Relating Logics of Justifications and Evidence

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Abstract: The paper relates evidence and justification logics, both philosophically and technically. On the philosophical side, it is suggested that the difference between the approaches to evidence in the two families of logics can be explained as a result of their focusing on two different notions of support provided by evidence. On the technical side, a justification logic with operators pertaining to both kinds of support is shown to be sound and complete with respect to a special class of awareness models. In addition, a realization theorem with respect to **K** is shown to hold for the logic.

Keywords: awareness logics, completeness, epistemic logic, evidence logics, justification logics, realization

1 Introduction

It is commonly assumed that normal modal epistemic logics (Fagin, Halpern, Moses, & Vardi, 1995; Hintikka, 1962; Meyer, 2001; Meyer & van der Hoek, 1995; van Benthem, 2011; van Ditmarsch, van der Hoek, & Kooi, 2008) focus on the *implicit beliefs* of an agent without being able to represent the *evidence* the agent might use to *justify* her beliefs. To represent evidence, it is argued, normal epistemic logics have to be extended. Two families of such extensions have recently risen into prominence: *justification logics* (Artemov, 2001, 2008, 2011) and *evidence logics* (Shi, 2013; van Benthem, Fernández-Duque, & Pacuit, 2012, 2014; van Benthem & Pacuit, 2011a, 2011b). Justification logics originate in provability logic and are, at least semantically, close to awareness logics of Fagin and Halpern (1988). Pieces of evidence are represented 'syntactically', as sets of formulas justified by the respective pieces, see (Artemov, 2012). Evidence logics build

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on an evidence-based interpretation of neighbourhood models for classical modal logics, see (Chellas, 1980), and are a combination of classical and normal modal logics. Pieces of evidence are represented 'semantically', as sets of worlds consistent with the respective pieces. The difference in their respective representations of evidence makes the investigation of their relationship and combinations rather interesting.[2]

This paper takes first steps to relate these two families of logics, both philosophically and technically. Simple versions of evidence and justification logics are outlined in Sections 2 and 3, respectively. Section 4 discusses the differences between evidence and justification logics. Three prima facie differences are pointed out and explained away. First, it is shown in Section 4.1 that the basic evidence logic discussed in Section 2 is sound and complete with respect to a class of models where 'pieces of evidence' are considered explicitly. Second, it is shown in the same section that evidence logics are consistent with a 'world-relative' construal of evidence. Third, it is argued in Section 4.2 that the difference between the renderings of evidence embodied in evidence and justification logics can be explained as a result of their focusing on two different *kinds of support* provided by pieces of evidence. This is an alternative to the provisional explanations of the difference known from the literature, which tend to point out 'different levels of analysis' as the main contrast, see (van Benthem et al., 2014, pp. 108, 132), for example. As a result, combinations of evidence and justification logics are a natural research program. However, only a very simple combination is provided here: Section 5 points out that a specific version of multi-dimensional awareness logic combines ideas related to evidence and justification logics. The main technical result is a completeness theorem for this combination, together with a realization theorem with respect to the modal logic **K**. The concluding Section 6 points out that a reformulation of evidence and justification logics in the framework of *term-modal logics*, see (Fitting, Thalmann, & Voronkov, 2001), is an interesting topic for further research.

[2]See (van Benthem et al., 2014, p. 132), for example. An interesting combination of justification logic with dynamic epistemic logic is put forward in (Baltag, Renne, & Smets, 2012, 2014).

2 A simple evidence logic

For sake of simplicity, only the basic evidence logic without dynamic and plausibility operators is discussed. The language $\mathcal{L}_E$ adds to the Boolean language the monadic operators $[E]$, $[B]$ and $[K]$. '$[E]\phi$' is read 'there is evidence supporting ϕ' (or 'the agent has evidence for ϕ'), '$[B]\phi$' means 'the agent believes that ϕ' and the operator $[K]$ is construed as a universal modality ('$[K]\phi$' may be read as 'the agent knows that ϕ').

The simplest models for $\mathcal{L}_E$ are *extended evidence models*

$$M^e = \langle W, R, E, V\rangle \tag{1}$$

where $\langle W, R, V\rangle$ is a one-dimensional Kripke model and E is an 'evidence relation' $E \subseteq W \times 2^W$. It is required that $\langle w, W\rangle \in E$ and $\langle w, \emptyset\rangle \notin E$ for all $w \in W$. Hence, extended evidence models are a combination of Kripke models with neighbourhood models, see (Chellas, 1980; Hansen, Kupke, & Pacuit, 2009). Sets $X \in E(w)$ represent the evidence available to the agent at w (agent's 'evidential state').[3] It is assumed in addition to the usual Boolean truth-conditions that ($\|\phi\|_{M^e} = \{w : M^e, w \models \phi\}$):

- $M^e, w \models [B]\phi$ iff $R(w) \subseteq \|\phi\|_{M^e}$
- $M^e, w \models [E]\phi$ iff there is $X \in E(w)$ such that $X \subseteq \|\phi\|_{M^e}$
- $M^e, w \models [K]\phi$ iff $\|\phi\|_{M^e} = W$

R is construed as an 'epistemic accessibility' relation and $[B]$ is the usual implicit belief operator. The literature on evidence logics does not offer a detailed explanation of the relation between pieces (sources) of evidence and sets $X \in E(w)$, but the following might be plausible. Let us assume that we have a set P of pieces of evidence. These might include 'propositional evidence' (i.e. statements such as 'There is a table in front of Alice') as well as non-propositional entities such as sense-experiences (Alice's visual experience of a table) etc.[4] Now assume that every $x \in P$ comes with $C(x) \subseteq W$, the set of worlds *consistent* with x. A preliminary explanation

[3] $E(w) = \{X : EwX\}$ and similarly for $R(w)$.

[4] It is customary in philosophy of science to assume that all evidence is propositional, see (Achinstein, 2001, 2010). On the other hand, the use of 'evidence' in epistemology is broader, including propositional as well as non-propositional entities. For nice examples, see (Feldman, 1988, 1995; Feldman & Conee, 1985), where sense-experiences are frequently cited as evidence. However, some epistemologists imply that evidence is exclusively propositional, see (Williamson, 2000), for example.

of the invoked notion of consistency might run as follows. If $x \in P$ is propositional, then $w \in C(x)$ iff x is true in w (our assumption is that w is *maximally* consistent—every statement consistent with w is true in w). If $x \in P$ is non-propositional (sense-experience, event, object etc.), then $w \in C(x)$ iff x exists (obtains) in w. Every $X \in E(w)$ can be seen as corresponding to some $x \in P$ in the sense that $X = C(x)$. The set $E(w)$ then corresponds to $P(w) \subseteq P$, the evidence available to the agent at w. $[E]\phi$ then holds in w iff there is a piece of evidence x such that x is available at w and ϕ holds in every world in which x holds (or exists, obtains, occurs etc.), i.e. x 'necessitates' ϕ.[5]

Definition 1 (van Benthem et al., 2012) *The Hilbert system $H(\mathbf{EL})$ is given by the following axiom schemes and rules:*

(A0) *Propositional tautologies in $\mathcal{L}_E$*

(A1) $\mathbf{S5}$ *axioms for* $[K]$

(A2) $\mathbf{K}$ *axioms for* $[B]$

(A3) $[E]\top$

(A4) $([E]\phi \wedge [K]\psi) \leftrightarrow [E](\phi \wedge [K]\psi)$

(A5) $[K]\phi \rightarrow [B][K]\phi$

(R1) *Modus Ponens*

(R2) $\phi \rightarrow \psi / [E]\phi \rightarrow [E]\psi$

(R3) $\phi/[o]\phi$ *for* $o \in \{K, B\}$

The basic evidence logic $\mathbf{EL}$ *is the set of formulas provable in* $H(\mathbf{EL})$.

Fact 1 (van Benthem et al., 2012) *For every $\phi \in \mathcal{L}_E$: $\phi \in \mathbf{EL}$ iff $M^e, w \models \phi$ for all pointed extended evidence models M^e, w.*

[5] This is in need of a deeper discussion. However, such a discussion is left out of the present paper, due to space limitations. We note that our preliminary characterisation of consistency loosely builds on Feldman's characterisation of the relation of 'necessitation' between pieces of evidence and propositions, see (Feldman, 1995).

3 A simple justification logic

The basic language of justification logic $\mathcal{L}_J$ adds to the Boolean language formulas of form $t : \phi$, where $t \in Tm$. The set of *justification terms* Tm is defined inductively over disjoint countable sets Var of justification variables and Con of justification constants (x, y, z range over Var and d, e range over Con):

- $Var \cup Con \subseteq Tm$
- if $s, t \in Tm$, then $s \cdot t \in Tm$ and $s + t \in Tm$

Hence, in addition to 'tags' for specific pieces of evidence (or justifications), $\mathcal{L}_J$ contains *operators*, which allow to build complex justification terms. Formulas $t : \phi$ are read 'ϕ is believed for reason t'.

Definition 2 *The Hilbert system $H(\mathbf{J})$ comprises of the following axioms and rules:*

(jA0) *Propositional tautologies*

(jA1) $s : (\phi \to \psi) \to (t : \phi \to (s \cdot t) : \psi)$

(jA2) $(s : \phi \vee t : \phi) \to (s + t) : \phi$

(jR1) *Modus Ponens*

(jR2) *For every axiom ϕ and any constants $e_1, \ldots, e_n$ infer that $e_n : e_{n-1} : \ldots : e_1 : \phi$*

The constant specification induced by $H(\mathbf{J})$*, $CS_{\mathbf{J}}$, is the set of all formulas of the form $e : \phi$, where e is a constant, provable in $H(\mathbf{J})$. The basic justification logic $\mathbf{J}$ is the set of $\mathcal{L}_J$-formulas provable in $H(\mathbf{J})$.*

The justification logic $\mathbf{J}$ has two interesting properties: it 'internalizes' its own proofs and it 'realizes' the basic normal modal logic $\mathbf{K}$.

Theorem 1 (Internalization; Brezhnev, 2000) *If $\phi \in \mathbf{J}$, then there is a t such that $t : \phi \in \mathbf{J}$.*

Theorem 2 ($\mathbf{K}$-realization; Brezhnev, 2000)
i) If $\phi \in \mathbf{K}$ then there is a formula $\phi^r \in \mathcal{L}_J$ such that ϕ^r results from ϕ by replacing occurrences of 'boxes' by justification operators and $\phi^r \in \mathbf{J}$.
ii) If $\phi \in \mathbf{J}$ and $\phi^\Box$ results from ϕ by replacing every occurrence of a justification operator by a 'box', then $\phi^\Box$ is a theorem of $\mathbf{K}$.

The usual models for $\mathcal{L}_J$ are *Fitting models*, see (Fitting, 2005):

$$M^f = \langle W, R, A, V \rangle \tag{2}$$

where $\langle W, R, V \rangle$ is a one-dimensional Kripke model and A is a function from $(W \times Tm)$ to 2^{Fm} such that:

- If $\phi \to \psi \in A(w, s)$ and $\phi \in A(w, t)$, then $\psi \in A(w, s \cdot t)$
- $A(w, s) \cup A(w, t) \subseteq A(w, s + t)$
- If $e : \phi \in CS_{\mathbf{J}}$, then $\phi \in A(w, e)$ for all $w \in W$

The truth-conditions of the Boolean fragment are as usual. Moreover:

- $M^f, w \models s : \phi$ iff i) $M^f, v \models \phi$ for all v such that Rwv and ii) $\phi \in A(w, s)$.

The relation R is construed in the usual way. The set $A(w, s)$ is seen as the set of formulas justified by s at w. This is a 'syntactic filter' akin to the awareness function of Fagin and Halpern (1988) with an extra parameter, the justification term.[6] Note that 't justifies ϕ' is world-relative: typically $A(w, t) \neq A(v, t)$ for $w \neq v$.

The operator '$\cdot$' corresponds to applying Modus Ponens (it is often called 'application'): If s justifies $\phi \to \psi$ and t justifies ϕ, then $s \cdot t$ justifies ψ. No other properties of '$\cdot$' are assumed.[7] The operator '$+$' ('sum' or 'weakening') corresponds to 'monotonic merging' of justifications: If $\phi \in A(w, s)$ or $\phi \in A(w, t)$, then $\phi \in A(w, s + t)$.[8] Formula $t : \phi$ holds at w iff the agent implicitly believes that ϕ and t justifies ϕ at w. Hence, the precise meaning of $t : \phi$ could be spelled out as 'agent's implicit belief that ϕ is justifiable by reference to t'. Note that justification logics do not work with the notion of some justifications being 'available'.[9]

Fact 2 (Fitting, 2005) $\phi \in \mathbf{J}$ *iff* ϕ *is valid in every Fitting model.*

[6]Of course, the function A can be replaced by a family of awareness functions $\{A_s\}_{s \in Tm}$. We will return to this suggestion later.

[7]In particular, '$\cdot$' is not assumed to be commutative (there are models with $A(w, s \cdot t) \neq A(w, t \cdot s)$), associative ($A(w, s \cdot (t \cdot t')) \neq A(w, (s \cdot t) \cdot t')$), nor idempotent ($A(w, s) \neq A(w, s \cdot s)$). Applying '$\cdot$' is 'non-monotonic', as it may lead to 'forgetting': there are models with $A(w, s) \nsubseteq A(w, s \cdot t)$.

[8]Commutativity, associativity and idempotence are not assumed, although $A(w, t) \subseteq A(w, t + t)$. 'Forgetting' is ruled out.

[9]However, availability of t at w could be mimicked (at least in the mono-agent case) by $A(w, t) \neq \emptyset$.

4 What is the difference?

A preliminary characterisation of the difference between evidence and justification logics is now at hand. There are at least three interesting points:

1. *Reference to pieces of evidence.* Evidence logics do not refer to specific pieces of evidence explicitly, 'syntactically' ($\mathcal{L}_E$) nor 'semantically' (extended evidence models). On the other hand, justification logics refer to 'justifications' both syntactically (justification terms in $\mathcal{L}_J$) and semantically (A in Fitting models).

2. *What does it mean to have evidence for ϕ?* In extended evidence models, 'there is evidence for ϕ' is a function of the proposition expressed by ϕ. On the other hand, the function A in Fitting models picks formulas directly, without reference to the propositions expressed.

3. *Relativity of justifications.* In Fitting models, 't justifies ϕ' is world-relative. On the other hand, it is not clear if this is the case in extended evidence models. Our explanation in Section 2 suggests that extended evidence models are consistent with a 'constant-evidence' explanation.

We show in Section 4.1 that direct reference to pieces of evidence, at least on the 'level of models', is easily added to evidence logics. However, adding 'tags' for pieces of evidence to $\mathcal{L}_E$ is more complicated and we leave it for another occasion. It is also shown that evidence logics are consistent with a 'world-relative' construal of evidence. Item 2 makes it tempting to conclude that justification logics 'go deeper', beyond the semantic level of propositions. Section 4.2 offers a different explanation, according to which the difference results from focusing on two different kinds of support.

4.1 Evidence logics and pieces of evidence

Extended evidence models can be 'safely' replaced by models that *do* invoke specific pieces of evidence. Neighbourhoods can be simulated by sets of binary relations.

Definition 3 *A* two-sorted *evidence model is a tuple*

$$\mathcal{M} = \langle W, R, \{R_i\}, S, V \rangle_{i \in G} \tag{3}$$

where $\langle W, R, \{R_i\}, V\rangle_{i\in G}$ *is a* $(|G|+1)$*-dimensional Kripke model and* S*:* $W \to 2^G$*. The truth-conditions for Boolean connectives,* $[B]\phi$ *and* $[K]\phi$ *are as in extended evidence models. Moreover:*

- $\mathcal{M}, w \models [E]\phi$ *iff there is* $i \in S(w)$ *such that* $R_i(w) \subseteq \|\phi\|_{\mathcal{M}}$

A two-sorted model is extendible *iff (i) for every* w*, there is* $i \in S(w)$ *such that* $R_i(w) = W$*, and (ii) if* $i \in S(w)$*, then* $R_i(w) \neq \emptyset$*.*

The set G is thought of as a set of pieces of evidence and every R_i is a binary relation of 'relative compatibility' corresponding to $i \in G$. R_iwv can be thought of as representing the fact that v is compatible with i, relatively to w. $S(w)$ is seen as the body of evidence available to the agent at w. Sets $R(w)$ can be seen as corresponding to the evidence the agent 'accepts' or 'trusts' at w. One can think of $R(w)$ as the intersection of a multitude of sets corresponding to specific 'accepted' pieces of evidence, but we shall not go into such details here. The models recognize two sorts of evidence, hence their name. Observe that no specific relation between the 'available' and the 'accepted' evidence is assumed.

Definition 4 *Let* $\mathcal{M} = \langle W, R, \{R_i\}, S, V\rangle_{i\in G}$ *be a two-sorted evidence model. The* extended copy *of* $\mathcal{M}$*,* $\mathcal{M}^* = \langle W, R, E^*, V\rangle$*, is obtained by defining:*

- $E^*(w) = \{X : X = R_i(w)$ *for some* $i \in S(w)\}$

Fact 3 *Let* $\mathcal{M}$ *be an extendible two-sorted model. Then* $\mathcal{M}^*$ *is an extended evidence model. Moreover, if* M^e *is an arbitrary extended evidence model, then there is an extendible two-sorted model* $\mathcal{M}$ *such that* $M^e = \mathcal{M}^*$*.*

Fact 4 *Let* $M^e = \mathcal{M}^*$*. Then* $\|\phi\|_{M^e} = \|\phi\|_{\mathcal{M}}$ *for every* $\phi \in \mathcal{L}_E$*.*

Theorem 3 *For all* $\phi \in \mathcal{L}_E$*:* $\phi \in$ **EL** *iff* $\mathcal{M}, w \models \phi$ *for all pointed extendible two-sorted models* $\mathcal{M}, w$*.*

Proof. Soundness is easily established by induction on the length of proofs. Completeness follows from Facts 1, 3 and 4. □

Note that the evidence-based interpretation of two-sorted models extends to simple multi-dimensional Kripke models $M = \langle W, \{R_i\}, V\rangle_{i\in G}$. In other words, normal multi-dimensional modal logics can be construed as simple logics of evidence, where 'agents' are replaced by 'pieces of evidence' and 'groups' by 'evidential states'.

4.2 Two kinds of support

Evidence logics are based on the notion that ϕ is supported by evidence iff there is an available piece of evidence that *necessitates* ϕ, i.e. every world in which the piece of evidence holds (or obtains) is a ϕ-world (call this the *propositional* notion of support). Consequently, if ϕ and ψ express the same proposition, then ϕ is propositionally supported iff ψ is. It is plain that this is not the case in the context of justification logics, where ϕ is supported by evidence (at w) iff $\phi \in A(w, t)$ for some t. It is possible that ϕ, ψ express the same proposition in a Fitting model and $\phi \in A(w, t)$, but $\psi \notin A(w, s)$ for all s.

The propositional notion of support is assumed by many sceptical arguments. A typical sceptical argument insists that a class of beliefs is not supported *in that* it is not necessitated by any evidence. For example, the sceptic argues that my belief that there is a table in front of me is not supported by my visual experiences, because it is possible for me to have the experiences in worlds where there is no table in front of me. For example, it is possible to see a table while hallucinating, while being a brain in a vat etc. These possibilities are not *ruled out* by our evidence.

Accordingly, the propositional notion of support seems unrealistic from an intuitive point of view. For example, when considering perceptual beliefs, one tends to consider visual experiences as evidence *par excellence*. There is a difference between our 'intuitive' understanding of evidence and the propositional notion of support. Some epistemologists make a similar point. For example, Feldman (1995) argues that the question whether one's belief in ϕ is *rational* is independent of the question whether one's evidence necessitates ϕ.

Example 1. Let $\top$ be any propositional tautology, let p be short for 'There is a table in front of Alice' and let x denote Alice's visual experience of a table in front of her. Intuitively, x supports p and x does not support $\top$. But, it is plain that x necessitates $\top$, for the sole reason that $\top$ is necessary. Moreover, x does not necessitate p, as the obvious sceptical counterexamples show.

One can explain the difference between evidence and justification logics in terms of the contrast between the propositional and the 'intuitive' notion of support. Evidence logics represent propositional support and justification logics focus on the independent intuitive notion. On this interpretation of the difference, one immediately sees the relevance of logics that *combine* these two approaches.

5 A simple two-sorted justification logic

A simple logic that represents the propositional as well as the 'intuitive' notion of support is introduced in the present section (5.1). The logic is shown to be sound and complete with respect to a class of multi-dimensional awareness models (5.2). In addition, it is shown that the logic realizes **K**, but the usual proofs of internalization fail (5.3).

5.1 The logic JE

The language $\mathcal{L}_{JE}$ is $\mathcal{L}_J$ extended with monadic operators $[E_t]$ and $[A_t]$ for every $t \in Tm$. $[E_t]\phi$ is read 't necessitates ϕ' and $[A_t]\phi$ is read 't weakly supports ϕ'. The justification formulas $t : \phi$ are read 't strongly supports ϕ'. Strong support is construed as a combination of necessitation (propositional support) and weak ('intuitive') support.

Definition 5 *The Hilbert system $H(\mathbf{JE})$ is given by the following axioms and rules:*

(Ax0) *Propositional tautologies*

(Ax1) $[E_t](\phi \to \psi) \to ([E_t]\phi \to [E_t]\psi)$

(Ax2) $([E_s]\phi \vee [E_t]\phi) \to ([E_{s\cdot t}]\phi \wedge [E_{s+t}]\phi)$

(Ax3) $([E_s]\phi \wedge [E_t]\psi) \to ([E_{s\cdot t}](\phi \wedge \psi) \wedge [E_{s+t}](\phi \wedge \psi))$

(Ax4) $[A_s](\phi \to \psi) \to ([A_t]\phi \to [A_{s\cdot t}]\psi)$

(Ax5) $([A_s]\phi \vee [A_t]\phi) \to [A_{s+t}]\phi$

(Ax6) $t : \phi \leftrightarrow ([E_t]\phi \wedge [A_t]\phi)$

(Ax7) $s : (\phi \to \psi) \to (t : \phi \to (s \cdot t) : \psi)$

(Ax8) $(s : \phi \vee t : \phi) \to (s + t) : \phi$

(Ru1) *Modus Ponens*

(Ru2) $\phi/[E_t]\phi$ *for all* $t \in Tm$

(Ru3) *For every axiom* ϕ *and any constants* $e_1, \ldots, e_n$ *infer that* $e_n : e_{n-1} : \ldots : e_1 : \phi$

The constant specification induced by $H(\mathbf{JE})$, $CS_{\mathbf{JE}}$, is the set of formulas of the form $e : \phi$ provable in $H(\mathbf{JE})$. The logic $\mathbf{JE}$ is the set of formulas provable in $H(\mathbf{JE})$.

Lemma 1 $\mathbf{J} \subseteq \mathbf{JE}$.

Lemma 2 *Axioms (Ax7) and (Ax8) are redundant, i.e. derivable from the other axioms.*

We note that Lemma 1 requires the inclusion of the 'redundant' axioms (Ax7) and (Ax8), since (Ru3) applies only to axioms. The constant specification induced by the Hilbert system without the two axioms does not contain $CS_{\mathbf{J}}$.

Definition 6 *A* common model *is a tuple*

$$\mathfrak{M} = \langle W, \{R_t\}, \{A_t\}, V\rangle_{t\in Tm} \tag{4}$$

where every $R_t \subseteq W^2$ and A_t: $W \to 2^{Fm(\mathcal{L}_{JE})}$. It is assumed that

- $R_{s\cdot t}, R_{s+t} \subseteq R_s \cap R_t$
- *If $\phi \to \psi \in A_s(w)$ and $\phi \in A_t(w)$, then $\psi \in A_{s\cdot t}(w)$*
- $A_s(w) \cup A_t(w) \subseteq A_{s+t}(w)$
- *If $e : \phi \in CS_{\mathbf{JE}}$, then $\phi \in A_e(w)$ for all $w \in W$*

The truth-conditions for the Boolean fragment are as usual. Moreover:

$\mathfrak{M}, w \models [E_t]\phi$ *iff* $R_t(w) \subseteq \|\phi\|_{\mathfrak{M}}$ *($*$)*

$\mathfrak{M}, w \models [A_t]\phi$ *iff* $\phi \in A_t(w)$ *($**$)*

$\mathfrak{M}, w \models t : \phi$ *iff ($*$) and ($**$)*
Common frames *and validity are defined in the usual way.*

Common models are a special class of multi-dimensional awareness models. For every $t \in Tm$, R_t represents the propositional support provided by t and A_t represents the weak ('intuitive') support. $R_t(w)$ is the set of worlds consistent with t relatively to w. $A_t(w)$ is the set of formulas weakly supported by t at w. Note that these are independent: there are models where $R_t(w) \subseteq \|\phi\|_{\mathfrak{M}}$ but $\phi \notin A_t(w)$ and vice versa. **EJ** can be seen as a justification logic that incorporates the notion of propositional support from evidence logic. However, the logic does not contain **EL** for the simple reason that it replaces the idea of 'quantifying' over pieces of evidence by operators expressing the propositional support provided by *specific* pieces of evidence.

5.2 Completeness

This section establishes the usual soundness and completeness results. Completeness is shown by the standard canonical model construction. We note that the strong justification logic **LP** was shown to be sound and complete with respect to a special class of common models in (Sedlár, 2013).

Theorem 4 (Soundness) *If $\phi \in \mathbf{JE}$, then ϕ is valid in every common frame.*

Proof. Induction on the length of proofs. □

Definition 7 (Canonical frame and model) *The* canonical frame *for* **JE** *is a structure $\mathfrak{F}^c = \langle W^c, \{R_t^c\}, \{A_t^c\}\rangle_{t \in Tm}$ where*

- *W^c is the set of maximal $\mathbf{JE}$-consistent sets of formulas $\Gamma, \Delta, \ldots$*
- *$\Gamma R_t^c \Delta$ iff $\Gamma_E(t) \subseteq \Delta$, where $\Gamma_E(t) = \{\phi : [E_t]\phi \in \Gamma\}$*
- *$\phi \in A_t^c(\Gamma)$ iff $[A_t]\phi \in \Gamma$*

The canonical model *for* **JE** *is $\mathfrak{M}^c = \langle \mathfrak{F}^c, V^c\rangle$, where $\Gamma \in V^c(p)$ iff $p \in \Gamma$.*

Lemma 3 (Frame Lemma) *The canonical frame is a common frame.*

Proof. As usual, we have to show that the canonical frame satisfies the frame conditions of Definition 6. First, it has to be shown that $\Gamma R_s^c \Delta$ and $\Gamma R_t^c \Delta$ if $\Gamma R_{s+t}^c \Delta$. If $\phi \in \Gamma_E(s)$, then $[E_s]\phi \in \Gamma$ and, by (Ax2) and propositional logic, $[E_{s+t}]\phi \in \Gamma$. By the assumption, $\phi \in \Delta$. The cases for $\Gamma R_t^c \Delta$ and $R_{s \cdot t}^c$ are similar.

Second, assume that $\phi \to \psi \in A_s^c(\Gamma)$ and $\phi \in A_t^c(\Gamma)$. $\psi \in A_t^c(\Gamma)$ follows from (Ax4). The fact that $A_s^c(\Gamma) \cup A_t^c(\Gamma) \subseteq A_{s+t}^c(\Gamma)$ is proven similarly by invoking (Ax5).

Third, assume that $e : \phi \in CS_{\mathbf{JE}}$. Then $e : \phi \in \mathbf{JE}$ and $e : \phi \in \Gamma$ for all $\Gamma \in W^c$. By (Ax6) and propositional logic, $[A_e]\phi \in \Gamma$ for all $\Gamma \in W^c$. Hence, $\phi \in A_e^c(\Gamma)$ for all Γ. □

Lemma 4 (Model Lemma) *The canonical model is a common model.*

Proof. The Lemma follows from Lemma 3 and the fact that $\phi \in \Gamma$ iff $\mathfrak{M}^c, \Gamma \models \phi$. The second claim (Truth Lemma) is shown by standard induction on the complexity of ϕ. The base case holds by definition and the cases for Boolean connectives are trivial.

Now assume that $[E_t]\phi \in \Gamma$. It follows that $\phi \in \Gamma_E(t)$ and, hence, $\phi \in \Delta$ for all $\Gamma_E(t) \subseteq \Delta$. Consequently, $\mathfrak{M}^c, \Gamma \models [E_t]\phi$. Conversely, assume that $[E_t]\phi \notin \Gamma$. Then $\Gamma_E(t) \cup \{\neg\phi\}$ is $\mathbf{JE}$-consistent and can be extended to a maximal $\mathbf{JE}$-consistent set Γ^*. It is plain that $\Gamma R^c_t \Gamma^*$ and $\phi \notin \Gamma^*$. Hence, $\mathfrak{M}^c, \Gamma \not\models [E_t]\phi$.

Next, $[A_t]\phi \in \Gamma$ iff $\phi \in A^c_t(\Gamma)$ (by definition) iff $\mathfrak{M}^c, \Gamma \models [A_t]\phi$. Now $t : \phi \in \Gamma$ iff $[E_t]\phi \in \Gamma$ and $[A_t]\phi \in \Gamma$ by (Ax6). The rest follows from the previous claims concerning $[E_t]\phi$ and $[A_t]\phi$. $\square$

Theorem 5 (Completeness) *If ϕ is valid in every common frame, $\phi \in \mathbf{JE}$.*

5.3 Realization and internalization

In this section, an 'operator' is any instance of '$[E_t]$', '$[A_t]$' and 't :', and a 'justification operator' is any instance of 't :'.

Lemma 5 *Let $\phi \in \mathcal{L}_{JE}$ and let $\phi^\Box$ be the result of replacing every occurrence of an operator in ϕ by an occurrence of the modal box '$\Box$'. If $\phi \in \mathbf{JE}$, then $\phi \in \mathbf{K}$.*

Proof. Simple induction on the length of $H(\mathbf{JE})$-proofs. Observe that the claim holds for every axiom of $H(\mathbf{JE})$ and the rules 'preserve the claim' as well. $\square$

Theorem 6 *If $\phi \in \mathbf{K}$ then there is a formula $\phi^r \in \mathcal{L}_{JE}$ such that ϕ^r results from ϕ by replacing occurrences of 'boxes' by occurrences of justification operators and $\phi^r \in \mathbf{JE}$.*

Proof. Follows from Theorem 2 and Lemma 1. $\square$

A corollary of these two results is that $\mathbf{JE}$ 'realizes' $\mathbf{K}$. However, the usual proof of the internalization property (see, e.g., Artemov, 2008) does not work. The reason is that there is no justification operator corresponding to the necessitation rule (Ru2). Moreover, other well-known techniques used when justification logic is combined with normal modal logics (see Artemov & Nogina, 2005a, 2005b for example) are not applicable in our context either.

6 Conclusion

The paper attempted to take first steps to relate evidence and justification logics. The main results are: (i) a completeness proof for the basic evidence logic with respect to a new class of models, where 'pieces of evidence' are invoked explicitly, (ii) completeness and realization proofs for a justification logic that incorporates some ideas form evidence logic (operators for propositional support). In a more philosophical vein, it has been suggested that (i) the difference between the rendering of evidence in justification and evidence logics can be explained as the result of their focusing on two distinct notion of support, (ii) even multi-dimensional normal modal logics can be seen as logics of evidence.

However, a natural goal is to extend $\mathbf{JE}$ at least with the operator $[E]$ of evidence logic. Of course, this could be done by adding neighbourhoods to common models. A more interesting approach is to construe $[E]$ as a quantifier over selected subsets of justification terms. Such a framework comes close to term-modal logics of Fitting et al. (2001), a version of first-order modal logic where modal operators are indexed by the terms of the language. This approach makes the introduction of predicates for and quantification over pieces of evidence relatively straightforward. However, this interesting project is beyond the scope of this paper.

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