Karl Weierstraß and the theory of Abelian and elliptic functions

Peter Ullrich

Universität Koblenz-Landau, Campus Koblenz,
Fachbereich 3: Mathematik / Naturwissenschaften,
Mathematisches Institut

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Influence on Weierstraß's curriculum vitae

Winter 1837/38: successful encounter with a problem on elliptic functions $\rightsquigarrow$ final decision to choose mathematics

1854: article "Zur Theorie der *Abel*schen Functionen" in the "Journal für die reine und angewandte Mathematik" ("Crelle's Journal") $\rightsquigarrow$

- promotion to "Oberlehrer" (senior teacher),
- honorary doctoral degree from the university at Königsberg,
- professorship at the "Gewerbeinstitut" at Berlin,
- extraordinary professorship at the university at Berlin, and
- membership in the Prussian academy of sciences

Influence on choice of areas of mathematical research

necessary ingredients to study elliptic and Abelian functions:

- the theory of functions of complex variables,
- in particular of several complex variables,
- power series,
- differential equations,
- the study of non-generic cases and the rôle of counterexamples,
- . . .

From Emil Lampe's obituary (1897)

"In dem Centrum aller Arbeiten von W e i e r s t r a ß stehen die A b e l'schen Functionen; man könnte sogar sagen, daß alle allgemeinen functionentheoretischen Untersuchungen von ihm nur zu dem Zwecke unternommen sind, um das Problem in Vollständigkeit und Klarheit zu lösen, das durch die Forderung der Darstellung der A b e l'schen Functionen jener Zeit gestellt war."

Elliptic and Abelian functions, 1

Problem. "Understand" the map

$$x \mapsto \int_{x_0}^{x} \frac{1}{\sqrt{P(t)}}\, dt$$

with $P(t)$ a polynomial in t

- of degree 3 or 4 (**elliptic case**)

solved by Carl Friedrich Gauß,
Niels Henrik Abel and Carl Gustav Jacob Jacobi (around 1830)

and, more general, (**Abelian case**)

- with $P(t)$ of degree 5 or 6 (**ultraelliptic case**)

solved by Adolph Göpel (1847) and Georg Rosenhain (1851)

Elliptic and Abelian functions, 2

and, even more general,

$$x \mapsto \int_{x_0}^{x} G(t)\,dt$$

- with

  $$G(t) = \frac{1}{\sqrt{P(t)}}$$

  with $P(t)$ of degree greater than 4 (**hyperelliptic case**)

  solved by Weierstraß (1849 / 1854)

- with $G(t)$ a rational function of an algebraic function of t (**general Abelian case**)

  solved by Bernhard Riemann (1857)
  and Weierstraß (1857 / 1869)

Elliptic functions

Let $P(t)$ be a polynomial of degree 3 or 4 and consider

$$x \mapsto \int_{x_0}^{x} \frac{1}{\sqrt{P(t)}}\, dt =: u(x).$$

Gauß, **Abel**, and **Jacobi**: Study the inverse function f of this map! This is an "elliptic function":

- a function of one complex variable u,
- which fulfils the differential equation

$$(f')^2 = P(f),$$

- is doubly periodic, and
- can be expressed as the quotient of two theta series (in particular of convergent power series) in u.

Weierstraß at Bonn

Abel (1829 / 1830)

For

$$x \mapsto \int_{x_0}^{x} \frac{1}{\sqrt{(1-t^2)(1-c^2t^2)}}\,dt =: u(x),$$

with c a parameter, the inverse function $\lambda(u)$ is the quotient of an odd and an even power series:

$$\lambda(u) = \frac{u + A_1u^3 + A_2u^5 + A_3u^7 + \dots}{1 + B_2u^4 + B_3u^6 + B_4u^8 + \dots}.$$

The coefficients $A_1, A_2, A_3, \dots$ *and* $B_2, B_3, B_4, \dots$ *are polynomials in* c^2.

Weierstraß (1837/38)

infers this directly from the differential equation

$$(\lambda'(u))^2 = (1-\lambda(u)^2)(1-c^2\lambda(u)^2).$$

Weierstraß at Münster

Abel's $\lambda(u)$ equals Christoph Gudermann's

- $\operatorname{sn} u$ ("sinus amplitudinis").

There were also

- $\operatorname{cn} u$ ("cosinus amplitudinis") and
- $\operatorname{dn} u := \sqrt{1 - c^2 \operatorname{sn}^2 u}$.

Weierstraß (1840, published 1894)
defines power series $Al_1(u)$, $Al_2(u)$, $Al_3(u)$ and $Al(u)$ similar to Abel's series for which

$$\operatorname{sn} u = \frac{Al_1(u)}{Al(u)}, \quad \operatorname{cn} u = \frac{Al_2(u)}{Al(u)}, \quad \text{and} \quad \operatorname{dn} u = \frac{Al_3(u)}{Al(u)}$$

holds.

Several complex variables

Jacobi (1834)
notes that already in the ultraelliptic case ($\deg P \in \{5, 6\}$) one is forced to consider

$$x \mapsto \left(\int_{x_0}^{x} \frac{1}{\sqrt{P(t)}}\, dt, \int_{x_0}^{x} \frac{t}{\sqrt{P(t)}}\, dt \right) =: u(x)$$

instead of only the first integral.

"[I]n hac quasi desparatione" he starts studying functions of several complex variables.

The Jacobi inversion problem

Jacobi (1835)
sets up the plan:

Describe the inverse function $x = x(u)$ again by means of theta series, but now

- theta series A, B and C of two variables and
- by the **quadratic** equation

$$A \cdot x^2 + B \cdot x + C = 0.$$

Furthermore: Generalize this to the hyperelliptic and the general Abelian case.

Göpel and Rosenhain

independently from each other

- Göpel (1846, published 1847) and
- Rosenhain (1847, published 1851):

solution of the Jacobi inversion problem for the ultraelliptic case

$$x \mapsto \int_{x_0}^{x} \frac{1}{\sqrt{P(t)}}\, dt$$

with $\deg P \in \{5, 6\}$

- Göpel: appraisal by Jacobi in his obituary
- Rosenhain: prize of the Paris academy

Difficulty: For higher degree of P there are "too many" theta functions as compared to hyperelliptic integrals.

Weierstraß 1849, 1854, and 1856

1849: "Beitrag zur Theorie der Abel'schen Integrale" (Braunsberg prospect),

1854: "Zur Theorie der *Abel*schen Functionen" (Crelle's Journal),

1856: "Theorie der *Abel*'schen Functionen" (Crelle's Journal)

solution of the Jacobi inversion problem for general hyperelliptic integrals

$$x \mapsto \int_{x_0}^{x} \frac{1}{\sqrt{P(t)}}\, dt$$

with deg P arbitrary

- increasing number of pages devoted to the necessary fundaments from analysis

preparation of a paper for the solution of the Jacobi inversion problem for the general Abelian case

Riemann 1857

1857: "Theorie der *Abel*'schen Functionen" (Crelle's Journal)

solution of the Jacobi inversion problem

- more by use of geometrical / topological means ("Riemann surfaces") than by algebraic / analytic means

Weierstraß withdraws his manuscript from the printing press and works on comparing his method with Riemann's,

in particular, examines the "non-obivous" arguments with Riemann:

- the Dirichlet principle,
- general singularities of complex algebraic curves.

Weierstraß lectures on Abelian functions from 1869 onwards

- line of thought as in the hyperelliptic case,
- method: study of algebraic functions with the help of means from analysis,
  e.g. instead of Riemann's "surfaces": analytic / algebraic configurations (generalization of analytic functions),
  algebraic definition of the "genus",
- definition of prime functions for the function field (analogy to number fields),
- published only in 1902 in Volume 4 of the "Mathematische Werke"

Elliptic functions revisited: $\wp$

In his lecture courses from the winter term 1862/63 onwards Weierstraß defines elliptic functions as solutions of an algebraic differential equation of first order and degree 3 or 4.

He reduces the differential equation to the form

$$\left(\frac{ds}{du}\right)^2 = 4s^3 - g_2 s - g_3$$

and defines $\wp$ as the (unique) function as the unique solution with pole at $u = 0$.

From this he infers that

$$\wp(u) = \frac{1}{u^2} + \sum_{(\mu,\nu)\in\mathbb{Z}\times\mathbb{Z}-\{(0,0)\}} \left(\frac{1}{(u-(\mu\omega_1+\nu\omega_2))^2} - \frac{1}{(\mu\omega_1+\nu\omega_2)^2}\right).$$