Gordon's Conjectures:
Pontryagin-van Kampen Duality and
Fourier Transform in Hyperfinite Ambience

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Abstract

Using the ideas of E. I. Gordon [Go1], [Go2] we present an approach, based on *nonstandard analysis* (NSA), to simultaneous approximation of locally compact abelian (LCA) groups and their duals by finite abelian groups, as well as to approximation of the Fourier transforms on various functional spaces over them by the discrete Fourier transform (DFT). In 2012 we proved the three Gordon's Conjectures (GC1–3) which were open since 1991 and are crucial both in the formulations and proofs of the LCA groups and Fourier transform approximation theorems. The proofs of GC1 and GC2 combine some methods of NSA with Fourier-analytic methods of *additive combinatorics*, stemming from the paper [GR] by Green and Ruzsa and the book [TV] by Tao and Vu. The proof of GC3 relies on a fairly general nonstandard version of the *Smoothness-and-Decay Principle.*

Our approach is based on representing LCA groups by triplets (G, G_0, G_{f}) where G is a hyperfinite abelian group, and $G_0 \subseteq G_{\mathrm{f}}$ are its external subgroups, intuitively viewed as the monad of infinitesimal elements and the galaxy of finite elements, respectively. It can be shown that every LCA group $\mathbf{G}$ is isomorphic to the *observable trace* or *nonstandard hull* $G^\flat = G_{\mathrm{f}}/G_0$ of such a triplet. The dual triplet of (G, G_0, G_{f}) is defined as $\big(\widehat{G}, G_{\mathrm{f}}^{\perp\!\!\!\perp}, G_0^{\perp\!\!\!\perp}\big)$, where $\widehat{G}$ is the group of all internal homomorphisms (characters) $G \to {}^*\mathbb{T}$, and the *infinitesimal annihilators*

$$G_{\mathrm{f}}^{\perp\!\!\!\perp} = \big\{\chi \in \widehat{G} \;\big|\; \forall\, a \in G_{\mathrm{f}} : \chi(a) \approx 1\big\}\,,$$
$$G_0^{\perp\!\!\!\perp} = \big\{\chi \in \widehat{G} \;\big|\; \forall\, a \in G_0 : \chi(a) \approx 1\big\}$$

of the subgroups G_{f}, G_0 consist of characters which are infinitesimally close to 1 on the galaxy G_{f} or continuous in the intuitive sense backed by NSA, respectively.

GC1 states that for $\mathbf{G} = G^\flat$ its dual group $\widehat{\mathbf{G}}$ is canonically isomorphic to the observable trace $\widehat{G}^\flat = G_0^{\perp\!\!\!\perp}/G_{\mathrm{f}}^{\perp\!\!\!\perp}$ of the dual triplet $\big(\widehat{G}, G_{\mathrm{f}}^{\perp\!\!\!\perp}, G_0^{\perp\!\!\!\perp}\big)$. It turns out to be equivalent to the *Triplet Duality Theorem* according to which the dual triplet $\big(G, G_0^{\perp\!\!\!\perp\perp\!\!\!\perp}, G_{\mathrm{f}}^{\perp\!\!\!\perp\perp\!\!\!\perp}\big)$ of the dual triplet $\big(\widehat{G}, G_{\mathrm{f}}^{\perp\!\!\!\perp}, G_0^{\perp\!\!\!\perp}\big)$ coincides with the original triplet (G, G_0, G_{f}).

GC2 states certain natural duality relation, partly akin to the *Uncertainty Principle*, between "normalizing coefficients" or "elementary charges" on both the triplets, by means of which the Haar measures on their nonstandard hulls can be defined using the *Loeb measure* construction.

Representing the pair of dual LCA groups $\mathbf{G}$, $\widehat{\mathbf{G}}$ by a pair of dual triplets enables to approximate the Fourier-Plancherel transform $\mathrm{L}^2(\mathbf{G}) \to \mathrm{L}^2\big(\widehat{\mathbf{G}}\big)$ by means of the hyperfinite dimensional DFT ${}^*\mathbb{C}^G \to {}^*\mathbb{C}^{\widehat{G}}$. GC3 states that such an approximation is infinitesimally precise almost everywhere. Essentially the same is true also for the Fourier transform $\mathrm{L}^1(\mathbf{G}) \to \mathrm{C}_0\big(\widehat{\mathbf{G}}\big)$ and even for the Fourier-Stieltjes transform $\mathrm{M}(\mathbf{G}) \to \mathrm{C}_{\mathrm{bu}}\big(\widehat{\mathbf{G}}\big)$ extending it, as well as for the generalized Fourier transforms $\mathrm{L}^p(\mathbf{G}) \to \mathrm{L}^q\big(\widehat{\mathbf{G}}\big)$, for any pair of adjoint exponents $1 < p \le 2 \le q < \infty$.

Standard interpretations of these results imply the existence of "arbitrarily good" approximations of all the above Fourier transforms on every LCA group $\mathbf{G}$ by the DFT on some finite abelian group G.

Depending on time, we will survey most of the above mentioned constructions and results.

References

[Go1] E. I. Gordon, *Nonstandard analysis and locally compact abelian groups*, Acta Appl. Math. **25** (1991), 221–239.

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