An introduction to Abstract Algebraic Logic

Parts IV

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Frege hiearchy

Definition

Let $\vdash$ be a logic.

1. $\vdash$ is selfextensional when for every n-ary connective f,
$$\text{if } \varphi_i \dashv\vdash \psi_i \text{ for } i = 1, \dots, n, \text{ then } f(\vec{\varphi}) \dashv\vdash f(\vec{\psi}).$$
2. $\vdash$ is Fregean when for every n-ary connective f and set of formulas Γ,
$$\text{if } \Gamma, \varphi_i \dashv\vdash \psi_i, \Gamma \text{ for } i = 1, \dots, n, \text{ then } \Gamma, f(\vec{\varphi}) \dashv\vdash f(\vec{\psi}), \Gamma.$$

- Remark: selfextensionality and Fregeanity are inherited by fragments.

Frege hierarchy

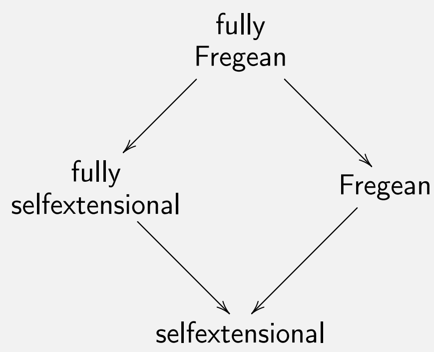
Definition

Let $\vdash$ be a logic.

1. $\vdash$ is fully selfextensional, when for every algebra $\boldsymbol{A}$ and every connective f,
$$\text{if } \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(a_i) = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(b_i), \text{ then } \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(f(\vec{a})) = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(f(\vec{b})).$$
2. $\vdash$ is fully Fregean, when for every algebra $\boldsymbol{A}$, every connective f and every set $F \subseteq A$,
$$\text{if } \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(F, a_i) = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(F, b_i), \text{ then } \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(F, f(\vec{a})) = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(F, f(\vec{b})).$$

- Remark: selfextensionality (Fregeanity) amounts to full selfextensionality (resp. full Fregeanity) restricted to the case where $\boldsymbol{A} = \boldsymbol{Fm}$.

Frege hierarchy



Frege hierarchy: examples

- Axiomatic extensions of **IPC** (e.g. **CPC**) are fully Fregean.
- Local modal logic $\vdash^{l}_{\mathbf{K}}$ is fully selfextensional, but not Fregean:

$$x, x \vee y \dashv\vdash^{l}_{\mathbf{K}} x, x \text{ but } x, \Box(x \vee y) \nvdash^{l}_{\mathbf{K}} \Box x.$$

- The same holds for Belnap-Dunn logic **BD**:

$$x, x \dashv\vdash_{\mathbf{BD}} x \vee y, x \text{ but } x, \neg x \nvdash_{\mathbf{BD}} \neg(x \vee y).$$

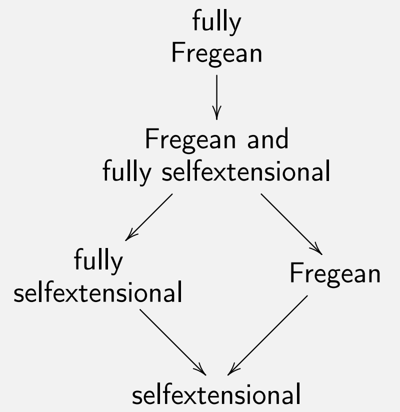
- The $\langle \neg, 1 \rangle$-fragment of **CPC** is Fregean, but not fully selfext.
- There are (artificial) selfextensional logic, neither fully selfextensional or Fregean.
- Łukasiewicz logic **Ł** is not selfextensional:

$$x \dashv\vdash_{\mathbf{Ł}} x * x \text{ but } \neg(x * x) \nvdash_{\mathbf{Ł}} \neg x.$$

- Global modal logic $\vdash^{g}_{\mathbf{K}}$ is not selfextensional:

$$x \dashv\vdash^{g}_{\mathbf{K}} x \wedge \Box x \text{ but } \Diamond x \nvdash^{g}_{\mathbf{K}} \Diamond(x \wedge \Box x).$$

Frege hierarchy



- Is it true that fully Fregean = Fregean + fully selfextensional?
- For logics with theorems yes.

Logics with a conjunction vs semilattice-based logics

Definition

A logic $\vdash$ has a conjunction if there is a formula $x \wedge y$ such that

$$x, y \vdash x \wedge y \quad x \wedge y \vdash x \quad x \wedge y \vdash y.$$

Definition

A logic $\vdash$ is semilattice-based if there is a class of algebras K with semilattice reduct such that

$$\gamma_1, \ldots, \gamma_n \vdash \varphi \iff \mathsf{K} \vDash (\gamma_1 \wedge \cdots \wedge \gamma_n) \leq \varphi$$

where $\leq$ is the meet-semilattice order.

- Lattice based logics have a conjunction (the converse is false).
- **CPC**, **IPC** and $\vdash^{l}_{\mathbf{K}}$ are lattice based logics.

Selfextensional logics with a conjunction

Theorem

Let $\vdash$ be a finitary selfextensional logic with a conjunction $\wedge$.

1. $\wedge$ is a semilattice operation in $\mathrm{Alg}^{*}(\vdash)$.
2. $\mathcal{F}i_{\vdash}\boldsymbol{A}$ is the class of semilattice filters of $\boldsymbol{A}$ (possibly with $\emptyset$), for every $\boldsymbol{A} \in \mathrm{Alg}^{*}(\vdash)$.
3. $\vdash$ is semilattice based on $\mathrm{Alg}^{*}(\vdash)$.
4. $\vdash$ is fully selfextensional.

Selfextensional logics with a conjunction: Gentzen systems

- Let $\vdash$ be a finitary selfextensional logic with a conjunction.
- Pick sequents of the form $\langle \gamma_1, \ldots, \gamma_n \rangle \rhd \varphi$, possibly with empty antecedent if $\vdash$ has theorems.
- Then define a Gentzen system $\vdash_{\mathbf{G}}$ as follows: structural rules plus

$$\frac{\emptyset}{\Gamma \rhd \varphi} \qquad \frac{\{x_i \rhd y_i, y_i \rhd x_i : i \leq n\}}{f(x_1, \ldots, x_n) \rhd f(y_1, \ldots, y_n)}$$

for all $\Gamma \cup \{\varphi\}$ s.t. $\Gamma \vdash \varphi$, and for all n-ary connectives f.

Theorem

$\vdash_{\mathbf{G}}$ is adequate for $\vdash$ is the sense that

$$\Gamma \vdash \varphi \Longleftrightarrow \emptyset \vdash_{\mathbf{G}} \Gamma \rhd \varphi.$$

Moreover, $\vdash_{\mathbf{G}}$ is algebraizable with equiv. alg. sem. $\mathbb{Q}(\mathsf{Alg}^*(\vdash))$.

Selfextensional logics with a conjunction: Gentzen systems

- Hence there are non-algebraizable fully selfextensional logics $\vdash$, which can be described by algebraizable Gentzen systems $\vdash_{\mathbf{G}}$.
- Thus $\vdash_{\mathbf{G}}$ (as opposed to $\vdash$) can be used to exploit bridge theorems w.r.t. $\mathsf{Alg}^*(\vdash)$.

Example

- Let $\mathbf{CPC}_{\wedge\vee}$ be the $\langle \wedge, \vee \rangle$-fragment of $\mathbf{CPC}$.
- Clearly $\mathbf{CPC}_{\wedge\vee}$ is finitary selfextensional with a conjunction.
- Then consider its algebraizable Gentzen system $\vdash_{\mathbf{G}_{\wedge\vee}}$, given by structural rules plus

$$\frac{\emptyset}{\varphi \rhd \varphi} \qquad \frac{\Gamma, \varphi, \psi \rhd \gamma}{\Gamma, \varphi \wedge \psi \rhd \gamma} \qquad \frac{\Gamma \rhd \varphi \quad \Gamma \rhd \psi}{\Gamma \rhd \varphi \wedge \psi}$$

$$\frac{\Gamma, \varphi \rhd \gamma \quad \Gamma, \psi \rhd \gamma}{\Gamma, \varphi \vee \psi \rhd \gamma} \qquad \frac{\Gamma \rhd \varphi}{\Gamma \rhd \varphi \vee \psi} \qquad \frac{\Gamma \rhd \varphi}{\Gamma \rhd \psi \vee \varphi}$$

Selfextensional logics with a conjunction: Gentzen systems

Example

- Then $\vdash_{\mathbf{G}_{\wedge\vee}}$ is algebraizable with equiv. algebraic semantics DL.
- Now, DL has some nice algebraic properties, e.g. EDPC: for all $\boldsymbol{A} \in \mathsf{DL}$,

$$\langle c, d \rangle \in \mathsf{Cg}(a, b) \Longleftrightarrow \text{ both } \begin{cases} c \wedge a \wedge b & = d \wedge a \wedge b \\ c \vee a \vee b & = d \vee a \vee b \end{cases}.$$

- We can apply the bridge with $\vdash_{\mathbf{G}_{\wedge\vee}}$ (but not with $\mathbf{CPC}_{\wedge\vee}$) and obtain that $\vdash_{\mathbf{G}_{\wedge\vee}}$ has the DDT:

$$\Delta, \langle \gamma_1, \ldots, \gamma_n \rangle \rhd \psi \vdash_{\mathbf{G}_{\wedge\vee}} \langle \delta_1, \ldots, \delta_m \rangle \rhd \varphi \Longleftrightarrow$$

$$\Delta \vdash_{\mathbf{G}_{\wedge\vee}} \begin{cases} \bigwedge \gamma_i \wedge \bigwedge \delta_j \wedge \psi \rhd \bigwedge \gamma_i \wedge \bigwedge \delta_j \wedge \psi \wedge \varphi \\ \bigwedge \gamma_i \vee \bigwedge \delta_j \rhd \bigwedge \gamma_i \vee (\varphi \wedge \bigwedge \delta_j) \end{cases}$$

Uniterm DDT

Definition

A logic $\vdash$ has the uniterm deduction theorem (u-DDT) if there exists a formula $x \to y$ such that for all $\Gamma \cup \{\psi, \varphi\}$,

$$\Gamma, \psi \vdash \varphi \Longleftrightarrow \Gamma \vdash \psi \to \varphi.$$

- $\mathbf{CPC}$, $\mathbf{IPC}$ and $\vdash^l_{\mathbf{K}}$ have the u-DDT.

Definition

Hilbert algebras are implicative subreducts of Heyting algebras. Equivalently they are algebras axiomatized by

$$x \to x \approx y \to y$$
$$(x \to x) \to x \approx x$$
$$x \to (y \to z) \approx (x \to y) \to (x \to z)$$
$$(x \to y) \to ((y \to x) \to y) \approx (y \to x) \to ((x \to y) \to x).$$

Hilbert-algebra-based logics

Definition

A logic $\vdash$ is Hilbert-algebra-based if there exists a class K of expanded Hilbert algebras such that

$$\gamma_1, \ldots, \gamma_n \vdash \varphi \iff \mathsf{K} \vDash \gamma_1 \to (\gamma_2 \to (\ldots (\gamma_n \to \varphi) \ldots)) \approx \top.$$

- Hilbert-algebra-based logics have the u-DDT w.r.t. $\to$.
- **CPC**, **IPC** and $\vdash^{l}_{\mathsf{K}}$ are Hilbert-algebra-based.

Theorem

Let $\vdash$ be finitary selfextensional logic with the u-DDT. w.r.t. $\to$. Then $\vdash$ is Hilbert-algebra based w.r.t. $\mathrm{Alg}^*(\vdash)$. Moreover, $\vdash$ is fully selfextensional and is described by an algebraizable Gentzen system.

- For finitary logics, in presence of conjunctions or u-DDT, selfextensionality implies full selfextensionality.

Definability of truth-sets

Definition

A logic $\vdash$ is assertional, when $\mathrm{Mod}^*(\vdash)$ is a class of matrices $\langle \boldsymbol{A}, F \rangle$ where F is singleton.

- If $\vdash$ is assertional, then it has a theorem $\top$ in variable x. Then truth is equationally definable in $\mathrm{Mod}^*(\vdash)$ by $x \approx \top$.

Theorem

1. A Fregean logic is assertional if and only if it has theorems.
2. If truth is equationally definable in $\mathrm{Mod}^*(\vdash)$, then $\vdash$ is fully selfextensional if and only if it is fully Fregean.

- Thus the Frege hierarchy collapses for finitary logics $\vdash$ for which truth is equationally definable in $\mathrm{Mod}^*(\vdash)$ and either with conjunction or the u-DDT.

Definability of logical equivalence

Theorem

Let $\vdash$ be a Fregean protoalgebraic logic. Then

1. $\vdash$ is equivalential.
2. If $\vdash$ is finitary, then it is fully Fregean.
3. If $\vdash$ has theorems, then it is (regularly) algebraizable.

- In case 3 (plus finitarity) the equivalent algebraic semantics have been characterized.

Definability of logical equivalence

Definition

Let K be a pointed quasi-variety.

1. K is congruence orderable if for every $\boldsymbol{A} \in \mathsf{K}$ and $a, b \in A$,
$$\mathrm{Cg}_{\mathsf{K}}(a, \top) = \mathrm{Cg}_{\mathsf{K}}(b, \top) \iff a = b.$$
2. K is relatively point regular if for every $\boldsymbol{A} \in \mathsf{K}$ and $\theta, \phi \in \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$, if $\top/\theta = \top/\phi$, then $\theta = \phi$.
3. K is Fregean if it is cong. orderable and rel. point-regular.

Theorem

Let $\vdash$ be a logic. TFAE:

1. $\vdash$ is finitary, protoalgebraic, Fregean and non-almost inc.
2. $\vdash$ is the $\top$-assertional logic of a Fregean quasi-variety K.

In this case, $\vdash$ is algebraizable with equiv. alg. sem. K.

u-DDT again

Definition

A Hilbert algebra with compatible operations is an algebra $\boldsymbol{A}$ with a Hilbert algebra reduct $\langle A, \to \rangle$ s.t. for every n-ary basic operation f,

$$\boldsymbol{A} \vDash (x \to y) \to ((y \to x) \to (f(\vec{z}, x, \vec{u}) \to f(\vec{z}, y, \vec{u}))) \approx \top$$

Theorem

Let $\vdash$ be a finitary Fregean protoalgebraic logics with the u-DDT. Then $\mathrm{Alg}^*(\vdash)$ is a variety of Hilbert algebras with compatible operations and $\vdash$ is algebraizable with equivalent algebraic semantics $\mathrm{Alg}^*(\vdash)$.