Weierstraß' Approximation Theorem (1885)

and his 1886 lecture course revisited

Festveranstaltung
200ster Geburtstag von Karl Weierstraß

Termin

31. Oktober 2015

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Almost three decades ago, in 1988, I published Karl Weierstraß' Berlin lecture course from the summer semester 1886 with detailed commentary. I based the edition on lecture notes taken by an author unknown to me in 1988.

HISTORIA MATHEMATICA **15** (1988), 299–310

Der Beweis des Weierstraßschen Approximationssatzes 1885 vor dem Hintergrund der Entwicklung der Fourieranalysis*

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Eine bisher unveröffentlichte Ausarbeitung der Weierstraßschen Vorlesung "Ausgewählte Kapitel aus der Funktionenlehre" vom Sommersemester 1886 sowie verschiedene Briefwechsel aus jener Zeit gestatten es, Weierstraß' Verhältnis zur Theorie der Funktionen reeller Variablen näher zu beleuchten. Es zeigt sich, daß Weierstraß den Beweis seines Approximationssatzes (1885) bewußt in die Traditionslinie der Fourieranalysis einordnete. Dies führte ihn zur Verallgemeinerung seines Satzes auf unstetige Funktionen mittels des Cantorschen Begriffs des (äußeren) Inhalts einer beschränkten Punktmenge.

Band 9 LEIPZIG

K. Weierstraß

Ausgewählte Kapitel aus der Funktionenlehre

Vorlesung, gehalten in Berlin 1886

Mit der akademischen Antrittsrede, Berlin 1857, und drei weiteren Originalarbeiten von K. WEIERSTRASS aus den Jahren 1870 bis 1880/86

Teubner-Archiv zur Mathematik

[1988]

In July 1885 Weierstraß presented a paper to the “Preußische Akademie der Wissenschaften” in Berlin, which was printed the same month in the *Sitzungsberichte* of the Academy. In the paper Weierstraß proved the following two theorems (WAS), which are basically equivalent and still cornerstones of analysis today:

I. Any function of a real variable which is continuous in a closed interval can be expanded in a uniformly convergent series of polynomials.
II. Any continuous function of a real variable with period 2 π can be expanded in a uniformly convergent series of finite trigonometric sums.

In his lecture course at Berlin University in the summer 1886 Weierstraß dealt with both real and complex function theory. The WAS was a kind of unifying principle and backbone in this lecture course.
During the course Weierstraß generalized WAS in two directions: finitely many variables and weakening of the condition of continuity to more general domains for the real variable, using Cantor’s recent (1884) notion of “content” (Inhalt) of a point set.

So what is new in my article in the Weierstraß volume?

- Extensive excerpts from the lectures and letters translated into English
- I have come closer to finding the authors of the 1886 lecture notes.
- Additional information on the role of the Berlin "Mathematische Verein", in creating the lecture notes
- A description of the lecture by the listener Ludwig Schlesinger
- Role of the amateur mathematician and businessman Carl Itzigsohn in the preservation of Weierstraß' lecture notes

(Most of the previous based on material from Mittag-Leffler-Institute!)

Finally:

I looked somewhat closer at the relation between Weierstraß and his student Georg Cantor, because the latter's notion of "content" (Inhalt) of a point set came up in the lectures.

Weierstraß

1886 Berlin lecture course "Selected chapters from the theory of functions"

S ?
Schlesinger
(2 handwr. copies
lost from Gießen UL,
and Djursholm)

LL
(Lohnstein lithograph,
handwr., Leipzig UL, Djursholm)

B
(F. Bennecke, handwr.
incomplete, Djursholm)

M
(handwr. copy by
C. Michaelis, Djursholm)

T
(free handwr.copy,
possession G. A. Thieme)

GG
(2 almost identical
handwr. copies, without
historical/method. remarks,
P. Günther, A. Gutzmer
Halle UL)

T'
(typescript, 5 copies, 3 Berlin UL,
1 Bielefeld UL, 1 Djursholm)

Rudolf Lohnstein
(1866-1935)

Either Theodor Lohnstein (to the right) or his twin brother Rudolf, born 6 June 1866, or both were the authors of the lecture notes.

Both later became prominent medical doctors; they had earlier shown interest in the more numerical aspects of W.s mathematics.

THEODOR LOHNSTEIN

(1866-1942)


1861 1911

Geschichte
des
Mathematischen Vereins
an der Universität Berlin.

Berlin 1911.

Former student of Weierstraß, the doctor of physics Carl Michaelis (1858-?) copied the lecture notes into Latin, because Mittag-Leffler could not read old German script.

The prominent German-Hungarian mathematician Ludwig Schlesinger (1864-1933) - son in law of Weierstraß' colleague and former student Lazarus Fuchs (1833-1902) - wrote on 26 September 1921 to Mittag-Leffler in Djursholm:

"I personally attended the Weierstraß lectures of 1886, and I have taken notes from them. … The lecture course in the summer of 1886 was … rather poorly attended … because it was not announced in the official list and because it began several weeks after the official start of the semester. This explains why the content of the lecture course has not become well known."

LUDWIG SCHLESINGER

To his student Sofia Kovalevskaya, Weierstraß wrote on 16 May, 1885:

“Riemann’s definition of $\int_b^a f(x)dx$, which was usually considered to be the most general possible, is <u>insufficient</u>. Let *f* (*x*) be a uniquely defined function of a real variable in the interval. We allow infinitely many points [Werthe = (literally) values] between *a*, *b* where *f* (*x*) is not defined at all. In addition there can be points of discontinuity [Unstetigkeits-Stellen] in countable or non-countable numbers. We only assume that in any arbitrarily small part of the interval *a* ... *b* there are points where the function exists and where the value of the function does not exceed a fixed limit. Then one can always find a definition which preserves all properties of the integral which follow from Cauchy’s and Riemann’s definitions. ”

He added:

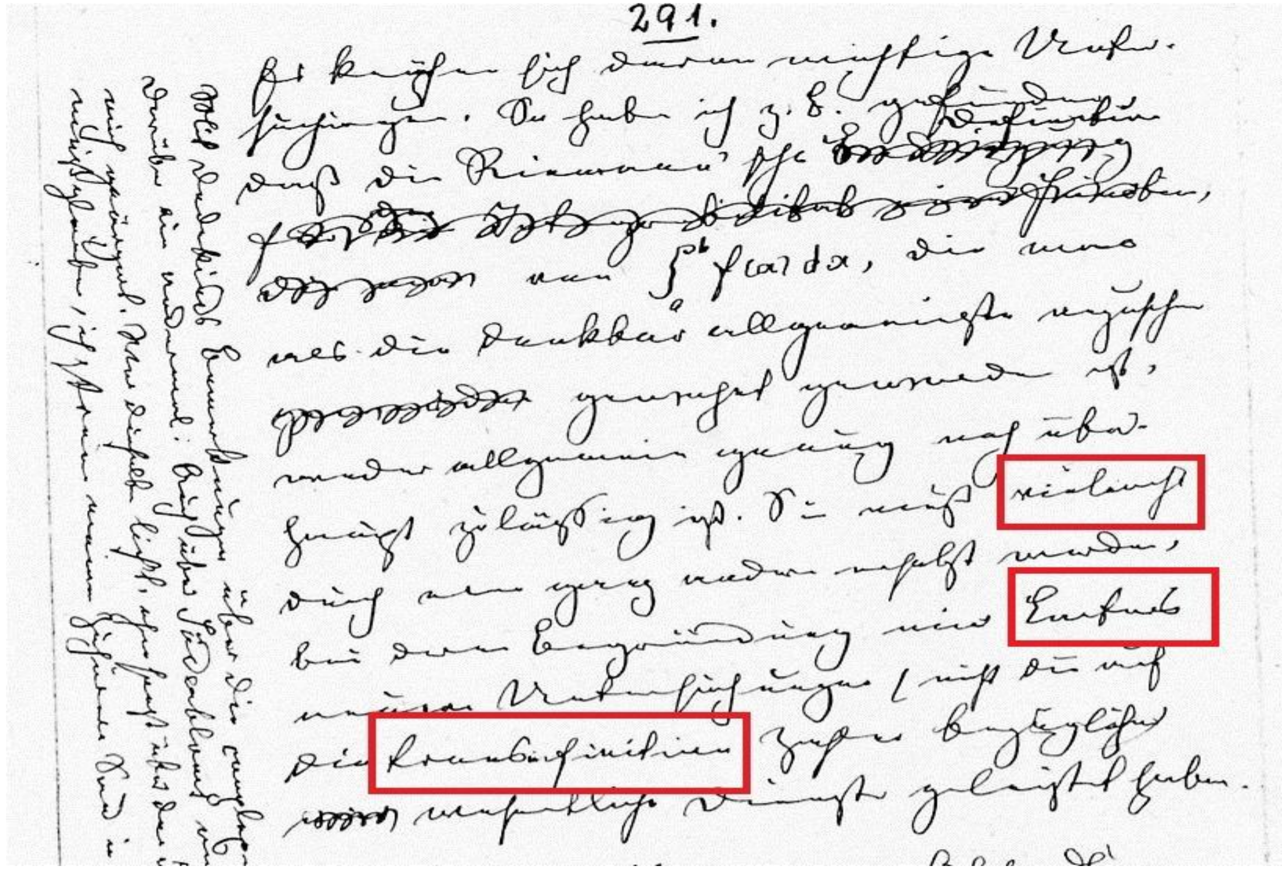
“One can conclude this very easily from Cantor’s notion of <u>content</u> [Inhalt] of an arbitrary point set, formulated in volume 4 of Acta.”

Two weeks later Weierstraß wrote to Hermann Amandus Schwarz on 28 May 1885:

> "In addition, I have to finish my paper on the representation of arbitrary functions by trigonometric series by the 25th of the coming month. The paper is connected to important investigations. I have e.g. found out that Riemann's definition of $\int_b^a f(x)dx$ which previously was considered the most general possible, is neither sufficiently general nor even acceptable. On the contrary [vielmehr], it has to be replaced by a totally different one, in which I have found substantial support in Cantor's recent investigations (not those related to transinfinitive numbers)."

The use of the wrong word "transinfinitive" for what Weierstraß' former student Georg Cantor had called "transfinite" seems to indicate that Weierstraß was less familiar with, or less convinced by, Cantor's more general (by some contemporaries even considered as too philosophical) parts of set theory, while he could make good use of Cantor's relatively more traditional notion of the "content" (Inhalt) of a point set.

Indeed in my original edition of 1988 I had replaced "transinfinit", which I had found in a typewritten transcript of the letter, by "transfinite", thus assuming a typo. But now for the article I wanted to be sure and I checked the original of the letter, which was provided to me in copy by Reinhard Bölling.

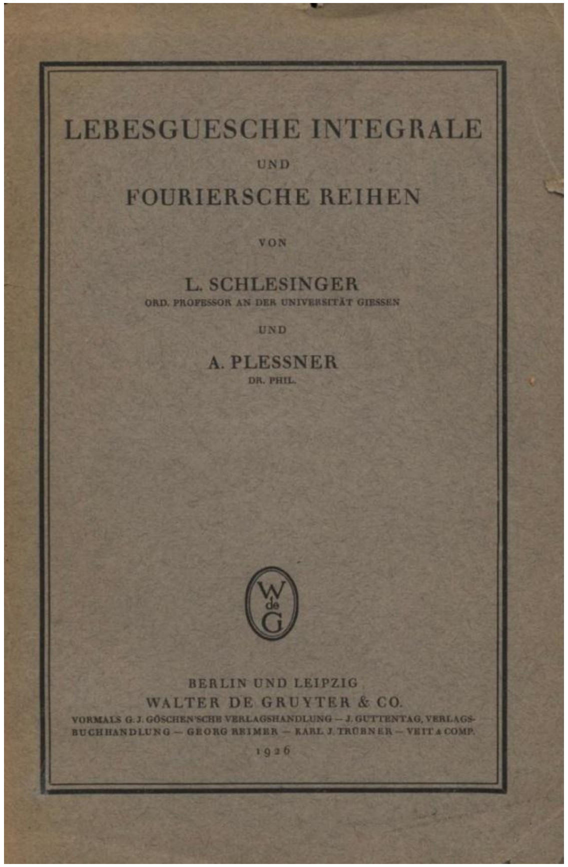

291.

We found out that Weierstrass had even used "transinfinitiv," which is even farther away from "transfinit".

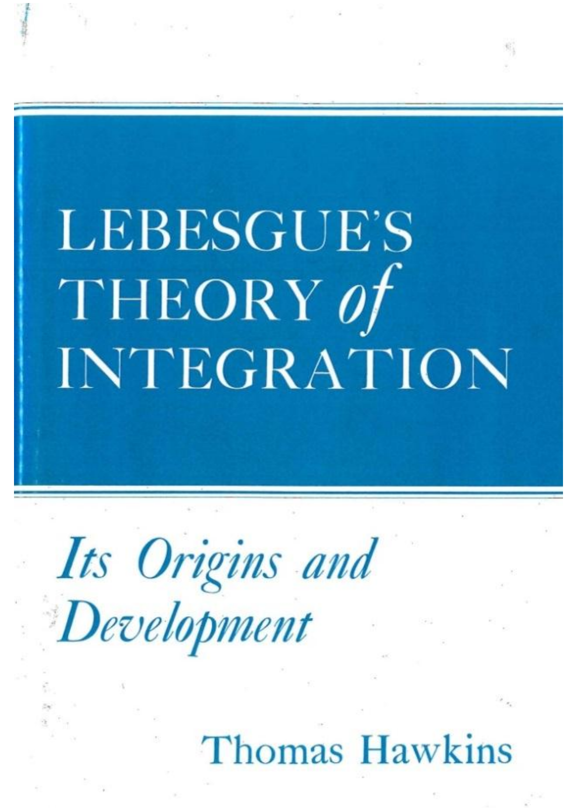
According to Bölling in his Cantor-Biography in the Jahresbericht of the DMV (1997, 70), Cantor had sent Weierstraß his first systematic, if introductory and partly philosophical paper on the theory of transfinite cardinal and ordinal numbers in December 1882, and Weierstraß had not expressed reservations.

In the resulting publication in the *Mathematische Annalen* (1883) one finds one of the most famous quotes by Cantor in which he makes a respectful nod to his teacher:

“The *essence* of *mathematics* lies precisely in its *freedom*…. Had Gauss, Cauchy, Abel, Jacobi, Dirichlet, Weierstrass, Hermite, and Riemann always been constrained to subject their new ideas to a metaphysical control, we should certainly not now enjoy the magnificent structure of the modern theory of functions.”



Somewhat contrary to the announcement in his letter to Schwarz, Weierstraß did not include his generalization of the integral in the 1885 publication. In fact he never published it.



Schlesinger, in his 1926 book with Plessner on the Lebesgue Integral, explained why Weierstraß' attempt to use Cantor's notion of "Inhalt" could not succeed in generalizing the WAS to non-continuous functions because it does not possess additivity (upper Darboux integral), contrary to what W. had written to Kovalevskaya. Also Hawkins in probably the best book on the history of the L-integral refers to W.-s failed attempt.

Carl Itzigsohn

(1840-before 1909)

W.s student and inveterate supporter, also using private means.

Redaktion: Dr. H. Potonié.

Verlag: Ferd. Dümmlers Verlagsbuchhandlung, Berlin SW. 12, Zimmerstr. 94.

XIII. Band. | Sonntag, den 20. Februar 1898. | Nr. 8.

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Ein Beitrag zur naturwissenschaftlichen Erkenntnisstheorie behufs Begründung der Sociologie auf Weierstrass'scher mathematischer Grundanschauung.

Von Mathematiker Carl Itzigsohn.

Die bisherige Behandlungsweise der Probleme der Sociologie bietet denjenigen, welche die Methode der exacten Wissenschaften gewöhnt sind, eine eigenartige Erscheinung dar. Ohne die wissenschaftliche Behandlung der aufgeworfenen Fragen in genügender Weise, geschweige denn erschöpfend, vollendet zu haben, geht man hier an die praktische Lösung, d. h. die Praxis meint hier das Recht zu haben, der Theorie voranzueilen. Es scheint somit, als habe man in der Sociologie das Vertrauen zur Theorie eingebüsst.

Die Mangelhaftigkeit einer derartigen Weise der Forschung springt klar ins Auge. Theoretisch wird so die einheitliche Behandlung einer Wissenschaft zur Unmöglichkeit. Dann aber hat auch die Verwendung dieser Art von Theorie für die Praxis ihre Schattenseiten: die Praxis zeigt die Absonderlichkeit, dass zur Begründung resp. zur Bekämpfung socialer Forderungen bald wirthschaftliche, bald ethische Gesichtspunkte in den Vordergrund geschoben werden, wie es eben dem Klassen- oder Partei-Egoismus ins Handwerk passt. Alle diese Missstände würden ver-