An introduction to Abstract Algebraic Logic

Parts III

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Leibniz operator

Definition

Given an algebra $\boldsymbol{A}$, the Leibniz operator is the map

$$\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathcal{P}(A) \to \mathrm{Con}\boldsymbol{A}$$

defined by the rule $F \longmapsto \boldsymbol{\Omega}^{\boldsymbol{A}} F$.

- Recall that:

$$\langle a, b \rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F \iff \text{for every unary pol. function } p \colon \boldsymbol{A} \to \boldsymbol{A}, \; p(a) \in F \text{ if and only if } p(b) \in F.$$

- The Leibniz operator (restricted to deductive filters) can be used to characterize interesting facts about logics, e.g. semantic characterization of algebraizability.

Leibniz operator in algebraizable logics

Theorem (semantic characterization of algebraizability)

Let $\vdash$ be a logic and K a generalized quasi-variety. TFAE:

1. $\vdash$ is algebraizable with equivalent algebraic semantics K.
2. For every algebra $\boldsymbol{A}$ there is $\Phi^{\boldsymbol{A}} \colon \mathcal{F}i_{\vdash}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ that commutes with endomorphisms σ in the sense that $\Phi^{\boldsymbol{A}}\sigma^{-1}F = \sigma^{-1}\Phi^{\boldsymbol{A}}F$ for every $F \in \mathcal{F}i_{\vdash}\boldsymbol{A}$.
3. There is a lattice isomorphism $\Phi \colon \mathcal{T}h(\vdash) \to \mathcal{T}h(\vDash_{\mathsf{K}})$ that commutes with substitutions σ in the sense that $\Phi\sigma^{-1}\Gamma = \sigma^{-1}\Phi\Gamma$ for every $\Gamma \in \mathcal{T}h(\vdash)$.

Moreover, $\Phi^{\boldsymbol{A}}$ can be always taken to be $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathcal{F}i_{\vdash}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$.

Leibniz operator in algebraizable logics

Theorem (semantic characterization of algebraizability)

Let $\vdash$ be a logic and K a generalized quasi-variety. TFAE:

1. $\vdash$ is algebraizable with equivalent algebraic semantics K.
2. $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathcal{F}i_{\vdash}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ is an iso that commutes with endomorphisms σ in the sense that $\sigma^{-1}\boldsymbol{\Omega}^{\boldsymbol{A}}F = \boldsymbol{\Omega}^{\boldsymbol{A}}\sigma^{-1}[F]$, for every algebra $\boldsymbol{A}$ and $F \in \mathcal{F}i_{\vdash}\boldsymbol{A}$.
3. $\boldsymbol{\Omega} \colon \mathcal{T}h(\vdash) \to \mathcal{T}h(\vDash_{\mathsf{K}})$ is an iso that commutes with substitutions σ.

- Thus the fact that the Leibniz operator is an iso preserving substitutions characterizes algebraizability.

Equivalential logics

Definition

Let $\vdash$ be a logic.

1. $\vdash$ is equivalential if there is a set of formulas $\Delta(x, y)$ such that for every model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$,

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F \iff \Delta(a, b) \subseteq F.$$

2. $\vdash$ is finitely equivalential if, moreover, Δ can be chosen finite.

- This idea abstract the Lindenbaum-Tarski process: **IPC** and **CPC** are equivalential with

$$\Delta(x, y) = \{x \to y, y \to x\}$$

i.e. if $\langle \boldsymbol{A}, F\rangle$ is a model of **IPC**, then

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F \iff \{a \to b, b \to a\} \subseteq F.$$

Equivalential logics: syntactic characterization

- Recall that:

Theorem (definability of Leibniz congruence)

Let $\vdash$ be a logic and $\Delta(x, y)$ be a set of formulas. TFAE:

1. For every model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$,

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F \iff \Delta^{\boldsymbol{A}}(a, b) \subseteq F.$$

2. The following inferences are valid in $\vdash$:

$$\emptyset \vdash \Delta(x, x) \quad \text{(Ref)}$$
$$x, \Delta(x, y) \vdash y \quad \text{(MP)}$$
$$\bigcup_{i \leq n} \Delta(x_i, y_i) \vdash \Delta(f(\vec{x}), f(\vec{y})) \quad \text{(Rep)}$$

for all connectives f of $\vdash$.

Equivalential logics: syntactic characterization

Theorem

A logic $\vdash$ is equivalential if and only if there exists a set $\Delta(x, y)$ of formulas such that:

$$\emptyset \vdash \Delta(x, x) \quad \text{(Ref)}$$
$$x, \Delta(x, y) \vdash y \quad \text{(MP)}$$
$$\bigcup_{i \leq n} \Delta(x_i, y_i) \vdash \Delta(f(\vec{x}), f(\vec{y})) \quad \text{(Rep)}$$

for all connectives f of $\vdash$.

Corollary

Every algebraizable logic is equivalential: if the algebraization of $\vdash$ is witnessed by the sets of formulas $\Delta(x, y)$ and of equations $E(x)$, then the equivalentiality of $\vdash$ is witnessed by Δ.

Equivalential logics: modal examples

- Recall that local modal consequence $\vdash^l_{\mathbf{K}}$ is not algebraizable.
- However it is equivalential with

$$\Delta(x, y) = \{\Box^n(x \to y), \Box^n(y \to x) : n \in \omega\}.$$

- Hint: apply syntactic characterization of equivalentiality to Δ.
- However $\vdash^l_{\mathbf{K}}$ is not finitely equivalential (hints: later on).
- $\vdash^l_{\mathbf{K4}}$ is finitely equivalential with

$$\Delta(x, y) = \{x \to y, y \to x, \Box(x \to y), \Box(y \to x)\}.$$

- $\vdash^l_{\mathbf{S4}}$ is finitely equivalential with

$$\Delta(x, y) = \{\Box(x \to y), \Box(y \to x)\}.$$

Equivalential logics: semantic characterization

Theorem

Let $\vdash$ be a logic. TFAE:

1. $\vdash$ is equivalential.
2. $\Omega^{\boldsymbol{A}} \colon \mathcal{F}i_{\vdash}\boldsymbol{A} \to \mathrm{Con}\boldsymbol{A}$ is monotone and commutes with endomorphisms σ in the sense that $\sigma^{-1}\Omega^{\boldsymbol{A}}F = \Omega^{\boldsymbol{A}}\sigma^{-1}[F]$, for every algebra $\boldsymbol{A}$ and $F \in \mathcal{F}i_{\vdash}\boldsymbol{A}$.
3. $\Omega \colon \mathcal{T}h(\vdash) \to \mathrm{Con}\boldsymbol{Fm}$ is monotone and commutes with substitutions σ.

Moreover, $\vdash$ is finitely equivalentialy if $\Omega^{\boldsymbol{A}} \colon \mathcal{F}i_{\vdash}\boldsymbol{A} \to \mathrm{Con}\boldsymbol{A}$ is continuous for every algebra $\boldsymbol{A}$.

- Remark: this provides a readily falsifiable characterization of equivalentiality.

Equivalential logics: class-operators characterization

Theorem

Let $\vdash$ be a logic.

1. $\vdash$ is equivalential iff $\mathrm{Mod}^*(\vdash)$ is closed under $\mathbb{S}$ and $\mathbb{P}$.
2. $\vdash$ is finitary finitely equiv. iff $\mathrm{Mod}^*(\vdash)$ is closed under $\mathbb{S}, \mathbb{P}, \mathbb{P}_{\mathrm{u}}$.

- An algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \neg, 0, 1\rangle$ is an orthALattice when $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded lattice such that

$$\neg(x \wedge y) \approx \neg x \vee \neg y \quad \neg\neg x \approx x$$
$$x \vee \neg x \approx 1 \quad x \wedge \neg x \approx 0.$$

- Let OL be the variety of ortholattices. Consider the logic

$$\Gamma \vdash_{\mathbf{OL}} \varphi \iff \text{for all } \boldsymbol{A} \in \mathrm{OL} \text{ and evaluation } v \colon \boldsymbol{Fm} \to \boldsymbol{A}$$
$$\text{if } v[\Gamma] = 1, \text{ then } v(\varphi) = 1.$$

- $\vdash_{\mathbf{OL}}$ is not equivalential, as $\mathrm{Mod}^*(\vdash_{\mathbf{OL}})$ is not closed under $\mathbb{S}$.

Equivalential logics: recap

Characterizations of equivalentiality for $\vdash$

- Syntactic: $\vdash$ satisfies the rules

$$\emptyset \vdash \Delta(x, x) \quad \text{(Ref)}$$
$$x, \Delta(x, y) \vdash y \quad \text{(MP)}$$
$$\bigcup_{i \leq n} \Delta(x_i, y_i) \vdash \Delta(f(\vec{x}), f(\vec{y})) \quad \text{(Rep)}$$

- Semantic: $\Omega^{\boldsymbol{A}} \colon \mathcal{F}i_{\vdash}\boldsymbol{A} \to \mathrm{Con}\boldsymbol{A}$ is monotone and commutes with endomorphisms.
- Class operators: $\mathrm{Mod}^*(\vdash)$ is closed under $\mathbb{S}$ and $\mathbb{P}$.

Protoalgebraic logics

Definition

A logic $\vdash$ is protoalgebraic if there is a set of formulas $\Delta(x, y, \vec{z})$ such that for every model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$,

$$\langle a, b\rangle \in \Omega^{\boldsymbol{A}}F \iff \Delta(a, b, \vec{c}) \subseteq F \text{ for all } \vec{c} \in A.$$

- First examples: All equivalential logics are protoalgebraic.

Protoalgebraic logics: characterizations

- Protoalgebraic logics can be characterized in different ways:

Theorem

Let $\vdash$ be a logic. TFAE:

1. $\vdash$ is protoalgebraic.
2. There exists a set of formulas $\Delta(x,y)$ such that

$$\emptyset \vdash \Delta(x,x) \qquad \text{(Ref)}$$
$$x, \Delta(x,y) \vdash y \qquad \text{(MP)}$$

3. $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathcal{F}i_{\vdash}\boldsymbol{A} \to \mathrm{Con}\boldsymbol{A}$ is monotone, for every algebra $\boldsymbol{A}$.
4. $\mathrm{Mod}^{*}(\vdash)$ is closed under $\mathbb{P}_{\mathrm{sd}}$.

- By 2 all logics having an implication-like connective are protoalgebraic, e.g. $\vdash_{\mathbf{OL}}$ with $\Delta(x,y) = \{\neg x \vee y\}$.

Protoalgebraic logics: parametrized local deduction theorem

Definition

1. $\vdash$ has the parametrized local deduction theorem (PLDDT) if there is a family of sets of formulas $\{\Phi_i(x,y,\vec{z}) : i \in I\}$ s.t.

$$\Gamma, \psi \vdash \varphi \iff \text{there is } i \in I \text{ and } \vec{\gamma} \text{ s.t. } \Gamma \vdash \Phi_i(\psi, \varphi, \vec{\gamma}).$$

2. $\vdash$ has the local contextual deduction theorem (LCDDT) if for every $n \in \omega$ there is a family of sets of formulas $\Psi_n = \{\Phi_i(x_1, \dots, x_n, y_1, y_2) : i \in I\}$ such that for every $\Gamma \cup \{\varphi, \psi\}$ in variables $x_1, \dots, x_n$,

$$\Gamma, \psi \vdash \varphi \iff \text{there is } \Phi_i \in \Psi_n \text{ s.t. } \Gamma \vdash \Phi_i(x_1, \dots, x_n, \psi, \varphi).$$

Theorem

$\vdash$ is protoalgebraic iff it has PLDDT iff it has LCDDT.

Protoalgebraic logics: a finite basis theorem

- Protoalgebraic logics (as opposed to algebraizable ones) are the definitive framework to state most bridge theorems.
- Moreover, they are amenable to provide generalizations of the deductively-related aspects of universal algebra:

Theorem

Let $\boldsymbol{A}$ be a finite algebra of finite type. If $\mathbb{V}(\boldsymbol{A})$ is congruence distributive, then it is finitely based.

Theorem

Let M be a finite set of finite matrices of finite type, which induces a protoalgebraic logic $\vdash$. If $\vdash$ is filter distributive, then it is finitely axiomatizable.

- Generalizations involving logical variants of "definable principal subcongruences" are available as well.

Equational definability of truth-sets

- Matrices $\langle \boldsymbol{A}, F\rangle$ are models of logics where

$$\boldsymbol{A} = \text{structured set of truth values}$$
$$F = \text{values representing truth}$$

- We say that F is the truth-set of the matrix $\langle \boldsymbol{A}, F\rangle$.

Definition

Let M be a class of matrices.

1. Truth is equationally definable in M if there is a set of equations $E(x)$ such that for every $\langle \boldsymbol{A}, F\rangle \in$ M,

$$F = \{a \in A : \boldsymbol{A} \vDash E(a)\}.$$

2. Truth is universally definable in M if there is a set of equations $E(x, \vec{z})$ such that for every $\langle \boldsymbol{A}, F\rangle \in$ M with $F \neq \emptyset$,

$$F = \{a \in A : \boldsymbol{A} \vDash E(a, \vec{c}) \text{ for every } \vec{c} \in A\}.$$

Equational definability of truth-sets: characterization

Theorem

For a logic $\vdash$ TFAE:

1. Truth is equationally (resp. universally) definable in $\mathrm{Mod}^*(\vdash)$.
2. $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathcal{F}i_{\vdash}\boldsymbol{A} \to \mathrm{Con}\boldsymbol{A}$ is completely order-reflecting (resp. over $\mathcal{F}i_{\vdash}\boldsymbol{A} \smallsetminus \{\emptyset\}$), for every algebra $\boldsymbol{A}$.
3. $\boldsymbol{\Omega} \colon \mathcal{T}h(\vdash) \to \mathrm{Con}\boldsymbol{Fm}$ is completely order-reflecting (resp. over $\mathcal{T}h(\vdash) \smallsetminus \{\emptyset\}$).

- Remark: Truth is equationally definable in $\mathrm{Mod}^*(\vdash)$ for all algebraizable logics $\vdash$: if the algebraization of $\vdash$ is witnessed by $\Delta(x, y)$ and $E(x)$, then $E(x)$ defines truth sets in $\mathrm{Mod}^*(\vdash)$.

Corollary

$\vdash$ is algebraizable iff it is equivalential and truth is equationally definable in $\mathrm{Mod}^*(\vdash)$.

Equational definability of truth-sets: examples

- Consider the $\langle \wedge, \vee, \neg, 0, 1\rangle$-fragment $\mathbf{IPL}^*$ of $\mathbf{IPC}$.
- An algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \neg, 0, 1\rangle$ is a pseudocomplemented lattice if it is a bounded lattice such that for every $a \in A$,

$$\neg a = \max\{c \in A : a \wedge c = 0\}.$$

- If $\langle \boldsymbol{A}, F\rangle \in \mathrm{Mod}^*(\vdash_{\mathbf{IPL}^*})$, then $\boldsymbol{A}$ is a pseudocomplemented distributive lattice and $F = \{1\}$.
- Hence truth is equationally definable in $\mathrm{Mod}^*(\vdash_{\mathbf{IPL}^*})$ by

$$E(x) = \{x \approx 1\}.$$

- However, $\mathbf{IPL}^*$ is not protoalgebraic (hint: disprove monotonicity of $\boldsymbol{\Omega}^{\boldsymbol{A}}$).

Implicit definability of truth-sets

Definition

Truth is implicitly definable in a class of matrices M, if the members of M are determined by their algebraic reducts, in the sense that if $\langle \boldsymbol{A}, F\rangle, \langle \boldsymbol{A}, G\rangle \in \mathrm{M}$, then $F = G$.

- Let $\mathbf{S4}^*$ be the $\langle \Box, 1\rangle$-fragment of $\vdash^{l}_{\mathbf{S4}}$. Truth is implicitly, but not equationally, definable in $\mathrm{Mod}^*(\vdash_{\mathbf{S4}^*})$.

Lemma

Truth is implicitly definable in $\mathrm{Mod}^*(\vdash)$ iff $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathcal{F}i_{\vdash}\boldsymbol{A} \to \mathrm{Con}\boldsymbol{A}$ is injective for every algebra $\boldsymbol{A}$.

- The injectivity of $\boldsymbol{\Omega}^{\boldsymbol{A}}$ cannot be equivalently restricted to theories $\mathcal{T}h(\vdash)$ unless the language of $\vdash$ is countable.

Beth-like definability theorem

- Classical Beth's theorem in 1st order logic states that implicit and explicit definability coincide.

Theorem

Let $\vdash$ be protoalgebraic. Truth is implicitly definable in $\mathrm{Mod}^*(\vdash)$ iff it is equationally definable.

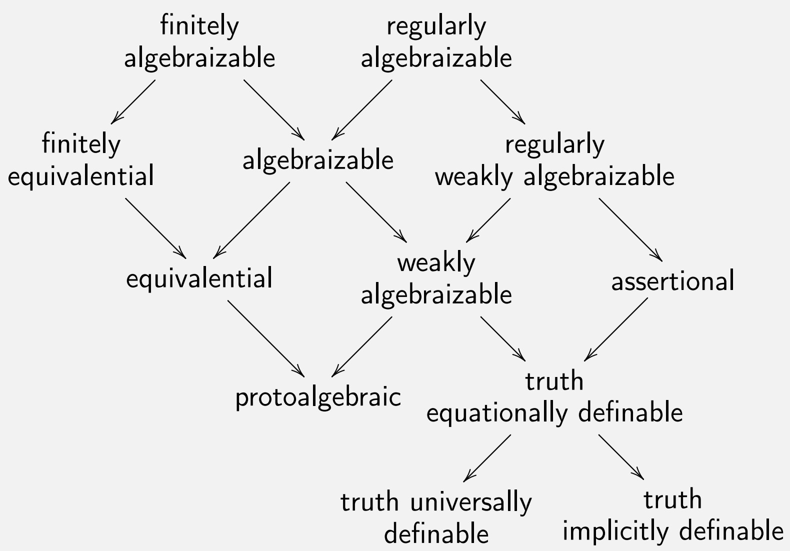
Definition

A logic $\vdash$ is weakly algebraizable when it is protoalgebraic and truth is equationally definable in $\mathrm{Mod}^*(\vdash)$.

Corollary

For a logic $\vdash$ TFAE: $\vdash$ is weakly algebraizable iff $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathcal{F}i_{\vdash}\boldsymbol{A} \to \mathrm{Con}\boldsymbol{A}$ is monotone and injective for every $\boldsymbol{A}$ iff $\boldsymbol{\Omega} \colon \mathcal{T}h(\vdash) \to \mathrm{Con}\boldsymbol{Fm}$ is monotone and injective.

Leibniz hierarchy



Miscellanea

Computational aspects:

- The problem of classifying logics presented by Hilbert calculi in the Leibniz hierarchy in undecidable.
- The problem of determining whether logics presented by a finite set of finite matrices of finite type belong to a given level of the Leibniz hierarchy if decidable but (in most cases) complete for **EXPTIME**.

Related topics:

- A hierarchy somehow parallel to the Leibniz one was introduced to focus on implication (as opposed to equivalence).
- Relations between the Leibniz and Maltsev hierarchy are being explored.