A^{b+c}(p)$. However, ugps has still a rather strong logical form: two existential quantifiers preceding an universal one. Hence one can consider changing the order of quantifiers to weaken the property:

(eventually) periodic sequences:

	globally				locally			
uniformly	$\exists b.$	$\exists c.$	$\forall A.$	$A^b(p) \equiv_L A^{b+c}(p)$	$\exists c.$	$\forall A.$	$\exists b.$	$A^b(p) \equiv_L A^{b+c}(p)$
parametrically	$\exists b.$	$\forall A.$	$\exists c.$	$A^b(p) \equiv_L A^{b+c}(p)$	$\forall A.$	$\exists b.$	$\exists c.$	$A^b(p) \equiv_L A^{b+c}(p)$

So, do standard non-classical propositional calculi, IPC in particular, have at least *plps* (*parametrically locally periodic sequences*)?[2] To begin with, we have an obvious observation:

Lemma 2. *Any locally finite logic has plps.*

It is, however, well-known that IPC is not locally finite: even in one propositional variable, there are infinitely many nonequivalent formulas. And one can show that (uniformly or parametrically) globally periodic sequences would be too much to expect, at least when formulas are allowed to contain other variables than p itself [19, §2]. But we do have

Theorem 3 ([19], Theorem 1.9). IPC *has the ulps property: for any A, there exists b s.t. $A^b(p) \equiv_{\mathsf{IPC}} A^{b+2}(p)$. Moreover, b is linear in the size of A.*

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[1] I would like to thank Albert Visser for attracting my attention to this work and for his comments on this abstract.

[2] Ruitenburg himself was using the term *finite order*.

In fact, Ruitenburg's theorem is effective: the proof provides an algorithm to compute b in question.[3] Moreover, as the periodic sequence property (in all its incarnations) transfers from sublogics to extensions in the same signature (just like local finiteness and unlike uniform interpolation), we also get that all superintuitionistic logics (*si-logics*) have ulps. This shows that unlike local finiteness, ulps does *not* guarantee the fmp, or even Kripke completeness.

As it turns out, however, finding other natural examples of logic enjoying plps without local finiteness is a very challenging task. First let us consider intuitionistic or classical normal modal logics (with $\Box$ only), with superscript $\cdot^{\sf cl}$ denoting the $\sf CPC$ propositional base:

Theorem 4. *A normal extension of* $\sf K4^{cl}$ *has plps iff it is locally finite.*

Corollary 5. *All extensions of* $\sf K_\Box^{int}$ *contained in either* $\sf S4Grz.3^{cl}$ (*including, for example,* $\sf K^{cl}$, $\sf K4^{cl}$, $\sf S4^{cl}$, $\sf T^{cl}$, $\sf K4_\Box^{int}$, $\sf T_\Box^{int}$, $\sf S4_\Box^{int}$, $\sf S4Grz.3_\Box^{int}$ *or* $\sf S4Grz.3_\Box^{int}$) *or* $\sf GL.3^{cl}$ (*including, for example,* $\sf GL^{cl}$, $\sf GL_\Box^{int}$ *or* $\sf GL.3_\Box^{int}$) *fail to have locally periodic sequences.*[4]

Some intuitionistic modal logics of computational interest have "degenerate" classical counterparts and hence Corollary 5 cannot be used to disprove they have periodic sequences. This includes $\sf S_\Box^{int} := K_\Box^{int} \oplus A \to \Box A$, i.e., the Curry-Howard logic of *applicative functors*, also known as *idioms* [14]. Its classical counterpart $\sf S^{cl}$ and all its two consistent proper extensions are finite logics enjoying ulps. In contrast, not only does $\sf S_\Box^{int}$ have uncountably many propositional extensions, but the failure of plps remains a common phenomenon among them:

Theorem 6. *No sublogic of* $\sf KM.3_\Box^{int}$, *also denoted as* $\sf KM_{lin}$ [4] *has parametrically locally periodic sequences; this in particular applies to* $\sf SL.3_\Box^{int} := S_\Box^{int} \oplus GL.3_\Box^{int}$, $\sf SL_\Box^{int} := S_\Box^{int} \oplus GL_\Box^{int}$ *or* $\sf S_\Box^{int}$.

To contrast this with Theorem 4, note that $\sf KM.3_\Box^{int}$, the propositional fragment of the logic of the Mitchell-Bénabou logic of the *topos of trees* [2, 4, 8], is prefinite (*pretabular*). Turning to substructural logics:

Theorem 7. *The product logic* Π, *the infinite valued Łukasiewicz logic* Ł_∞ *or the logic of the* heap model *of* $\sf BBI$ (*boolean logic of bunched implications* [3, 15, 17, 18]) *fail to have plps. Consequently, the property fails in all their sublogics, including* $(\sf In-)FL_{(ew)}$, *multiplicative-additive fragment of linear logic* $\sf MALL$ (*and its intuitionistic fragment* $\sf IMALL$) *and fuzzy logics like* $\sf BL$ *or* $\sf MTL$.[5]

Presently, I am running out of ideas how to obtain an example of a natural non-locally-finite logic with plps which is not a si-logic. Here are the remaining lines of attack I can think of:

Open Problem 1. *Do any extensions of the relevance logic* $\sf R$ *have periodic sequences without being locally finite? How about the propositional lax logic* $\sf PLL_\Box^{int}$?

For the latter case, note that si-logics can be identified with extensions of $\sf PPL_\Box^{int}$ satisfying $p \leftrightarrow \Box p$, so the question here is if Ruitenburg's result can be extended in a *nontrivial* way. And at any rate, we need an in-depth algebraic investigation why plps tends to collapse to local finiteness so often—and why varieties of Heyting algebras do not follow the trend.

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[3]A formally verified proof in the Coq proof assistant allowing computation of b using either programming features of Coq itself or via extraction to other languages is available at `git://git8.cs.fau.de/ruitenburg1984`, with a web front end at `https://git8.cs.fau.de/redmine/projects/ruitenburg1984`.

[4]The reader is referred to the extensive literature [1, 8, 20–23] for basic information about intuitionistic modal logics, including axiomatizations of systems mentioned in this theorem.

[5]See Galatos et al. [5] for substructural systems mentioned in the statement of this theorem.

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Yes/No Formulae as a Description of Theories of Intuitionistic Kripke Models

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Abstract

The notion of logical equivalence still remains one of the most interesting subjects of investigation. In many logical systems the question that arises is how to describe the theory of a considered structure by means of a single formula.

We transfer this problem to intuitionistic first-order logic and consider Kripke semantics. By a Kripke model for a first-order language L we define a structure $\mathcal{K} = (K, \leq, \{K_\alpha \colon \alpha \in K\}, \Vdash_\mathcal{K})$ (for a more general definition see [2]). To any node $\alpha \in K$ there is assigned a classical first-order structure K_α for L, and for any two nodes $\alpha, \alpha' \in K$ we require that

$$\alpha \leq \alpha' \Rightarrow K_\alpha \subseteq K_{\alpha'}.$$

The forcing relation $\Vdash_\mathcal{K}$ on $\mathcal{K}$ is defined in the standard way, inductively over the construction of a formula (see [1], [2]).

Since quantifiers $\forall$ and $\exists$ are not mutually definable, and implication refers to all nodes accessible from a certain node, as a measure of formula's complexity we consider the *characteristic* of a formula (see [1]). We say that characteristic of a formula φ, $char(\varphi)$, equals $(^\rightarrow p, ^\forall q, ^\exists r)$ whenever there are p nested implications, q nested universal quantifiers and r nested existential quantifiers in φ.

Given two Kripke models $\mathcal{K} = (K, \leq, \{K_\alpha \colon \alpha \in K\}, \Vdash_\mathcal{K})$ and $\mathcal{M} = (M, \leq, \{M_\beta \colon \beta \in M\}, \Vdash_\mathcal{M})$, we consider a relation of logical equivalence with respect to all formulae of characteristic not greater than $(^\rightarrow p, ^\forall q, ^\exists r)$. For nodes $\alpha \in K$, $\beta \in M$ and any sequences $\overline{a}$ and $\overline{b}$ of elements of structures K_α and M_β respectively, we define a relation $\equiv_{p,q,r}$ as follows

$$(\alpha, \overline{a}) \equiv_{p,q,r} (\beta, \overline{b}) : \iff (\alpha \Vdash_\mathcal{K} \varphi[\overline{a}] \Leftrightarrow \beta \Vdash_\mathcal{M} \varphi[\overline{b}])$$

for all formulae $\varphi(\overline{x})$ with $char(\varphi) \leq (^\rightarrow p, ^\forall q, ^\exists r)$.

Since intuitionistic connectives differ significantly from the classical ones, one might expect a more complex solution of the aforementioned problem. We confine our considerations to a class of strongly finite Kripke models. We say that Kripke model $\mathcal{K}$ is *strongly finite* if and only if both the frame and first-order structures assigned to the nodes are finite. Moreover, the finite signature of language L is considered with no function symbols.

Under these assumptions we construct so-called *Yes Formulae* and *No Formulae* which describe theory of a node, the former will encode positive information and the latter – negative information of a node. For a strongly finite Kripke model $\mathcal{K} = (K, \leq, \{K_\alpha \colon \alpha \in K\}, \Vdash_\mathcal{K})$, its node $\alpha \in K$ and a sequence $\overline{a}$ of elements of the structure K_α, we introduce a symbol

$$Y_{p,q,r}^{\alpha, \overline{a}}$$

to denote a formula of characteristic not greater than $(^\rightarrow p, ^\forall q, ^\exists r)$ that is forced at α by $\overline{a}$. Similarly, a formula of characteristic at most $(^\rightarrow p, ^\forall q, ^\exists r)$ that is refuted at α by $\overline{a}$ is denoted by

$$N_{p,q,r}^{\alpha, \overline{a}}.$$

Formulas $Y_{p,q,r}^{\alpha,\overline{a}}$ and $N_{p,q,r}^{\alpha,\overline{a}}$ are defined inductively over $p, q, r \geq 0$ in the following way:

$$Y_{0,0,0}^{\alpha,\overline{a}}(\overline{x}) = \big(\bigwedge\{\varphi\colon char(\varphi) = (^{\rightarrow}0, {}^{\forall}0, {}^{\exists}0), \alpha \Vdash \varphi(\overline{a})\}\big)(\overline{x})$$

$$N_{0,0,0}^{\alpha,\overline{a}}(\overline{x}) = \big(\bigvee\{\varphi\colon char(\varphi) = (^{\rightarrow}0, {}^{\forall}0, {}^{\exists}0), \alpha \nVdash \varphi(\overline{a})\}\big)(\overline{x})$$

$$Y_{p+1,q,r}^{\alpha,\overline{a}}(\overline{x}) = \bigwedge_{\alpha' \geq \alpha} (N_{p,q,r}^{\alpha',\overline{a}} \rightarrow Y_{p,q,r}^{\alpha',\overline{a}})(\overline{x})$$

$$N_{p+1,q,r}^{\alpha,\overline{a}}(\overline{x}) = \bigvee_{\alpha' \geq \alpha} (Y_{p,q,r}^{\alpha',\overline{a}} \rightarrow N_{p,q,r}^{\alpha',\overline{a}})(\overline{x})$$

$$Y_{p,q+1,r}^{\alpha,\overline{a}}(\overline{x}) = \forall_y \bigvee_{\alpha' \geq \alpha} \bigvee_{a \in K_{\alpha'}} Y_{p,q,r}^{\alpha',\overline{a}a}(\overline{x}, y)$$

$$N_{p,q+1,r}^{\alpha,\overline{a}}(\overline{x}) = \bigvee_{\alpha' \geq \alpha} \bigvee_{a \in K_{\alpha'}} \forall_y N_{p,q,r}^{\alpha',\overline{a}a}(\overline{x}, y)$$

$$Y_{p,q,r+1}^{\alpha,\overline{a}}(\overline{x}) = \bigwedge_{a \in K_\alpha} \exists_y Y_{p,q,r}^{\alpha,\overline{a}a}(\overline{x}, y)$$

$$N_{p,q,r+1}^{\alpha,\overline{a}}(\overline{x}) = \exists_y \bigwedge_{a \in K_\alpha} N_{p,q,r}^{\alpha,\overline{a}a}(\overline{x}, y)$$

For a strongly finite Kripke model $\mathcal{K}$, its node $\alpha \in K$ and a sequence $\overline{a}$ of elements of K_α, by $Th_{p,q,r}(\alpha, \overline{a})$ we denote a set of all formulae of characteristic not greater than $(^{\rightarrow}p, {}^{\forall}q, {}^{\exists}r)$ forced at α by $\overline{a}$, and by $\widetilde{Th}_{p,q,r}(\alpha, \overline{a})$ we will mean a set of all formulae of characteristic at most $(^{\rightarrow}p, {}^{\forall}q, {}^{\exists}r)$ refuted at α by $\overline{a}$. We claim that

$$Y_{p,q,r}^{\alpha,\overline{a}} \vdash Th_{p,q,r}(\alpha, \overline{a}) \quad \text{and} \quad N_{p,q,r}^{\alpha,\overline{a}} \vdash \widetilde{Th}_{p,q,r}(\alpha, \overline{a}).$$

Using this fact, we can characterise the notion of (p,q,r)-equivalence, $\equiv_{p,q,r}$. Consider strongly finite Kripke models $\mathcal{K}$ and $\mathcal{M}$, and nodes $\alpha \in K$, $\beta \in M$. Let $\overline{a}$ and $\overline{b}$ be sequences of elements of worlds K_α and M_β respectively. For $p, q, r \geq 0$,

$$(\alpha, \overline{a}) \equiv_{p,q,r} (\beta, \overline{b})$$

if and only if

$$\beta \Vdash_\mathcal{M} Y_{p,q,r}^{\alpha,\overline{a}}(\overline{b}) \quad \text{and} \quad \beta \nVdash_\mathcal{M} N_{p,q,r}^{\alpha,\overline{a}}(\overline{b}).$$

Hence, logical equivalence between strongly finite rooted Kripke models $(\mathcal{K}, \alpha)$ and $(\mathcal{M}, \beta)$ can be described as follows:

$$(\mathcal{K}, \alpha) \equiv_{p,q,r} (\mathcal{M}, \beta)$$

if and only if

$$\beta \Vdash_\mathcal{M} Y_{p,q,r}^{\alpha,\overline{a}}(\overline{b}) \quad \text{and} \quad \beta \nVdash_\mathcal{M} N_{p,q,r}^{\alpha,\overline{a}}(\overline{b})$$

for all $p, q, r \geq 0$ and all sequences $\overline{a}$ of K_α and $\overline{b}$ of M_β.

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Bunched Hypersequent Calculi for Distributive Substructural Logics and Extensions of Bunched Implication Logic

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The algebraic semantics for distributive Full Lambek logic DFL is the class of algebras $\mathbf{A} = (A, \wedge, \vee, \otimes, /, \backslash, \mathbf{1}, \mathbf{0})$ such that $(A, \wedge, \vee)$ is a distributive lattice (i.e. $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$), and $(A, \otimes, \mathbf{1})$ is a monoid, and $\mathbf{0}$ is an arbitrary element of A, and $x \otimes y \leq z$ iff $x \leq z/y$ iff $y \leq x \backslash z$ for all $x, y, z \in A$. Meanwhile the algebraic semantics for the logic of Bunched Implication BI is the class of Heyting algebras equipped with an additional monoidal operation $\otimes$ and associated implications $/$ and $\backslash$ satisfying $x \otimes y \leq z$ iff $x \leq z/y$ iff $y \leq x \backslash z$. Thus BI has the intuitionistic implication $\rightarrow$ *and* the multiplicative left $/$ and right $\backslash$ implications. Here we propose a new proof calculus formalism called *bunched hypersequents* which can be used to study those subclasses of these algebras that satisfy suitable inequalities. In particular, we construct *analytic proof calculi* such that the inequalities that hold on the class of algebras are precisely those that can be proved *in* the proof calculus in a finite number of steps. Here the term *analytic* means that the proofs in the proof calculus need only contain subterms of the inequality to be proved. In the language of proof-theory, such proof calculi are said to have the *subformula property*. The subformula property (and the ensuing restriction on the space of possible proofs) is crucial for using the calculus for investigating various logical properties such as decidability, complexity, interpolation, conservativity, standard completeness [10], and for developing automated deduction procedures.

Gentzen [7] presented the first analytic calculi, for classical and intuitionistic logic, in his *sequent calculus* formalism. For example, his sequent calculus for intuitionistic logic consists of a small number of unary and binary rules (functions) on sequents; a *sequent* has the form $X \Rightarrow A$ where X is a ;-separated list of formulas and A is a formula. By repeated application of the rules, complicated sequents can be proved (derived) starting from initial sequents of the form $p \Rightarrow p$ such that $B_1; \ldots; B_n \Rightarrow A$ is derivable iff $B_1 \wedge \ldots \wedge B_n \rightarrow A$ is a theorem of intuitionistic logic (i.e. the corresponding inequality is valid on the class of Heyting algebras). This calculus can be used to give direct proofs of e.g. consistency (there is no derivation of $\Rightarrow \bot$) and optimal complexity bounds for the derivability relation.

Unfortunately there are many logics which do not support an analytic treatment in the sequent calculus formalism. The reason is that the form of the proof rules in that formalism are too restrictive. In the last three decades this has led to the introduction of many other formalisms of varying expressivity; prominent examples include the hypersequent [14, 1], display calculus [2] (viewed from a more algebraic perspective as residuated frames [6]), labelled calculus [16, 12] and bunched sequent calculus [5, 11]. The reason for the numerous different formalisms is the tradeoff that exists between an expressive formalism which yields an analytic treatment of many different logics and the difficulty in using such a formalism to prove metalogical properties. As a slogan: typically, the formalism most amenable for proof-theoretic investigation of a logic is the simplest formalism which supports its analyticity.

For distributive substructural logics (including relevant logics)—the logics that are of interest here—bunched sequent calculi, also known as Dunn-Mints systems [5, 11], have been proposed

as a means of developing an analytic formulation. In this formalism, sequents have the form $X \Rightarrow A$ where A is a formula and X is a list of formulas with *two* list separators: (";") is the list separator corresponding to the logical connective $\wedge$ and (",") is the list separator corresponding to $\otimes$. Bunched calculi have also been employed to define analytic calculi for the logic of Bunched Implication BI [15]. This logic has been used to reason about dynamic data structures [13] and is a propositional fragment of (intuitionistic) separation logic. Note that although these logics can be formalised using the more powerful formalism of display calculi, the advantage of using a simpler formalism is evident, e.g., when searching for proofs of decidability and complexity of the logic (see [8, 3, 9]).

In this paper we introduce a new proof theoretic framework called bunched hypersequents. Bunched hypersequents extend the bunched sequents by adding a hypersequent structure. In analogy with its extension of traditional sequents, we consider a non-empty set of bunched sequents rather than just a single bunched sequent. This structure allows the definition of new rules which apply to several bunched sequents simultaneously thus increasing the expressive power of the bunched sequent framework. Although a bunched hypersequent is a more complex data structure than a bunched sequent, it is nevertheless a simple and natural extension, retaining many of the useful properties of the sequent calculus (recall the slogan).

The expressive power of the new formalism is demonstrated by introducing analytic bunched hypersequent calculi for a large class of extensions of distributive Full Lambek calculus DFL. The extensions are obtained by suitably extending the procedure in [4] for transforming Hilbert axioms into structural rules. We then consider the case of extensions of the logic of bunched implication BI. Extensions of BI by a certain class of axioms including restricted weakening and restricted contraction are obtained.

Our attempt to extend the BI calculus to obtain a simple analytic calculus for BBI (boolean BI; known to be undecidable) met with a surprising obstacle. While a hypersequent structure extending the bunched calculus for BI *can* be defined (and hence also logics extending BI via the exploitation of the hypersequent structure), there are technical difficulties associated with the interpretation of hypersequent structure at intermediate points of the derivation. In response, we turn the investigation on its head and formulate an analytic hypersequent calculus for a consistent extension of BI which derives a limited boolean principle. The properties of this logic, including its decidability problem, invite further investigation.

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First-order cologic for profinite structures

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1 Logic for LFP Categories

The domain of a first-order structure M is a (typically infinite) set. First-order logic provides a finitary syntax for describing properties of M by way of how M is constructed from finite pieces, i.e. as the directed colimit of all finite subsets of M. Explicitly, a first-order formula describes a property of a tuple of elements of M, and quantifiers allow us to explore how this tuple can be expanded to larger finite tuples. This perspective on the expressive power of first-order logic is elegantly captured by the Ehrenfeucht–Fraïssé game.

A locally finitely presentable (LFP) category is one in which every object is a directed colimit of objects which are finitary in a precise sense. In direct analogy with ordinary first-order logic for Set, we develop a logic for describing properties of an object M in an LFP category (possibly expanded by extra "finitary" structure) by way of how M is constructed from finitary pieces.

To be more precise, an object x in a category $\mathcal{D}$ is called finitely presentable if the functor $\mathrm{Hom}_{\mathcal{D}}(x, -)$ preserves directed colimits. The category $\mathcal{D}$ is called locally finitely presentable if it is cocomplete, every object is a directed colimit of finitely presentable objects, and the full subcategory $\mathcal{C}$ of finitely presentable objects is essentially small. We call $\mathcal{D}$ the category of *domains* and $\mathcal{C}$ the category of *variable contexts*, and we fix a set $\mathcal{A}$ of isomorphism representatives for the objects of $\mathcal{C}$, called *arities*.

Then a *signature* $\mathcal{L}$ consists of a set of relation symbols with associated arities from $\mathcal{A}$, together with a finitary endofunctor $F\colon \mathcal{D} \to \mathcal{D}$, and an $\mathcal{L}$-*structure* is an object M in $\mathcal{D}$, given with an F-algebra structure $\eta\colon F(M) \to M$, and interpretations of the relation symbols: given an arity $n \in \mathcal{A}$ and an object $M \in \mathcal{D}$, an n-tuple from M is just an arrow $n \to M$, and an n-ary relation is a subset of $\mathrm{Hom}(n, M)$.

We can now describe the logic $\mathrm{FO}(\mathcal{D}, \mathcal{L})$: For an arity n and a variable context x, an n-term in x is a map $n \to T(x)$, the *term algebra* (i.e. free F-algebra) on x. An atomic formula is an equality between two n-terms or an n-ary relation symbol applied to an n-term. General formulas are built from atomic formulas by ordinary Boolean combinations and by quantifiers: for each arrow $f\colon x \to y$ between contexts, we associate a universal and existential quantifier $\exists_f$ and $\forall_f$ which quantify over extensions of x-tuples to y-tuples, respecting f. Of course there is a completely natural semantics for evaluation of terms and satisfaction of formulas in L-structures.

2 The first-order translation

To each M in $\mathcal{D}$, we associate the finite-limit preserving presheaf $\mathrm{Hom}_{\mathcal{D}}(-, M)\colon \mathcal{A}^{\mathrm{op}} \to \mathsf{Set}$. In fact, by Gabriel-Ulmer duality (see [1]), $\mathcal{D}$ is equivalent to the category $\mathrm{Lex}(\mathcal{A}^{\mathrm{op}}, \mathsf{Set})$ of finite-limit preserving presheaves on $\mathcal{A}$. Such presheaves can be viewed as models for a certain (ordinary) first-order theory, in a language with a sort for each object in $\mathcal{A}$. Extending this equivalence from objects of $\mathcal{D}$ to $\mathcal{L}$-structures, we obtain the following theorem.

Theorem 1. *For every LFP category $\mathcal{D}$ and signature $\mathcal{L}$, there is an ordinary multi-sorted first-order signature $\mathrm{PS}(\mathcal{D}, \mathcal{L})$ and a theory T_{PS} in this signature, so that the category of $\mathcal{L}$-structures is equivalent to the category of models of T_{PS}. Further, there is an explicit satisfaction-preserving translation from formulas in $\mathrm{FO}(\mathcal{D}, \mathcal{L})$ to first-order $\mathrm{PS}(\mathcal{D}, \mathcal{L})$-formulas.*

This interpretation of $\mathrm{FO}(\mathcal{D}, \mathcal{L})$ in ordinary first-order logic allows us to easily import theorems and notions (compactness, Löwenheim-Skolem, interpretability, stability, etc.) from first-order model theory.

3 Cologic

Whenever $\mathcal{B}$ is a category with finite limits, the category $\mathsf{pro}-\mathcal{B}$ (the formal completion of $\mathcal{B}$ under codirected limits) is co-LFP, i.e. $(\mathsf{pro}-\mathcal{B})^{\mathrm{op}}$ is LFP. Then the logic $\mathrm{FO}((\mathsf{pro}-\mathcal{B})^{\mathrm{op}}, \mathcal{L})$ expresses properties of "cotuples" from an object M, i.e. maps $M \to x$, where $x \in \mathcal{B}$. For example, a cotuple from a Stone space S (an object of $\mathsf{Stone} = \mathsf{pro}-\mathsf{FinSet}$) is a continuous map from S to a finite discrete space, or equivalently a partition of M into clopen sets. And a cotuple from a profinite group G (an object of $\mathsf{pro}-\mathsf{FinGrp}$) is a group homomorphism from G to a finite group.

These logics provide a unified framework for the model theory of profinite structures, with connections to several independent bodies of work. I will mention a few:

1. Projective (or Dual) Fraïssé theory, as developed by Irwin and Solecki [3] and recently reformulated in terms of corelations by Panagiotopoulos [5]. The dual ultrahomogeneity exhibited by projective Fraïssé limits can be expressed by $\forall\exists$ sentences in the logic $\mathrm{FO}(\mathsf{Stone}, \mathcal{L})$.

2. The "cologic" of profinite groups (e.g. Galois groups), developed by Cherlin, van den Dries, and Macintyre [2] and by Chatzidakis, which plays an important role in the model theory of PAC fields. This logic is presented in a multi-sorted first-order framework, which is essentially equivalent to the first-order translation of Theorem 1, applied to $\mathrm{FO}(\mathsf{pro}-\mathsf{FinGrp}, \emptyset)$.

3. The theory of coalgebraic logic, in the special case of cofinitary functors on Stone spaces (see, e.g. [4]), is exactly the theory of equationally defined classes in $\mathrm{FO}(\mathsf{Stone}, \mathcal{L})$, since $\mathcal{L}$-structures are coalgebras for cofinitary functors. This theory has connections to modal logic; for example, when the functor F is the Vietoris functor, $\mathrm{FO}(\mathsf{Stone}, \mathcal{L})$ embeds modal logics on descriptive general frames.

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The Frame of the p-Adic Numbers and a p-Adic Version of the Stone-Weierstrass Theorem in Pointfree Topology

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The connection between topology and lattice theory began to be exploited after the work of Marshall Stone. The fact that the lattice of open sets of a topological space contains plenty of information about the topological space indicates that a complete lattice, satisfying the distributive law $a \wedge \bigvee S = \{a \wedge s \mid s \in S\}$, deserves to be studied as a "generalized topological space". In this sense, frames (locales) generalize the notion of topological spaces and frame homomorphisms (localic maps) generalize the notion of continuous functions; that is, pointfree topology is an abstract lattice approach to topology. The algebraic nature of a frame allows its definition by generators and relations. Joyal [5] used this to introduced the frame of the real numbers; the idea is to take the set of open intervals with rational endpoints for the basic generators. Later, Banaschewski [1] studied this frame with a particular emphasis on the pointfree extension of the ring of continuous real functions and provided a pointfree version of the Stone-Weierstrass Theorem. We are interested in the field of the p-adic numbers $\mathbb{Q}_p$ and the ring of continuous p-adic functions. $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic absolute value $|\cdot|_p$, which satisfies $|x + y|_p \leq \max\{|x|_p, |y|_p\}$ (i.e., it is nonarchimedean, see [3] and [4]). In particular, $\mathbb{Q}_p$ is 0-dimensional, completely regular, and locally compact. In [2], Dieudonné proved that the ring $\mathbb{Q}_p[X]$ of polynomials with coefficients in $\mathbb{Q}_p$ is dense in the ring $\mathcal{C}(F, \mathbb{Q}_p)$ of continuous functions defined on a compact subset F of $\mathbb{Q}_p$ with values in $\mathbb{Q}_p$, and Kaplansky [6] extended this result by proving that if $\mathbb{F}$ is a nonarchimedean valued field and X is a compact Hausdorff space, then any unitary subalgebra $\mathcal{A}$ of $\mathcal{C}(X, \mathbb{F})$ which separates points is uniformly dense in $\mathcal{C}(X, \mathbb{F})$. We define the frame of $\mathbb{Q}_p$ and we give a p-adic version of the Stone-Weierstrass theorem in pointfree topology.

To specify the frame of $\mathbb{Q}_p$ by generators and relations, we consider the fact that the open balls centered at rational numbers generate the open subsets of $\mathbb{Q}_p$ and thus we think of them as the basic generators; we consider the (lattice) properties of these balls to determine the relations these elements must satisfy. Let $\mathcal{L}(\mathbb{Q}_p)$ be the frame generated by the elements $B_r(a)$, where $a \in \mathbb{Q}$ and $r \in |\mathbb{Q}| := \{p^{-n}, n \in \mathbb{Z}\}$, subject to the following relations:

(Q1) $B_s(b) \leq B_r(a)$ whenever $|a - b|_p < r$ and $s \leq r$,

(Q2) $B_r(a) \wedge B_s(b) = 0$ whenever $|a - b|_p \geq r \vee s$,

(Q3) $1 = \bigvee \{B_r(a) : a \in \mathbb{Q}, \ r \in |\mathbb{Q}|\}$,

(Q4) $B_r(a) = \bigvee \{B_s(b) : |a - b|_p < r, \ s < r, \ b \in \mathbb{Q}, \ s \in |\mathbb{Q}|\}$.

We prove that $\mathcal{L}(\mathbb{Q}_p)$ is the pointfree counterpart of $\mathbb{Q}_p$; that is, $\mathcal{L}(\mathbb{Q}_p)$ is a spatial frame whose space of points is homeomorphic to $\mathbb{Q}_p$. In particular, we show with pointfree arguments that $\mathcal{L}(\mathbb{Q}_p)$ is 0-dimensional, completely regular, and continuous.

As in the real case, from the well-known adjunction between frames and topological spaces (see, e.g., [7]), we have a natural isomorphism $\mathbf{Top}(X, \mathbb{Q}_p) \cong \mathbf{Frm}(\mathcal{L}(\mathbb{Q}_p), \Omega(X))$, for a topological space X. This provides a natural extension of the classical notion of a continuous p-adic

function: *a continuous p-adic function* on a frame L is a frame homomorphism $\mathcal{L}(\mathbb{Q}_p) \to L$. We denote the set of all continuous *p*-adic functions on a frame L with $\mathcal{C}_p(L)$, and we show that it is a $\mathbb{Q}_p$-algebra under the following operations:

$$(f + g)\big(B_r(a)\big) = \bigvee \left\{ f\big(B_{s_1}(b_1)\big) \wedge g\big(B_{s_2}(b_2)\big) \mid B_{s_1 \vee s_2}\langle b_1 + b_2\rangle \subseteq B_r\langle a\rangle \right\}$$

$$(f \cdot g)\big(B_r(a)\big) = \bigvee \left\{ f\big(B_{s_1}(b_1)\big) \wedge g\big(B_{s_2}(b_2)\big) \mid B_t\langle b_1 \cdot b_2\rangle \subseteq B_r\langle a\rangle \right\},$$

where $t = \max\left\{ p^{-1}rs,\, s|a|_p,\, r|b|_p \right\}$.

If X is compact Hausdorff and $f \in \mathcal{C}(X, \mathbb{Q}_p)$, then $||f|| = \sup\{|f(x)|_p\}$ defines a nonarchimedean norm on $\mathcal{C}(X, \mathbb{Q}_p)$. In our case, we show that if L is a compact regular frame, then $||h|| = \inf\left\{ p^{-n} : n \in \mathbb{Z},\, h\big(B_{p^{-n+1}}(0)\big) = 1 \right\}$ defines a nonarchimedean norm on $\mathcal{C}_p(L)$.

Recall that if X is compact Hausdorff then X is 0-dimensional iff $\mathcal{C}(X, \mathbb{Q}_p)$ separates points, thus we assume that X is 0-dimensional; in the pointfree context, we assume that L is a compact 0-dimensional frame. Additionally, if X is compact Hausdorff and 0-dimensional, then each $f \in \mathcal{C}(X, \mathbb{Q}_p)$ can be approximated by a linear combination of $\mathbb{Q}_p$-characteristic functions of clopen subsets. Thus, if $\mathcal{A}$ is a unitary subalgebra of $\mathcal{C}(X, \mathbb{Q}_p)$ such that its closure contains these $\mathbb{Q}_p$-characteristic functions, then $\mathcal{A}$ is dense in $\mathcal{C}(X, \mathbb{Q}_p)$. It can be shown (see [6]) that this is the case whenever $\mathcal{A}$ separates points. Therefore, we extend the notion of a $\mathbb{Q}_p$-characteristic function of a clopen subset, showing that if u is a complemented element (with complement u') in L, then the function $\chi_u : \mathcal{L}(\mathbb{Q}_p) \to L$ defined on generators by

$$\chi_u\big(B_r(a)\big) = \begin{cases} 1 & \text{if } |a|_p < r \text{ and } |1 - a|_p < r, \\ u & \text{if } |a|_p \geq r \text{ and } |1 - a|_p < r, \\ u' & \text{if } |a|_p < r \text{ and } |1 - a|_p \geq r, \\ 0 & \text{otherwise,} \end{cases}$$

is a frame homomorphism. We show that these elements are precisely the idempotents in $\mathcal{C}_p(L)$ and we extend the notion of a subalgebra in $\mathcal{C}(X, \mathbb{Q}_p)$ that separates points to the pointfree context as follows: Given a compact 0-dimensional frame L, we say that a unitary subalgebra $\mathcal{A}$ of $\mathcal{C}_p(L)$ *separates points* if $\overline{\mathcal{A}}$ contains the idempotents of $\mathcal{C}_p(L)$.

Finally, we provide the following pointfree version of the Stone-Weierstrass Theorem: Let L be a compact 0-dimensional (regular) frame and let $\mathcal{A}$ be a unitary subalgebra of $\mathcal{C}_p(L)$ which separates points, then $\mathcal{A}$ is uniformly dense in $\mathcal{C}_p(L)$.

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Neighborhood-Kripke product of modal logics [*]

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There are many ways to combine modal logics. Some of them are syntactical and some semantical. The simplest syntactical combination is the fusion. The fusion of two unimodal logics L_1 and L_2 is the minimal bimodal logic containing axioms from L_1 rewritten with $\square_1$ and axioms from L_2 rewritten with $\square_2$. Notation: $L_1 * L_2$.

The product of two modal logics is a semantical way to combine logics. The product of two modal logics is the logic of the class of all products of semantical structures of the corresponding logics. Such construction based on the product of Kripke frames was introduced by Shehtman in 1978 [10]. Later in 2006 van Benthem et al. [1] introduced a similar construction based on product of topological spaces[1].

Neighborhood semantics is a generalization of the Kripke semantics and the topological semantics. It was introduced independently by Dana Scott [9] and Richard Montague [7]. The product of neighborhood frames was introduced by Sano in [8]. The product of topological spaces from [1] is a particular case of the product of the neighborhood product for $S4$-frames. Several paper was studying neighborhood products [5, 6].

A recent paper by Kremer proposed a mixed space-frame product and proved a general completeness result for $S4$ and Horn axiomatized extensions of logic D.

In this work we generalize Kremer's results to neighborhood-Kripke frames product.

A *(normal) neighborhood frame* (or an n-frame) is a pair $\mathfrak{X} = (X, \tau)$, where X is a nonempty set and $\tau : X \to 2^{2^X}$ such that $\tau(x)$ is a filter on X for any x. The function τ is called the *neighborhood function* of $\mathfrak{X}$, and sets from $\tau(x)$ are called *neighborhoods of x*.

A *Kripke frame* is a pair tuple (X, R), where X is a non-empty set and $R \subseteq X \times X$ is a relation on X.

A *valuation* on a Kripke (n-) frame is a function $V : PV \to 2^X$. For a Kripke (n-) frame $\mathfrak{X}$ and a valuation V pair $(\mathfrak{X}, V)$ is called a Kripke (neighborhood) model.

The truth for models define in the usual way see [2] and [3].

A *neighborhood-Krike frame* is a triple (X, τ, R) such that (X, τ) is a n-frame and (X, R) is a Kripke frame. The notion of truth uses neighborhood structure for $\square_1$ and Kripke structure for $\square_2$. For n-frame $\mathfrak{X}_1 = (X_1, \tau_1)$ and Kripke frame $F_2 = (X_2, R_2)$ the product of them is a neighborhood-Krike frame $\mathfrak{X}_1 \times F_2 = (X_1 \times X_2, \tau_1', R_2')$ such that for $(x, y) \in X_1 \times X_2$

$$U \in \tau_1'(x, y) \iff \exists V \in \tau_1(x) \, (V \times \{y\} \subseteq U)$$
$$R_2'(x, y) = \{(x, y') \mid y R_2 y'\}$$

A logic of a frame or a class of frames is all the formulas that are true at all points in all models of these frames.

A logic L is called an *PTC-logic* if it can be axiomatized by closed formulas and formulas of the type $\Diamond^m \square p \to \square^n p$, $n, m \geq 0$. (see [4]).

[1] "Product of topological spaces" is a well-known notion in Topology but it is different from what we use here (for details see [1])

A logic L is called an *HTC-logic* (from Horn preTransitive Closed logic) if it can be axiomatized by closed formulas and formulas of the type $\Box p \to \Box^n p$, $n \geq 0$. These formulas correspond to universal strict Horn sentences (see [4]).

For two normal modal logics L_1 and L_2 the nk-product of them is the logic of all products of n-frames of logic L_1 and Kripke frames of logic L_2. Notation: $\mathsf{L}_1 \times_{nk} \mathsf{L}_2$.

Our main result is the following

Theorem 1. *For any HTC logic L_1 and PTC logic L_2*

$$
\begin{aligned}
\mathsf{L}_1 \times_{nk} \mathsf{L}_2 \quad &= \mathsf{L}_1 * \mathsf{L}_2 + com_{12} + chr + \Delta_1 + \Delta_2, \ \textit{where} \\
com_{12} \quad &= \Box_1\Box_2 p \to \Box_2\Box_1 p, \\
chr \quad &= \Diamond_1\Box_2 p \to \Box_2\Diamond_1 p, \\
\Delta_1 \quad &= \{\phi \to \Box_2\phi \mid \phi \textit{ is closed and } \Box_2\textit{-free}\}, \\
\Delta_2 \quad &= \{\psi \to \Box_1\psi \mid \psi \textit{ is closed and } \Box_1\textit{-free}\}.
\end{aligned}
$$

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On Kripke completeness of modal and superintuitionistic predicate logics with equality

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We consider first-order normal modal and superintuitionistic predicate logics in a signature with only predicate letters and perhaps with equality. A logic is defined in a standard way, as a certain set of formulas, cf. [2], sec. 2.6.

Every logic L without equality has the minimal extension $L^=$ with equality ([2], sec. 2.14.). It is well-known that completeness of L in the standard Kripke semantics does not imply the completeness of $L^=$. So there is a natural question — how to axiomatize the logic with equality characterized by Kripke frames for L. As we show, quite often (but not always) this is done by the extensions $L^{=d} := L^= + DE$ in the intuitionistic case and $L^{=c} := L^= + CE$ in the modal case, where

$$DE := \forall x \forall y (x = y \lor \neg(x = y)) \text{ (the axiom of decidable equality)},$$
$$CE := \forall x \forall y (\Diamond(x = y) \supset x = y) \text{ (the axiom of closed equality)}.$$

Here we deal with two kinds of semantics: the semantics of predicate Kripke frames ($\mathcal{K}$) and the semantics of Kripke frames with equality ($\mathcal{KE}$) (equivalent to the semantics of Kripke sheaves); cf. [2], sections 3.2, 3.5, 3.6. Recall that a *predicate Kripke frame* (PKF) over a propositional Kripke frame $F = (W, R)$ is a pair (F, D), where $D = (D_u)_{u \in W}$ is a family of non-empty expanding domains (uRv implies $D_u \subseteq D_v$). A *predicate Kripke frame with equality* (KFE) is a triple $(F, D, \asymp)$, where (F, D) is a PKF and $\asymp = (\asymp_u)_{u \in W}$ is a family of expanding equivalence relations $\asymp_u \subseteq D_u \times D_u$ (uRv implies $\asymp_u \subseteq \asymp_v$). The notions of validity in these semantics are standard. The set of formulas valid in a PKF or a KFE $\mathbf{F}$ is called the *logic of* $\mathbf{F}$ (modal or superintuitionistic) and denoted by $\mathbf{ML}(\mathbf{F})$ or $\mathbf{IL}(\mathbf{F})$, or by $\mathbf{ML}^=(\mathbf{F})$ or $\mathbf{IL}^=(\mathbf{F})$ for logics with equality.

The logics of a class of frames $\mathcal{C}$ are $\mathbf{ML}^{(=)}(\mathcal{C}) := \bigcap \{\mathbf{ML}^{(=)}(\mathbf{F}) \mid \mathbf{F} \in \mathcal{C}\}$, $\mathbf{IL}^{(=)}(\mathcal{C}) := \bigcap \{\mathbf{IL}^{(=)}(\mathbf{F}) \mid \mathbf{F} \in \mathcal{C}\}$; these logics are called *Kripke ($\mathcal{K}$-) complete* if $\mathcal{C}$ is a class of PKFs, *Kripke sheaf ($\mathcal{KE}$-) complete* if $\mathcal{C}$ is a class of KFEs.

Note that a KFE $(W, R, D, \asymp)$ validates CE iff its reflexive transitive closure $(W, R^*, D, \asymp)$ validates DE iff

$$\forall u, v \in W \, \forall a, b \in D_u \, (uR^*v \,\&\, a \asymp_v b \Rightarrow a \asymp_u b).$$

So CE and DE are obviously valid in every PKF, since a PKF can be regarded as a KFE, in which $\asymp_u$ are the identity relations.

Usually $\mathcal{KE}$-completeness transfers from L to $L^=$ and $L^{=d}$ (or $L^{=c}$); cf. [2], theorems 3.8.3, 3.8.4, 3.8.7, 3.8.8 for the details.

Proposition 1. (1) Suppose $\mathbf{F} \models CE$ is a KFE over a propositional frame F, F^* is the reflexive transitive closure of F and one of the following conditions holds: (i) F^* is an **S4**-tree; (ii) F^* is directed; (iii) $\mathbf{F}$ has a constant domain.

Then there exists a PKF $\mathbf{F}'$ such that $\mathbf{ML}^=(\mathbf{F}') = \mathbf{ML}^=(\mathbf{F})$.

(2) The same holds for the intuitionistic case and $\mathbf{F} \Vdash DE$.

Hence we obtain

Theorem 1. (1) Suppose L is a $\mathcal{K}$-complete modal predicate logic of one of the following types: (i) L is complete w.r.t. frames over trees; (ii) $L \vdash \Diamond\Box p \supset \Box\Diamond p$; (iii) $L \vdash \forall x \Box P(x) \supset \Box \forall x P(x)$ (the Barcan formula). Then $L^{=c}$ is $\mathcal{K}$-complete.

(2) Suppose L is a $\mathcal{K}$-complete superintuitionistic predicate logic of one of the following types: (i) L is complete w.r.t. frames over trees; (ii) $L \vdash J \ (= \ \neg p \vee \neg\neg p)$; (iii) $L \vdash CD \ (= \ \forall x(P(x) \vee q) \supset \forall x P(x) \vee q)$. Then $L^{=d}$ is $\mathcal{K}$-complete.

Remark. Recall that $L = \mathbf{QH} + CD + J$ is Kripke incomplete [1]. We do not know if $L^{=d}$ is Kripke complete in this case.

However, not every KFE validating DE is equivalent to a PKF. This allows us to construct Kripke complete logics L, for which $L^{=d}$ is Kripke incomplete.

Consider the weak De Morgan law

$$J_2 := \neg(p_0 \wedge p_1 \wedge p_2) \supset \neg(p_0 \wedge p_1) \vee \neg(p_0 \wedge p_2) \vee \neg(p_1 \wedge p_2),$$

and the frame $F_0 := (W_0, \leq)$, with $W_0 := \{u_0\} \cup \{u_{ij} \mid i, j \in \{1, 2\}\}$, which is a poset with the root u_0 and $(u_{ij} < u_{i'j'})$ iff $(i < i')$. Then F_0 validates J_2, but not J. $\mathbf{IL}(\mathcal{K}F_0)$ denotes the superintuitionistic logic of all PKFs over F_0 (which coincides with the logic of all KFEs over F_0).

Theorem 2. Let L be a predicate logic such that $\mathbf{QH} + J_2 \subseteq L \subseteq \mathbf{IL}(\mathcal{K}F_0)$. Then the logic $L^{=d}$ is Kripke incomplete.

We do not know if the segment mentioned in Theorem 2 contains finitely axiomatizable Kripke complete logics.

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Positive Coalgebraic Logic

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Partially ordered structures are ubiquitous in theoretical computer science: knowledge representation, abstract interpretation in static analysis, resource modelling, protocol or access rights modelling in formal security, etc, the list of applications is enormous. Being able to formally reason about transition systems over posets therefore seems important. The natural formalism to reason about transition systems is undoubtedly the class of *modal* logics, but most are tailored to transition structures over *sets*. This is a direct consequence of the fact that most modal logics are *boolean*. *Positive modal logic* is the exception, and is most naturally interpreted in partially ordered Kripke structures (see for example [2, 6]).

Arguably, the most natural and powerful framework to study boolean modal logics in uniform and systematic way, is the theory of *Boolean Coalgebraic Logics* (henceforth BCL, see e.g. [3]). In its abstract flavour, it is parametrised by an endofunctor $L : \mathbf{BA} \to \mathbf{BA}$ which builds modal terms over a boolean structure, an endofunctor $T : \mathbf{Set} \to \mathbf{Set}$ which builds the transition structures over which the modal terms are to be interpreted, and a natural transformation $\delta : L\mathcal{P} \to \mathcal{P}T$ (where $\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{BA}$ is the powerset functor) which implements the interpretation by associating sets of acceptable successors states to each modal term over a predicate. This data, and the dual adjunction between $\mathbf{Set}$ and $\mathbf{BA}$, is traditionally summarized in the following diagram

$$L \, \stackrel{\frown}{(} \, \mathbf{BA} \, \underset{\mathcal{P}}{\overset{\mathcal{S}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \, \mathbf{Set}^{\mathrm{op}} \, \stackrel{\frown}{)} \, T^{\mathrm{op}} \tag{1}$$

where $\mathcal{S}$ is the functor sending a boolean algebra to the set of its ultrafilters.

To develop an equally powerful framework for reasoning about transition structures over posets, it seems natural to study *Positive Coalgebraic Logics* (henceforth PCL). In fact, work in this direction has already started, see for example [7, 1]. We pursue this work further and present PCL in full generality, i.e. at the same level of generality as its boolean counterpart. Moreover, given the close kinship between the two theories, we will show that the wheel does not have to be re-invented every time, and that many BCLs have a canonical positive fragment which inherits useful properties of its boolean parent. In fact, adapting well-known situations from the boolean to the positive setting is one of the guiding principles of this work.

Let us sketch the main features of the theory of PCLs and its relationship with BCLs. First of all, whilst the mathematical universe hosting the theory of BCLs is ordinary category theory, the most natural environment to discuss PCLs is category theory *enriched over* $\mathbf{Pos}$, the category of posets and monotone maps. Indeed, on the model side the category $\mathbf{Pos}$ is naturally enriched over itself, while on the syntax side we will consider endofunctors $L' : \mathbf{DL} \to \mathbf{DL}$ over the category of distributive lattice, which is also $\mathbf{Pos}$-enriched. Moreover, whilst boolean modal

logics are axiomatized by equations, represented categorically as *coequalizers*, the axioms of standard positive modal logic (see [5, 2, 6]) are given by inequations, which are very naturally interpreted as *coinserters*, the **Pos**-enriched analogue of coequalizers. In this **Pos**-enriched framework, we have an analogue of Diagram (1) given by

$$L'\left(\begin{array}{c}\end{array}\right.\mathbf{DL}\underset{\mathcal{P}'}{\overset{\mathcal{S}'}{\rightleftarrows}}\perp\ \mathbf{Pos}^{\mathrm{op}}\left.\begin{array}{c}\end{array}\right)(T')^{\mathrm{op}}$$

where $\mathcal{S}'$ is the functor sending a distributive lattice to the poset of its prime filters (under inclusion), and $\mathcal{P}'$ is the functor sending a poset to the distributive lattice of its upsets.

Coalgebras over posets. Transition structures over posets will be formalized as coalgebras for an endofunctor $T' : \mathbf{Pos} \to \mathbf{Pos}$. Here, we are already confronted with a situation which perfectly captures the philosophy of this work. How do we choose such a functor? In practice both our requirements and our intuition are guided by examples of endofunctors $T : \mathbf{Set} \to \mathbf{Set}$, for example non-deterministic computations modelled as coalgebras for the powerset functor $\mathsf{P} : \mathbf{Set} \to \mathbf{Set}$, or models of graded modal logic as coalgebras for the multiset functor $\mathsf{B} : \mathbf{Set} \to \mathbf{Set}$. The solution is to adapt these well-know functors to posets. We use the *posetification* procedure developed in [1] and define for each **Set**-endofunctor T its posetification by $T' : \mathbf{Pos} \to \mathbf{Pos}$ by $T' = \mathrm{Lan}_{\mathsf{D}}\mathsf{D}T$, where D is the functor sending a set to its discrete poset. It was shown in [1] that T' can be computed using certain coinserters, and we now have a whole repertoire of **Set**-endofunctors for which we have computed the posetification: the powerset functor, the neighbourhood functor, the monotone neighbourhood functor, the multiset functor, etc, as well as a grammar to combine them.

Syntax. The syntax of a positive coalgebraic logic will be given by a locally monotone (i.e. **Pos**-enriched) endofunctor $L' : \mathbf{DL} \to \mathbf{DL}$. Whilst [1] focused defining such functors directly from the semantics, here we once again focus on adapting existing boolean logics. This leads us to an operation which is dual to that of posetification, and which we call *positivisation*: given an endofunctor $L : \mathbf{BA} \to \mathbf{BA}$, we define its positivisation $L' : \mathbf{DL} \to \mathbf{DL}$ by $L' = \mathrm{Ran}_{\mathsf{U}}\mathsf{U}L$ where $\mathsf{U} : \mathbf{BA} \to \mathbf{DL}$ is the obvious forgetful functor. The positivisation of a **BA**-endofunctor can be computed explicitly via *inserters*. We can distinguish two classes of positivisation: those for which the natural transformation $\beta : L'\mathsf{U} \to \mathsf{U}L$ given by universality of the right (enriched) Kan extension is an iso, and all the others. The former correspond to boolean logics $L : \mathbf{BA} \to \mathbf{BA}$ which have a *monotone* presentation. We have computed the positivisation of the functors defining the boolean modal logics with (i) no axioms, (ii) monotonicity only, (iii) the standard axioms of modal logic, and (iv) the axioms of graded modal logic.

Semantics. Following our guiding philosophy, we would like to canonically turn a boolean coalgebraic logic (L, T, δ) with nice properties (for example completeness) into a positive coalgebraic logic (L', T', δ'), hopefully with equally nice properties. First we need to build a semantic natural transformation $\delta' : L'\mathcal{P}' \to \mathcal{P}'T'$, from the transformation $\delta : L\mathcal{P} \to \mathcal{P}T$. By combining the posetification and the positivisation procedures described above, and the properties of $\mathcal{P}'$, one can build a transformation $\delta' : L'\mathcal{P}' \to \mathcal{P}'T'$ from δ in a universal way. Moreover, if δ and $\beta : L'\mathsf{U} \to \mathsf{U}L$ are component-wise injective, so is δ', in other words completeness transfers from the boolean logic to its canonical positive fragment. Similarly, strong completeness via the coalgebraic Jónsson-Tarski theorem – which is equivalent to the adjoint (or mate) $\hat{\delta}$ of δ being component-wise split epi [8, 9, 4] – transfers from a boolean coalgebraic logic to its positive fragment. The transfer of expressivity is more involved.

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A multi-valued framework for coalgebraic logics over generalised metric spaces

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Introduction. It is by now generally acknowledged that coalgebras for a Set-functor unify a wide variety of dynamic systems [16]. The classical study of their behavior and behavioral equivalence is based on qualitative reasoning – that is, Boolean, meaning that two systems (the systems' states) are bisimilar (equivalent) or not. But in recent years there has been a growing interest in studying the behavior of systems in terms of quantity. There are situations where one behaviour is smaller than (or, is simulated by) another behaviour, or there is a measurable distance between behaviours in terms of real numbers, as it was done in [15, 18]. This can be achieved by enlarging the coalgebraic set-up to the category of (small) enriched $\mathscr{V}$-categories $\mathscr{V}\text{-}\mathsf{cat}$ [10] ($\mathscr{V}$ is a commutative quantale), which subsumes both ordered sets and (generalised) metric spaces [12].

Coalgebras over generalised metric spaces. The project of developing multi-valued logic for coalgebras on $\mathscr{V}\text{-}\mathsf{cat}$ has started in [1] by extending functors $H : \mathsf{Set} \to \mathsf{Set}$ (and more generally Set-functors which naturally carry a $\mathscr{V}$-metric structure) to $\mathscr{V}\text{-}\mathsf{cat}$-functors. In this talk, we shall briefly outline the extension procedure: using the density of the discrete functor $D : \mathsf{Set} \to \mathscr{V}\text{-}\mathsf{cat}$, we apply H to the $\mathscr{V}$-nerve of a $\mathscr{V}$-category, and then take an appropriate quotient in $\mathscr{V}\text{-}\mathsf{cat}$. If H preserves weak pullbacks, then the above can be obtained using Barr's relation lifting in a form of "lowest-cost paths" (see also[18, Ch. 4.3], [9]). For example, the extension of the powerset functor yields the familiar Pompeiu-Hausdorff metric, if the quantale is completely distributive.

A logical framework. The next step, following the well-established tradition in coalgebraic logics (see e.g. [14]), is to seek for a contravariant $\mathscr{V}\text{-}\mathsf{cat}$-enriched adjunction - on top of which to develop coalgebraic logics– involving, on one side, a category of spaces Sp, and on the other side, a category of algebras Alg, obtained eventually by restricting the adjunction $\mathscr{V}\text{-}\mathsf{cat}^{\mathsf{op}} \underset{[-,\mathscr{V}]}{\overset{[-,\mathscr{V}]}{\underset{\top}{\rightleftarrows}}} \mathscr{V}\text{-}\mathsf{cat}$. Moreover, we would want for Alg be a variety in the "world of $\mathscr{V}$-categories", at least monadic over $\mathscr{V}\text{-}\mathsf{cat}$. In *classical (Boolean) coalgebraic logics* (no enrichment), this is achieved by taking Sp to be Set, and Alg to be the category of Boolean algebras (see e.g. [7]). One step further, the case of the simplest quantale $\mathscr{V} = 2$ targets *positive coalgebraic logics* [2], from an order-enriched point of view, by choosing Sp to be the category of posets and monotone maps, and Alg to be the category of bounded distributive lattices – which is a finitary *ordered* variety [4].

In the present work we focus on the unit interval quantale $\mathscr{V} = [0,1]$, endowed with the usual order, the Łukasiewicz tensor given by truncated sum $r \otimes s = \max(0, r + s - 1)$, with

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unit $e = 1$ and internal hom (residual) $[r, s] = \min(1 - r + s, 1)$. Our original motivation to do so came from (at least) the following reason: the unit interval naturally carries an MV-algebra structure. Recall that the MV-algebras are the models for Łukasiewicz multi-valued logic, and that their variety is generated by [0,1] [5, 6]. As the propositional (Boolean) logic is the base for the usual coalgebraic logic, we looked for a connection between coalgebras based on $[0, 1]$-categories (that is, "bounded-by-1" quasi-metric spaces) and multi-valued logics. However, we shall explain in the talk that MV-algebras are not adequate for our purpose, and propose a different solution instead, detailed below.

An alternative to MV-algebras. The logical connection we therefore propose uses an adaptation of the Priestley duality as in [8]. We introduce the notion of a *distributive lattice with adjoint pairs of $\mathscr{V}$-operators* (dlao($\mathscr{V}$)) as a bounded distributive lattice $(A, \wedge, \vee, 0, 1)$, endowed with a family of adjoint operators $(r \odot - \dashv \pitchfork (r, -) : A \to A)_{r \in \mathscr{V}}$, such that the conditions below are satisfied for all $r, r' \in \mathscr{V}$ and all $a, a' \in A$:

$$
\begin{aligned}
1 \odot a &= a & (r \otimes r') \odot a &= r \odot (r' \odot a) \\
0 \odot a &= 0 & (r \vee r') \odot a &= (r \odot a) \vee (r' \odot a) \\
\pitchfork (1, a) &= a & \pitchfork (r \otimes r', a) &= \pitchfork (r, \pitchfork (r', a)) \\
\pitchfork (0, a) &= 1 & \pitchfork (r \vee r', a) &= \pitchfork (r, a) \wedge \pitchfork (r', a)
\end{aligned}
$$

Notice that by adjointness $r \odot -$ preserves finite joins and $\pitchfork (r, -)$ preserves finite meets. A morphism of dlao($\mathscr{V}$) is a bounded distributive lattice map preserving all the adjoint operators $r \odot -$ and $\pitchfork (r, -)$. Let $\mathsf{DLatAO}(\mathscr{V})$ be the ordinary category of distributive lattices with adjoint pairs of $\mathscr{V}$-operators (notice that $\mathsf{DLatAO}(\mathscr{V})$ is an algebraic category).

Each dlao($\mathscr{V}$) A becomes a $\mathscr{V}$-category [3, 13] with $\mathscr{V}$-homs $A(a, a') = \bigvee \{r \in [0, 1] \mid r \odot a \leq a'\} = \bigvee \{r \in [0, 1] \mid a \leq \pitchfork (r, a')\}$, and each dlao($\mathscr{V}$)-morphism is also a $\mathscr{V}$-functor. The $\mathscr{V}$-categories thus obtained are antisymmetric, finitely complete and cocomplete [17]. Consequently, $\mathsf{DLatAO}(\mathscr{V})$ is a $\mathscr{V}$-cat-category, and it follows that the forgetful functor $\mathsf{DLatAO}(\mathscr{V}) \to \mathscr{V}$-$\mathsf{cat}$ is monadic $\mathscr{V}$-cat-enriched.

The ordinary dual category to $\mathsf{DLatAO}(\mathscr{V})$ can be obtained by adapting the arguments in [8]: an object is a Priestley space $(X, \tau, \leq)$, endowed with a family of ternary relations $(R_r)_{r \in \mathscr{V}}$, which satisfy, besides the topological conditions from [8, pp. 184-185], the requirements that R_1 is the order relation on X, and that $R_r \circ R_{r'} = R_{r \otimes r'}$ and $R_r \vee R_{r'} = R_{r \vee r'}$ hold. The morphisms are continuous bounded maps [8, Section 2.3]. Denote by $\mathsf{RelPriest}(\mathscr{V})$ the resulting category. Then the dual equivalence $\mathsf{RelPriest}(\mathscr{V})^{\mathsf{op}} \cong \mathsf{DLatAO}(\mathscr{V})$ is obtained by restricting the usual Priestley duality.

Using the above duality, we can transport the $\mathscr{V}$-cat-category structure on $\mathsf{RelPriest}(\mathscr{V})$, thus rendering the duality $\mathsf{RelPriest}(\mathscr{V})^{\mathsf{op}} \cong \mathsf{DLatAO}(\mathscr{V})$ $\mathscr{V}$-cat-enriched. The $\mathscr{V}$-cat-category structure such exhibited on $\mathsf{RelPriest}(\mathscr{V})$ does not say too much at first sight. To gain more insight, we use the lax-algebra framework of [9], in the context of $(T, \mathscr{V})$-categories, where T is a monad on Set which laxly distributes over the $\mathscr{V}$-valued powerset monad. We shall see that each relational Priestley space $(X, \tau, \leq, (R_r)_{r \in \mathscr{V}})$ is in fact a $\mathscr{V}$-compact topological space [11] – an algebra for the extension of the ultrafilter monad to $\mathscr{V}$-cat (see [9, Ch. III.5.2] for the cases $\mathscr{V} = \mathbb{2}$ and $\mathscr{V} = [0, \infty]$). The duality $\mathsf{RelPriest}(\mathscr{V})^{\mathsf{op}} \cong \mathsf{DLatAO}(\mathscr{V})$ can now be seen as a $\mathscr{V}$-cat-duality between a category of certain compact $\mathscr{V}$-topological spaces (in particular $\mathscr{V}$-categories) and a category of algebraic $\mathscr{V}$-categories. In future work, more properties of the above duality are planned to be investigated.

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Some properties of zero divisor graphs of lattices

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Beck [3] introduced the notion of coloring in a commutative ring R as follows. Let G be a simple graph whose vertices are the elements of R and two distinct vertices x and y are adjacent in G if $xy = 0$ in R.

Nimbhorkar et al. [8] introduced a graph for a meet-semilattice L with 0, whose vertices are the elements of L and two distinct elements $x, y \in L$ are adjacent if and only if $x \wedge y = 0$. They correlated properties of semilattices with coloring of the associated graph. A nonzero element $a \in L$ is called a zero-divisor if there exists a nonzero $b \in L$ such that $a \wedge b = 0$. We denote by $Z(L)$ the set of all zero-divisors of L. We associate a graph $\Gamma(L)$ to L with vertex set $Z^*(L) = Z(L) - \{0\}$, the set of nonzero zero-divisors of L and distinct $x, y \in Z^*(L)$ are adjacent if and only if $x \wedge y = 0$ and call this graph as the zero-divisor graph of L. In a meet-semilattice L with 0, a nonzero element $a \in L$ is called an atom if there is no $x \in L$ such that $0 < x < a$.

MAIN RESULTS

Lemma 1. *Let L be a complemented distributive lattice. An element $b \in L$ is an atom in L iff $b^{'}$ is the unique end adjacent to b in $\Gamma(L)$.*

Lemma 2. *Let $L \neq C_2$ be a complemented distributive lattice. Then atoms in L are precisely the vertices in $\Gamma(L)$ which are adjacent to an end.*

We recall that C_2 denotes the two element chain.

Lemma 3. *Let $L \neq C_2$ be a complemented distributive lattice. The complement $a^{'}$ of $a \in L$ is also a complement of of a in $\Gamma(L)$. Hence $\Gamma(L)$ is uniquely complemented.*

Lemma 4. *If $\Gamma(L)$ splits into two subgraphs X and Y via a then a is an atom of L.*

However, the converse of Lemma 4 need not hold.

Lemma 5. *If $\Gamma(L)$ splits into two subgraphs X and Y via a then $a \leq x$ for every $x \in L - Z(L)$.*

Lemma 6. *For any lattice L with 0, $L - Z(L)$ is a dual ideal.*

Theorem 1. *Let L be a finite lattice. If $\Gamma(L)$ splits into two subgraphs X and Y via a, then either X or Y is a set of isolated vertices.*

Theorem 2. *If $\Gamma(L)$ splits into two subgraphs X and Y via a, then $N(a)$ is a maximal element in the set $\{N(x) | x \in \Gamma(L)\}$.*

The converse need not hold.

Theorem 3. *If $a - x$ is an edge in $\Gamma(L)$ and a, x are not pendant vertices then the edge $a - x$ is contained in a 3-cycle or a 4-cycle.*

Theorem 4. *Every pair of non-pendant vertices in $\Gamma(L)$ is contained in a cycle of length less than or equal to 6.*

The following example shows that 6 is the best possible bound.

Example 1. *Let L be the lattice of all positive divisors of $n = 4620$ with divisibility as the partial order. Then a, b are adjacent in $\Gamma(L)$ iff the greatest common divisor of a, b is 1. Consider $a = 30$ and $b = 154$. Then a, b are non-pendant vertices in $\Gamma(L)$ and these are contained in the 6-cycle $30 - 7 - 5 - 154 - 3 - 11 - 30$ but not in a cycle of smaller length. Moreover, this cycle is not unique. $30 - 7 - 3 - 154 - 5 - 11 - 30$ is another cycle.*

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A SIMPLE RESTRICTED PRIESTLEY DUALITY FOR BOUNDED DISTRIBUTIVE LATTICES WITH AN ORDER-INVERTING OPERATION

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Introduction. Bounded distributive lattices with a single unary order-inverting operation form an algebraic semantics (in a technical sense of Blok-Jónsson equivalence, cf. [4]) for a logic of a minimal negation on top of the classical disjunction and conjunction. This logic was investigated in [8], and found particularly useful for analysing various forms of negation occurring in natural languages. It is quite easy to give a natural sequent system for that logic, and prove cut elimination.

Although Priestley-like dualities for distributive-lattice-based algebras are many and varied, they are either very general and quite complex (e.g., [1] or [3]), or not quite as general as needed here (e.g., [6] or [7]). Canonical extensions, which of course cover our case and a topological duality can be extracted from them (not without some work, see e.g., [5]), are a significantly different setting.

Apart from the connection to the logic of minmal negation, I choose to work with a single unary order-inverting operation only for simplicity. Generalising to any number of unary order-inverting or order-preserving operations is completely straightforward, and generalisations to operations of arbitrary arities should not be difficult either. However, generality and naturalness seem to be contravariant here.

Algebras. Let BDLN (*bounded distributive lattices with negation*) stand for the class of all algebras $\mathbf{A} = \langle A; \wedge, \vee, \neg, 0, 1 \rangle$ such that $\langle A; \wedge, \vee, 0, 1 \rangle$ is a bounded distributive lattice, and $\neg$ is a unary operation on A satisfying the quasiequation

$$x \le y \Rightarrow \neg y \le \neg x \tag{$\star$}$$

which states that $\neg$ is an order-inverting operation. It is easily shown that BDLN is a variety, axiomatised by adding any one of

$$\neg x \vee \neg y \le \neg(x \wedge y)$$
$$\neg(x \vee y) \le \neg x \wedge \neg y$$

to the identities defining bounded distributive lattices.

Dual spaces. Some notation first. For a Priestley space P, we write $\mathrm{Clup}(P)$ for the set of clopen upsets of P. For any ordered set P, we write $\mathcal{O}(P)$ for the set of downsets (order ideals) of P. Any order-preserving map $h \colon P \to Q$ between ordered sets P and Q can be naturally lifted to the setwise inverse map $h^{-1} \colon \mathcal{P}(Q) \to \mathcal{P}(P)$ taking each $X \in \mathcal{P}(Q)$ to $h^{-1}(X) \in \mathcal{P}(P)$. It maps upsets to upsets and downsets to downsets. The lifting can be iterated to $(h^{-1})^{-1} \colon \mathcal{P}(\mathcal{P}(P)) \to \mathcal{P}(\mathcal{P}(Q))$. We will write $\overline{h}$ for this double lifting.

As expected, we will now define a category of Priestley spaces with an additional structure. The objects are pairs $\big(P, \ \mathcal{N} \colon P \to \mathcal{O}(\mathrm{Clup}(P))\big)$, such that:

(1) P is a Priestley space.
(2) $\mathrm{Clup}(P)$ is the set of clopen upsets of P.
(3) $\mathcal{O}(\mathrm{Clup}(P))$ is the set of downsets of $\mathrm{Clup}(P)$.

(4) $\mathcal{N}\colon P \to \mathcal{O}(\mathrm{Clup}(P))$ is an order-preserving map, such that for every $X \in \mathrm{Clup}(P)$, the set $\{p \in P\colon X \in \mathcal{N}(p)\}$ is clopen.

Since the domain and range of the map $\mathcal{N}\colon P \to \mathcal{O}(\mathrm{Clup}(P))$ are completely determined by P, from now on we will write $(P, \mathcal{N}_P)$ for the objects. One may find it convenient to think of $\mathcal{N}$ as associating a system of non-topological neighbourhoods to any point in P. If P is finite, then $(P, \mathcal{N}_P)$ is just P together with an order-preserving map from P to the set of downsets of (the poset of) upsets of P. If P is a singleton there are precisely three such objects, and their dual algebras generate the three minimal subvarieties of BDLN.

Let $(P, \mathcal{N}_P)$ and $(Q, \mathcal{N}_Q)$ be objects, and let $h\colon P \to Q$ be a continuous map. Since h is continuous, the map $h^{-1}\colon \mathrm{Clup}(Q) \to \mathrm{Clup}(P)$ is well defined. It follows that the double lifting $\overline{h}$ is also well defined as a map from $\mathcal{O}(\mathrm{Clup}(P))$ to $\mathcal{O}(\mathrm{Clup}(Q))$. It is easy to verify that, for a $W \in \mathcal{O}(\mathrm{Clup}(P))$, we have $\overline{h}(W) = \{U \in \mathrm{Clup}(Q)\colon h^{-1}(U) \in W\}$.

Now we can define morphisms. A morphism from $(P, \mathcal{N}_P)$ to $(Q, \mathcal{N}_Q)$ is a continuous map h such that the diagram

$$
\begin{array}{ccc}
P & \xrightarrow{\ \ h\ \ } & Q \\
\Big\downarrow{\mathcal{N}_P} & & \Big\downarrow{\mathcal{N}_Q} \\
\mathcal{O}(\mathrm{Clup}(P)) & \xrightarrow{\ \ \overline{h}\ \ } & \mathcal{O}(\mathrm{Clup}(Q))
\end{array}
$$

commutes. The category we have just defined will be called *Priestley neighbourhood systems*, or $\mathbb{PNS}$.

Theorem 1. *The categories* BDLN *(with homomorphisms) and* $\mathbb{PNS}$ *are dually equivalent.*

Indeed, this duality is an instance of a restricted Prestley duality, in the sense of [2]. Several existing dualities can be obtained as special cases.

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