Leaky conical surfaces: spectral asymptotics and isoperimetric properties

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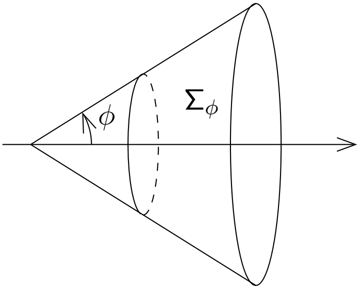
Part I. Hamiltonian

Geometric setting: circular conical surfaces

Circular conical surface in $\mathbb{R}^d$ with opening angle $\phi \in (0, \pi/2]$

$$\Sigma_\phi := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_d^2 = \cot^2 \phi (x_1^2 + x_2^2 + \cdots + x_{d-1}^2), x_d > 0\}$$

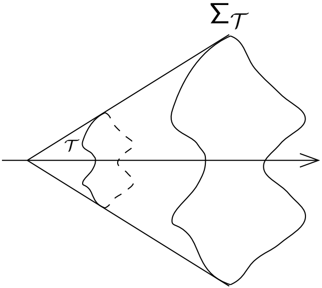


$\Sigma_{\pi/2}$ – hyperplane; Σ_ϕ for $d > 3$ – hypercone.

Geometric setting: general conical surfaces in $\mathbb{R}^3$

General conical surface in $\mathbb{R}^3$

- $\mathcal{T}$ – C^2-smooth loop on the unit sphere $\mathbb{S}^2$.
- $\Sigma_{\mathcal{T}} := \{r\mathcal{T} : r > 0\}$ – conical surface with base $\mathcal{T}$.



$\mathcal{T}$ – a circle of length $L \in (0, 2\pi] \Longrightarrow \Sigma_{\mathcal{T}} = \Sigma_\phi$ with $\phi = \arcsin(\frac{L}{2\pi})$.

Hamiltonian

Notations

(i) $d \geq 3$ – space dimension and $\Sigma \subset \mathbb{R}^d$ – a conical surface.

(ii) $\alpha > 0$ – coupling constant.

Schrödinger operator with δ-interaction of strength α supported on Σ

$\mathsf{H}_{\alpha,\Sigma} := -\Delta - \alpha\delta(x - \Sigma)$ on $\mathbb{R}^d$.

Two ways of a rigorous definition for $\mathsf{H}_{\alpha,\Sigma}$

- Semibounded form $H^1(\mathbb{R}^d) \ni u \mapsto \|\nabla u\|^2_{L^2(\mathbb{R}^d)} - \alpha \|u|_\Sigma\|^2_{L^2(\Sigma)}$ in $L^2(\mathbb{R}^d)$ is represented by $\mathsf{H}_{\alpha,\Sigma}$; $u|_\Sigma$ – the restriction of u onto Σ.
- Self-adjoint extension of $\mathsf{S}u = -\Delta u$, $\mathrm{dom}\,\mathsf{S} = \{u \in H^2(\mathbb{R}^d) : u|_\Sigma = 0\}$

Conical structure of Σ $\Rightarrow$ $\mathsf{H}_{\alpha,\Sigma} \cong \frac{\alpha^2}{4}\mathsf{H}_{2,\Sigma}$. Set $\alpha = 2$ and drop the index.

Motivations

Motivation from physics

(i) H_Σ models a 'leaky' quantum system wherein a particle is confined to Σ but the tunnelling between different parts of Σ is not neglected.

(ii) Quantum graphs and waveguides do not explain the tunnelling!

Motivation from spectral geometry

Characterise the spectrum of H_Σ in terms of Σ!

This question can be asked for various shapes of Σ

(i) See Exner-Kovařík-15 and the references therein.

(ii) Universal description of spectrum for general Σ can hardly be found!

Why conical surfaces?

General surfaces $\supset$ asymptotically flat surfaces

Exner-Ichinose-01, Exner-Kondej-02, Brown-Eastham-Wood-08, Duchene-Raymond-14, Pankrashkin-16,...

Asymptotically flat = unbounded surface with vanishing curvatures at ∞.

- ✓ Local deformation of the hyperplane.
- × Graph of $x \mapsto \sin x$.

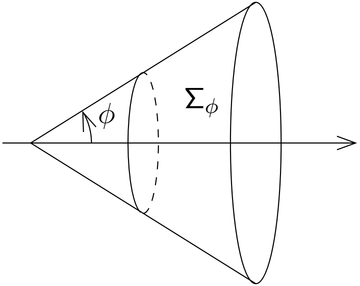
Description of spectra for asymptotically flat Σ – still a hard task!

(i) Existence of geometrically induced bound states is completely understood only in 2-D (Exner-Ichinose-01).

(ii) Partial results for 3-D (Exner-Kondej-02).

asymptotically flat surfaces $\supset$ conical surfaces $\supset$ circular conical surfaces

Part II. Spectral properties for circular conical surfaces



- J. Behrndt, P. Exner, and V. L., *Schrödinger operators with δ-interactions supported on conical surfaces*, J. Phys. A: Math. Theor. **47** (2014), 355202, 16 p.
- V. L. and T. Ourmières-Bonafos, *On the bound states of Schrödinger operators with δ-interactions on conical surfaces*, Comm. in PDE **41** (2016), 999–1028.

Continuous spectrum

We agree that $\mathsf{H}_\phi := \mathsf{H}_{\Sigma_\phi}$.

Theorem (Behrndt-Exner-L-14, $d = 3$)

$\sigma_{\rm ess}(\mathsf{H}_\phi) = [-1, \infty)$.

Variables can be almost separated far away from the vertex of $\Sigma_\phi \Rightarrow$
$\sigma_{\rm ess}(\mathsf{H}_\phi) = \sigma_{\rm ess}(\mathsf{H}_{\pi/2}) = [-1, \infty)$.

Making this observation rigorous

- $\sigma_{\rm ess}(\mathsf{H}_\phi) \supset [-1, \infty)$ – Weyl's singular sequences.
- $\sigma_{\rm ess}(\mathsf{H}_\phi) \subset [-1, \infty)$ – Neumann bracketing.

Theorem (Bruneau-Popoff-15, $d > 3$)

$\sigma_{\rm ess}(\mathsf{H}_\phi) = [-1, \infty)$.

Different method of the proof.

Discrete spectrum

Theorem (Behrndt-Exner-L-14, $d = 3$)

$\#\sigma_{\rm d}(\mathsf{H}_\phi) = \infty$ *if* $\phi \in (0, \pi/2)$ *and* $\#\sigma_{\rm d}(\mathsf{H}_\phi) = 0$ *if* $\phi = \pi/2$.

These eigenvalues accumulate to $E = -1$.

Proof via construction of test functions & min-max principle

- Make use of functions which are employed in Brezis-Marcus-97 to show sharpness of Hardy inequality.
- The strategy goes back to the proof of $\#\sigma_{\rm d} = \infty$ for ${}_2$He in Kato-51.

Theorem (L-Ourmieres-Bonafos-16, $d > 3$)

$\sigma(\mathsf{H}_\phi) = \sigma_{\rm ess}(\mathsf{H}_\phi) = [-1, \infty)$ *and* $\#\sigma_{\rm d}(\mathsf{H}_\phi) = 0$.

Proof relies on rotational invariance of Σ_ϕ & separation of variables

- $\sigma(\mathsf{H}_\phi) = \cup_{m=0}^{\infty} \sigma(\mathsf{H}_{\phi,m})$; $\mathsf{H}_{\phi,m}$ – fibre operators on $\mathbb{R}^2_+$.
- $\inf \sigma(\mathsf{H}_{\phi,m}) \geq -1$.

Eigenvalue counting function

Eigenvalues of H_ϕ repeated with multiplicities

$E_1(\mathsf{H}_\phi) \le E_2(\mathsf{H}_\phi) \le \cdots \le E_k(\mathsf{H}_\phi) \le \cdots < -1$

$\mathcal{N}_{-1-E}(\mathsf{H}_\phi)$ = number of eigenvalues of H_ϕ below the point $-1-E$.

$$\boxed{\mathcal{N}_{-1-E}(\mathsf{H}_\phi) := \#\{k \in \mathbb{N} \colon E_k(\mathsf{H}_\phi) < -1-E\}.}$$

Behaviour of $\mathcal{N}_{-1-E}(\mathsf{H}_\phi)$ is non-trivial for $d = 3$

- $\#\sigma_{\rm d}(\mathsf{H}_\phi) = \infty \Rightarrow \lim_{E\to 0+} \mathcal{N}_{-1-E}(\mathsf{H}_\phi) = \infty$.
- How fast is $\mathcal{N}_{-1-E}(\mathsf{H}_\phi)$ growing?

- Behrndt-Exner-L-14 – an estimate for $\mathcal{N}_{-1-E}(\mathsf{H}_\phi)$ from one side.
- Aim of L-Ourmieres-Bonafos-16 – to obtain more on $\mathcal{N}_{-1-E}(\mathsf{H}_\phi)$.

Main theorem on spectral asymptotics

Theorem (L-Ourmiéres-Bonafos-16)

$\mathcal{N}_{-1-E}(\mathsf{H}_\phi) \sim \frac{\cot\phi}{4\pi}|\ln E|$ *as* $E \to 0+$.

$\mathsf{S}_c = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{c}{x^2}$ on $(1,\infty)$ + Dirichlet BC at $x = 1$

Spectral properties of S_c (Kirsch-Simon-87)

- $\sigma_{\rm ess}(\mathsf{S}_c) = [0,\infty)$.
- $\#\sigma_{\rm d}(\mathsf{S}_c) = \infty$ for $c > 1/4$.
- $\mathcal{N}_{-E}(\mathsf{S}_c) = \#\{k \in \mathbb{N} \colon E_k(\mathsf{S}_c) < -E\} \sim \frac{1}{2\pi}\sqrt{c - \frac{1}{4}}|\ln E|$,

Spectral asymptotics of H_ϕ and of S_c are related

$\mathcal{N}_{-1-E}(\mathsf{H}_\phi) \sim \mathcal{N}_{-E}(\mathsf{S}_{1/(4\sin^2\phi)}) \sim \frac{1}{2\pi}\sqrt{\frac{1}{4\sin^2\phi} - \frac{1}{4}}|\ln E| = \frac{\cot\phi}{4\pi}|\ln E|$.

Proof by domain decomposition methods

Comparison operators via Dirichlet-Neumann bracketing

$\mathsf{H}^-_{\phi,E} \le \mathsf{H}_\phi \le \mathsf{H}^+_{\phi,E}$

The geometry of the bracketings depends on the spectral parameter.

$$\boxed{\mathcal{N}_{-1-E}(\mathsf{H}^-_{\phi,E}) \ge \mathcal{N}_{-1-E}(\mathsf{H}_\phi) \ge \mathcal{N}_{-1-E}(\mathsf{H}^+_{\phi,E})}.$$

Technical estimates for $\mathcal{N}_{-1-E}(\mathsf{H}^\pm_{\phi,E})$

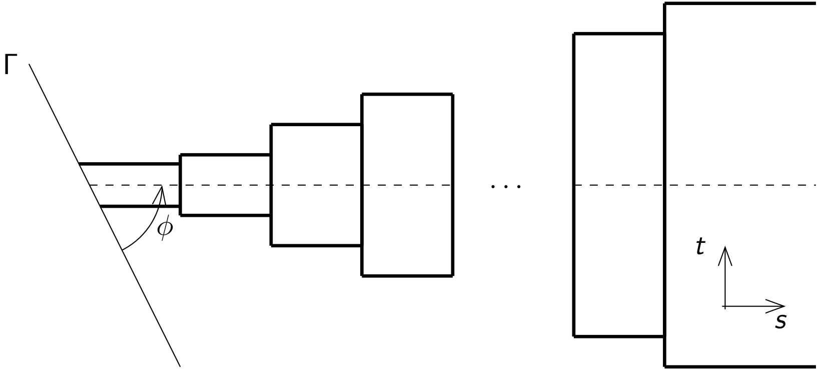
$\frac{\cot\phi}{4\pi} \le \liminf_{E\to 0+} \frac{\mathcal{N}_{-1-E}(\mathsf{H}^+_{\phi,E})}{|\ln E|} \le \limsup_{E\to 0+} \frac{\mathcal{N}_{-1-E}(\mathsf{H}^-_{\phi,E})}{|\ln E|} \le \frac{\cot\phi}{4\pi}.$

In these estimates spectral asymptotics by Kirsch and Simon is used.

Main difficulty

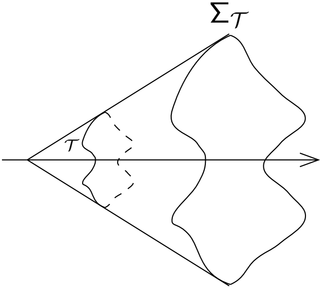
To invent a suitable domain decomposition for $\mathsf{H}^\pm_{\phi,E}$.

Maya-pyramid-like tiling



To get the decomposition that is used for $\mathsf{H}^{-}_{\phi,E}$ rotate the figure around Γ.
The number of the boxes $\rightarrow +\infty$ as $E \nearrow -1$, their sizes vary appropriately, and on their boundaries Neumann $\mathrm{b.c.}$ is imposed.

Part III. Spectral properties for general conical surfaces



- P. Exner and V. L., *A spectral isoperimetric inequality for cones*, arXiv:1512.01970.

Qualitative spectral properties

We agree that $\mathsf{H}_{\mathcal{T}} := \mathsf{H}_{\Sigma_{\mathcal{T}}}$.

Theorem (Bruneau-Popoff-15)

$\sigma_{\rm ess}(\mathsf{H}_{\mathcal{T}}) = [-1, \infty)$

$\inf \sigma_{\rm ess}(\mathsf{H}_{\mathcal{T}}) < -1$ if $\mathcal{T}$ has corner points; *e.g.* a polygon on $\mathbb{S}^2$.

Theorem (Exner-L-15)

$\sigma_{\rm d}(\mathsf{H}_{\mathcal{T}}) \neq \varnothing$ *if* $|\mathcal{T}| < 2\pi$.

If $|\mathcal{T}| = 2\pi$ then $\sigma_{\rm d}(\mathsf{H}_{\mathcal{T}}) = \varnothing$ for $\mathcal{T}$ being equator of $\mathbb{S}^2$.

Open questions

(i) $\#\sigma_{\rm d}(\mathsf{H}_{\mathcal{T}}) = \infty$, $|\mathcal{T}| < 2\pi$, as for circular case?

(ii) $|\mathcal{T}| \geq 2\pi$, $\mathcal{T}$ not equator, $\sigma_{\rm d}(\mathsf{H}_{\mathcal{T}}) \neq \varnothing$?

Isoperimetric inequality

Optimization problem

$\mathcal{T} \mapsto E_1(\mathsf{H}_\mathcal{T}) = \max! + \text{constraint } |\mathcal{T}| = L \in (0, 2\pi) \qquad (\star)$

Theorem (Exner-L-15)

The optimizer for the problem $(\star)$ *is a circle on the unit sphere.*

Circular cone maximizes the 1st eigenvalue among all cones with fixed base length!

This theorem belongs to a family of optimization results

Most famous: the ball minimizes the 1st eigenvalue of Dirichlet Laplacian among domains of fixed volume (Faber-23, Krahn-25).

Isoperimetric inequality: method of the proof

$\mathcal{C}, \mathcal{T} \subset \mathbb{S}^2$, $|\mathcal{C}| = |\mathcal{T}| < 2\pi$, $\mathcal{C}$ – circle. $\gamma_\mathcal{C}, \gamma_\mathcal{T} \colon [0, |\mathcal{C}|] \to \mathbb{S}^2$, $|\dot\gamma_\mathcal{C}| = |\dot\gamma_\mathcal{T}| = 1$

A bijection between $\Sigma_\mathcal{T}$ and $\Sigma_\mathcal{C}$

$$\mathcal{M}\colon \Sigma_\mathcal{T} \to \Sigma_\mathcal{C}, \quad \mathcal{M}(x) := |x| \cdot \gamma_\mathcal{C}(\gamma_\mathcal{T}^{-1}(x/|x|)).$$

Spectral analysis of $\mathsf{H}_\mathcal{C}$, $\mathsf{H}_\mathcal{T}$ reduces to analysis of operator-valued functions in $L^2(\Sigma_\mathcal{C})$ and $L^2(\Sigma_\mathcal{T})$, resp.

Next step inspired by (Exner-05, Exner-Harrell-Loss-06)

Proving $E_1(\mathsf{H}_\mathcal{T}) \le E_1(\mathsf{H}_\mathcal{C})$ reduces to showing for $k = (-E_1(\mathsf{H}_\mathcal{C}))^{1/2}$

$$\int_{\Sigma_\mathcal{T}^2} \widetilde\psi(x) G_k(x-y) \widetilde\psi(y) \mathrm{d}\sigma(x)\mathrm{d}\sigma(y) \ge \int_{\Sigma_\mathcal{C}^2} \psi(x) G_k(x-y)\psi(y)\mathrm{d}\sigma(x)\mathrm{d}\sigma(y);$$

ψ – trace on $\Sigma_\mathcal{C}$ of the ground-state of $\mathsf{H}_\mathcal{C}$, $\widetilde\psi = \psi \circ \mathcal{M}$, $G_k(x) = \frac{e^{-k|x|}}{4\pi|x|}$.
The latter inequality follows from mean-chord inequality (Lükő-66).

Novelty: ψ is unknown, only its positivity and symmetry are used.

Three related spectral problems

Domains

- $\mathcal{C}_\phi = \{(x,y,z) \in \mathbb{R}^3 \colon z > \cot\phi (x^2+y^2)^{1/2}\}$,
- $\mathcal{L}_\phi = \{(x,y,z) \in \mathbb{R}^3 \colon z - \cot\phi (x^2+y^2)^{1/2} \in (0,1)\}$.

(magnetic) Laplace operator on $\mathcal{L}_\phi$ + Dirichlet B.C.

Duclos-Krejčiřík-Exner-01, Exner-Tater-10, Dauge-Raymond-Ourmiéres-Bonafos-15, Krejčiřík-VL-Ourmiéres-Bonafos-16,...

Laplace operator on $\mathcal{C}_\phi$ + Robin B.C.

Levitin-Parnovski-08, Pankrashkin-15, Bruneau-Popoff-15, Bruneau-Pankrashkin-Popoff-16

Magnetic Laplace operator on $\mathcal{C}_\phi$ + Neumann B.C.

Bonnaillie-Noël-Dauge-Popoff-16, Raymond-16 and the references therein.

Summary

Circular conical surface

- Spectrum is described on the qualitative level.
- Spectral asymptotics.

General conical surfaces

- Spectrum is partially described on the qualitative level.
- Isoperimetric inequality.

Next challenges

- A lot of open questions for general conical surfaces.
- Other classes of asymptotically flat surfaces much less understood.

Thank you for your attention!