Counterexamples in the Work of Karl Weierstraß

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Outline

Figure : Opernplatz und Universität, Berlin 1860 (Borcher)

A function for counterexamples

- In today's mathematics, students meet counterexamples early, in order to show the precise range of a definition.
- In the mid-nineteenth century, however, definitions lacked the formal character that we now ascribe to them, in our post-Hilbert era.
- Instead, definitions for the most part were treated as *descriptive*, more like dictionary definitions.

A Definition from the Oxford English Dictionary

dog, *n.*[1]

Text size: A A

View as: Outline | Full entry

Quotations: Show all | Hide all Keywords: On | Off

Pronunciation: Brit. /dɒg/ , U.S. /dɔg/ , /dag/

Forms: ... (Show More)

Etymology: Origin unknown.... (Show More)

I. The animal.

1.

a. A domesticated carnivorous mammal, *Canis familiaris* (or *C. lupus familiaris*), which typically has a long snout, an acute sense of smell, non-retractile claws, and a barking, howling, or whining voice, widely kept as a pet or for hunting, herding livestock, guarding, or other utilitarian purposes.

Thesaurus »
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Weierstraß and counterexamples

- Much of the thrust of Weierstraß' work to make analysis rigorous strikes at unwarranted *assumptions*.
- In his work, counterexamples are frequently constructed for the specific purpose of improving, or rejecting, arguments.
- The examples we look at today will all have that direction, and all aim at the work of Riemann.
  1. The existence of functions that minimize certain integrals (critique of the Dirichlet principle)
  2. The existence of functions that are everywhere continuous but not differentiable on any interval.
  3. The existence of functions that cannot be continued analytically across "natural boundaries."
- All of these had particular importance in his own work and have become classic in several senses. We discuss only the first two.

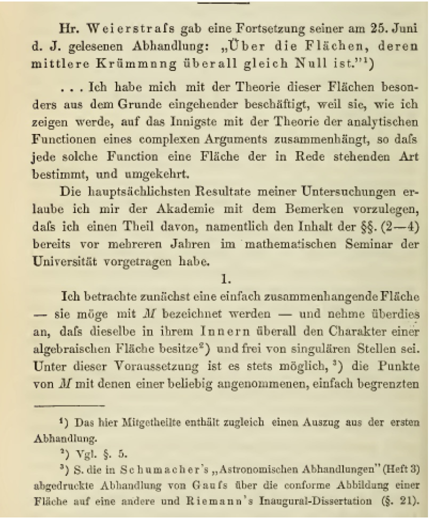
Riemann's mathematics and rigour

> *Auch wir jungen Mathematiker hatten damals sämtlich das Gefühl, als ob die Riemannschen Anschauungen und Methoden nicht mehr der strengen Mathematik der Euler, Lagrange, Gauß, Jacobi, Dirichlet u.a. angehörten – wie dies ja stets der Fall zu sein pflegt, wenn eine neue große Idee in die Wissenschaft eingreift, welche erste Zeit braucht, um in den Köpfen der lebenden Generation verarbeitet zu werden. So wurden die Leistungen der Göttinger Schule von uns, zum Teil wenigstens, nicht so geschätzt...*

Leo Koenigsberger, 1919, discussing the Berlin of the1860s

Weierstraß and Existence questions

- Already in 1861, Weierstrass had worked on minimal surfaces, where one seeks functions that minimize the integral expressing surface area.
- As usual he returned to this area with a critical eye, carefully examining his assumptions in the 1866 publication of this work.
- But his first detailed critique was of the Dirichlet Principle.



Hr. Weierstrafs gab eine Fortsetzung seiner am 25. Juni d. J. gelesenen Abhandlung: „Über die Flächen, deren mittlere Krümmnng überall gleich Null ist."[1])

. . . Ich habe mich mit der Theorie dieser Flächen besonders aus dem Grunde eingehender beschäftigt, weil sie, wie ich zeigen werde, auf das Innigste mit der Theorie der analytischen Functionen eines complexen Arguments zusammenhängt, so dafs jede solche Function eine Fläche der in Rede stehenden Art bestimmt, und umgekehrt.

Die hauptsächlichsten Resultate meiner Untersuchungen erlaube ich mir der Akademie mit dem Bemerken vorzulegen, dafs ich einen Theil davon, namentlich den Inhalt der §§. (2—4) bereits vor mehreren Jahren im mathematischen Seminar der Universität vorgetragen habe.

1.

Ich betrachte zunächst eine einfach zusammenhangende Fläche — sie möge mit M bezeichnet werden — und nehme überdies an, dafs dieselbe in ihrem Innern überall den Charakter einer algebraischen Fläche besitze[2]) und frei von singulären Stellen sei. Unter dieser Voraussetzung ist es stets möglich, [3]) die Punkte von M mit denen einer beliebig angenommenen, einfach begrenzten

[1]) Das hier Mitgetheilte enthält zugleich einen Auszug aus der ersten Abhandlung.

[2]) Vgl. §. 5.

[3]) S. die in Schumacher's „Astronomischen Abhandlungen" (Heft 3) abgedruckte Abhandlung von Gaufs über die conforme Abbildung einer Fläche auf eine andere und Riemann's Inaugural-Dissertation (§. 21).

Dirichlet on the Dirichlet Principle I

- Despite Dirichlet's reputation for rigour (not only Koenigsberger but also Jacobi) his text on this leaves room for questions.
- Even the initial statement of the theorem that the Dirichlet problem has a solution is vague to our eyes.

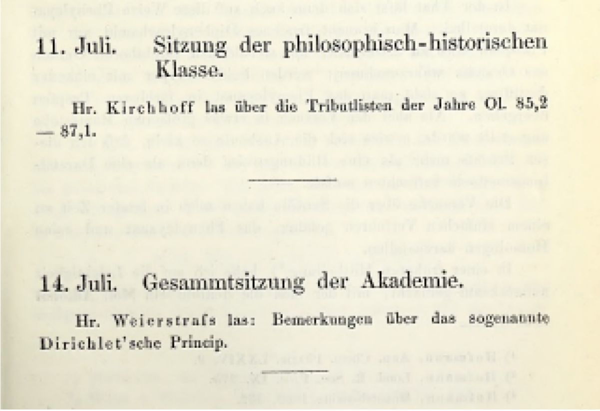
Ist irgend eine endliche Fläche gegeben, so kann man dieselbe stets, aber nur auf eine Weise, so mit Masse belegen, dass das Potential in jedem Punkte der Fläche einen beliebig vorgeschriebenen (nach der Stetigkeit sich ändernden) Werth hat. (Dirichlet lectures 1856, transcribed by Dedekind, quoted by Weierstraß)

Dirichlet - The principle

The proof reveals the point to which Weierstraß was later to object:

> *Wir beweisen den Satz, indem wir von einer rein mathematischen Evidenz ausgehen. Es ist in der That einleuchtend, dass unter allen Functionen u, welche überall nebst ihren ersten Derivirten sich stetig in t ändern und auf der Begrenzung von t die vorgeschriebenen Werthe annehmen, es eine (oder mehrere) geben muss, für welche das durch den ganzen Raum t ausgedehnte Integral*
>
> $$U = \int \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right\} dt$$
>
> *seinen kleinsten Werth erhält.*

1870: An announcement



11. Juli. Sitzung der philosophisch-historischen Klasse.

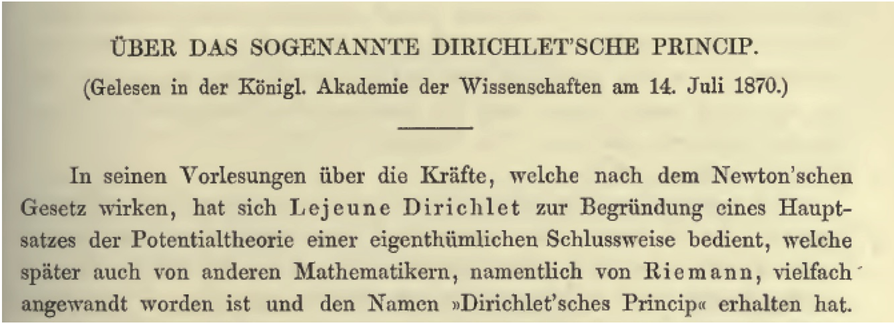
Hr. Kirchhoff las über die Tributlisten der Jahre Ol. 85,2 — 87,1.

14. Juli. Gesammtsitzung der Akademie.

Hr. Weierstrafs las: Bemerkungen über das sogenannte Dirichlet'sche Princip.

Figure : From the *Monatsberichte*, July 1870.

Riemann and the Weierstraß critique



ÜBER DAS SOGENANNTE DIRICHLET'SCHE PRINCIP.

(Gelesen in der Königl. Akademie der Wissenschaften am 14. Juli 1870.)

In seinen Vorlesungen über die Kräfte, welche nach dem Newton'schen Gesetz wirken, hat sich Lejeune Dirichlet zur Begründung eines Hauptsatzes der Potentialtheorie einer eigenthümlichen Schlussweise bedient, welche später auch von anderen Mathematikern, namentlich von Riemann, vielfach angewandt worden ist und den Namen »Dirichlet'sches Princip« erhalten hat.

Figure : "... namentlich von Riemann"

Publication and Reception of Weierstraß' Example

- Weierstraß lectured on the counterexample on July 14, 1870 at this Academy.
- The presentation was noted in the *Monatsberichte* (p. 575) but only the title appears, and the details were published only with his collected works in the 1890s.
- Nevertheless information about it spread rapidly.
- Both Carl Neumann and H. A. Schwarz devised alternative methods for proving existence of solutions to the Dirichlet problem, for example.
- The perceived importance of Riemann's results was part of the reason for the quick reception.

Weierstraß' Counterexample I

Let $\phi(x)$ be a real single-valued function of a real variable x such that ϕ and its derivative are continuous in $(-1, 1)$ and that $\phi(-1) = a, \phi(1) = b, a \neq b$.
If the Dirichlet "Schlussweise" were correct then among such ϕ there would be one that would minimize the integral

$$J = \int_{-1}^{1} \left(x \frac{d\phi(x)}{dx} \right)^2 dx.$$

Now, the [greatest] lower bound of this integral on the interval is 0. For if one chooses for example

$$\phi(x) = \frac{a+b}{2} + \frac{b-a}{2} \frac{\arctan \frac{x}{\varepsilon}}{\arctan \frac{1}{\varepsilon}},$$

where ε is an arbitrary positive value, this function fulfils the conditions: in particular note the endpoint values.

Weierstraß' Counterexample II

Now

$$J < \int_{-1}^{1} (x^2 + \varepsilon^2) \left(\frac{d\phi(x)}{dx} \right)^2 dx,$$

and

$$\frac{d\phi(x)}{dx} = \frac{b-a}{2 \arctan \frac{1}{\varepsilon}} \cdot \frac{\varepsilon}{x^2 + \varepsilon^2},$$

yielding

$$J < \varepsilon \frac{(b-a)^2}{(2 \arctan \frac{1}{\varepsilon})^2} \int_{-1}^{1} \frac{\varepsilon dx}{x^2 + \varepsilon^2}$$

and hence

$$J < \frac{\varepsilon}{2} \frac{(b-a)^2}{\arctan \frac{1}{\varepsilon}}.$$

Clearly then the lower bound is zero.But J can't attain that bound: if $J = 0$, then $\phi'(x) = 0$ on the interval, and ϕ is constant. But $\phi(-1) = a$ and $\phi(1) = b$ with $a \neq b$.

Continuous Nowhere-differentiable Functions

- Riemann's example, lost, was produced in lectures in 1861 or possibly earlier.
- We know this from Weierstraß, who heard oral testimony from some who had attended Riemann's lectures.
- Weierstraß turned to this in 1872. He says: even the most rigorous of mathematicians (his examples are Gau§, Cauchy, and Dirichlet) assumed that a single-valued continuous function will have a first derivative except "an einzelnen Stellen" where it can be "unbestimmt oder unendlich gross".
- Riemann's example, according to Weierstraß, is the function

$$\sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2}$$

- Riemann's proof had not survived, and Weierstraß had not been able to prove it himself.

18. Juli. Gesammtsitzung der Akademie.

Hr. Weierstraſs las über stetige Funktionen ohne bestimmte Differentialquotienten.

Figure : A not-very-informative announcment, 1872

The publication of Weierstraß' example

- Originally just an announcement appeared in the *Monatsberichte*
- The first published version appeared in a paper of Paul du Bois-Reymond in 1875.
- We will omit details, though I'll display the argument.
- The familiarity of the style and notation of the argument to a student of today is naturally one of the most important of Weierstraß' legacies.

The Weierstraß Example

- The example he provides is the (continuous) function

$$f(x) = \sum_{n=0}^{\infty} b^n \cos(a^n x\pi)$$

where x is real, a is an odd integer and $0 < b < 1$.
- If the product ab is too great, differentiability will fail. Weierstraß chooses the constants so that $ab > 1 + \frac{3}{2}\pi$.
- Let x_0 be a fixed real. Then there is an integer α_m such that

$$x_{m+1} = a^m x_0 - \alpha_m \in \left(-\frac{1}{2}, \frac{1}{2}\right]$$

- It turns out that we can choose m sufficiently large that $x' < x_0 < x''$ where

$$x' = \frac{\alpha_m - 1}{a^m} \text{ and } x'' = \frac{\alpha_m + 1}{a^m}$$

and the interval (x', x'') can thus be made as small as we wish.

The Weierstraß Example continued

- Weierstraß calculated the differential quotient from the left and right directly.
- For the left, Weierstraß splits the resulting sum into two parts:

$$\sum_{n=0}^{m-1}\left((ab)^n \frac{\cos(a^n x'\pi)-\cos(a^n x_0\pi)}{a^n(x'-x_0)}\right)+\sum_{n=0}^{\infty}\left(b^{m+n}\cdot\frac{\cos(a^{m+n}x'\pi)-\cos(a^{m+n}x_0\pi)}{x'-x_0}\right).$$

- Using trigonometric identities and the fact that a is an odd integer, he obtains by manipulating inequalities that the differential quotient from the left can be written

$$\frac{f(x')-f(x_0)}{x'-x_0}=(-1)^{\alpha_m}(ab)^m\cdot\eta\left(\frac{2}{3}+\varepsilon\frac{\pi}{ab-1}\right)$$

and from the right, we have the opposite sign:

$$\frac{f(x'')-f(x_0)}{x''-x_0}=-(-1)^{\alpha_m}(ab)^m\cdot\eta_1\left(\frac{2}{3}+\varepsilon_1\frac{\pi}{ab-1}\right)$$

where $\eta>1$ and $\varepsilon\in(-1,1)$.

The Example Concluded

- Now choose the constants a and b in such a way that $ab > 1 + \frac{3}{2}\pi$, we have immediately that

$$\frac{2}{3} > \frac{\pi}{ab-1}$$

  which ensures that the expression in the rightmost bracket in each term, in the first case $\frac{2}{3} + \varepsilon\frac{\pi}{ab-1}$, remains positive.
- Hence the left and right differential quotients increase without bound as m increases, but have opposite sign.
- "Hieraus ergiebt sich unmittelbar, dass $f(x)$ an der Stelle $(x = x_0)$ weder einen bestimmtem endlichen, noch auch einen bestimmten unendlich grossen Differentialquotienten besitzt."

Concluding Remarks

- Counterexamples such as these contributed materially to the renown of the Weierstraß approach.
- They attracted attention to his work both inside and outside his "school".
- This was not least because they were critical of the work of Riemann.
- The examples themselves are worthy of someone for whom pedagogy was a primary aim, easily learned, extended, and retold.
- Indeed, as several of the researchers here today have argued, Weierstraß aimed at a foundation of analysis that would endure.
- These examples and the kind of argument they exemplify provided evidence of the value, indeed superiority, of his methods.