Adaptive methods for PDE-eigenvalue problems

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SEMINAR ON NUMERICAL ANALYSIS

Modelling and Simulation of Challenging Engineering Problems

January 18-22, 2010 Nové Hrady

Basic Notation

Definition

Let

$$\|u\|_{L_p(\Omega)} = \left(\int_\Omega |u|^p \right)^{\frac{1}{p}} \quad \text{for} \quad 1 \le p < \infty.$$

Then the Lebesgue space $L_p(\Omega)$ is defined as

$$L_p(\Omega) := \{u : \|u\|_{L_p(\Omega)} < \infty\}.$$

Definition

The Sobolev space based on $L_2(\Omega)$ is denoted by $\mathcal{H}^m(\Omega)$

$$\mathcal{H}^m(\Omega) := \{\varphi \in L_2(\Omega) : \partial^\alpha \varphi \in L_2(\Omega) \ \forall \ |\alpha| \leq m\},$$

with corresponding norm

$$\|\varphi\|_m := \|\varphi\|_{\mathcal{H}^m(\Omega)} = \|u\|_m = \|u\|_{m,\Omega} = \left\{ \sum_{|\alpha|\leq m} \|\partial^\alpha \varphi\|^2_{L_2(\Omega)} \right\}^{\frac{1}{2}},$$

and seminorm

$$|\varphi|_m := |\varphi|_{\mathcal{H}^m(\Omega)} = \left\{ \sum_{|\alpha|=m} \|\partial^\alpha \varphi\|^2_{L_2(\Omega)} \right\}^{\frac{1}{2}}.$$

Example

- m = 0, $\mathcal{H}^0(\Omega) = L_2(\Omega)$

$$\|u\|_{\mathcal{H}^0(\Omega)} = \|u\|_0 = \|u\|_{0,\Omega} = \|u\|_{L_2(\Omega)}.$$

- m = 1, $\mathcal{H}^1(\Omega)$

$$\|u\|_{\mathcal{H}^1(\Omega)} = \|u\|_1 = \|u\|_{1,\Omega} := \left\{\|u\|^2_{L_2(\Omega)} + \|\nabla u\|^2_{L_2(\Omega)}\right\}^{\frac{1}{2}},$$

$$|u|_{\mathcal{H}^1(\Omega)} = |u|_1 = |u|_{1,\Omega} := \left\{\|\nabla u\|^2_{L_2(\Omega)}\right\}^{\frac{1}{2}}.$$

[I. Babuška & J. Osborn, Eigenvalue Problems, In: Ciarlet, P.G. & Lions, J.L.: Handbook of Numerical Analysis, Vol. II, Elsevier Science Publishers B.V., North-Holland, 1991, 641–787.]

[G. Strang & G.J. Fix, An analysis of the finite element method, Prentice-Hall, 1973.]

Classical and Variational formulation

Definition

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d = 1, 2, \ldots$ and $\mathcal{A}, \mathcal{B}$ are a (non-)selfadjoint second-order elliptic operator and a symmetric, positive definite operator, respectively. The *classical formulation* of the eigenvalue problem is

$$\mathcal{A}u = \lambda \mathcal{B}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Then the eigenpair $(\lambda, u) \in \mathbb{R} \times V$ satisfies the *variational formulation*

$$a(u, v) = \lambda b(u, v) \quad \forall v \in V.$$

where $a : V \times V \to \mathbb{F}$, $b : H \times H \to \mathbb{R}$ are bilinear (sesquilinear in the complex case) forms, generated by $\mathcal{A}$ and $\mathcal{B}$, respectively, and $V \subset H \subset V^*$.

Remark

The bilinear (sesquilinear) form $a(\cdot,\cdot)$ is assumed to be bounded in V and V-elliptic, that is

$$|a(u,v)| \leq C_1 \|u\|_V \|v\|_V \quad \forall u,v \in V, \quad \text{Re } a(u,u) \geq C\|u\|_V \quad \forall u \in V,$$

and the bilinear (sesquilinear) form $b(\cdot,\cdot)$ is assumed to be bounded in H, i.e.,

$$|b(u,v)| \leq C_2 \|u\|_H \|v\|_H \quad \forall u,v \in H,$$

where $C_1, C_2, C > 0$.

Remark

The bilinear (sesquilinear) forms induce the corresponding norms

$$|||u||| := \|u\|_a = a(u,u)^{\frac{1}{2}}, \quad u \in V, \quad \text{and} \quad \|u\|_b := b(u,u)^{\frac{1}{2}}, \quad u \in H,$$

where $|||.||| \simeq \|.\|_V$ and $\|.\|_b \simeq \|.\|_H$.

Discrete eigenvalue problem

For a finite-dimensional subspace $V_h \subseteq V$, the eigenvalue problem:
Determine a non-trivial eigenpair $(\lambda, u) \in \mathbb{R} \times \mathcal{H}^1_0(\overline{\Omega})$ with $\|u\|_{L_2(\Omega)} = 1$ s.t.

$$a(u, v) = \lambda b(u, v) \quad \forall v \in V,$$

is approximated by the *discrete eigenvalue problem*:
Determine a non-trivial eigenpair $(\lambda_h, u_h) \in \mathbb{R} \times V_h)$ with $\|u_h\|_{L_2(\Omega)} = 1$ s.t.

$$a(u_h, v_h) = \lambda b(u_h, v_h) \quad \forall v_h \in V_h.$$

[I. Babuška & J. Osborn, Eigenvalue Problems, In: Ciarlet, P.G. & Lions, J.L.: Handbook of Numerical Analysis, Vol. II, Elsevier Science Publishers B.V., North-Holland, 1991, S. 641-787.]

[E.M. Garau and P. Morin and C. Zuppa, Convergence of adaptive finite element methods for eigenvalue problems, Preprint, arXiv:0803.0365v1, 2008.]

Definition (Regular triangulation, [Ver96])

The family of triangulations $\mathcal{T}_h, h > 0$ of Ω, which satisfies the following conditions:

- any two triangles in $\mathcal{T}_h$ share at most a common edge or a common vertex (in 2D),
- the minimal angle of all triangles in the whole family $\mathcal{T}_h$ is bounded away from zero,

is called regular.

[M. Ainsworth and J.T. Oden, A Posteriori Error Estimation in Finite Element Analysis, John Wiley & Sons, Inc. 2000.]

[R. Verfürth, A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques, Wiley and Teubner,New York, Stuttgart, 1996.]

The Ritz-Galerkin approximation

Let $\mathcal{T}_h$ be the partition of $\bar{\Omega}$ into elements and let P_p denote the set of continuous piecewise polynomial functions of total degree $p \geq 1$, which vanish on the boundary of Ω.
The *Ritz-Galerkin* discretization is given by

$$a(u_h, v_h) = \lambda_h b(u_h, v_h) \quad \text{for all } v_h \in V_h^p,$$

where $V_h^p \subset V$ is finite element space with dimension $dim\ V_h = n_h$, i.e.

$$V_h^p(\Omega) := \{v \in V : v|_T \in P_p, \ \text{for all } T \in \mathcal{T}_h\}.$$

The generalized algebraic eigenvalue problem

Let $\{\varphi_1^h, \ldots, \varphi_{n_h}^h\}$,be a basis for a the finite dimensional space V_h. Since globally the solution u_h is determined by its values at the n_h grid points of $\mathcal{T}_h$, it can be written as

$$u_h = \sum_{i=1}^{n_h} u_{h,i}\varphi_i^h.$$

Then discretized problem can be written as a *generalized eigenvalue problem* of the form

$$\mathbf{A}_h \mathbf{u}_h = \lambda_h \mathbf{B}_h \mathbf{u}_h,$$

where

$$\mathbf{A}_h := [a(\varphi_i^h, \varphi_j^h)]_{1 \le i,j \le n_h}, \mathbf{B}_h := [b(\varphi_i^h, \varphi_j^h)]_{1 \le i,j \le n_h}, \text{ and } \mathbf{u}_h = [u_{h,i}]_{1 \le i \le n_h}.$$

Selfadjoint eigenvalue problem

Laplace eigenvalue problem

Determine a non-trivial eigenpair $(\lambda, u) \in \mathbb{R} \times \mathcal{H}_0^1(\overline{\Omega})$ with $\|u\|_{L_2(\Omega)} = 1$ such that

$$-\Delta u = \lambda u \text{ in } \Omega \qquad \text{and} \qquad u = 0 \text{ on } \partial\Omega.$$

The weak formulation

Determine a non-trivial eigenpair $(\lambda, u) \in \mathbb{R} \times V$, with $b(u, u) = 1$ s.t.

$$a(u, v) = \lambda b(u, v) \quad \text{for all } v \in V$$

with

$$a(u, v) := \int_\Omega \nabla u \nabla v dx, \quad b(u, v) := \int_\Omega uv dx.$$

Remark

- The bilinear form $a(.,.)$ is elliptic, continuous and symmetric in V,
- The bilinear form $b(.,.)$ is continuous, symmetric and positive definite, and hence induces a norm $\|.\|_b := b(.,.)^{1/2}$ on H,
- $V := \mathcal{H}_0^1(\Omega)$ with seminorm $|||.||| := |.|_{\mathcal{H}^1(\Omega)} \simeq \|.\|_{\mathcal{H}_0^1(\Omega)}$,
- $H := L_2(\Omega)$ with norm $\|.\|_b := \|.\|_{L_2(\Omega)}$.

For the above model problem $|||.||| = a(.,.)^{1/2}$ and $\|.\| = \|.\|_{L_2(\Omega)}$.

Discrete Laplace eigenvalue problem

It is known, see e.g. that Laplace problem has a countable set of real eigenvalues, [BO91]

$$0 < \lambda_1 \leq \lambda_2 \leq \dots$$

and corresponding eigenfunctions

$$u_1, u_2, \dots, \quad \text{such that} \quad b(u_i, u_j) = \delta_{i,j}.$$

The associated algebraic eigenvalue problems is given by

$$\mathbf{A}_h \mathbf{u}_h = \lambda_h \mathbf{B}_h \mathbf{u}_h,$$

where $\mathbf{A}_h$ and $\mathbf{B}_h$ are symmetric and positive definite matrices.
The algebraic generalized eigenvalue problem has a finite set of eigenvalues

$$0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \dots \leq \lambda_{n_h,h}$$

and corresponding eigenvectors

$$\mathbf{u}_{1,h}, \mathbf{u}_{2,h}, \dots, \mathbf{u}_{n_h,h}, \quad \text{such that} \quad \mathbf{u}_{i,h}^T \mathbf{B}_h \mathbf{u}_{j,h} = \delta_{i,j}.$$

Minimum-maximum principle

It follows from the *Courant-Fischer min-max* theorem [SF73, Dem97] that

$$\lambda_i \leq \lambda_{i,h} \ \text{ for all } i = 1, \ldots, n_h,$$

and if $\mathcal{T}_h$ is any refinement of $\mathcal{T}_H$, i.e., $H > h$, then

$$0 \leq \lambda_{i,h} \leq \lambda_{i,H}, \quad i = 1, \ldots, n_H.$$

[I. Babuška & J. Osborn, Eigenvalue Problems, In: Ciarlet, P.G. & Lions, J.L.: Handbook of Numerical Analysis, Vol. II, Elsevier Science Publishers B.V., North-Holland, 1991, S. 641-787.]

[G. Strang & G.J. Fix, An analysis of the finite element method, Prentice-Hall, 1973.]

Non-selfadjoint eigenvalue problem

Convection diffusion eigenvalue problem

Determine for a non-trivial eigenpair $(\lambda, u) \in \mathbb{C} \times \mathcal{H}_0^1(\overline{\Omega}; \mathbb{C}) \cap \mathcal{H}^2(\Omega)$ with $\|u\|_{L_2(\Omega)} = 1$ such that

$$-\Delta u + \beta \cdot \nabla u = \lambda u \text{ in } \Omega \qquad \text{and} \qquad u = 0 \text{ on } \partial\Omega,$$

where $\beta \in \mathbb{R}^2$ is divergence free, i.e., $\int_\Omega v \, div(\beta) dx = 0, \forall v \in \mathcal{H}_0^1(\Omega; \mathbb{C})$.

The weak formulation

Determine a non-trivial eigenpair $(\lambda, u) \in \mathbb{C} \times V$, with $b(u, u) = 1$ s.t.

$$a(u, v) + c(u, v) = \lambda b(u, v) \quad \text{for all} \quad v \in V$$

with

$$a(u, v) := \int_\Omega \nabla u \nabla \bar{v} dx, \quad c(u, v) := \int_\Omega \beta \cdot \nabla u \bar{v} dx, \quad b(u, v) := \int_\Omega u \bar{v} dx.$$

The dual eigenvalue problem

Determine a non-trivial dual eigenpair $(\lambda^\star, u^\star) \in \mathbb{C} \times V$ with $b(u^\star, u^\star) = 1$ such that

$$a(w, u^\star) + c(w, u^\star) = \overline{\lambda^\star} b(w, u^\star).$$

Note that the primal and dual eigenvalues are connected by $\lambda = \overline{\lambda^\star}$.

[V. Heuveline & R. Rannacher, A posteriori error control for finite element approximations of elliptic eigenvalue problems, Adv. Comp. Math., 15(2001), 107–138.]

[C. Carstensen & J. Gedicke, A Posteriori Error estimators for Non-Symmetric Eigenvalue Problems, DFG Research Center Matheon, Preprint 659, 2009]

[C. Carstensen, J. Gedicke, V. Mehrmann & A. Miedlar, An adaptive homotopy approach for non-selfadjoint eigenvalue problems, In preparation.]

Remark

- The bilinear form $a(.,.) + c(.,.)$ is elliptic and continuous in V,
- The bilinear form $b(.,.)$ is continuous, symmetric and positive definite, and hence induces a norm $\|.\|_b := b(.,.)^{1/2}$ on H,
- $V := \mathcal{H}^1_0(\Omega)$ with seminorm $|||.||| := |.|_{\mathcal{H}^1(\Omega)} \simeq \|.\|_{\mathcal{H}^1_0(\Omega)}$,
- $H := L_2(\Omega)$ with norm $\|.\|_{L_2(\Omega)}$.

For the above model problem $|||.||| = (a(.,.) + c(.,.))^{1/2}$ and $\|.\| = \|.\|_{L_2(\Omega)}$.

The generalized primal and dual eigenvalue problems

$$(A_\ell + C_\ell)\mathbf{u}_\ell = \lambda_\ell B_\ell \mathbf{u}_\ell \qquad \text{and} \qquad \mathbf{u}_\ell^\star (A_\ell + C_\ell) = \lambda_\ell^\star \mathbf{u}_\ell^\star B_\ell,$$

where A_ℓ is s.p.d. stiffness matrix, C_ℓ nonsymmetric convection matrix and B_ℓ s.p.d. mass matrix.
The smallest eigenvalue of this problem is proved to be simple (real) and well separated, [Eva00].

Error estimators

Definition (Error estimator)

Given a norm $\|.\|$, an approximation η to an error $\|e\| = \|u - u_h\|$ is called an *error estimator*.

Definition (A priori error estimator)

A quantity η is called a priori error estimator if it can not be extracted from the computed numerical solution and the given data of the problem, i.e., regularity conditions of the exact solution are required.

Definition (A posteriori error estimator, [Car04])

A computable quantity η is called a posteriori error estimator if it can be extracted from the computed numerical solution and the given data of the problem,i.e., u_h, known domain Ω and its boundary $\partial\Omega$.

A priori error estimators

- give asymptotic rates of convergence as the mesh parameter h tends to zero (rough information on the asymptotic behavior of errors),
- give information about stability of various solvers,
- require regularity conditions of the solution which are in general not available, i.e., because of singularities,
- based on the stability properties of the 'discrete' operator,
- insufficient since they only yield information on the asymptotic behavior,
- not computable.

[M. Ainsworth and J.T. Oden, A Posteriori Error Estimation in Finite Element Analysis, John Wiley & Sons, Inc. 2000.]

[R. Verfürth, A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques, Wiley and Teubner,New York, Stuttgart, 1996.]

A posteriori error estimators

- can be extracted from the numerical solution and the given data of the problem, which make them computable,
- are less expensive to calculate than the computation of the numerical solution,
- are based on the stability properties of the 'continuous' operator,
- have global upper bounds which are sufficient to obtain a numerical solution with the accuracy below a prescribed tolerance,
- have local upper and lower bounds for the true error in a user-specified norm, i,e., $|||.|||$,
- employ information about the continuous problem.

[M. Ainsworth and J.T. Oden, A Posteriori Error Estimation in Finite Element Analysis, John Wiley & Sons, Inc. 2000.]

[R. Verfürth, A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques, Wiley and Teubner,New York, Stuttgart, 1996.]

Efficiency index and asymptotic exactness

Definition (Efficiency index, [AO00])

Let $\|e\|$ be the global error in the energy norm and η be the global error estimator then the ratio

$$\theta = \frac{\eta}{\|e\|},$$

is called a global *efficiency index*.

Definition (Asymptotic exactness, [AO00])

An error estimator η is called *asymptotically exact* if

$$\lim_{h \to 0} \theta = 1.$$

Reliable and efficient error estimator

Definition (Reliability, [Car04])

An estimator η is called reliable if

$$\|e\| \leq C_{rel}\eta + h.o.t_{rel},$$

with a constant $C_{rel} > 0$ independent of the mesh-size h.

Definition (Efficiency, [Car04])

An estimator η is called *efficient* if

$$\eta \leq C_{eff}\|e\| + h.o.t._{eff},$$

with a constant $C_{eff} > 0$ independent of the mesh-size h.

A "good" a posteriori error estimator

Definition (A "good" a posteriori error estimator)

An error estimator η is called *good* if it is reliable and efficient, i.e.,

$$C_1 \|e\| \leq \eta \leq C_2 \|e\|,$$

with $C_1 = \frac{1}{C_{rel}}$ and $C_2 = C_{eff}$.

[M. Ainsworth and J.T. Oden, A Posteriori Error Estimation in Finite Element Analysis, John Wiley & Sons, Inc. 2000.]

[C. Carstensen, Some remarks on the history and future of averaging techniques in a posteriori finite element error analysis, Z. Angew. Math. Mech. 84 (2004),3–21.]

Classification of the a posteriori error estimators

- residual error estimators $\equiv$ explicit error estimators,
- solution of local problems $\equiv$ implicit error estimators,
- hierarchical error estimators $\equiv$ multilevel error estimators,
- averaging error estimators $\equiv$ recovery-based error estimators.

[S.C. Brenner and C. Carstensen, Finite Element Methods in Encyclopedia of Computational Mechanics, Vol. I, (E. Stein and R. de Borst and T.J.R. Huges eds.), John Wiley and Sons Inc., New York, 2004, 73–114.]

[M. Ainsworth and J.T. Oden, A Posteriori Error Estimation in Finite Element Analysis, John Wiley & Sons, Inc. 2000.]

[R. Verfürth, A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques, Wiley and Teubner,New York, Stuttgart, 1996.]

Residual based error estimators (Explicit error estimators)

- involve a direct computation using available data, i.e., residuals in the current approximation,
- typically consist of local norms of explicitely given interior residuals (volume residuals) and edge residuals (jump residuals).

The residual error representation formula, [BC04]

$$Res(v) = \sum_{T \in \mathcal{T}} \int_T R_T \cdot v dx - \sum_{E \in \mathcal{E}} \int_E R_E \cdot v ds \in V^*$$

with volume residual R_T and the jump residual R_E.

[S.C. Brenner and C. Carstensen, Finite Element Methods in Encyclopedia of Computational Mechanics, Vol. I, (E. Stein and R. de Borst and T.J.R. Huges eds.), John Wiley and Sons Inc., New York, 2004, 73–114.]

Residual based error estimators (Explicit error estimators)

Explicit residual-based estimator, [BC04]

$$\eta^2 := \sum_{T \in \mathcal{T}} h_T^2 \|R_T\|_{L_2(T)}^2 + \sum_{E \in \mathcal{E}} h_E \|R_E\|_{L_2(E)}^2$$

- volume residual how well the finite element approximation satisfies the PDE on the interior of the domain,
- edge residual depends on the jumps in the numerical approximation at the element boundaries and reflects the regularity of the approximation.

[R. Verfürth, A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques, Wiley and Teubner,New York, Stuttgart, 1996.]

[R.G. Durán and C. Padra and R. Rodríguez, A posteriori error estimates for the finite element approximation of eigenvalue problems, Math. Mod. Meth. Appl. Sci. 13(2003), 1219–1229.]

Solution of local problems (implicit error estimator)

- based on a local norms of the local solutions for error estimation,
- uses the residuals indirectly and generally involves the solution of of an algebraic system of equations with the residual terms on the right-hand side, i.e., solve a local analog of the residual equation using a higher order finite element approximation and use a suitable norm of the solution as error estimator.

[S.C. Brenner and C. Carstensen, Finite Element Methods in Encyclopedia of Computational Mechanics, Vol. I, (E. Stein and R. de Borst and T.J.R. Huges eds.), John Wiley and Sons Inc., New York, 2004, 73–114.]

[M. Ainsworth and J.T. Oden, A Posteriori Error Estimation in Finite Element Analysis, John Wiley & Sons, Inc. 2000]

[R. Verfürth, A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques, Wiley and Teubner,New York, Stuttgart, 1996]

Hierarchical error estimators (multilevel error estimators)

- concern at least two meshes $\mathcal{T}_H$ and $\mathcal{T}_h$ with associate discrete space $V_H \subset V_h \subset V$ and two discrete solutions,
- bound the error $u - u_H$ by evaluating the residual of u_H with respect to certain basis functions of another finite element space V_h which consists of higher order finite elements or corresponds to a refinement $\mathcal{T}_h$ of $\mathcal{T}_H$.

[S.C. Brenner and C. Carstensen, Finite Element Methods in Encyclopedia of Computational Mechanics, Vol. I, (E. Stein and R. de Borst and T.J.R. Huges eds.), John Wiley and Sons Inc., New York, 2004, 73–114.]

[M. Ainsworth and J.T. Oden, A Posteriori Error Estimation in Finite Element Analysis, John Wiley & Sons, Inc. 2000]

[R. Verfürth, A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques, Wiley and Teubner,New York, Stuttgart, 1996]

Averaging error estimators (gradient recovery estimators)

- focus on one mesh and one known low-order approximation and the difference to a piecewise polynomial value in a finite-dimensional subspace of higher polynomial degrees and a more restrictive continuity conditions than those generally satisfied by approximation, i.e., take a picewise smooth approximation and approximate it by some globally continuous piecewise polynomials of higher degree [Car04],
- they do not require any residual or partial differential equation,
- use some local extrapolation or averaging technique for error estimation.

[C. Carstensen, Some remarks on the history and future of averaging techniques in a posteriori finite element error analysis, Z. Angew. Math. Mech. 84 (2004),3–21.]
[S.C. Brenner and C. Carstensen, Finite Element Methods in Encyclopedia of Computational Mechanics, Vol. I, (E. Stein and R. de Borst and T.J.R. Huges eds.), John Wiley and Sons Inc., New York, 2004, 73–114.]

A posteriori error control

Error estimation

- termination with prescribed tolerance $TOL > 0$, idealized stopping criteria $\|e\| \leq TOL$,
- the error $\|e\|$ is unknown, it is replaced by its upper bound which leads to $C_{rel}\eta + h.o.t._{rel} \leq TOL$,
- global upper bounds are sufficient to obtain a numerical solution with the accuracy below a prescribed tolerance TOL.

Optimal use of resources

- minimal work for a prescribed accuracy,
- maximal accuracy for a prescribed work.

[W. Bangerth & R. Rannacher, Adaptive Finite Element Methods for Differential Equations, Lectures in Mathematics ETH Zürich, Birkhäuser, Basel, 2003.]

[S.C. Brenner and C. Carstensen, Finite Element Methods in Encyclopedia of Computational Mechanics, Vol. I, (E. Stein and R. de Borst and T.J.R. Huges eds.), John Wiley and Sons Inc., New York, 2004, 73–114.]

The Adaptive Finite Element Method

The adaptive finite element method (AFEM) generates a sequence of nested triangulations $\mathcal{T}_0, \mathcal{T}_1, \ldots$ with corresponding nested spaces

$$V_0 \subseteq V_1 \subseteq \ldots \subseteq V_\ell \subset V.$$

A typically loop of the AFEM consists of the four steps

$$\text{Solve} \longrightarrow \text{Estimate} \longrightarrow \text{Mark} \longrightarrow \text{Refine}.$$

AFEM algorithm

- SOLVE given the current triangulation compute the finite element solution,
- ESTIMATE check the accuracy of the finite element solution using refinement indicators,
- MARK based on refinement indicators identify the elements, edges or patches in the current mesh which need to be refined (or coarsened), additionally apply the closure algorithm to ensure that the resulting triangulation is regular,
- REFINE generate new triangulation and corresponding data.

  [S.C. Brenner and C. Carstensen, Finite Element Methods in Encyclopedia of Computational Mechanics, Vol. I, (E. Stein and R. de Borst and T.J.R. Huges eds.), John Wiley and Sons Inc., New York, 2004, 73–114.]

Goals for adaptive algorithms

Goals

- refine the discretization near the critical regions, i.e., place more grid-points where the solution is less regular,
- assure a good balance between the refined and un-refined regions such that the overall accuracy is optimal,
- the mesh refinement is automatic and controlable.

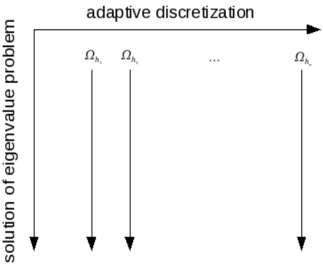
Solutions

- automatic mesh adaptation according to certain refinement strategies based on the (local) refinement indicators extracted from the a posteriori error estimators, which are global and involves constants and higher-order terms,
- local upper and lower bounds for the true error i.e., $\|e\|$, are necessary to ensure that the grid is correctly refined,i.e., a numerical solution with a prescribed tolerance is obtained using a minimal number of grid-points,
- the efficient estimator is required to provide practical criteria to control an adaptive refinement algorithm and to construct a stopping criteria for the iterative solver.

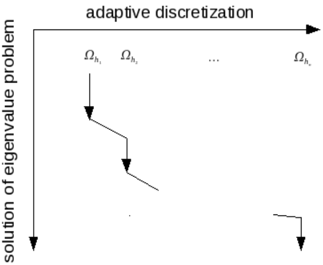
[S.C. Brenner and C. Carstensen, Finite Element Methods in Encyclopedia of Computational Mechanics, Vol. I, (E. Stein and R. de Borst and T.J.R. Huges eds.), John Wiley and Sons Inc., New York, 2004, 73–114.]

Standard AFEM versus AFEMLA

Solve → Estimate → Mark → Refine



Standard AFEM



AFEMLA

How we can incorporate the solution of the algebraic eigenvalue problem (AEVP) into adaptation process?

Notation

- Roman letters will denote the functions (i.e. u_H),
- **Bold** letters for the coordinate vectors (i.e. $\mathbf{u}_H$),
- H, (or h) the diameter of the coarse (or fine) element (i.e. $H > h$),
- $\lambda_{i,H}$ (or $\lambda_{i,h}$) the eigenvalues of the discretized algebraic eigenvalue problem associated with the space V_H (or V_h),
- $\tilde{\lambda}_{i,H}$ (or $\tilde{\lambda}_{i,h}$) approximation of $\lambda_{i,H}$ (or $\lambda_{i,h}$) computed by an iterative eigenvalue solver in finite precision arithmetic,
- P prolongation matrix, $P : V_H \rightarrow V_h$,
- $(\hat{\lambda}_h, \hat{\mathbf{u}}_h)$ the eigenpair obtained from the prolongation of the eigenvector $\tilde{\mathbf{u}}_h$ to the fine space V_h,
- The corresponding eigenfunctions are denoted in a similar fashion.

The AFEMLA algorithm

Solve:

- compute eigenpair $(\tilde{\lambda}_H, \tilde{\mathbf{u}}_H)$ on coarse mesh,
- use iterative solver i.e. Krylov subspace method,
- do not solve the problem very accurately, stop after k steps or when the desired *tol* is reached.

Estimate:

- prolongate $\tilde{\mathbf{u}}_H$ from the coarse mesh $\mathcal{T}_H$ to the fine mesh $\mathcal{T}_h$,
- compute residual vector $\hat{\mathbf{r}}_h$ and identify all its large coefficients and corresponding basis functions (nodes),
- if the i-th entry in the residual vector is large, then the i-th basis function has a huge influence on the solution, namely its support should be further investigated [Kam07].

Mark and Refine: mark elements and refine the mesh.

The AFEMLA algorithm

Does the residual provide sufficient information for the refinement procedure?

Error estimates for the eigenvalues

small residual vector

$$\stackrel{Q1?}{\Longrightarrow} \text{good approximation of the discretized eigenpair } (\tilde{\lambda}_H, \tilde{\mathbf{u}}_H)$$

$$\stackrel{Q2?}{\Longrightarrow} \text{good approximation of the PDE eigenpair } (\lambda, u)$$

Q1: yes

- residual errors can be transformed to the backward errors [BDD$^+$00, Dem97, Wat07],
- eigenvalues are well-conditioned.

Q2: yes, if

- saturation assumption holds i.e. $\lambda_h - \lambda \leq \beta(\lambda_H - \lambda),\ \beta \in (0,1)$ [Ney02].

Computable bounds = backward error analysis + saturation assumption

Error bounds for eigenvalues

We can obtain bounds for the following errors

- $|\tilde{\lambda}_H - \lambda_H|$, $|\hat{\lambda}_h - \lambda_h|$, $|\tilde{\lambda}_H - \hat{\lambda}_h|$, $|\tilde{\lambda}_H - \lambda_h|$, $|\lambda_H - \hat{\lambda}_h|$ and $|\lambda_H - \lambda_h|$,
- $|\lambda_H - \lambda|$, $|\lambda_h - \lambda|$, $|\tilde{\lambda}_H - \lambda|$ and $|\hat{\lambda}_h - \lambda|$,

i.e.

$$\begin{aligned} |\tilde{\lambda}_H - \lambda| &\leq \frac{1}{1-\beta}\left(\|\mathbf{r}_H\|_2\|B_H^{-1}\|_2 + \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2\|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} + \|\hat{\mathbf{r}}_h\|_2\|B_h^{-1}\|_2\right) \\ &+ \|\mathbf{r}_H\|_2\|B_H^{-1}\|_2. \end{aligned}$$

Numerical examples

PDE formulation

$$-\Delta u = \lambda u \in \Omega \quad \text{and} \quad u = 0 \quad \text{on} \quad \partial\Omega$$

Discrete variational formulation

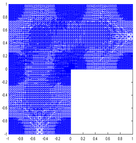
$$a(u_H, v_H) = \lambda_H (u_H, v_H), \quad \forall v_H \in V_H \subset V.$$

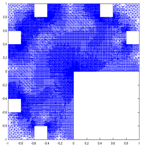
GEVP

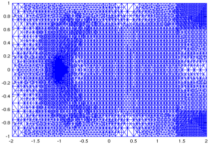
$$A_H \mathbf{u}_H = \lambda_H B_H \mathbf{u}_H$$

where A_H and B_H are symmetric and positive definite matrices.

Domains







AFEMLA versus AFEM with uniformly refined meshes

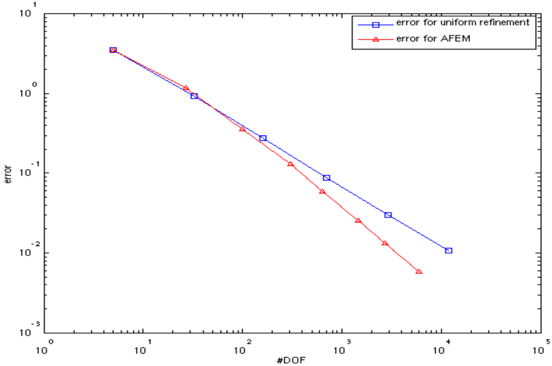


Figure: Convergence history on the L-shape domain.

#DOF 5961, CPU time 2.14 sec., error 10^{-3} #DOF 12033, CPU time 7.6 sec., error 10^{-2}

$\lambda_1 \approx 9.6397$

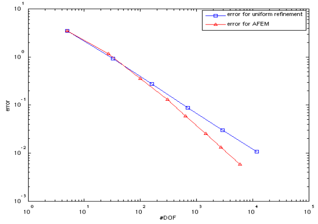


Figure: Convergence history.

Table: Approximation of the smallest eigenvalue.

ref. level	#DOF	$\tilde{\lambda}_1$	$\lvert\lambda_1 - \tilde{\lambda}_1\rvert$
1	5	13.1992	3.5595
2	27	10.8173	1.1775
3	99	9.9982	0.3584
4	306	9.7721	0.1323
5	641	9.6982	0.0585
6	1461	9.6652	0.0255
7	2745	9.6528	0.0131
8	5961	9.6455	0.0058

More eigenvalues - refinement based on all residual vectors

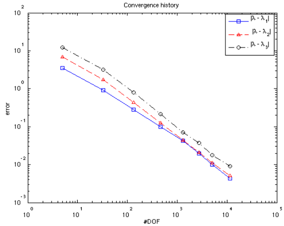


Figure: Convergence history.

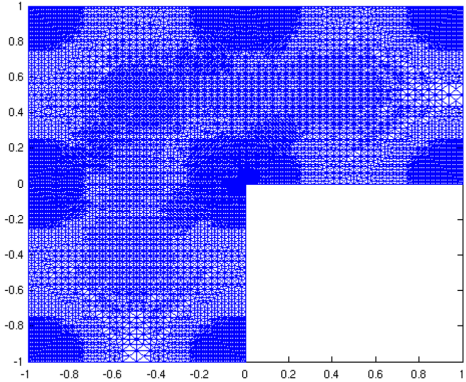


Figure: Mesh.

The smallest eigenvalue obtained on a slit domain

Table: AFEM with uniformly refined meshes.

ref. level	#DOF	$\tilde{\lambda}_1$	CPU time (s)
1	2	5.1429	0.29
2	19	3.8704	0.04
3	101	3.5444	0.04
4	457	3.4538	0.11
5	1937	3.4253	0.52
6	7969	3.4150	3.99
7	32321	3.4109	120.53

Table: AFEMLA.

ref. level	#DOF	$\tilde{\lambda}_1$	CPU time (s)
1	2	5.1429	0.02
2	12	3.9949	0.01
3	46	3.6020	0.02
4	148	3.4895	0.04
5	354	3.4462	0.09
6	723	3.4271	0.15
7	1496	3.4173	0.34
8	3030	3.4125	0.81

AFEMLA meshes for more complicated domains

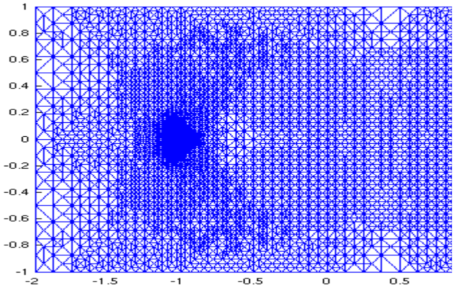


Figure: Slit domain.

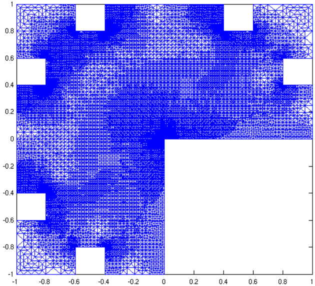


Figure: L-shape domain with holes.

Conclusion

Current results

- AFEMLA balances the cost of adaptation of the mesh with the costs for iterative solver for AEVP,
- adaptation process is based on error estimates which incorporate discretization errors, approximation errors in EV solver and roundoff errors,
- AFEMLA approach makes the adaptation process much more efficient with guaranteed computable bounds in the algebraic part,
- AEVP does not have to be solved to full accuracy, if the analytic approach with the standard assumption of solving the algebraic problem exactly converge, then either does our extended approach,
- complete error estimates for the eigenfunctions.

Motivation and the model problem

Convection-diffusion eigenvalue problem:

$$-\Delta u + \beta \cdot \nabla u = \lambda u \text{ in } \Omega \qquad \text{and} \qquad u = 0 \text{ on } \partial\Omega$$

Discrete weak primal and dual problem:

$$a(u_\ell, v_\ell) + c(u_\ell, v_\ell) = \lambda_\ell b(u_\ell, v_\ell) \quad \text{for all } v_\ell \in V_\ell,$$
$$a(w_\ell, u_\ell^\star) + c(w_\ell, u_\ell^\star) = \overline{\lambda_\ell^\star} b(w_\ell, u_\ell^\star) \quad \text{for all } w_\ell \in V_\ell.$$

Generalized algebraic eigenvalue problem:

$$(A_\ell + C_\ell)\mathbf{u}_\ell = \lambda_\ell B_\ell \mathbf{u}_\ell \qquad \text{and} \qquad \mathbf{u}_\ell^\star (A_\ell + C_\ell) = \lambda_\ell^\star \mathbf{u}_\ell^\star B_\ell$$

We want to compute the eigenvalue with the smallest real part, which is simple and well separated [Eva00].

Homotopy method

Homotopy for the model eigenvalue problem:

$$\mathcal{H}(t) = (1-t)\mathcal{L}_0 + t\mathcal{L}_1 \quad \text{for } t \in [0,1],$$

where $\mathcal{L}_0 u := -\Delta u$ and $\mathcal{L}_1 u := -\Delta u + \beta \cdot \nabla u$.

Homotopy for the discretized model eigenvalue problem:

$$\mathcal{H}_\ell(t) = (1-t)A_\ell + t(A_\ell + C_\ell) = A_\ell + tC_\ell.$$

The homotopy, discretization and approximation error

Homotopy error:

$$|\lambda(1) - \lambda(t)| \lesssim (1-t)\|\beta\|_{L^\infty(\Omega)}\|u\| = \nu, \qquad \text{[BE03]}.$$

Discretization error:

$$|\lambda(t) - \lambda_\ell(t)| \lesssim \sum_{T \in \mathcal{T}_\ell} \left(\eta_\ell^2(T) + \eta_\ell^{\star 2}(T)\right), \qquad \text{[HR01, CG09b]}.$$

Approximation error:

$$|\lambda_\ell(t) - \tilde{\lambda}_\ell(t)| + |\lambda_\ell^\star(t) - \tilde{\lambda}_\ell^\star(t)| \leq \mu_\ell,$$

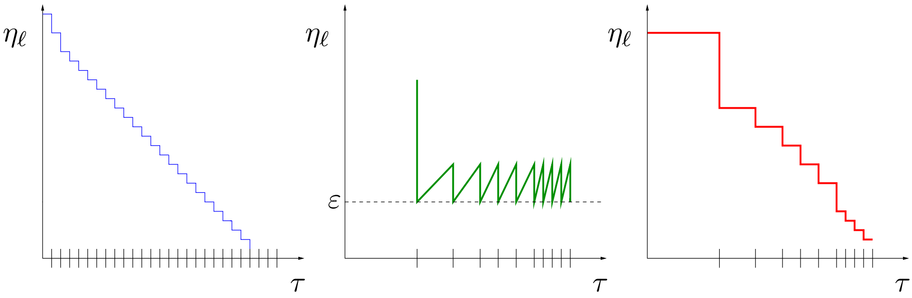
Different adaptive homotopy algorithms

Solve → Estimate → Mark → Refine

Algorithm 1: fixed step size in t, adaptive in grid size h always to be below homotopy error.

Algorithm 2: the adaptive step sizes τ in t, adaptation in h to be below fixed tolerance ε.

Algorithm 3: overcome the drawback of a fixed step size in Algorithm 1 and a fixed discretization control in Algorithm 2, adaptive step sizes τ in t, adaptation in h to be below homotopy error.



Algorithm 1 **Algorithm 2** **Algorithm 3**

η_ℓ - discretization error (on level ℓ)

ν - homotopy error

τ - step size in t

ε - desired tolerance

t	$\eta_\ell(t)$	$\nu_\ell(t)$	$\mu_\ell(t)$	est. error
0.0000	23.0488	95.6367	0.37689	119.06242
0.1000	23.3493	86.3575	0.00851	109.71530
0.2000	24.3235	77.5194	0.00760	101.85051
0.3000	26.0821	68.9273	0.00881	95.01820
0.4000	28.7994	60.3875	0.01145	89.19842
0.5000	32.6795	51.7102	0.01559	84.40527
0.6000	37.9426	42.7114	0.02117	80.67515
0.7000	10.2087	33.3835	0.41788	44.01011
0.8000	12.9786	23.4116	0.01615	36.40632
0.9000	6.6371	12.5056	0.48146	19.62412
1.0000	0.0005	0.0000	0.00004	0.00054

Table: The discretization $\eta_\ell(t)$, the homotopy $\nu_\ell(t)$ and the iteration $\mu_\ell(t)$ error estimators for all homotopy steps t concerning Algorithm 1.

$\lambda \approx 44.739208802205724$

t	$\tilde{\lambda}_\ell(t)$	$\frac{\lvert\lambda_\ell(1)-\tilde{\lambda}_\ell(t)\rvert}{\lvert\lambda_\ell(1)\rvert}$	#DOF	CPU time
0.0000	22.86580	0.48891	25	0.22
0.1000	23.01734	0.48552	25	0.25
0.2000	23.47366	0.47532	25	0.27
0.3000	24.23970	0.45820	25	0.29
0.4000	25.32407	0.43396	25	0.31
0.5000	26.73963	0.40232	25	0.33
0.6000	28.50439	0.36288	25	0.35
0.7000	30.96072	0.30797	57	0.44
0.8000	34.25674	0.23430	57	0.47
0.9000	39.10394	0.12596	109	0.53
1.0000	44.73751	0.00004	71870	75.19

Table: The approximation $\tilde{\lambda}_\ell(t)$, the relative error $\frac{\lvert\lambda_\ell(1)-\tilde{\lambda}_\ell(t)\rvert}{\lvert\lambda_\ell(1)\rvert}$, the number of degrees of freedom (#DOF) and the CPU time for all homotopy steps t concerning Algorithm 1.

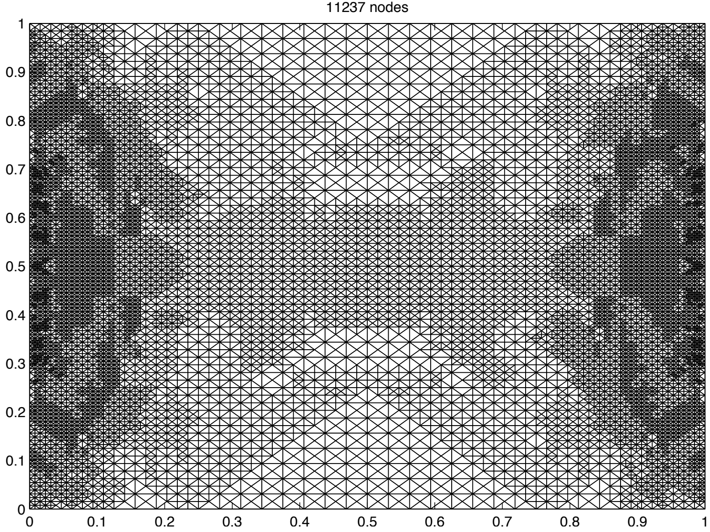


Figure: Final mesh for Algorithm 1.

t	$\eta_\ell(t)$	$\nu_\ell(t)$	$\mu_\ell(t)$	est. error
0.0000	0.0006	88.8633	0.0001485	88.86401
0.5000	0.0005	50.9794	0.0000009	50.97985
1.0000	0.0008	0.0000	0.0000136	0.00077

Table: The discretization $\eta_\ell(t)$, the homotopy $\nu_\ell(t)$ and the iteration $\mu_\ell(t)$ error estimators for all homotopy steps t concerning Algorithm 2.

$\lambda \approx 44.739208802205724$

t	$\tilde{\lambda}_\ell(t)$	$\frac{\lvert\lambda_\ell(1)-\tilde{\lambda}_\ell(t)\rvert}{\lvert\lambda_\ell(1)\rvert}$	#DOF	CPU time
0.0000	19.74171	0.55874	9761	8.74
0.5000	25.98896	0.41910	19023	62.44
1.0000	44.73651	0.00006	29700	108.27

Table: The approximation $\tilde{\lambda}_\ell(t)$, the relative error $\frac{|\lambda_\ell(1)-\tilde{\lambda}_\ell(t)|}{|\lambda_\ell(1)|}$, the number of degrees of freedom (#DOF) and the CPU time for all homotopy steps t concerning Algorithm 2.

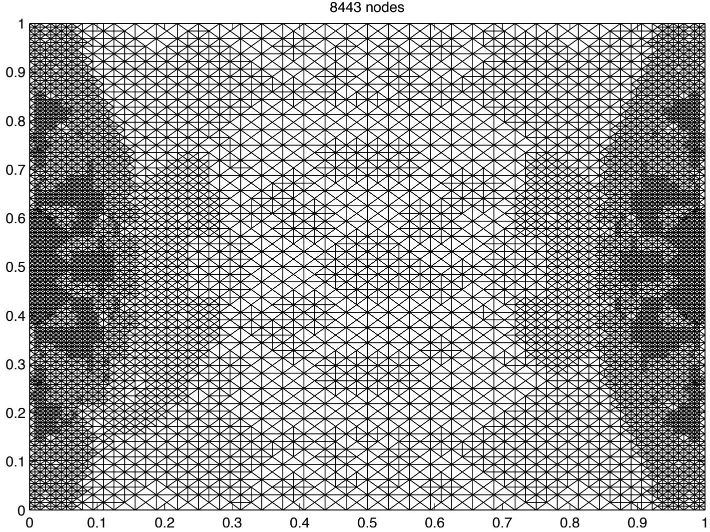


Figure: Final mesh for Algorithm 2.

$\lambda \approx 44.7392088022057 24$

t	$\eta_\ell(t)$	$\nu_\ell(t)$	$\mu_\ell(t)$	est. error
0.0000	23.0271	95.6366	0.2701265	118.93382
0.5000	32.6896	51.7112	0.0843690	84.48512
0.7500	11.6020	28.5244	0.4515713	40.57800
0.8750	6.7380	15.4099	0.4711298	22.61912
0.9375	7.8500	7.9782	0.0272551	15.85547
0.9688	3.2088	4.0697	0.2891100	7.56762
0.9844	1.2060	2.0673	0.4278706	3.70119
0.9922	0.4560	1.0380	0.0004539	1.49451
0.9961	0.4602	0.5202	0.0029006	0.98322
0.9980	0.1864	0.2608	0.0012530	0.44843
0.9990	0.0707	0.1305	0.0204610	0.22162
0.9995	0.0282	0.0653	0.0003639	0.09386
0.9998	0.0282	0.0326	0.0001766	0.06105
0.9999	0.0106	0.0163	0.0001521	0.02703
1.0000	0.0007	0.0000	0.0000243	0.00073

Table: The discretization $\eta_\ell(t)$, the homotopy $\nu_\ell(t)$ and the iteration $\mu_\ell(t)$ error estimators for all homotopy steps t concerning Algorithm 3.

t	$\tilde{\lambda}_\ell(t)$	$\frac{\lvert\lambda_\ell(1)-\tilde{\lambda}_\ell(t)\rvert}{\lvert\lambda_\ell(1)\rvert}$	#DOF	CPU time
0.0000	22.86578	0.48891	25	0.76
0.5000	26.73866	0.40234	25	1.20
0.7500	32.54928	0.27247	55	1.55
0.8750	38.00079	0.15062	107	2.18
0.9375	40.73818	0.08943	107	3.07
0.9688	42.39339	0.05243	197	4.01
0.9844	43.77023	0.02166	385	6.06
0.9922	44.13547	0.01349	715	9.74
0.9961	44.32847	0.00918	715	16.59
0.9980	44.58151	0.00352	1398	23.57
0.9990	44.65025	0.00199	2494	37.14
0.9995	44.68298	0.00126	4848	66.70
0.9998	44.69522	0.00098	4848	119.47
0.9999	44.72311	0.00036	8785	175.75
1.0000	44.73615	0.00007	55235	226.87

Table: The approximation $\tilde{\lambda}_\ell(t)$, the relative error $\frac{\lvert\lambda_\ell(1)-\tilde{\lambda}_\ell(t)\rvert}{\lvert\lambda_\ell(1)\rvert}$, the number of degrees of freedom (#DOF) and the CPU time for all homotopy steps t concerning Algorithm 3.

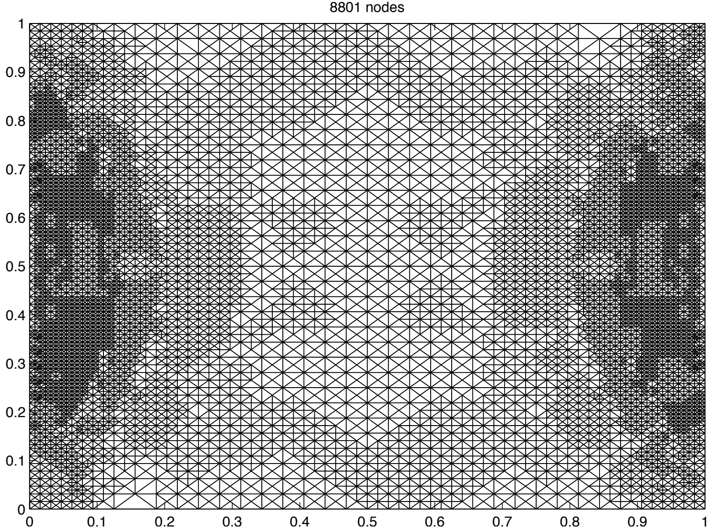


Figure: Final mesh for Algorithm 3.

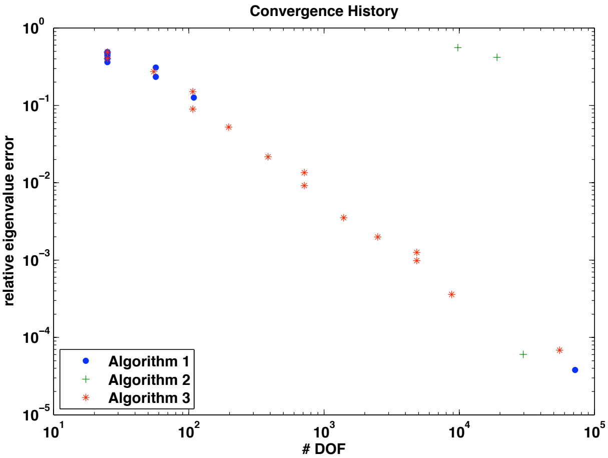


Figure: Convergence history of Algorithm 1, 2 and 3 with respect to #DOF.

$\frac{|\lambda_\ell(1) - \tilde{\lambda}_\ell(t)|}{|\lambda_\ell(1)|}$ - relative eigenvalue error.

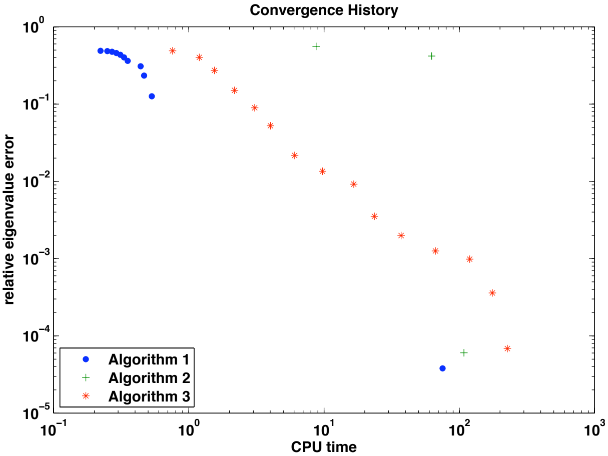


Figure: Convergence history of Algorithm 1, 2 and 3 with respect to CPU time.

Conclusion

Current results

- adaptive homotopy approach with simple step size control,
- balancing the discretization, homotopy and approximation error.

Ongoing work

- bifurcation,
- ill-conditioned problems,
- more complicated model problems.

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