Weierstrass, monodromy and normal forms.

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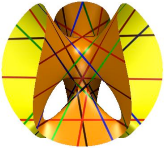

Outline

1. Preamble= Vorwort
2. Weierstraß from the point of view of semi-Profanes
3. Monodromy theorem
4. Monodromy and normal forms

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Please allow me to introduce myself..(Sympathy for the devil?)

Madams and gentlemen, sehr verehrte Damen und Herren,
I am honoured to be here at the Berlin-Brandenburgische Akademie der Wissenschaften and, as a member of the Göttinger Akademie der Wissenschaften, I rejoyce in conveying the greetings of the rival Academy, according to the best academic tradition.

This rivalry dates back to the 19th century, when the two centres of mathematics were competing for supremacy.

Let me cite in this context Constantin Caratheodory, who wrote so in the preface of his two volume-book on 'Funktionentheorie'.

The Academies of Göttingen and Berlin

Caratheodory wrote:

‘ The genius of B. Riemann (1826-1865) intervened not only to bring the Cauchy theory to a certain completion, but also to create the foundations for the geometric theory of functions. At almost the same time, K. Weierstraß(1815-1897) took up again the .. idea of Lagrange’s (whose bold idea was to develop the entire theory on the basis of power series), on the basis of which he was able to arithmetize Function Theory and to develop a system that in point of rigor and beauty cannot be excelled.

‘During the last third of the 19th Century the followers of Riemann and those of Weierstraß formed two sharply separated schools of thought.

The schools of Göttingen and Berlin

However, in the 1870's Georg Cantor (1845-1918) created the Theory of Sets. .. With the aid of Set Theory it was possible for the concepts and results of Cauchy's and Riemann's theories to be put on just as firm basis as that on which Weierstraß ' theory rests, and this led to the discovery of great new results in the Theory of Functions as well as of many simplifications in the exposition.'

Let me now introduce myself: I am a mathematician active in research in the fields of algebra, geometry and complex analysis, and many of the problems I deal with are within the glorious path initiated in the 19th century by giants of mathematics such as Weierstraß, Riemann and others.

From Florence to Pisa, then Göttingen, now Bayreuth (but, unlike Wagner, got no Festspielhaus)

I am also honoured to be here on equal footing with many distinguished international and German colleagues who are renowned historians of mathematics.

I was the successor of Aldo Andreotti and Enrico Bombieri in Pisa, the successor of Hans Grauert in Göttingen, and of Michael Schneider in Bayreuth. These friends and esteemed colleagues had a big influence on me.

I have always been interested in the history of algebraic geometry and complex analysis, and I collect old mathematics books, and sometimes I even find the time to glimpse through them!

Florentine roots and Humanism

Florentine roots means first of all to have an addition towards jokes and sarcasm: also Aldo Andreotti was born in Florence and one of his most beautiful papers begins with a lovely quotation from Benedetto Marcello's " Il Teatro alla moda".

‘ In primo luogo non dovra' il Poeta **moderno** aver letti, né legger mai gli Autori antichi Latini o Greci. Imperocché nemmeno gli antichi Greci o Latini hanno mai letti i **moderni**.

This means: ‘A modern poet shall never read or ever have read the ancient Latin or Greek authors, since these have never read the modern'.
The real meaning is: we are fascinated by the history of ideas, but we do not like the game of prophets, wizards, and those who pretend to draw new rabbits from an old hat.

Florentine roots and Humanism

Florentine people look up proudly at the Cupola del Brunelleschi, recall that he was the first to develop (1415) the method of central perspective, which was later expounded in Leon Battista Alberti's book 'De Pictura' (1435), and in Piero Della Francesca's book 'De prospectiva pingendi' (1474).
Florentines do not believe in modernism; once Martin Grötschel gave a talk in Bayreuth, and with Florentine sarcasm I told him: 'The mathematics you are talking about (Minkovski's Geometry of numbers) is not the mathematics of last century, it is the mathematics of the previous century, the 19-th century!'
Did he realize how big a compliment was this from a Florentine?

Laudator temporis acti?

While getting old, I remember my High school teachers (Gymnasiallehrer) in Florence. They did research and their lectures at the classical Liceo-Ginnasio ‘Michelangelo’ were like University lectures, and some of them left us half way when they were offered University Chairs.
Also in Germany we should miss the times when Gymnasiallehrer as Weierstraß did revolutionary research while being school teachers, and before moving to University chairs!
Everybody knows that Weierstraß was also professor of calligraphy, as documented by the beautiful notation he invented for his Weierstraß ’ $\wp$ function!
Humanists believe that excessive specialization and early separation of careers are dangerous.

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Weierstraß is ubiquitous?

When one starts to study mathematics, he is soon confronted with the name of Weierstraß :

- Weierstraß' theorem on extrema, theorem of Bolzano-Weierstraß , Weierstraß approximation ..(in real analysis),
- Weierstraß' uniform convergence theorem, theorem of Casorati-Weierstraß' , Weierstraß' infinite products, Weierstraß' majorant theorem, Weierstraß ' $\wp$ function ..(in function theory)
- Weierstraß ' preparation theorem, Weierstraß ' factorization theorem, periodic functions ..(in several complex variables)
- Weierstraß -Erdman criterion, .. (in the calculus of variations)

Weierstraß' Werke

As we said earlier, Weierstraß' was a teacher, and this is reflected also in his collected works:
Volumes 1-2-3 are devoted to his published articles, while

- volume 4 is entitled: 'Lectures on the theory of transcendental Abelian functions"
- volume 5 is entitled: 'Lectures on the theory of elliptic functions"
- volume 6 is entitled: 'Lectures on the applications of elliptic functions"
- volume 7 is entitled: 'Lectures on the calculus of variations".

These had a long lasting influence: Oskar Bolza for instance created real analysis in the U.S.A.!

Weierstraß' Legacy

We see therefore that most of his legacy was transmitted through 'Mitschriften' (Lecture Notes) taken by his students, among them the most brilliant mathematicians Adolf Hurwitz, Hermann Amandus Schwarz (Cantor, Kovaleskaya and Mittag Leffler were, by the way, Weierstraß' students).

Of course this fact makes the work of Weierstraß an incredible source of inspiration for historical quests, especially from the point of view I described above: finding the history of ideas, their birth and creation.

In the essay I wrote for the volume
'Karl Weierstraß (1815-1897)- Aspects of his Life and Work
I considered four such quests.

Some quests from Weierstraß' Legacy

1. When was the statement of the monodromy theorem first fully formulated (resp. : proven)? How did the study of monodromy (i.e., polydromy) groups evolve?
2. When did first appear the normal form for elliptic curves
$$y^2 = x(x-1)(x-\lambda)?$$
3. Weierstraß and Riemann proved the 'Jacobi inversion theorem' for hyperelliptic integrals; today the theorem is geometrically formulated through the concept of the 'Jacobian variety $J(C)$' of an algebraic curve C: when did this formulation clearly show up (and so clearly that, ever since, everybody was talking only in terms of the Jacobian variety)?
4. Which is the history of the theorem of linearization of systems of exponents for Abelian functions?

Some answers to the above quests

1. According to Ullrich, the full statement of the Monodromy theorem for simple connected domains is contained in the 'Mitschrift' of Killing (Weierstraß lectures in the summer term 1868), even if the proof is not given in a complete way.
2. The several normal forms for elliptic curves and their monodromies shall be described in the last section.
3. The word 'Jacobian variety $J(C)$' of a curve C clearly showed up in the title of a famous paper by Ruggero Torelli (1913), and was probably first used in 1907 in the work of Enriques and Severi.
4. The theorem of linearization of systems of exponents for Abelian functions was proven by Appell and Humbert for $n \leq 2$, and $\forall n$ by Fabio Conforto only in 1942. The whole theory simplified drastically with the L^2-methods of complex analysis, developed by Lars Hörmander in the 1960's.

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Monodromy and Polydromy

There are several famous monodromy theorems, the classical one and many modern ones.
The classical one is easier to understand, it revolves around the concept of function, and distinguishes between a 'monodromic' (i.e., 'single valued') function, and a 'polydromic' (or 'multiple valued') function.
Typical examples of monodromic functions are the 'functions' in modern (Cantor's) sense:

$$z \mapsto z^2,\ z \mapsto e^z := \Sigma_n \frac{z^n}{n!},\ z \mapsto cos(z) := \frac{1}{2}(e^{iz} + e^{-iz}).$$

Monodromy and Polydromy

Examples of polydromic functions are

$$z \mapsto \sqrt{z}, z \mapsto \sqrt{(1-z^2)(1-k^2z^2)},$$

$$z \mapsto log(z) := \int_1^z \frac{dt}{t},\ z \mapsto arcsin(z) := \int_0^z \frac{dt}{\sqrt{(1-t^2)}}$$

$$z \mapsto \mathcal{L}^{-1}(z) := \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

where the last function is the Legendre elliptic integral.

Monodromy and Polydromy

The meaning of the words is easily understood if we recall that in Greek $\mu o \nu o \sigma$ means ‘single’, $\pi o \lambda y$ means ‘many’, and $\delta \rho o \mu \epsilon \iota \nu$ means ‘to run’. So, a function is polydromic if running around some closed path we end up, by analytic continuation, with a value different from the beginning one.
Hurwitz writes in his notes of Weierstraß ’ lectures, chapter 10, page 97:
‘Läßt sich für einen Punkt nur ein einziges Funktionenelement aufstellen, so heißt die Funktion eindeutig, in entgegengesetzten Falle mehrdeutig.’

The Monodromy theorem

The monodromy theorem gives a sufficient condition for the analytic continuation to be single valued (monodromic).

Theorem

(**Monodromy Theorem, Der Monodromie Satz.**) *Let two continuous paths* $\gamma(s), 0 \leq 1$ *and* $\delta(s), 0 \leq 1$ *be given, which have the same end points* $\gamma(0) = \delta(0), \gamma(1) = \delta(1)$ *and which are homotopic, i.e., there is a continuous deformation of* γ *and* δ *given by a continuous function* $F(s,t),\ 0 \leq s,t \leq 1$ *such that* $F(s,0) = \gamma(s),\ F(s,1) = \delta(s)$. *Then analytic continuation along* γ *yields the same result as analytic continuation along* δ.
In particular, if we have a function f *which admits analytic continuation over the whole domain* Ω, *and the domain* Ω *is simply connected, then* f *extends to a monodromic (single valued) function.*

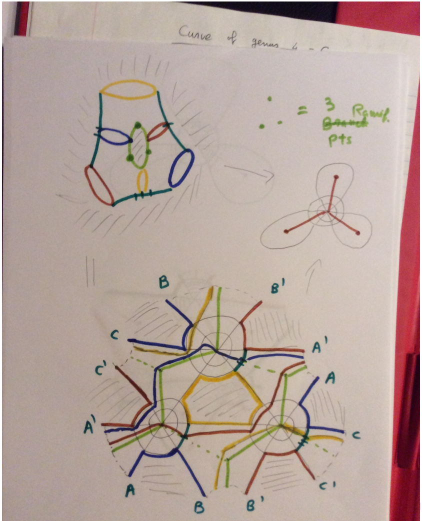
Explicit appearance of the Monodromy theorem

The first book source where I was able to find the full statement and a complete proof of the monodromy theorem is in the section ‘Geometrische Funktionentheorie’ added by Richard Courant (editor of the book) to the lectures by Adolf Hurwitz on ‘Funktionentheorie’ (1922).
It is to be found on page 348, in chapter 5, entitled ‘Analytic continuation and Riemann surfaces’.
There are two main ingredients in the monodromy theorem: first, the concept of analytic continuation, second, the topological idea of a simply connected domain.
It is no coincidence that Courant formulates the theorem for Riemann surfaces: in our opinion the monodromy theorem represents the ideal marriage of the ideas of Weierstraß with those of Riemann.

Polydromy was later called Monodromy !

All the later ‘monodromy theorems’ were indeed theorems where indeed the polydromy, i.e. the lack of monodromy, was explicitly calculated.
The story reminds me of the latin word ‘Lucus’ for a forest, where the word ‘Lucus’ stems from the word ’Lux= light’.
Honoratus Maurus explained: ‘Lucus a non lucendo’, ’it is called in this way because there is no light’!?!
The research on monodromies is still ongoing, and too complicated to report on it: let me just observe that it leads to beautiful pictures.
The next picture is related with the most symmetrical elliptic curve!

The Fermat Handlebody



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The Weierstraß equation

The equation

$$y^2 = 4x^3 - g_2x - g_3$$

is called the Weierstraß equation of an elliptic curve.
Writing the elliptic curve as

$$\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau),$$

where τ is a point in the upper half plane $\{\tau \in \mathbb{C} | Im(\tau) > 0\}$, g_m is an automorphic form of weight m, i.e. we have the transformation formula for a change of $\mathbb{Z}$-basis in the lattice:

$$g_m(\tau) = g_m(\frac{a\tau + b}{c\tau + d})(c\tau + d)^{-2m}.$$

The pseudo Weierstraß normal form

In number theory one prefers the 'normal form'

$$y^2 = x(x-1)(x-\lambda), \lambda \neq 0, 1,$$

in which the three roots of the polynomial $4x^3 - g_2 x - g_3$ are numbered, and brought, via a unique affine transformation of $\mathbb{C}$, to be equal to $0, 1, \lambda$. This normal form is essentially explained in the collected works of Weierstraß (vol. 6, page 136); chapter 13 is dedicated to the degree two transformation leading to the Legendre normal form:

$$y^2 = (x^2-1)(x^2-a^2), a \neq 1, -1.$$

The Inoue normal form

A variant of the Legendre normal form has played an important role in the discovery of new algebraic surfaces done by Inoue and in our joint work with Ingrid Bauer and Davide Frapporti; it is the normal form:

$$y^2 = (x^2 - 1)(x^2 - b^4).$$

Now, one is simply taking a square root b of a: what is all this fuss about ?

The importance of the Inoue normal form is to give an algebraic formula, on the given family of elliptic curves, for the action of the group $(\mathbb{Z}/2)^3$ acting by sending

$$z \mapsto \pm z + \frac{1}{2}\omega, \ \omega \in \Omega/2\Omega \cong (\mathbb{Z}/2)^2.$$

Monodromy of the normal forms

Another main difference among the normal forms comes out if we view the variables a, b, λ as parameters: then we get covering spaces and different monodromies.

Recall that two elliptic curves are isomorphic if and only if they have the same value of the invariant j, where

$$j(\lambda) := \frac{4}{27}\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2} = \frac{g_2^3}{g_2^3-27g_3^2}.$$

One can view $a(\tau), b(\tau), \lambda(\tau), j(\tau)$ as functions on the upper half plane $\{\tau \in \mathbb{C} | Im(\tau) > 0\}$.

Each of these function is left invariant by a well determined group of transformations.

Monodromy groups of the normal forms

To this purpose we consider the subgroups of $\mathbb{P}SL(2,\mathbb{Z})$ given, for $m = 1, 2, 4$, and setting $m' := MIN(m, 2)$, by:

$$\Gamma_{2,2m} := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \middle| \begin{array}{cc} \alpha \equiv 1 \mod 2m' & \beta \equiv 0 \mod 2m, \\ \gamma \equiv 0 \mod 2, & \delta \equiv 1 \mod 2 \end{array} \right\}.$$

To the chain of inclusions

$$\Gamma_{2,8} < \Gamma_{2,4} < \Gamma_2 < \mathbb{P}SL(2,\mathbb{Z})$$

corresponds a chain of fields of invariants

$$\mathbb{C}(b) \supset \mathbb{C}(a) \supset \mathbb{C}(\lambda) \supset \mathbb{C}(j),$$

where each inclusion is a Galois extension, of respective degrees 2,2,6 (but the compositions are not all Galois).

Normal forms and their use

There is no normal form where only appears the variable j.
The Weierstraß form is not a normal form, while

$$y^2 = x(x-1)(x-\lambda), \lambda \neq 0, 1,$$

is the normal form for elliptic curves given together with an isomorphism of the group of 2-torsion points with $(\mathbb{Z}/2)^2$, and is the most used nowadays in number theory.
The Legendre normal form is the most used in applications, but the Inoue normal form is extremely important: it give an algebraic formula, on the given family of elliptic curves, for the action of the group $(\mathbb{Z}/2)^3$ acting by sending

$$z \mapsto \pm z + \frac{1}{2}\omega, \ \omega \in \Omega/2\Omega \cong (\mathbb{Z}/2)^2.$$

In case you asked the question

The action of the group $(\mathbb{Z}/2)^3$ acting by sending

$$z \mapsto \pm z + \frac{1}{2}\omega, \ \omega \in \Omega/2\Omega \cong (\mathbb{Z}/2)^2$$

is described by complicated algebraic formulae for the Inoue normal form, I refer to my essay (to be found on arXiv).

One has to reduce to the form

$$w^2 = (x^2 - 1), t^2 = (x^2 - a^2),$$

where each of the variables x, w, t can be multiplied by ± 1.