ACLT: Algebra, Categories, Logic in Topology
- Grothendieck's generalized topological spaces (toposes)

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1. Sheaves

"Sheaf = continuous set-valued map"

TACL = Topology, Algebra, Categories *in* Logic

e.g. Stone spaces, Boolean algebras, Stone duality *in* modal logic

TAC give new angles on logic, its syntax and semantics

TACL

ACL T

ACLT = Algebra, Categories, Logic *in* Topology

Discrete maths *in* continuous maths

ACL give new angles on topology and continuity

Have already seen this happen in algebraic topology!

Specific new angle here: Grothendieck

"A topos is a generalized topological space"

Grothendieck used them to generalize sheaf cohomology

e.g. "space of sets", "space of groups"
- proper classes, but also have non-discrete "topology"

Need ACL to understand Grothendieck's generalized topology
ACL develop in unexpected directions to make this work
Aim of course: give overview of those unexpected directions
- and *why* they are needed.

Overall story

Open = continuous map valued in truth values
- Theorem: open = map to Sierpinski space $

Sheaf = continuous set-valued map
- no theorem here - "space of sets" not defined in standard topology
- motivates definition of local homeomorphism
- each fibre is discrete
- somehow, fibres vary continuously with base point

Can define topology by defining sheaves
- opens are the subsheaves of 1

But why would you do that?
- much more complicated than defining the opens

Generalized spaces (Grothendieck toposes)

But why would you do that?
- much more complicated than defining the opens

Grothendieck discovered generalized spaces
- there are not enough opens
- you have to use the sheaves
- e.g. spaces of sets, or rings, of local rings
- set-theoretically - can be proper classes
- generalized topologically:
- specialization order becomes specialization *morphisms*
- continuous maps must be *at least* functorial and preserve filtered colimits
- cf. Scott continuity

Outline

"Space" = space of models of a geometric theory
- geometric maths = colimits + finite limits
- constructive
- includes free algebras, finite powersets
- but not exponentials, full powersets
- only a fragment of elementary topos structure
- fragment preserved by inverse image functors

cf. unions, finite intersections of opens

Space represented by classifying topos
= geometric maths generated by a generic point (model)

"continuity = geometricity"
- a construction is continuous if can be performed in geometric maths
- continuous map between toposes = geometric morphism
- geometrically constructed space = bundle, point |-> fibre
- "fibrewise topology of bundles"

Some adjustments to ACL

Algebra - used as in point-free topology (e.g. locales), but -
- Lindenbaum algebras become *categories* - of sheaves
- universal algebra general enough to cover partial operators
  (for essentially algebraic theories)

Categories
- categorical logic
- categorical structure to express mathematics being used

Logic (first order, many sorted)
- *geometric* theories, as needed for point-free topology
- infinitary disjunctions (cf. infinitary unions of open sets)
- sequent presentation of theories
  (negation not a connective - cf. no complements of open sets)
- constructive logic becomes important

Outline of course

1. Sheaves: Continuous set-valued maps

2. Theories and models: Categorical approach to many-sorted first-order theories.

3. Classifying categories: Maths generated by a generic model

4. Toposes and geometric reasoning: How to "do generalized topology".

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1. Sheaves

Local homeomorphism viewed as continuous map base point |-> fibre (stalk)

Alternative definition via presheaves

Idea: sheaf theory = set-theory "parametrized by base point"

Constructions that work fibrewise
- finite limits, arbitrary colimits
- cf. finite intersections, arbitrary unions for opens
- preserved by pullback

Interaction with specialization order

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2. Theories and models

(First order, many sorted)

Theory = signature + axioms
Context = finite set of free variables
Axiom = sequent

Models in Set
- and in other categories

Homomorphisms between models

Geometric theories

Propositional geometric theory => topological space of models.

Generalize to predicate theories?

3. Classifying categories

Geometric theories may be incomplete
- not enough models in Set
- category of models in Set doesn't fully describe theory

Classifying category - e.g. Lawvere theory
= stuff freely generated by generic model
- there's a universal characterization of what this means

For finitary logics, can use universal algebra
- theory presents category (of appropriate kind) by generators and relations

For geometric logic, classifying topos is constructed by more ad hoc methods.

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Constructive!
No choice
No excluded middle

4. Toposes and geometric reasoning

Classifying topos for T represents "space of models of T"

It is "geometric mathematics freely generated by generic model of T"

Map = geometric morphism
= result constructed geometrically from generic argument

Bundle = space constructed geometrically from generic base point
- fibrewise topology

Arithmetic universes for when you don't want to base everything on Set

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1. Sheaves

Local homeomorphism viewed as continuous map base point |-> fibre (stalk)

Alternative definition via presheaves

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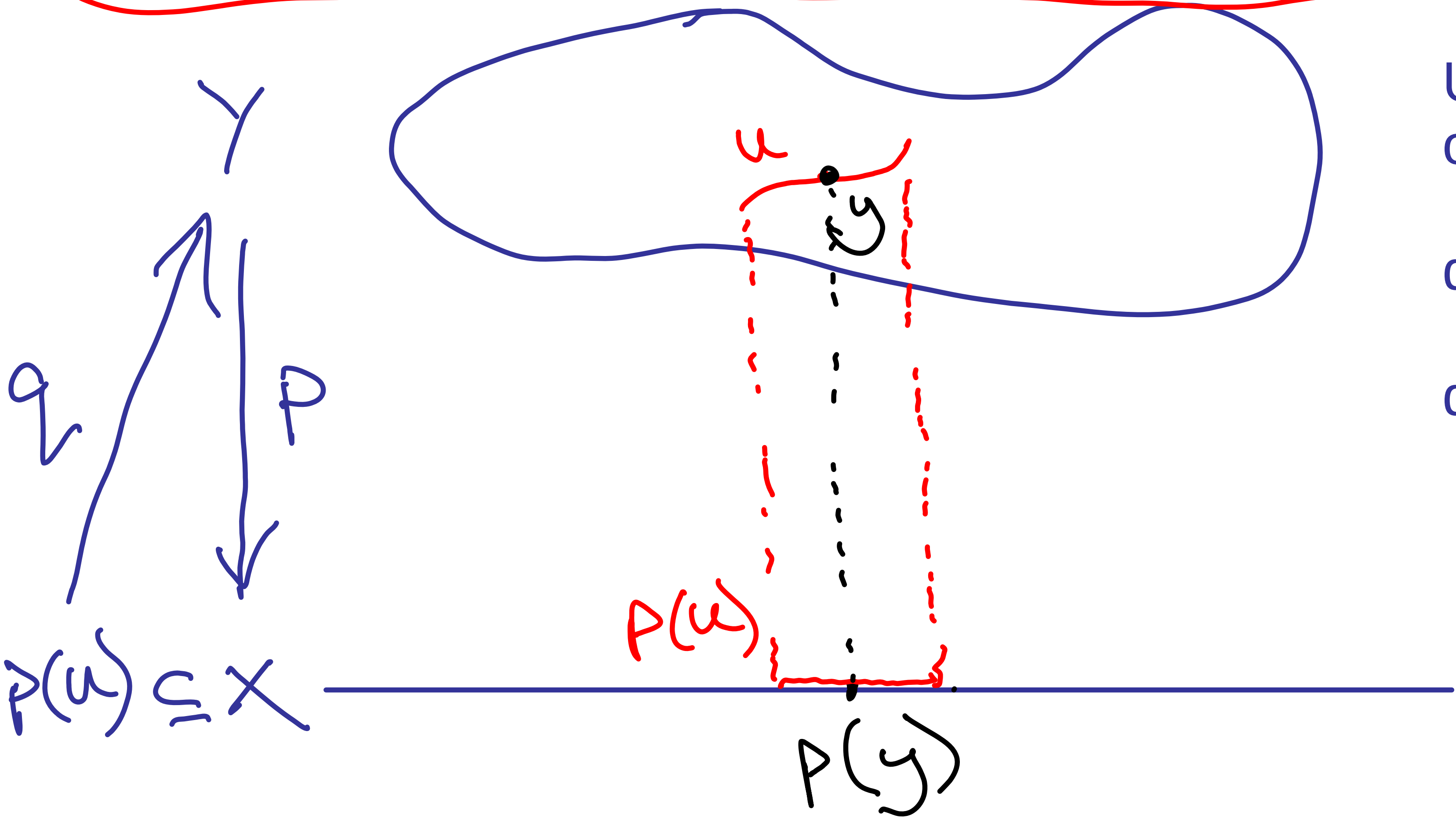
Interaction with specialization order

Local homeomorphisms

Call an open U p-basic if it has this property
[non-standard terminology]

Let p: Y -> X be a continuous map.

p is a local homeomorphism if every y in Y has an open neighbourhood U such that p maps U homeomorphically to an open in X.



U is open image of partial section

q: p(U) -> Y

qp = Id_p(U)

Examples

Constant sheaves for set S

$$X \times S \xrightarrow{\pi_1} X$$

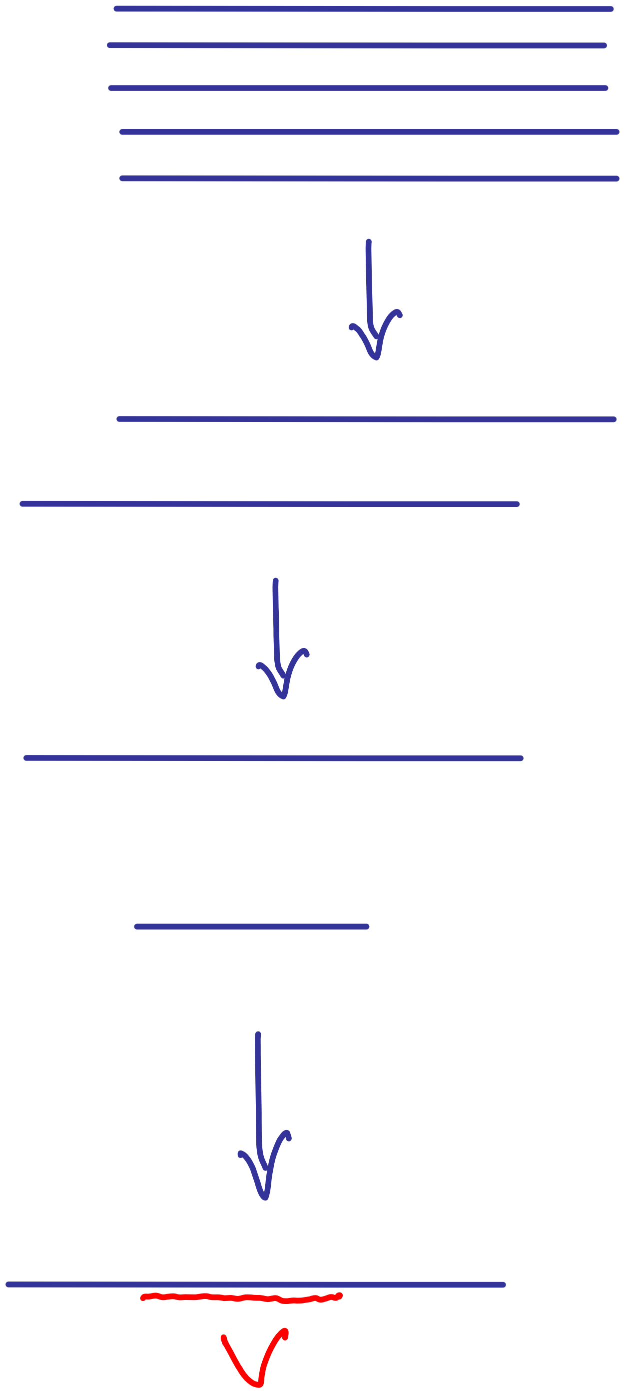
Every fibre is isomorphic to S.

Special case, S = 1

$$X \xrightarrow{\mathrm{Id}_X} X$$

Open inclusion V -> X

$$V \longrightarrow X$$



Local homeomorphism $p: Y \to X$ as continuous set-valued map

Every fibre (stalk) is discrete (in subspace topology of Y).

$y \in p^{-1}\{x\} \Rightarrow \{y\} = U \cap p^{-1}\{x\}$, open in $p^{-1}\{x\}$

Hence fibre is genuine set, not just a detopologized space.

Existence in fibre spreads out to neighbourhood of x (using U).

Equality in fibre also spreads out.
- Given U and U' for same y, so equal in fibre, use intersection to show equality over some open neighbourhood of x.

Hence view "local homeomorphism" as attempt to define continuity of a map X -> {sets}

Not a topological space in ordinary definition!

Special case: Y discrete iff diagonal Y -> YxY is open.

Theorem

p is a local homeomorphism iff p is open, and so is the diagonal

$$Y \xrightarrow{\Delta} Y \times_X Y$$

kernel pair of p,
pullback

$$\begin{tikzcd} Y \times_X Y \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ Y \arrow[r, "p"] & X \end{tikzcd}$$

Subspace of YxY
comprising the pairs (y,y')
such that p(y) = p(y')

$$\Delta(y) = (y,y)$$

$\Rightarrow$: Easy to see p is open.
For Δ, given Δ-basic U
show $\Delta(U) = (U \times U) \cap (Y \times_X Y)$

$\Leftarrow$: $\Delta(Y)$ is open. Given y,
find open U with $y \in U$,
$(U \times U) \cap (Y \times_X Y) \subseteq \Delta(Y)$

Then $p: U \to p(U)$ is bijection. p open $\Rightarrow$ its inverse is continuous & its image is open.

Specialization order

X a topological space, x, y points

y **specializes** x (x less than y in specialization order)
if every open neighbourhood of x also contains y.

$x \sqsubseteq y$

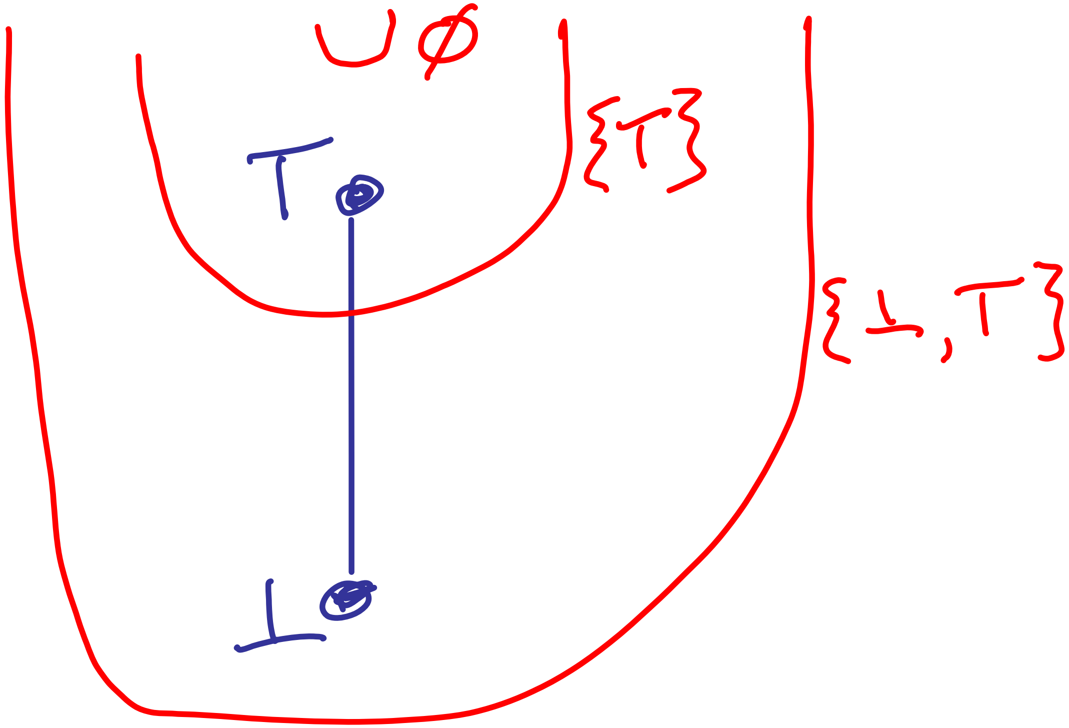
Get preorder on points.

e.g. **Sierpinki space $**: 2 points, 3 open sets

Hasse diagram for specialization order suffices to define topology.

NB Every open is **up-closed** in specialization order.

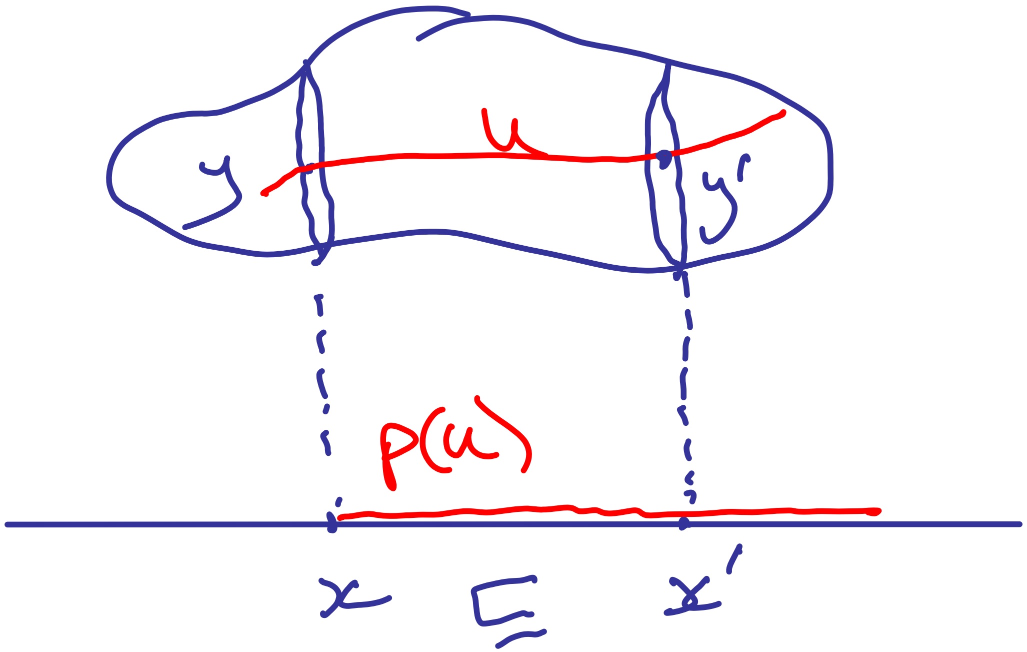
NB Continuous maps preserve specialization order.



Specialization order and local homeomorphisms

Theorem Let p: Y -> X be a local homeomorphism.
Suppose p(y) = x and x' specializes x.
Then there is a unique y' specializing y such that p(y') = x'.

Proof
(1) Given y, choose p-basic neighbourhood U
(2) x, hence also x', are in p(U),
- so find unique y' in p(U) with p(y') = x'.
(3) y' specializes y,
(4) and is the unique such with p(y') = x'.



Specialization order and local homeomorphisms

Theorem Let p: Y -> X be a local homeomorphism.
Suppose p(y) = x and x' specializes x.
Then there is a unique y' specializing y such that p(y') = x'.

$y \mapsto y'$

Hence: for $x \sqsubseteq x'$, get function $p^{-1}\{x\} \to p^{-1}\{x'\}$

(1) Continuous maps preserve specialization order.
(2) For a local homeomorphism, "base point |-> fibre" is supposed to be continuous set-valued map.

We shall see:
- functions between sets are like specialization order on {sets}
- but they are not an order
- they are specialization morphisms
- Still, local homeomorphisms transform specialization order between base points to specialization morphisms between fibres.

Generalized spaces will involve category theory in lots of ways!

Morphisms between local homeomorphisms

= commuting triangles of maps

$$\begin{tikzcd} Y_1 \arrow[rr, "g"] \arrow[dr, "p_1"'] & & Y_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Get a category LHom_X of local homeomorphisms with base space X.

Fact It's monic iff g is an open inclusion

p_1 a subsheaf of p_2

Changing the base space - pullback

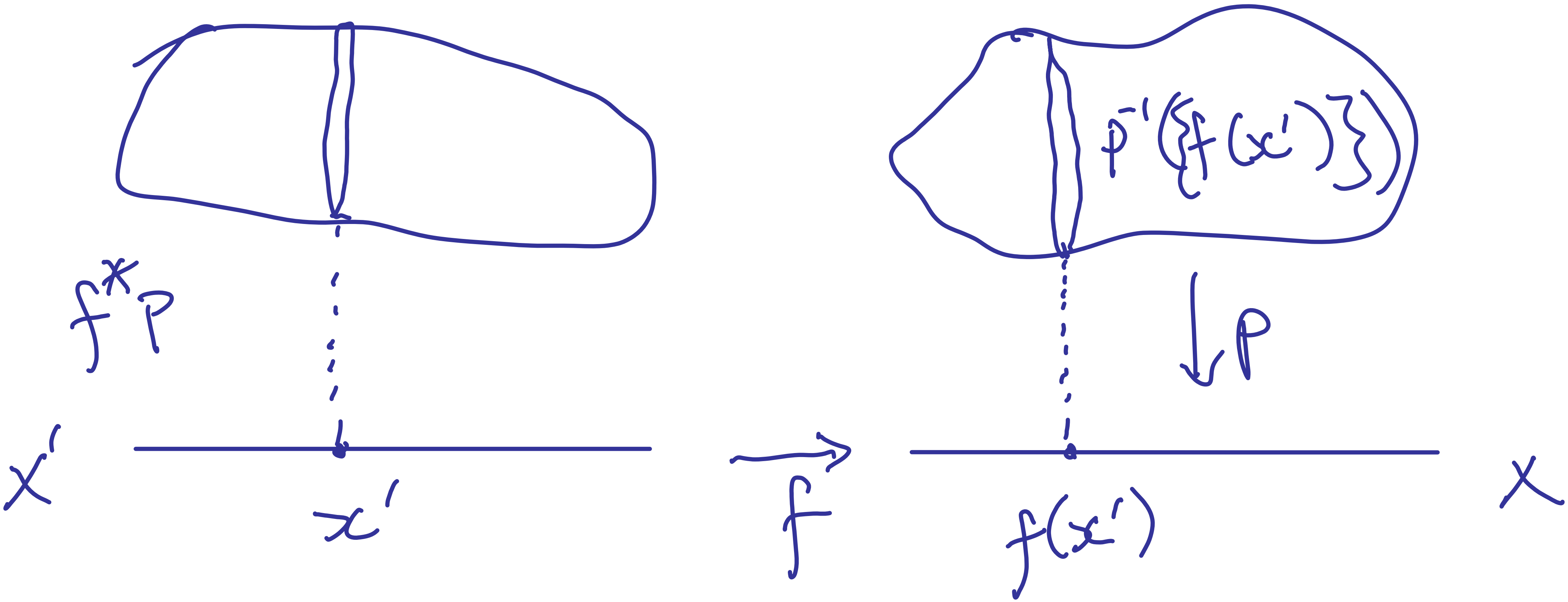
$$f^*Y = Y \times_X X' \xrightarrow{p^*f} Y$$

$$\begin{array}{ccc} f^*Y & \xrightarrow{p^*f} & Y \\ {\scriptstyle f^*p}\downarrow & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{f} & X \end{array}$$

Suppose f: X' -> X is a continuous map.
For each local homeomorphism p: Y -> X,
its pullback is a local homeomorphism over X.

$$(f^*p)^{-1}(\{x'\}) \cong p^{-1}(\{f(x')\})$$

Y



Presheaves

Let X be a topological space, and Ω(X) its lattice of opens.
A presheaf on X is a contravariant functor F: Ω(X) -> Set.
It has the pasting property if the following holds.

Suppose V_i $(i \in I)$ is a family of opens, and $a_i \in F(V_i)$ for each i, and $F(V_i \cap V_j \subseteq V_i)(a_i) = F(V_i \cap V_j \subseteq V_j)(a_j)$ in $F(V_i \cap V_j)$ for each i, j.

Then there is a unique $a \in F(\bigcup_i V_i)$ s.t. $a_i = F(V_i \subseteq \bigcup_i V_i)(a)$ for all i

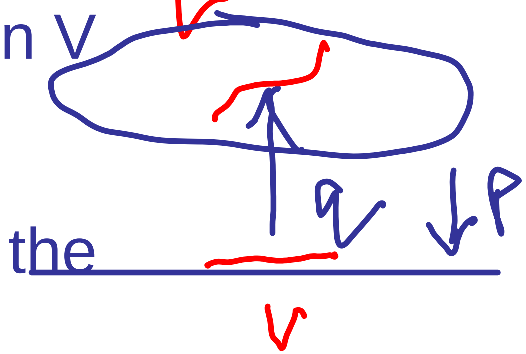
Theorem

Presheaves on X with pasting are equivalent to local homeomorphisms with base space X.

$q : V \to Y$

$p \circ q = \mathrm{Id}_V$

Given local homeomorphism p: Y -> X: define F(V) to be the set of continuous local sections of p defined on V - p-basic opens whose image is V.



Given F, define the stalk of F at x to be the colimit of the sets F(V) taken over open neighbourhoods V of x. Take Y = the disjoint union of the stalks, and then the elements of the F(V)'s provide a base of opens.

Make two categories, of presheaves and local homeomorphisms, show the two constructions above are functorial, and give an equivalence of categories.

Sheaves

Usual definition: sheaf = presheaf with pasting

Since these are equivalent to local homeomorphisms, I shall use the term sheaf ambiguously, to refer to either.

Local homeomorphisms bring out the idea of continuous set-valued map, which I want to emphasize.

Pasting presheaves, more technical, show the notion depends only on the lattice $\Omega(X)$, not on the points X.

Constructions on sheaves that work fibrewise
(1) Terminal sheaf

$$\begin{tikzcd} Y \arrow[r, dashed, "p"] \arrow[dr, "p"'] & X \arrow[d, "id"] \\ & X \end{tikzcd}$$

Terminal sheaf over X is identity map.

This is a local homeomorphism.

Any other sheaf factors uniquely through it.

It is the terminal sheaf over X.

Its fibres are all singletons, i.e. terminal sets.

Terminality works fibrewise

Constructions on sheaves that work fibrewise
(2) Pullbacks

$$
\begin{tikzcd}
W \arrow[r, dashed] \arrow[rr, bend left] \arrow[dr] \arrow[drrr, bend right] & Y_1 \times_{Y_3} Y_2 \arrow[r] \arrow[d] & Y_2 \arrow[dr, "g_2"] \arrow[ddr, "p_2"'] & \\
 & Y_1 \arrow[rr, "g_1"'] \arrow[drr, "p_1"] & & Y_3 \arrow[d, "p_3"] \\
 & & & X
\end{tikzcd}
$$

Pullback of Y's done for ordinary spaces
- ignoring X

It still works as pullback for triangles over X, as needed for LHom_X

Consider W = 1
- so W -> X picks out a point x of X
- and W -> Y_i picks out an element of fibre of p_i
- deduce fibres of pullback are pullbacks of fibres.
- pullback works fibrewise

Exercise (harder!)

Verify that the pullback as constructed is still a local homeomorphism over X.

Constructions on sheaves that work fibrewise
(2a) Binary products

$$
\begin{tikzcd}
W \arrow[r, dashed] \arrow[rr, bend left] \arrow[dr, bend right] \arrow[drr, bend right] & Y_1 \times_X Y_2 \arrow[r] \arrow[d] & Y_2 \arrow[d, "p_2"] \\
& Y_1 \arrow[r, "p_1"'] & X
\end{tikzcd}
$$

Special case of pullback
Take Y_3 = X

$$
\begin{tikzcd}
W \arrow[r, dashed] \arrow[rr, bend left] \arrow[dr] \arrow[ddrr, bend right] & Y_1 \times_{Y_3} Y_2 \arrow[r] \arrow[d] & Y_2 \arrow[d, "g_2"] \arrow[dd, bend left, "p_2"] \\
& Y_1 \arrow[r, "g_1"] \arrow[dr, "p_1"'] & Y_3 \arrow[d, "p_3"] \\
& & X
\end{tikzcd}
$$

Fibres are products of fibres

Pullback for "fibrewise product" is often called fibred product

Constructions on sheaves that work fibrewise
(1+2) Finite limits

Putting 1 and 2 together:

LHom_X has all finite limits.

They are constructed fibrewise.

Not infinite limits, though -

e.g. ???

Constructions on sheaves that work fibrewise
(3) Arbitrary coproducts (disjoint unions)

Similar to pullbacks

Coproduct of Y's done for ordinary spaces
- ignoring X

It still works as coproduct for triangles over X, as needed for LHom_X

Fibres in coproduct are coproducts of fibres.

Straightforward to show it's a local homeomorphism

$i \in I$

$$\begin{tikzcd} Y_i \arrow[r] \arrow[rr, bend left] \arrow[dr, "p_i"'] & \coprod Y_i \arrow[r, dashed] \arrow[d] & W \arrow[dl] \\ & X & \end{tikzcd}$$

Constructions on sheaves that work fibrewise
(4) Quotients by equivalence relations

$$
\begin{tikzcd}
E \arrow[r, hook] & Y \times_X Y \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\pi_1"'] & Y \arrow[r] \arrow[d, "p"] & Y/E \arrow[dl, "q"] \\
 & & X &
\end{tikzcd}
$$

E is -
- open in fibred product Y x_X Y
- an equivalence relation on each fibre
- also on whole of Y

Y/E topologized by opens in Y that are unions of equivalence classes.

q is a local homeomorphism
Given p-basic U for Y, can fatten it to U', q-basic open of Y/E

$$U' = \bigcup \{ V \mid V \text{ p-basic}, \ p(V) \subseteq p(U), \ (V \times U) \cap (Y \times_X Y) \subseteq E \}$$

$$= \text{smallest open for } Y/E \text{ containing } U$$

Constructions on sheaves that work fibrewise
(5) Intersections and unions of subsheaves

Intersection is pullback

$$
\begin{tikzcd}
 & Y_1 \arrow[r, "f_1"] & Y \arrow[dd, "p"] \\
Y_1 \cap Y_2 \arrow[ur] \arrow[r] & Y_2 \arrow[ur, "f_2"'] & \\
 & & X
\end{tikzcd}
$$

For union:

- Form coproduct
- Take image factorization
(quotient of kernel pair)

$$
\begin{tikzcd}
 & Y_1 \amalg Y_2 \arrow[r, dashed] \arrow[dr, dashed] & Y_1 \cup Y_2 \arrow[d, dashed] \\
Y_1 \arrow[ur] \arrow[rr, "f_1"'] & & Y \arrow[d, "p"] \\
 & Y_2 \arrow[uu] \arrow[ur, "f_2"'] & X
\end{tikzcd}
$$

Constructions on sheaves that work fibrewise
(6) Coequalizers

$$\begin{tikzcd}
R \arrow[rr] & & Y_2 \times_X Y_2 \arrow[d, shift left] \arrow[d, shift right] \\
Y_1 \arrow[u] \arrow[urr, "\langle f,g \rangle"] \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[drr, "p_1"'] & & Y_2 \arrow[d, "p_2"] \\
& & X
\end{tikzcd}$$

Using previous constructions

- Pair f and g to make <f,g>
- Take image factorization
to make R
It's fibrewise relation on Y_2,
- though not equivalence relation
- Generate reflexive, symmetric
relation S
(use coproducts and image factorization)
- Use pullbacks to get S^n for each n
- Take coproduct over n
- Its image is the transitive closure of S
= equivalence relation generated by R
- Quotient to get the coequalizer

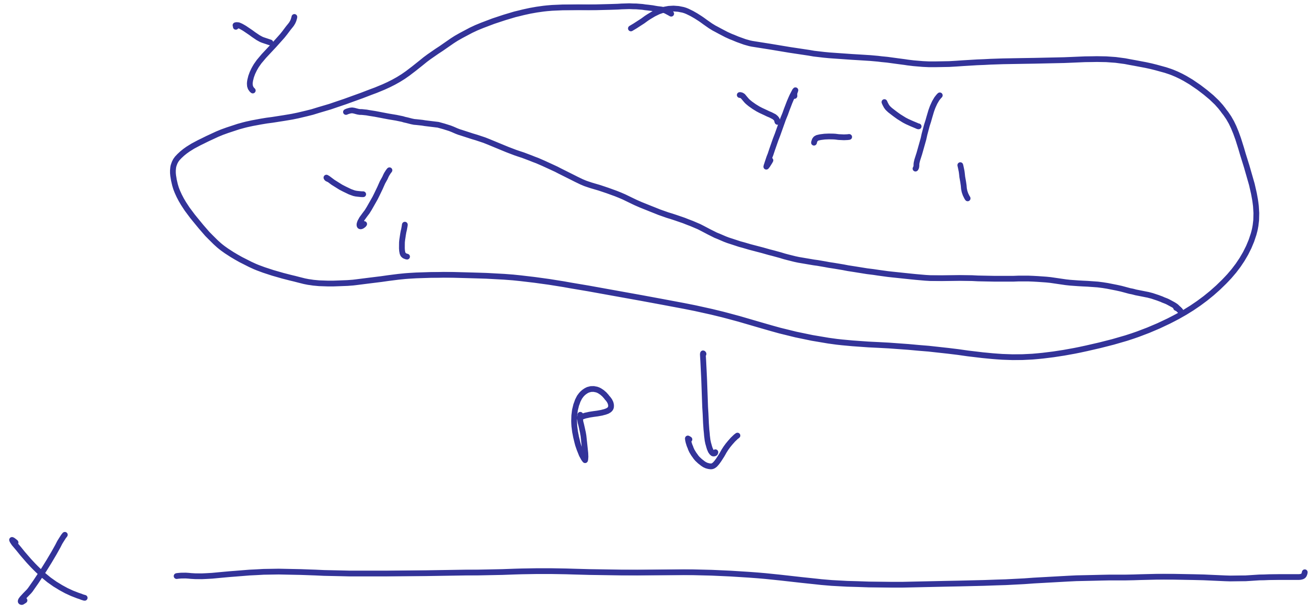
Constructions on sheaves that don't work fibrewise

(1) Complements of subsheaves

$$
\begin{tikzcd}
Y_1 \arrow[r, hook] & Y \arrow[d] \\
 & X
\end{tikzcd}
$$

Can define Y - Y_1

It's biggest subsheaf disjoint from Y_1



But subsheaves are *open*

Y - Y_1 is *interior* of complement of Y_1

Along boundary, points get lost.

This is intuitionistic negation.

Logic of sheaves is not classical.

Example

Take constant sheaf 1 (Y = X).
Subsheaf = open U of X.

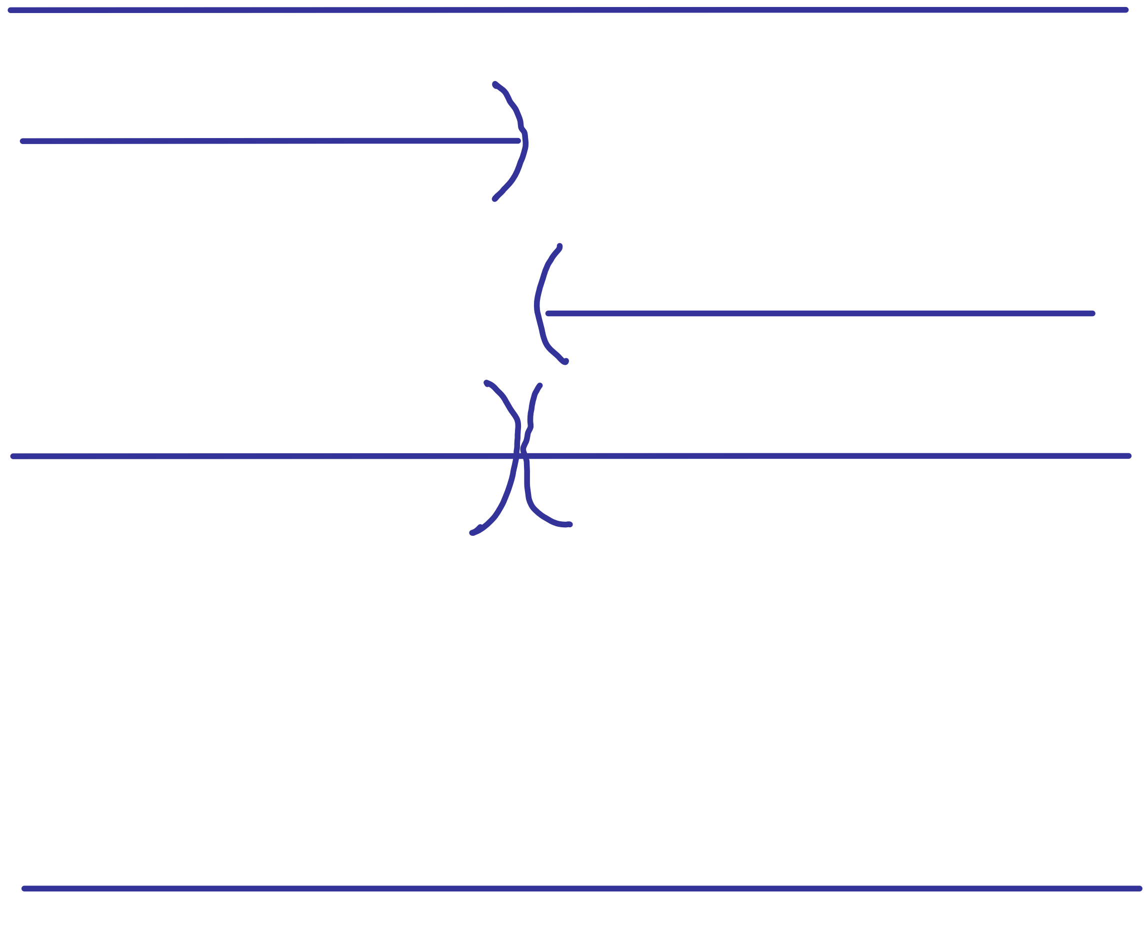
Sheaf 1

Subsheaf U

Subsheaf 1-U = $\neg U$

Subsheaf $U \vee \neg U$

$\neq 1$



Excluded middle fails!

X

Other examples

Other constructions can be done, but not fibrewise.
e.g. exponentials (function spaces)
e.g. powersheaves

Idea

If a sheaf is a set-valued map on X:
Working with sheaves (over X) should be "just like" working with sets
- but with parameters x in X everywhere

This can be made to work. But -

1. Can only use certain constructions on the "parametrized sets".
2. We think of them the constructions categorically, rather than as sets of elements.
3. For sheaves, it comes down to finite limits, arbitrary colimits, and whatever can be expressed in those terms.
4. The trick is to work within this constrained, non-classical mathematics. Then everything is "automatically continuous".
5. There are some other constructions too, but they don't work fibrewise, and they're not classical either.

Toposes

The sheaves we have seen were for an ordinary "ungeneralized" space X.

Grothendieck noticed that the finite limits, arbitrary colimits were the constructions needed for some cohomology theories, and he invented toposes as the categories where he could carry this out.

Thus they were the "categories of sheaves for generalized spaces".

Further reading

Sheaves
Mac Lane and Moerdijk "Sheaves in geometry and logic"

Introduction to connection with geometric logic
Vickers "Fuzzy sets and geometric logic"