Weierstrass Institute for Applied Analysis and Stochastics

Christian Bayer, Peter K. Friz, Benjamin Stemper

Pricing under rough volatility

Leibniz Association

OPTION PRICING

Equity derivatives pricing

With the equity derivatives market being a multiple of global GDP, the availability of efficient pricing and hedging techniques of derivatives (in particular European options) is essential for financial intermediaries around the world.

Fundamental question: How to model instantaneous stochastic volatility consistently across all strikes and maturities?

- Model needs to reproduce empirical implied volatility surface (Fig. 1)!
- An important proxy for the surface is given by the term structure of the at-the-money (ATM) volatility skew (Fig. 2).

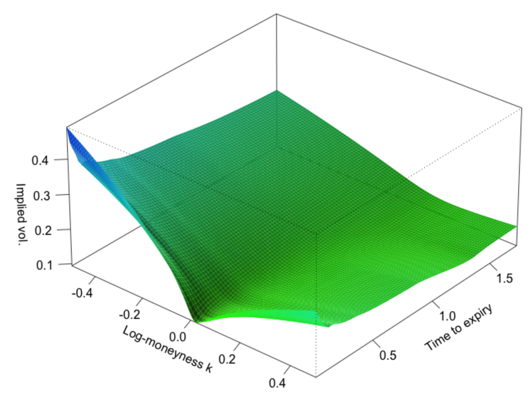

***Figure 1:** SPX Implied vol surface on 14/08/14.*

Implied volatility: Volatility parameter σ needed in Black-Scholes formula in order to match market or model price.

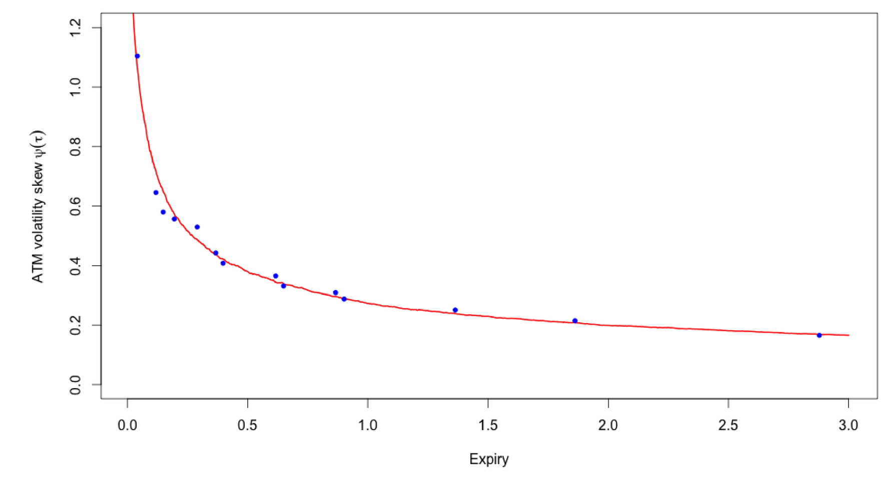

***Figure 2:** SPX ATM vol skew on 20/08/14.*

Vol skew: Let log-strike k and time to maturity τ, then:

$$\psi(\tau) = \left| \frac{\partial}{\partial k} \sigma_{BS}(k, \tau) \right|_{k=0}$$

ROUGH BERGOMI: THEORY

Two common problems

Diffusive stochastic volatility models commonly have two deficiencies:

- They lack a good agreement with market data under both the physical as well as the pricing measure.
- They fail to reproduce the power-law behaviour of the volatility skew for small times to maturity (Fig. 2).

Revelation: Volatility is rough

There is strong empirical evidence that log realized volatility has Hoelder regularity much less than 1/2 (typically around 0.1). This is a ubiquitous phenomenon observed across thousands of equities and indices [GJR14, BLP16]. Hence, model log-volatility as a fractional Brownian motion with Hurst parameter $H < 1/2$.

Background on VIX

- Realized variance: $w_{t,T} = \int_t^T v_s ds$
- Forward variance: $\xi_t(u) = E_t v_u$ for $t \leq u$ may in essence be observed in the markets via variance swaps.
- CBOE introduced volatility index

$$VIX_t \approx \sqrt{\frac{1}{\Delta} E_t w_{t,t+\Delta}} \approx \sqrt{v_t}.$$

The rough Bergomi model

For S_t the S&P 500 Index, (Z, W) a two-dim. BM with $d\langle Z, W \rangle = \rho \in (-1, 1)$ and Riemann-Liouville fBm $\hat{W}$ given by $\hat{W}_t = \sqrt{2H} \int_0^t (t-s)^{H-1/2} dW_s$, consider

$$dS_t = \sqrt{v_t} S_t dZ_t$$
$$v_t = \xi_0(t) \mathcal{E}\left(\eta \hat{W}_t\right)$$

Non-Markovianity resolved

The instantaneous variance v_t inherits non-Markovianity from fBM:

$$\xi_t(u) = \mathbb{E}\left[v_u \mid \mathcal{F}_t\right] \neq \mathbb{E}\left[v_u \mid v_t\right]$$

for $u > t$ which is very bad from a simulation point of view. Fortunately, $\xi_t(u)$ may be observed in the market (via variance swaps).

EMPIRICS

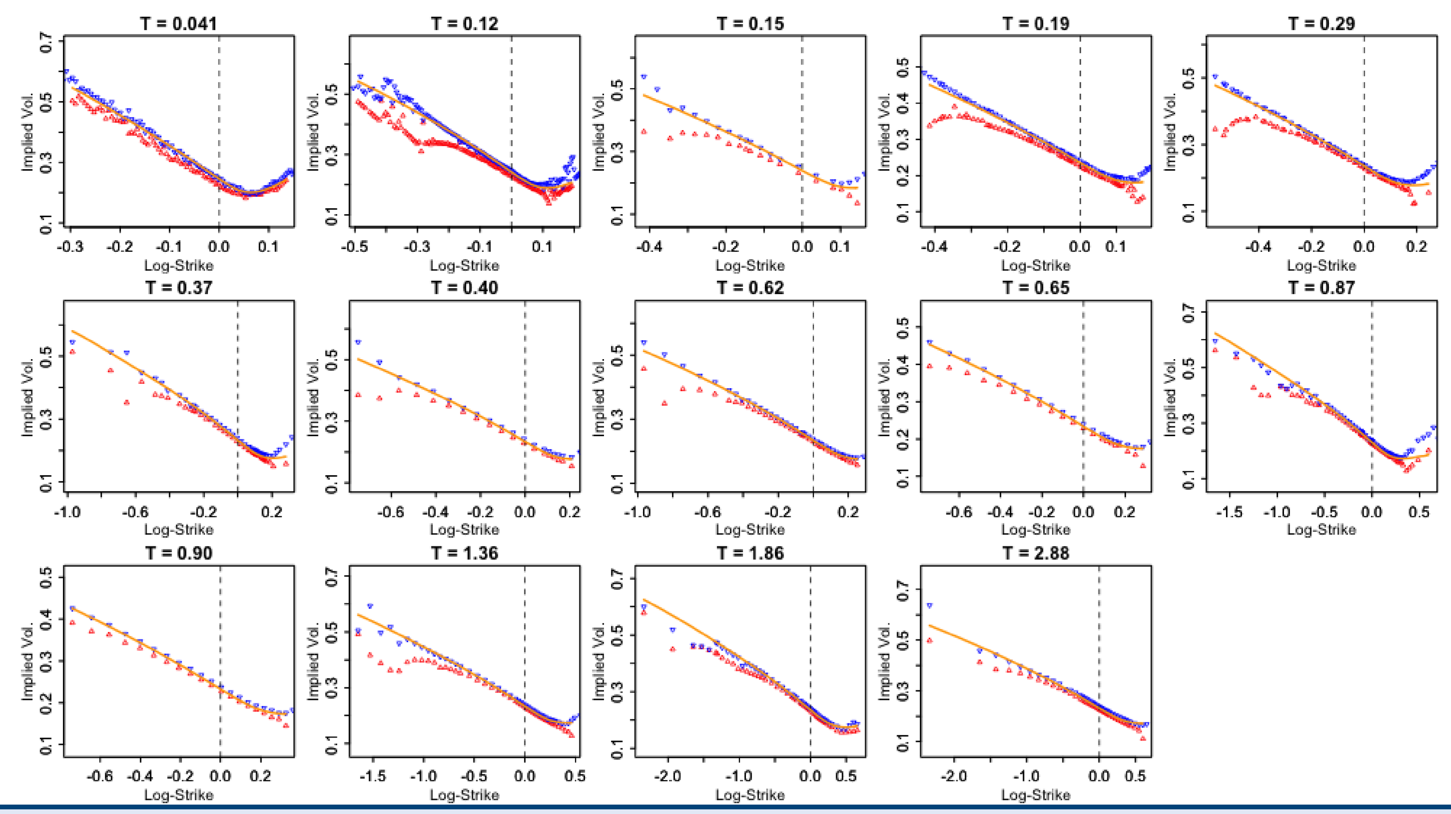

***Figure 3:** SPX smiles on 4/02/2010: Red and blue points represent bid and offer SPX implied volatilites; orange smiles are obtained from rBergomi simulations.*

Simulation results

Rough Bergomi achieves a fantastic fit to real data! It captures

- the power-law behaviour of the SPX vol skew near zero (Fig. 2)
- the geometry of SPX smiles (Fig. 3)
- the SPX ATM implied volatility term structure

with only three parameters: the Hurst parameter H of the fBM, vol of vol η and correlation parameter ρ governing the leverage effect!

NEXT

Two ongoing projects in rough vol framework

- Development of small-time asymptotic formulae for the implied volatility term structure etc. (with A. Gulisashvili (Ohio) and B. Horvath (Imperial))
- A novel pricing algorithm via the theory of regularity structures (with P. Gassiat (Dauphine) and J. Martin (Berlin))

References

[GJR14]: Gatheral, Jaisson, Rosenbaum. Volatility is rough. Preprint, 2014. arXiv:1410.3394.

[BFG16]: Bayer, Friz, Gatheral. Pricing under rough volatility. Quant. Finance, 16(6):887–904, 2016.

[BLP16]: Bennedsen et al. . Decoupling the short- and long-term behavior of stochastic volatility. Preprint, 2016. arXiv:1610.00332.

Funding

We gratefully acknowledge financial support through DFG Research Grants BA5484/1 and FR2943/2.

contact: C. Bayer, P. K. Friz, B. Stemper · WIAS · Mohrenstr. 39, 10117 Berlin · +49 30 20372 0 · bayer@wias-berlin.de, friz@wias-berlin.de, stemper@wias-berlin.de · www.wias-berlin.de