Hyperintensional logics for everyone

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Abstract. We introduce a general representation of unary hyperintensional modalities and study various hyperintensional modal logics based on the representation. It is shown that the major approaches to hyperintensionality known from the literature, that is state-based, syntactic and structuralist approaches, all correspond to special cases of the general framework. Completeness results pertaining to our hyperintensional modal logics are established.

Keywords: Awareness logic, Hyperintensionality, Hyperintensional logic, Hyperintensional modalities, Impossible worlds, Modal logic, Non-Fregean logic, Structured propositions.

1 Introduction

The possible-worlds framework has provided semantics for various formal languages as well as large portions of natural language. The general strategy is to represent semantic contents of expressions by *intensions*, i.e. functions from possible worlds (usually taken as unanalysed indices) to extensions. The specific kind of extension depends on the kind of expression at hand. For instance, extensions are truth-values $0, 1$ in the case of sentences, individuals (from some fixed domain) in the case of names and n-ary relations in

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the case of n-ary predicates. *Modalities* are seen as expressing properties of (or relations between) intensions; the corresponding intensions are functions from possible worlds to relations on intensions. In the case of unary sentential modalities, an equivalent approach is to take functions from possible worlds to sets of *propositions* (i.e. to sets of sets of possible worlds).

A prominent example of this approach is the Montague–Scott semantics for modal logic [38, 46, 47, 11, 40]. In the MS semantics, a modal formula $\Box F$ says that $[\![F]\!]$, the proposition expressed by the formula F, has the property expressed by $\Box$. This property is represented by a function N from possible worlds to sets of propositions. So, $\Box F$ is true in a world w iff $[\![F]\!] \in N(w)$.

The problem with this picture is that some natural-language modalities do not express properties of propositions. Take *epistemic* modalities, for example. Sentences '$2 + 3 = 5$' and '110119 is a prime number' express the same proposition (the set of all possible worlds), but mutual substitution of these sentences in the scope of an epistemic modality is not guaranteed to preserve truth value. Simply put, it is possible for some agent, say John, to believe that $2+3 = 5$ without also believing that 110119 is a prime number. This would be impossible if 'John believes that' expressed a property of propositions.

Modalities that express properties of sentential contents grained finer than sentential intensions are known as *hyperintensional* modalities. Semantic frameworks where hyperintensional modalities can be represented are clearly of both practical and theoretical value. Although research in the area dates back at least to the 1970s (see [14], for example), it is still quite active today (see [27, 30], for example).[1] Central among such frameworks—and ones of interest not only to philosophers, but also to computer scientists, for example—are models for formal modal languages.

A framework of this kind is developed in this article. What sets it aside from earlier approaches is that it represents fine-grained sentential contents explicitly without assuming a specific theory of sentential content. In our framework, hyperintensional modalities express properties of abstract contents, represented by members of an arbitrary set. In addition, it is shown that frameworks of all three major kinds known from the literature—state-based, syntactic and structuralist frameworks—are special cases obtained from our framework by replacing the abstract representation of content by a more specific class of objects.

The benefits of studying the general framework in addition to the more specific ones are twofold. The framework allows to develop hyperinten-

[1] For a brief historical overview, see Chap. 1.1 of [17], for example.

sional modal logics independently of any specific theory of semantic content. Hence, it is useful for those who want to use such logics without committing to any one particular theory of content. At the same time, however, our framework offers a relatively simple common ground for proponents of rival semantic theories. Logics developed within the framework are *for everyone*—'engineers' who need a formalization of hyperintensional modalities, but are uninterested in philosophical questions, and 'philosophers' of various denominations who wish to add to their specific theory of content a formalization of modal reasoning that is also readily comprehensible to their semantic rivals. For the latter, the framework might also be a helpful tool for distinguishing questions of logic from questions of philosophical semantics.

The article also takes first steps in the investigation of hyperintensional modal logics based on the general framework. Our main technical result is a general completeness theorem for hyperintensional modal logics with an equivalence connective representing identity of content. These logics can be seen as modal extensions of the 'sentential calculus with identity' studied by Bloom and Suzsko [9, 10]. It is also shown that two versions of the compositionality principle correspond to two classes of axioms. We shall concentrate on the propositional fragment of the framework, both for the sake of simplicity and because the issue of hyperintensionality arises already on the propositional level. Extensions to first order and beyond are a natural research topic, yet one that is left for another occasion.

The article is structured as follows. Section 2 discusses MS semantics in more detail and Section 3 outlines the state of the art in the research on hyperintensional semantic frameworks. Section 4 introduces a hyperintensional generalisation of the MS semantics and the modal logic generated by the framework. The logic is virtually identical to classical propositional logic. Section 5 shows that hyperintensional frameworks of all three major kinds known from the literature are special cases of the general framework. It is also shown that special variants of these approaches generate the same modal logic as the general framework. Hyperintensional modal logics with an equivalence connective representing identity of content are studied in Section 6.

2 Montague–Scott semantics

For now, we use the ordinary propositional modal language $\mathcal{L}$ comprising a denumerable set At of atomic formulas (representing propositionally simple sentences such as 'John is happy'), the set of Boolean connectives $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$ and a pair of modal operators $\Box$ and $\Diamond$; the set of connectives of $\mathcal{L}$ is denoted $Con_{\mathcal{L}}$.[2] Symbols 'p', 'q' etc. stand for arbitrary atomic formulas and 'F', 'G' etc. (possibly with subscripts) for arbitrary formulas. The set of $\mathcal{L}$-formulas is defined as usual and denoted $Fm_{\mathcal{L}}$ (or just Fm).

Montague–Scott models (MS models) are tuples $\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle$. The components are as follows. W is a non-empty set, informally seen as a set of possible worlds. N is a function from W to sets of propositions on W (i.e. sets of subsets of W). Informally, N corresponds to a property of propositions and $N(w)$ is the extension of this property in world w. Distinct informal interpretations of the framework differ in the reading of N. Speaking generally, we say that propositions $X \in N(w)$ are 'distinguished' in w. Finally, $[\![\cdot]\!]$ is a function that assigns to every formula F a proposition $[\![F]\!]$ on W. Informally, $[\![F]\!]$ is the proposition expressed by 'F'. The function is required to satisfy the following conditions:

- $[\![\neg F]\!] = W \setminus [\![F]\!]$
- $[\![F \wedge G]\!] = [\![F]\!] \cap [\![G]\!]$
- $[\![F \vee G]\!] = [\![F]\!] \cup [\![G]\!]$
- $[\![F \rightarrow G]\!] = (W \setminus [\![F]\!]) \cup [\![G]\!]$
- $[\![F \leftrightarrow G]\!] = ((W \setminus [\![F]\!]) \cup [\![G]\!]) \cap ((W \setminus [\![G]\!]) \cup [\![F]\!])$
- $[\![\Box F]\!] = \{w \ ; [\![F]\!] \in N(w)\}$
- $[\![\Diamond F]\!] = \{w \ ; (W \setminus [\![F]\!]) \notin N(w)\}$

Hence, Boolean connectives correspond to the usual set-theoretic operations on the set of propositions. Importantly, $w \in [\![\Box F]\!]$ iff $[\![F]\!]$ is distinguished in w, so $\Box F$ may generally be read as 'the proposition expressed by "F" is distinguished'. Similarly, $w \in [\![\Diamond F]\!]$ iff $(W \setminus [\![F]\!])$ is not distinguished, so $\Diamond F$ generally means that the complement of the proposition expressed by 'F' is not distinguished. A formula F is valid in a class $\{\mathcal{M}_i \ ; i \in I\}$ of MS models iff $[\![F]\!] = W_i$ for every model $\mathcal{M}_i$ in the class. Note that $F \rightarrow G$ is valid in a class of models iff $[\![F]\!] \subseteq [\![G]\!]$ in every model in the class; and $F \leftrightarrow G$ is valid in a class of models iff $[\![F]\!] = [\![G]\!]$ in every model in the class.

Now consider the following four inference rules:

$$\frac{F}{\Box F}(\text{RN}) \quad \frac{F_1 \wedge F_2 \rightarrow G}{\Box F_1 \wedge \Box F_2 \rightarrow \Box G}(\text{RR}) \quad \frac{F \rightarrow G}{\Box F \rightarrow \Box G}(\text{RM}) \quad \frac{F \leftrightarrow G}{\Box F \leftrightarrow \Box G}(\text{RE})$$

[2] It is natural in a hyperintensional setting to consider all the usual Boolean connectives as primitive. For the sake of simplicity, we do this already when discussing MS models.

Only (RE) preserves validity in every class of MS models. This is clear—if $[\![F]\!] = [\![G]\!]$, then $[\![F]\!] \in N(w)$ iff $[\![G]\!] \in N(w)$ for each w. (RM) preserves validity in every class of models where $N(w)$ is closed under supersets for all w; (RR) requires, in addition, that $N(w)$ be closed under (binary and hence finitary) intersections; and (RN) requires that $W \in N(w)$ for all w. Hence, all four rules together preserve validity in every class of MS models where every $N(w)$ is a non-empty filter on W.[3] The set of formulas valid in the class of all such models is the basic *normal* modal logic **K**.[4] The set of formulas valid in the class of all MS models is the basic *classical* modal logic **E**. Other classical logics between these two are defined in the obvious way, but we shall not discuss them. The interested reader is referred to [47, 11, 40].

MS models offer a rich and versatile framework that can be adjusted for specific interpretations of '$\Box$', '$\Diamond$' and 'distinguished'. For instance, if $\Box$ is read as 'John believes that', then $N(w)$ is seen as the set of propositions believed by John in w and $\Box F$, 'John believes that F', is true in w iff the proposition expressed by F is among the propositions believed by John in w. On this picture, the logic **K** comes with the assumption that John's beliefs are always closed under arbitrary consequence. This is quite a strong assumption, so relaxing some of the closure conditions seems appropriate. For instance, lifting the assumption that $W \in N(w)$ means that John is not always required to believe all valid F; lifting closure under intersections means that John is not always required to 'pool' his beliefs together (as $\Box F \wedge \Box G \to \Box(F \wedge G)$ is in general not valid in models without the closure condition). However, (RE) is an immutable feature of the framework. As a result, the MS framework cannot represent hyperintensional modalities.

3 Three major hyperintensional frameworks

MS semantics incorporates three important assumptions:

(A1) Modalities express properties of semantic contents of sentences.

[3]The definition of logical consequence over a class of models is standard: Γ entails F iff, for all models in the class, $\bigcap_{G\in\Gamma} [\![G]\!] \subseteq [\![F]\!]$. Using $\mathcal{L}$, consequence relations over all classes of MS models have the property that Γ entails F iff there is a finite $\Gamma' \subseteq \Gamma$ such that $(\bigwedge_{G\in\Gamma'} G) \to F$ is valid in the given class. For such classes, (RR) entails that $N(w)$ is closed under consequence. (RM) says that every $N(w)$ is closed under single-premise consequence and (RN) entails closure under zero-premise consequence, i.e. validity.

[4]Stronger normal modal logics correspond to adding more assumptions. For instance, the logic **T** is determined by models where $w \in N(w)$ for all w. Obviously, $\Box F \to F$ is valid in any such model.

(A2) Semantic contents of sentences are sets of states.

(A3) All semantically relevant states are possible worlds.

(A similar list appears in [49].)

Semantic frameworks for hyperintensional modalities divide, roughly, into three categories that can be characterised by reference to (A1) – (A3). Firstly, *state-based* approaches retain (A1) and (A2), but lift the assumption (A3). Hence, sentential contents are sets of states that may contain outlandish states, or impossible worlds, that invalidate some of our dear logical, mathematical, or metaphysical principles [41, 42, 2, 43, 12, 13, 26, 36]. On this approach, John may believe that $2 + 3 = 5$ without also believing that 110119 is a prime number because the two sentences may be assigned two different sets of states, the union of which contains a state where $2 + 3 = 5$ but 110119 is composite, or a state where 110119 is prime but $2 + 3 \neq 5$ (the sets contain the same possible worlds, they differ only with respect to the impossible ones).[5] The state-based approaches have found a number of interesting applications and the logical theory arising from the framework is both simple and fairly well-understood.[6]

Secondly, *syntactic* approaches retain (A2) and (A3), but lift (A1). Hyperintensional modalities are explained not (only) by reference to sentential contents, but (also) to sentences themselves. A good example is the framework of [20]. On their view, epistemic modalities pertain to properties of propositions *and* properties of sentences. Intuitively, if John believes that 110119 is a prime number, then he has to be in some kind of cognitive relation ('awareness') to the sentence '110119 is a prime number'. This leads to a natural explanation of why John may very well believe that $2+3 = 5$ without also believing that 110119 is prime—he is aware of '$2+3 = 5$' but not of '110119 is a prime number' (the reason might be that he lacks the concept of primeness or that he has never thought of 110119, for example). *Justification logics* [1, 25] are another example of this approach. They build on the idea that, in order to know that $2 + 3 = 5$, John has to be in a certain relation to the proposition expressed by '$2 + 3 = 5$' but, in addition, he must have access to a reliable justification of the fact that the proposition expressed by '$2 + 3 = 5$' is true. As with the state-based approaches, the logical theory arising from the syntactical approaches is simple and well-understood.

[5] For instance, assume that different arithmetical laws hold in these states.

[6] An account of hyperintensionality using impossible worlds has been put forward recently by [27]; see also [44]. [3, 4] gives a general overview of the most important topics related to impossible worlds.

	MS	State-based	Syntactic	Structuralist
(A1)	•	•		•
(A2)	•	•	•	
(A3)	•		•	•

Figure 1: MS and the major hyperintensional frameworks.

Thirdly, *structuralist* approaches retain (A1) and (A3), but lift (A2).[7] On the structuralist picture, semantic contents of sentences are structured abstract entities quite unlike sets of states or sentences. Such contents (i) correspond, to some degree, to the structure of sentences expressing them, but are not syntactic in nature; and (ii) determine, but are not identical with, intensions of these sentences. One and the same intension may be determined by various distinct structured contents and several distinct sentences (for instance, synonymous sentences of different languages) may express the same structured content. Hence, John may believe that $2 + 3 = 5$ without believing that 110119 is prime because the two corresponding sentences express different structured contents and so he may be in the belief-relation with the former content without being in the relation with the latter one as well. Specific structuralist theories disagree as to the exact nature of structured contents. Some take them to be Russellian propositions, i.e. ordered tuples consisting of individuals, properties, and relations [45, 49, 33]; some see them as tuples of intensions [37, 15]; others claim that structured contents correspond to repeated applications of certain basic logical functions [54]; and yet others argue that contents are not set-theoretical in nature, but procedural [50, 17].

Figure 1 illustrates the differences between the three approaches to hyperintensions with respect to (A1) – (A3). Relations between the state-based approach on one side and the syntactic approach on the other are well-understood; see [53, 48], but also [21, chapter 9], for example. Similar observations connecting the state-based and syntactic approaches to the structuralist ones have not been provided yet. In what follows, we generalise MS semantics to a framework that enables such a comparison—prominent state-based, syntactic and structuralist approaches are special cases of the general framework.

The debate about the 'right' theory of fine-grained sentential content is one of the ongoing philosophical battles of today. As we show in the remainder of the article, hyperintensional modal logic—logic with opera-

[7] (A3) is retained in the sense that structuralist frameworks typically do not include other kinds of states in addition to possible worlds.

tors representing modalities pertaining to these fine-grained contents—can be developed without taking sides in the battle. The semantic framework introduced in the next section represents fine-grained sentential content explicitly, but does not subscribe to any of the rival theories of content. What is more, as shown in Section 5, the logic generated by special cases of the framework corresponding to specific state-based, syntactic and structuralist approaches to content is identical to the logic generated by the most general version of the framework. Nevertheless, (special cases of) the framework can be used by proponents of specific theories of content to articulate logics arising from their semantic views in a manner comprehensible to their philosophical rivals. The study of such special cases can lead proponents of specific theories of content to postulate various relations and operations on the contents that are to be reflected in the formal modal language. First steps in a general study of such modal logics are carried out in Section 6, where hyperintensional modal logics with an equivalence connective representing identity of content are investigated.

Remark 3.1. Hyperintensional frameworks based on versions of the truthmaker semantics (see [22, 23, 24] and also [35], for example) can be seen as instances of state-based approaches. They are not discussed in detail here since their primary aim is not a hyperintensional formalisation of *modalities.* The theory of topic-sensitive intensional modals developed by Berto (see [6, 7, 5, 8]) yields a framework for hyperintensional modalities. However, it falls outside the scope of our framework—'topics' are not seen as proposition-determining contents. In Berto's framework, for example, the topic of p is identical to the topic of $\neg p$. Nevertheless, a generalization of our framework that covers the topic-sensitive framework, and also some modal extensions of truthmaker frameworks, is a topic of ongoing work.

4 MS generalised

In this section we study a hyperinensional generalisation of the MS framework.

4.1 From MS models to pre-models and back

A *pre-model* is a tuple

$$\mathfrak{P} = \langle W, C, O, N_C, I \rangle$$

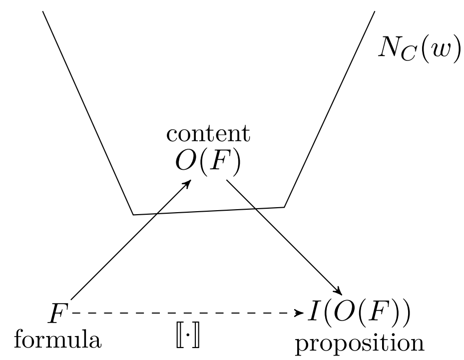

Figure 2: MS models generalised.

specified as follows. W and C are non-empty sets. O is a function from formulas to C. N_C is a function that assigns to every $w \in W$ a subset of C. Finally, I is a function that assigns to every $c \in C$ a proposition $I(c) \subseteq W$.

Every MS model is a pre-model. To see this, let us take any MS model and define C to be the power set 2^W of W, O as $[\![\cdot]\!]$ and I as the identity function on 2^W. N_C is just N.

To get an intuitive grasp on pre-models, C can be seen as the set of *semantic contents* of sentences represented by formulas and O as the function that assigns contents to sentences ('content function'). N_C is a function that assigns a set of contents distinguished in w to every possible world $w \in W$ (hence, N_C represents a property of contents). Finally, I is a function that assigns propositions to contents, not to formulas directly ('intension function'). The function represents the intuitive idea that propositions are determined by contents. The function $[\![\cdot]\!]$ assigning propositions to formulas may be defined as the composition of O and I. Figure 2 provides an illustration.

Seeing MS models as a special case of pre-models underlines their adherence to the assumptions (A1) – (A3). The set W is seen as a set of possible worlds, (A3), and C is the power set of W, (A2). Moreover, truth conditions of modal formulas are stated in terms of the function N_C, (A1). At the same time, the general framework of pre-models allows to dispense with (combinations of) (A1) – (A3) easily. Before studying such variations, let us focus on some specific sub-classes of pre-models.

4.2 Hyperintensional models

A *hyperintensional model* (or model, for short) is a pre-model $\mathfrak{M}$ where the function $[\![\cdot]\!]_{\mathfrak{M}}$, defined by $[\![F]\!]_{\mathfrak{M}} = I(O(F))$, satisfies the semantic conditions assumed in MS models for all Boolean connectives and, moreover,

(IB) $[\![\Box F]\!]_{\mathfrak{M}} = \{w \,;\, O(F) \in N_C(w)\}$

(ID) $[\![\Diamond F]\!]_{\mathfrak{M}} = \{w \,;\, O(\neg F) \notin N_C(w)\}$

We will often omit the subscript and write only $[\![F]\!]$ if the model is clear from the context. The set of $\mathcal{L}$-formulas valid in all hyperintensional models will be denoted as **H**.

A hyperintensional model is *weakly compositional* iff it satisfies the

Weak Compositionality Principle (wCP). If G results from replacing an occurrence of a direct subformula F' of F by an occurrence of G', then $O(F') = O(G')$ only if $O(F) = O(G)$.

A hyperintensional model is *strongly compositional* iff it satisfies the

Strong Compositionality Principle (sCP). If G results from replacing an occurrence of a direct subformula F' of F by an occurrence of G', then $O(F) = O(G)$ iff $O(F') = O(G')$.

The weaker (wCP) implies that $O(p) = O(q)$ only if $O(r \to p) = O(r \to q)$; (sCP) entails also that $O(r \to p) = O(r \to q)$ only if $O(p) = O(q)$. Since injectivity of O is consistent with both (wCP) and (sCP)—see the proof of Theorem 4.2 below, for example—neither principle entails that $O(p) = O(p \wedge p)$ or that $O(p \wedge q) = O(q \wedge p)$.[8]

It can be shown that every weakly compositional model represents 'the meaning of a compound expression as a function of the meanings of its parts'. This dictum is often cited as a vague statement of the compositionality principle; see [28], for example.

Proposition 4.1. *In every weakly compositional model, $O(F_1) = O(F_2)$ iff, for every G, $O(G(F_1)) = O(G(F_2))$ (where $G(F_2)$ is the result of replacing an occurrence of F_1 in G by an occurrence of F_2).*

Proof. The right-to-left implication is trivial (let $G = F_1$). The left-to-right implication is established as follows. If G does not contain any occurrences of F_1, then we are done. So assume that G contains some occurrences of F_1 and let $G(F_2)$ be the result of replacing one of the occurrences by F_2. Let

[8] We note that even though neither compositionality principle excludes the possibility that, say $O(\neg p) = O(p)$, there is no hyperintensional model where $O(\neg p) = O(p) = c$. If this were the case, then $I(c) = W \setminus I(c)$, but this is not possible if W is non-empty.

H be the subformula of G such that the replaced occurrence of F_1 was a direct subformula occurrence of H. The claim follows from (wCP). □

In fact, a hyperintensional model is weakly compositional *if, and only if* it satisfies the equivalence expressed in Proposition 4.1.

It is clear that MS models correspond to weakly compositional hyperintensional models. They are not strongly compositional, though; it is possible to have $[\![F]\!] \cap [\![G_1]\!] = [\![F]\!] \cap [\![G_2]\!]$ without $[\![G_1]\!] = [\![G_2]\!]$, or $[\![\Box F]\!] = [\![\Box G]\!]$ without $[\![F]\!] = [\![G]\!]$.

Nevertheless, strong compositionality (sCP) might seem plausible as a principle governing meaning since it embodies an assumption of 'no surprises' – if a substitution is made in a statement without change in meaning, then the substituted expressions must have had the same meaning.[9] Equivalently, substituting an expression in a sentence with an expression having a different meaning gives you a sentence with a different meaning. We do not, however, commit to either compositionality principle.

Theorem 4.2. *A complete axiomatization of* **H** *is obtained by adding to any axiomatization of classical propositional logic (based on axiom schemata) the Duality Axiom*

(DA) $\Diamond F \leftrightarrow \neg\Box\neg F$

The same holds for the set of $\mathcal{L}$-formulas valid in every weakly compositional model and every strongly compositional model.

Proof. Soundness follows from the definition of a hyperintensional model. Completeness is established by means of a canonical model construction. Consider a structure $M = \langle W, C, O, N_C, I\rangle$ where W is the set of maximal consistent theories (with respect to the selected axiomatisation of classical logic extended with (DA)) and

- $C = Fm_{\mathcal{L}}$
- O is the identity relation on $Fm_{\mathcal{L}}$
- $N_C(\Gamma) = \{F \,;\, \Box F \in \Gamma\}$
- $I(F) = \{\Gamma \,;\, F \in \Gamma\}$

[9]Consider, for example, the sentences 'John is not a bachelor' and 'John is not an unmarried man'. One might be inclined to say that they have the same meaning *because* 'John is a bachelor' and 'John is an unmarried man' have the same meaning.

It is easily established that M is a strongly compositional model. In particular (ID) is established as follows:

$$\begin{aligned}\llbracket \Diamond F \rrbracket = I(\Diamond F) = \{\Gamma\ ;\ \Diamond F \in \Gamma\} = \\ = \{\Gamma\ ;\ \neg\Box\neg F \in \Gamma\} = \{\Gamma\ ;\ \Box\neg F \notin \Gamma\} = \\ = \{\Gamma\ ;\ \neg F \notin N_C(\Gamma)\} = \{\Gamma\ ;\ O(\neg F) \notin N_C(\Gamma)\}\end{aligned}$$

□

Theorem 4.2 shows that, unsurprisingly, the logic of all hyperintensional models is not very interesting—it is classical propositional logic extended by the Duality Axiom.[10] It is noteworthy, however, that neither version of the compositionality principle affects the modal logic of the given class of models. This follows from the fact that, in $\mathcal{L}$, we cannot express that $O(F) = O(G)$ in a given model.[11]

One might argue that the hyperintensional logic based on one's interpretation of the modalities needs to add some interesting modal axioms to classical propositional logic. If the interpretation is epistemic, for instance, one might want to represent the idea that the hyperintensional epistemic attitudes represented by $\Box$ are 'closed under' certain simple inference rules. One way to achieve this is to take a set Λ of inference rules (pairs $\langle \Gamma, F \rangle$ where $\Gamma \subseteq Fm$) and require that, for all w, $N_C(w)$ be closed under the rules in Λ, i.e. if $\langle \Gamma, F \rangle \in \Lambda$ and $O(G) \in N_C(w)$ for all $G \in \Gamma$, then $O(F) \in N_C(w)$. For every such Λ with only finite Γs, the hyperintensional Λ-logic (the set of formulas valid in every hyperintensional model where N_C is closed under Λ) can be axiomatised by adding $\bigwedge_{G\in\Gamma} \Box G \to \Box F$ as axioms; such a logic, of course, will no longer be closed under substitution.[12]

Another way to get non-trivial hyperintensional modal logics is to assume some relations on C and extend the language $\mathcal{L}$ accordingly. We shall return to this in Section 6. Before that, we show that the approaches to hyperintensionality known from the literature can be formalised using hyperintensional models.

[10] Note that a related schema $\Box F \leftrightarrow \neg\Diamond\neg F$ is not valid. The reason is that we may have models where $O(F) \neq O(\neg\neg F)$.

[11] Note that it is possible to have $\llbracket \Box F \rrbracket = \llbracket \Box G \rrbracket$ while $O(F) \neq O(G)$.

[12] The completeness argument utilises a canonical model similar to the one presented in the proof of Theorem 4.2. The only difference is that the universe of the model is the set of maximal consistent theories with respect to the extended axiomatisation.

5 The three major hyperintensional frameworks as special cases

This section shows that special classes of hyperintensional models subsume state-based, syntactic and structuralist approaches to hyperintensionality.

5.1 State-based approaches

Let us define a *state-based model* as a hyperintensional model where $C \subseteq 2^S$ for some set $S \supseteq W$ and $I(c) = c \cap W$ for all $c \in C$. Validity and entailment are defined as for MS models.

It is not hard to see that the logic of all state-based models is identical to the logic of all hyperintensional models.

Proposition 5.1. *The set of $\mathcal{L}$-formulas valid in all state-based models is* **H**.

Proof. We have to show only that if a formula has a hyperintensional countermodel, then it has a state-based countermodel. Assume that F has a hyperintensional countermodel. By Theorem 4.2 and the Lindenbaum Lemma, there is a maximal consistent theory (relatively to the axiomatisation of **H**) not containing F. Now consider the following variant of the canonical model defined in the proof of Theorem 4.2:

- W is the set of all maximally consistent theories
- C is the set of all subsets of the power set of $Fm_{\mathcal{L}}$, that is, $2^{2^{Fm_{\mathcal{L}}}}$
- $O(F) = \{\Sigma \subseteq Fm_{\mathcal{L}} ; F \in \Sigma\}$
- $N_C(\Gamma) = \{O(F) ; \Box F \in \Gamma\}$
- $I(\{\Sigma_i ; i \in I\}) = \{\Gamma \in W ; \Gamma \in \{\Sigma_i ; i \in I\}\}$

(Hence, $I(O(F))$ is the set of $\Gamma \in W$ such that $F \in \Gamma$.) It is clear that this is a state-based hyperintensional model and that F is not valid in this model. □

Let us define a *Rantala model* as a tuple $M = \langle W, S, R, V\rangle$, where $W \subseteq S$, $R \subseteq S^2$ and $V : Fm \to 2^S$ such that $V(F) \cap W$ behaves like $[\![F]\!]$ in MS models and, in addition, for all $w \in W$,

$$w \in V(\Box F) \iff \forall s \in S(Rws \implies s \in V(F))$$
$$w \in V(\Diamond F) \iff \exists s \in S(Rws \;\&\; s \notin V(\neg F))$$

Let $[\![F]\!] = W \cap V(F)$. F is valid in a Rantala model iff $[\![F]\!] = W$. Rantala models are a generalisation of the models used by [41, 42], due to [52, 53].[13]

Every Rantala model corresponds to a state-based hyperintensional model. To see this, take any Rantala model M and define $M^* = \langle W, C, O, N_C, I\rangle$ such that $C = 2^S$, $O = V$, $N_C(w) = \{S' \subseteq S \,;\, \forall s : Rws \implies s \in S'\}$ and $I(c) = c \cap W$. It is clear that M^* is a state-based hyperintensional model and that $[\![F]\!]_M = [\![F]\!]_{M^*}$ for all F. In a sense, we may even say that M *is* a state-based hyperintensional model, that is, we may view M and M^* as the same model viewed from two perspectives.[14]

A *Kripke state-based* model is a state-based model such that $C = 2^S$ and every $N_C(w)$ is a principal filter on 2^S, i.e. for every w there is $w_C \in C$ such that $X \in N_C(w)$ iff $w_C \subseteq X$. Every Kripke state-based model is a Rantala model. To see this, define Rws iff $s \in w_C$ and $V = O$. We may stipulate that $\{x \,;\, Rsx\} = \emptyset$ for $s \in S \setminus W$.

[53] has shown that Rantala models simulate various kinds of models for hyperintensional epistemic modalities.[15] The observations of this section imply that, by the same token, this holds for hyperintensional models.

We note that one of the main applications of the models introduced by [41] is to provide semantics for modal logics with a restricted necessitation rule. The idea is to take a normal modal logic and restrict necessitation so that it applies only to formulas from a given set Ω:

$$\frac{F}{\Box F} \text{ if } F \in \Omega$$

We conclude this section by showing how such restrictions can be achieved using hyperintensional state-based models.

An Ω-*model* is a state-based model such that

- If $F \in \Omega$, then $O(F) \subseteq W \cup \{\omega\}$ and $\omega \in O(F)$, for some fixed $\omega \in (S \setminus W)$; and
- $W \cup \{\omega\} \in N_C(w)$ for all $w \in W$.

[13] Wansing defines these models over a language where only $\neg, \wedge, \Box$ are primitive, a detail we will pass over.

[14] Observe that M^* is not necessarily weakly compositional. The reason is that $V(F) \setminus W$ in Rantala models does not depend on the structure of F. This may be a reason for proponents of weak compositionality to refute Rantala models as an adequate representation of hyperintensional modalities.

[15] In particular, Wansing proves this for Levesque's logic of implicit and explicit belief [36], Fagin and Halpern's logic of awareness, logic of general awareness and logic of local reasoning [20] and van der Hoek and Meyer's logic of awareness and principles [51].

It is clear that if $F \in \Omega$ is valid in an Ω-model, i.e. $[\![F]\!] = W$, then $O(F) = W \cup \{\omega\}$. Hence, $O(F) \in N_C(w)$ for all w and $\Box F$ is valid in the model. Observe also that validity in Ω-models is preserved by a restricted version of (RE):

$$\frac{F \leftrightarrow G}{\Box F \leftrightarrow \Box G} \text{ if } F, G \in \Omega$$

If $F \leftrightarrow G$ is valid, then $[\![F]\!] = [\![G]\!]$, so if $F, G \in \Omega$, then $O(F) = O(G) = X \cup \{\omega\}$ for some $X \subseteq W$.

5.2 Syntactic approaches

A *purely syntactic* model is a hyperintensional model where $C = Fm$, i.e. $O(F)$ is a formula. The informal interpretation of such models may use the idea that $O(F)$ is a 'canonical' representation of the content of F (so contents are not represented directly). As an example, take $O(p) = q \wedge \neg r$, where p represents 'John is a bachelor', q represents 'John is a man' and r represents 'John is married'. Another interpretation could build on the idea that $O(F)$ is a ground, a reason or a justification for adopting the belief that F recognized by some fixed agent. On this reading, if $O(p) = q \wedge \neg r$, then the agent will believe p only if she is in some kind of a cognitive relation, represented by N_C, to $q \wedge \neg r$.

An *identity model* is a purely syntactic model where O is the identity relation on Fm. Identity models build on the idea that modalities express properties of formulas. Accounts of (mostly epistemic) natural language modalities along these lines were put forward by [39], [34] and [19], for example.

Proposition 5.2. *The set of $\mathcal{L}$-formulas valid in all purely syntactic models is* **H**.

The syntactic framework that has had perhaps the most impact are awareness models of [20]. The language $\mathcal{L}_2$ contains two box operators $\Box_1, \Box_2$ and two diamond operators $\Diamond_1$ and $\Diamond_2$; the set of $\mathcal{L}_2$-formulas is denoted as Fm_2. A *Fagin–Halpern* model is $M = \langle W, R, A, V \rangle$, where $R \subseteq W^2$, $A : W \to 2^{Fm_2}$ and $V : Fm_2 \to 2^W$ such that $V(F)$ satisfies the MS-conditions for Boolean F and

$$\begin{aligned}
w \in V(\Box_1 F) &\iff \forall v \in W(Rwv \implies v \in V(F)) \\
w \in V(\Diamond_1 F) &\iff \exists v \in W(Rwv \;\&\; v \notin V(F)) \\
w \in V(\Box_2 F) &\iff F \in A(w) \\
w \in V(\Diamond_2 F) &\iff \neg F \notin A(w)
\end{aligned}$$

The generalisation of this framework to MS models is straightforward—take N instead of R and assume

$$w \in V(\Box_1 F) \iff V(F) \in N(w)$$
$$w \in V(\Diamond_1 F) \iff V(\neg F) \notin N(w)$$

Taking inspiration from the interpretation put froward in [20], the operator $\Box_1$ represents a property of propositions ('implicit belief') and the operator $\Box_2$ represents a property of formulas ('awareness'); similarly for $\Diamond_1$ and $\Diamond_2$. The 'official' hyperintensional box modality of Fagin and Halpern is defined by $\Box F = \Box_1 F \wedge \Box_2 F$; we may define also $\Diamond F = \Diamond_1 F \wedge \Diamond_2 F$.

An *identity model for* $\mathcal{L}_2$ is $\mathfrak{M} = \langle W, C, O, N_C, N_I, I\rangle$ where $C = Fm_2$, $O = Id(Fm_2)$ and the new component $N_I : W \to 2^{2^W}$. We write 'I' instead of the subscript '1' and 'C' instead of '2'. The definitions of $[\![\Box_C F]\!]$ and $[\![\Diamond_C F]\!]$ are as before and

$$[\![\Box_I F]\!] = \{w \,;\, [\![F]\!] \in N_I(w)\}$$
$$[\![\Diamond_I F]\!] = \{w \,;\, [\![\neg F]\!] \notin N_I(w)\}$$

It is clear that every Fagin–Halpern model corresponds to an identity model for $\mathcal{L}_2$. Conversely, every identity model for $\mathcal{L}_2$ where N_I is a principal filter on 2^W corresponds to a Fagin–Halpern model. Note that every identity model is strongly compositional.

5.3 Structuralist approaches

The common feature of structuralist approaches to semantic content is the idea that contents are structured entities of some kind. It is clear that such approaches can be formalised using hyperintensional models—just take as C the set of structured entities of your favourite kind. We illustrate this point on simplified versions of two structuralist approaches to sentential content, namely, the neo-Russelian approach [49, 31, 32] and the procedural approach of Transparent Intensional Logic [50, 17].[16]

We first outline a simplified neo-Russelian approach to sentential content and show that it is embodied in a special class of hyperintensional models. Let IND be a set of 'individuals' and PRP a set of 'individual properties', where each $P \in$ PRP is assigned a fixed number $r(P) \geq 1$, called the

[16] A full discussion of these structuralist approaches is not provided mainly because of space limitations and proportionality considerations. Nevertheless, a more detailed account of how (fragments of) these structuralist accounts of sentential content can be formalised using hyperintensional models is an interesting topic of future research.

'arity' of P. Let $a, b, \ldots$ range over IND. In addition, we distinguish unary 'modal properties' *box* and *dia*. (Properties in PRP are individual properties while *box* and *dia* are properties of sentential contents.) To retain our level of generality, you can think of *box* as a generic property along the lines of 'distinguished'; if a more specific interpretation is desired, we can take 'is believed by John' for some specific John. Similar interpretations apply to *dia*. We do not need to provide a detailed account of properties and individuals; we only point out that they are generally thought of as distinct from individual and predicate intensions. Finally, let OPR be the set of Boolean truth functions (on $\{0, 1\}$), *neg*, *con* etc.

This machinery can be used to define a special 'structuralist' kind of hyperintensional model. *The set of neo-Russellian contents NRC* is the smallest set such that

1. $\left\langle P, \left\langle a_1, \ldots, a_{r(P)} \right\rangle \right\rangle \in \mathit{NRC}$ if $P \in$ PRP and $\left\{ a_1, \ldots, a_{r(P)} \right\} \subseteq$ IND (such members of *NRC* are called 'atomic');
2. $\langle \mathit{neg}, c \rangle \in \mathit{NRC}$ if $c \in \mathit{NRC}$; and $\langle \mathit{op}, \langle c_1, c_2 \rangle \rangle$ if $c \in \mathit{NRC}$ and $\mathit{op} \in$ OPR is binary;
3. $\langle \mathit{mod}, c \rangle \in \mathit{NRC}$ if $c \in \mathit{NRC}$ for $\mathit{mod} \in \{\mathit{box}, \mathit{dia}\}$.

A hyperintensional model is *neo-Russellian* iff $C = \mathit{NRC}$ and O satisfies the following conditions: 1. $O(p)$ is atomic; 2. $O(\neg F) = \langle \mathit{neg}, O(F) \rangle$, $O(F \wedge G) = \langle \mathit{con}, \langle O(F), O(G) \rangle \rangle$ etc.; 3. $O(\Box F) = \langle \mathit{box}, O(F) \rangle$ and $O(\Diamond F) = \langle \mathit{dia}, O(F) \rangle$.[17]

A similar strategy can be applied to a neo-Fregean structuralist account of sentential meaning along the lines of the procedural semantics of Transparent Intensional Logic [50, 17]; for some recent discussions of the framework, see [16, 29, 18]. In TIL, sentential meanings are identified with abstract structured procedures called *constructions* which, when 'executed', 'construct' possible-world propositions (constructions of propositions or propositional constructions). The framework is hyperintensional as one proposition may be constructed by multiple distinct constructions. Meanings of sentential connectives such as negation $\neg$ or conjunction $\wedge$ are constructions of Boolean functions. On the other hand, meanings of hyperintensional sentential modalities, such as 'John believes that', are constructions of functions from *sentential constructions* to propositions. The idea is that 'John believes that' expresses a property of sentential constructions (meanings of

[17] We need to be careful here as, for instance, if *mod* would be identical to the intension of the modal operator, i.e. to N_C, then the present definition of neo-Russelian models would be circular.

sentences), not of truth values or propositions. The proposition constructed by applying the meaning of 'John believes that' to the meaning of 'Jim is a bachelor', for instance, is the set of possible worlds in which John is in the cognitive relation of belief to the construction expressed by 'Jim is a bachelor'. For a more detailed discussion of TIL, we refer the reader to [17] and the articles [16, 29, 18].

Taking inspiration from TIL, one may outline the following representation of sentential meaning. Take an algebra $\mathbf{P} = \langle P, \{\circ^{\mathbf{P}} ; \circ \in Con_{\mathcal{L}}\}\rangle$ of '*procedures*' containing a subset AP of atomic procedures. Informally, procedures $\xi \in P$ are seen as abstract structured entities that 'produce' possible-world propositions; moreover, the operations $\circ^{\mathbf{P}}$ correspond to procedures yielding members of P when applied to (an appropriate number of) members of P.

A special case of this general framework, one that might be considered close in spirit to the approach of TIL, is as follows.[18] Take a countable set of unary function symbols $\{\alpha_i\}_{i\in\mathbb{N}} \cup \{Hyp\}$ and the set of (unary and binary) functor symbols $\left\{\neg^f, \wedge^f, \vee^f, \rightarrow^f, \leftrightarrow^f\right\}$. The sets of *function expressions* and *construction expressions* are defined by mutual induction as follows (x is a fixed variable):

- $\lambda x.\alpha_i(x)$ is a function expression for all α_i
- If $\lambda x.\varphi(x)$ and $\lambda x.\psi(x)$ are function expressions, then so are $\lambda x.(\neg^f\varphi)(x)$ and $\lambda x.(\varphi \circ^f \psi)(x)$ for all $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$
- If $\lambda x.\varphi(x)$ is a function expression, then ${}^0[\lambda x.\varphi(x)]$ is a construction expression
- If ξ is a construction expression, then $\lambda x.(Hyp\xi)(x)$ is a function expression

For example,

$${}^0[\lambda x.(Hyp{}^0[\lambda x.(\alpha_1 \wedge^f \alpha_2)(x)] \wedge^f \neg^f \alpha_2)(x)]$$

is a construction expression.

Informally, function expressions express, in usual lambda notation, functions from a fixed set of possible worlds to the truth values *true* and *false*. The function symbols in $\{\alpha_i\}_{i\in\mathbb{N}}$ give rise to a set of 'basic' functions of

[18] We reiterate that it is not our intention here to provide a faithful account of (a fragment of) TIL. The special case discussed in the text aims at providing only a hint of how TIL might be represented within our framework. Developing this hint to a complete account is a task that needs to be carried out in a separate article.

this kind and the functor symbols give rise to descriptions of 'Boolean combinations' of functions; for instance $\lambda x.(\neg^f \varphi)(x)$ is a function that assigns *true* to an argument iff $\lambda x.\varphi(x)$ assigns *false* to the argument (similarly for other 'Boolean' functors). Construction expressions are seen as representing abstract procedures that 'lead to' or 'construct' functions expressed by function expressions. Most informal explanations of TIL constructions can be used to make sense of these procedures (as our notation suggests), but we do not claim that the procedures *are* TIL constructions as defined in [17]. Finally, *Hyp* represents hyperintensional sentential modalities in the following sense. For each construction expression ${}^0[\lambda x.\varphi(x)]$, $\lambda x.(\mathit{Hyp}\,{}^0[\lambda x.\varphi(x)])(x)$ represents the function that assigns *true* to worlds in which the *abstract procedure* expressed by the construction expression ${}^0[\lambda x.\varphi(x)]$ has the property associated with the hyperintensional modality in question. For example, if we want to represent the modality expressed by 'John believes that', then

$$(\lambda x.(\mathit{Hyp}\,{}^0[\lambda x.\varphi(x)])(x))(w) = \mathit{true}$$

iff John believes that ${}^0[\lambda x.\varphi(x)]$ in w. In turn, ${}^0[\lambda x.\varphi(x)]$ can be seen as expressing the meaning of some natural-language sentence such as 'Jim is a bachelor'.

To simplify notation, we may omit the original superscripts, indicate the function/construction distinction by round/square brackets, write the variable x in a subscript and λx in a superscript after the corresponding closing round bracket. Hence, ${}^0[\lambda x.(\mathit{Hyp}\,{}^0[\lambda x.(\alpha_1 \wedge^f \alpha_2)(x)] \wedge^f \neg^f \alpha_2)(x)]$ simplifies to

(1) $$[(\mathit{Hyp}[(\alpha_1 \wedge \alpha_2)_x^{\lambda x}] \wedge \neg\alpha_2)_x^{\lambda x}]$$

It is clear that the set of construction expressions can be seen as an algebra of procedures in the sense defined above. In order to demonstrate this, we need to define the operations $\circ^{\mathbf{P}}$ for $\circ \in \mathit{Con}_{\mathcal{L}}$. We do this as follows:

$$\begin{aligned} \neg^{\mathbf{P}}[\varphi_x^{\lambda x}] &= [(\neg\varphi)_x^{\lambda x}] \\ [\varphi_x^{\lambda x}] \circ^{\mathbf{P}} [\psi_x^{\lambda x}] &= [(\varphi \circ \psi)_x^{\lambda x}] \qquad \text{for all binary connectives of } \mathcal{L} \\ \Box^{\mathbf{P}}[\varphi_x^{\lambda x}] &= [(\mathit{Hyp}[\varphi_x^{\lambda x}])_x^{\lambda x}] \\ \Diamond^{\mathbf{P}}[\varphi_x^{\lambda x}] &= [(\neg\mathit{Hyp}[(\neg\varphi)_x^{\lambda x}])_x^{\lambda x}] \end{aligned}$$

A *hyperintensional* **P**-*model* is a hyperintensional model where $C = P$, $O(P) \in AP$ and O is a homomorphism from the $\mathcal{L}$-formula algebra to $\mathbf{P}$.

Informally, formulas of $\mathcal{L}$ represent sentences of a (fragment of a) natural language, $O(F)$ is the procedure expressed by the sentence represented by F (we may say 'the procedure expressed by F' for the sake of simplicity) and $I(\xi)$ for $\xi \in P$ is the proposition produced by the procedure ξ. Hence, $I(O(F)) = [\![F]\!]$ is the proposition produced by the procedure expressed by F.

A *procedural model* is a **P**-model where **P** is the special case discussed above and $O(p_i) = [(\alpha_i)^{\lambda x}_x]$ for all $i \in \mathbb{N}$. Hence, in a procedural model, $O(\Box(p_1 \wedge p_2) \wedge \neg p_2)$ is (1).

Instead of proving that **H** is the logic of all procedural models and all **P**-models, we prove a general result about a class of hyperintensional models in the next subsection.

5.4 A note on fully distinguishing models

Let X be any set and let $o : Fm_{\mathcal{L}} \to X$ be an injective function. A *fully distinguishing model* is a structure $M = \langle W, X, o, N_X, I \rangle$ such that

- W is the set of maximally consistent theories with respect to some axiomatisation of classical propositional logic with the Duality Axiom
- $N_X(\Gamma) = \{o(F) \; ; \Box F \in \Gamma\}$
- $I(o(F)) = \{\Gamma \; ; F \in \Gamma\}$ and $I(x) = \emptyset$ for $x \notin Rng(o)$

Lemma 5.3. *Each fully distinguishing model is a strongly compositional hyperintensional model such that the set of formulas valid in the model is* **H**.

Proof. I is well-defined since $o(F) = o(G)$ only if $F = G$. The fact that $I(o(\cdot))$ satisfies the conditions required for $[\![\cdot]\!]$ in hyperintensional models follows from the fact that members of W are maximally consistent theories of CPL+DA. The model is strongly compositional since, again, $o(F) = o(G)$ only if $F = G$. The rest follows from Theorem 4.2 as it implies that $F \in \mathbf{H}$ iff $F \in \Gamma$ for all $\Gamma \in W$. □

Theorem 5.4. *If a class of hyperintensional models contains a fully distinguishing model, then the set of formulas valid in each model in the class is* **H**.

Proof. By Lemma 5.3, if $\mathfrak{M}$ is fully distinguishing and $F \notin \mathbf{H}$, then F is not valid in $\mathfrak{M}$. □

Corollary 5.5. **H** *is the set of formulas valid in all* **P***-models, all procedural models and all neo-Russelian models, but also in all state-based models and all purely syntactic models.*

This section has shown that approaches to modelling hyperintensional modalities based on quite distinct assumptions and intuitions are all special cases of our general framework. Moreover, we have shown that, even if rather different 'philosophically', these approaches are equivalent from a 'logical' point of view.

6 Modal logics with strong equivalence

We have seen that the classes of all hyperintensional models, all weakly compositional models and all strongly compositional models have the same class of valid $\mathcal{L}$-formulas, namely, **H**. This section introduces logics over an extended language $\mathcal{L}_{\equiv}$ and shows that the three classes of models yield distinct extensions of **H**. Some logics introduced in this section can be seen as *modal non-Fregean logics*, i.e. modal extensions of the 'sentential calculus with identity' studied by [9, 10].[19]

The language $\mathcal{L}_{\equiv}$ extends $\mathcal{L}$ with a binary connective '$\equiv$' and a unary operator '$\Box_U$'. Formulas $F \equiv G$ say that F and G have the same content. The operator $\Box_U$, interpreted as the universal modality, is included for technical reasons, see the proof of Theorem 6.1. Nevertheless, formulas $\Box_U F$ can be interpreted informally as stating that F is necessary in the sense that $[\![F]\!] = W$. The fact that $\Box$ is a hyperintensional modality corresponds to satisfiability of $\Box_U(F \leftrightarrow G) \wedge \neg(\Box F \leftrightarrow \Box G)$.

We require from now on that every hyperintensional model for $\mathcal{L}_{\equiv}$ satisfies the conditions

$$[\![F \equiv G]\!] = \begin{cases} W & \text{if } O(F) = O(G); \\ \emptyset & \text{otherwise.} \end{cases}$$

$$[\![\Box_U F]\!] = \begin{cases} W & \text{if } [\![F]\!] = W; \\ \emptyset & \text{otherwise.} \end{cases}$$

The set of $\mathcal{L}_{\equiv}$ formulas valid in each hyperintensional model is denoted as **HI**.

[19] Bloom and Suszko add to classical propositional logic a new equivalence connective '$\equiv$' representing *identity of content* as opposed to mere identity of truth value represented by classical equivalence. For details of their approach, the reader is referred to [9, 10].

(PC)	Propositional tautologies in $\mathcal{L}_\equiv$
(DA)	$\Diamond F \leftrightarrow \neg\Box\neg F$
(E0)	$F \equiv F$
(E1)	$F \equiv G \to G \equiv F$
(E2)	$(F \equiv G \wedge G \equiv H) \to F \equiv H$
(E3)	$F \equiv G \to ((F \leftrightarrow G) \wedge (\Box F \leftrightarrow \Box G))$
(U1)	$\Box_U F \to F$
(U2)	$\Box_U F \to \Box_U \Box_U F$
(U3)	$F \to \Box_U \neg \Box_U \neg F$
(U4)	$\Box_U (F \to G) \to (\Box_U F \to \Box_U G)$
(U5)	$\dfrac{F}{\Box_U F}$
(EU)	$\neg\Box_U (F \equiv G) \to \Box_U \neg (F \equiv G)$
(MP)	Modus Ponens

Figure 3: The proof system HIP.

Figure 3 lists the axioms and rules of the proof system HIP. Proofs in HIP are defined as usual.

Theorem 6.1. HIP *is a complete axiomatisation of* **HI**.

Proof. All the axioms of HIP are clearly valid. Take, for example, (E3). If $w \in [\![F \equiv G]\!]$, then $O(F) = O(G)$ by the definition of hyperintensional model. But then, of course, $[\![F]\!] = [\![G]\!]$ and $O(F) \in N_C(v)$ iff $O(G) \in N_C(v)$ for all $v \in W$.

Completeness is established by a canonical model construction. Assume that H is not provable. By the Lindenbaum Lemma, there is a maximal consistent theory Γ_0 containing $\neg H$. Let us define a relation $\sim$ on maximal consistent theories by $\Gamma_1 \sim \Gamma_2$ iff, for all F, $\Box_U F \in \Gamma_1$ only if $F \in \Gamma_2$. Axioms (U1) – (U3) ensure that $\sim$ is an equivalence relation.

Now define the canonical model as follows. The set W is the equivalence class, under $\sim$, of Γ_0. Define a relation $\equiv_W$ on Fm by

$$F \equiv_W G \iff \forall \Gamma \in W (F \equiv G \in \Gamma).$$

It is clear that $\equiv_W$ is an equivalence relation; denote as $[F]$ the equivalence class of F under $\equiv_W$. Now let $C = \{[F]\ ;\ F \in Fm\}$ and $O : F \mapsto [F]$. The intension function I is defined by $I : [F] \mapsto \{\Gamma \in W \,;\, F \in \Gamma\}$. In addition, let $N_C : \Gamma \mapsto \{[F] \,;\, \Box F \in \Gamma\}$. Note that, thanks to (E3), both I and N_C are well-defined.

It remains to be shown that the canonical model is a hyperintensional model. On Boolean inputs, the canonical $[\![\cdot]\!]$ behaves as it should thanks to (PC). Next, $\Box F \in \Gamma$ iff $[F] \in N_C(\Gamma)$ by the definition of N_C. Consequently, $[\![\Box F]\!] = \{\Gamma \,;\, O(F) \in N_C(\Gamma)\}$. $[\![\Diamond F]\!] = \{\Gamma \,;\, O(\neg F) \notin N_C(\Gamma)\}$ holds thanks to (DA). Next, assume $O(F) = O(G)$, i.e. $[F] = [G]$. This means that $F \equiv_W G$, so $F \equiv G \in \Gamma$ for all $\Gamma \in W$. This means that $[\![F \equiv G]\!] = W$. If $O(F) \neq O(G)$, then $F \equiv G \notin \Delta$ for some $\Delta \in W$. This means that $\neg\Box_U(F \equiv G) \in \Gamma_0$. By (EU), $\neg(F \equiv G) \in \Gamma$ for all $\Gamma \in W$, so $[\![F \equiv G]\!] = \emptyset$. Finally, assume that $\neg F \in \Delta$ for some $\Delta \in W$. It follows that $\neg\Box_U F \in \Gamma_0$. Since $\neg\Box_U F \to \Box_U\neg\Box_U F$ is a theorem,[20] $\Box_U\neg\Box_U F \in \Gamma_0$. Hence, $\neg\Box_U F \in \Gamma$ for all $\Gamma \in W$, i.e. $[\![\Box_U F]\!] = \emptyset$. To show that $[\![F]\!] = W$ implies $[\![\Box_U F]\!] = W$, assume that $\neg\Box_U F \in \Gamma$. Note that

$$D = \{G \,;\, \Box_U G \in \Gamma_0\} \cup \{\neg F\}$$

is consistent. If not, then $G_1 \wedge \ldots \wedge G_n \to F$ is provable and so $\Box_U G_1 \wedge \ldots \wedge \Box_U G_n \to \Box_U F$ is provable. But then, $\Box_U F \in \Gamma_0$ and, by (U2), $\Box_U\Box_U F \in \Gamma_0$. But this would mean that $\Box_U F \in \Gamma$, which contradicts our assumption. So, by the Lindenbaum Lemma, there is a maximally consistent $\Delta \supseteq D$ in W. Consequently, $[\![F]\!] \neq W$.

Hence, the canonical model is a hyperintensional model such that $[\![H]\!] \neq W$ for our unprovable H. □

For each model $\mathfrak{M}$, define an equivalence relation on formulas by $F \equiv_{\mathfrak{M}} G$ iff $F \equiv G$ is valid in $\mathfrak{M}$. We note that there are models $\mathfrak{M}$ such that $\equiv_{\mathfrak{M}}$ is not a congruence on the set of formulas. This makes our framework more general than a straightforward modal extension of the non-Fregean version of classical propositional logic by [9, 10]; in the latter system, a related relation on formulas is a congruence for each model. To obtain 'proper' modal non-Fregean logic, we have to focus on a narrower class of models, namely, weakly compositional models.

Let $H(F/G)$ denote the result of replacing an occurrence of F in H by an occurrence of G. If H does not contain any occurrences of F, then $H(F/G)$ is H.

[20] Since $\dfrac{F \to G}{\Box_U F \to \Box_U G}$ is an admissible rule, as can be shown using (U4) and (U5).

Proposition 6.2. $\mathfrak{M}$ *is weakly compositional iff every formula of the form*

(W) $\quad F \equiv G \to H \equiv H(F/G)$

is valid in $\mathfrak{M}$*.*

Proof. Assume that $\mathfrak{M}$ is weakly compositional. If $w \in [\![F \equiv G]\!]$, then $O(F) = O(G)$. The claim follows by repeated application of (wCP).

If $\mathfrak{M}$ is not weakly compositional, then there is a formula H with a direct subformula F such that $O(F) = O(G)$ for some G, but $O(H) \neq O(H(F/G))$. But then $[\![F \equiv G]\!] = W$ and $[\![H \equiv H(F/G)]\!] = \emptyset$. □

Let us denote the set of formulas valid in all weakly compositional models as **HIW**.

Theorem 6.3. $F \in$ **HIW** *iff* F *is a theorem of the proof system* WHIP *that results from adding (W) as an axiom schema to* HIP*.*

Proof. This can be shown using the canonical model construction from the proof of Theorem 6.1 (but this time with (W) as an axiom). The canonical model is weakly compositional by Proposition 6.2. □

Proposition 6.4. $\mathfrak{M}$ *is strongly compositional iff every formula of the form*

(S) $\quad F \equiv G \leftrightarrow H \equiv H(F/G)$

is valid in $\mathfrak{M}$*.*

Proof. Assume that $\mathfrak{M}$ is strongly compositional. If $w \in [\![F \equiv G]\!]$, then $O(F) = O(G)$. The claim follows by repeated application of (sCP). If $w \in [\![H \equiv H(F/G)]\!]$, then $O(H) = O(H(F/G))$. The claim again follows by repeated application of (sCP).

If $\mathfrak{M}$ is not strongly compositional, then either it is not weakly compositional or there is a formula H with a direct subformula F such that $O(H) = O(H(F/G))$ for some G, but $O(F) \neq O(G)$. In the former case, (W) is invalid and so is (S). In the latter case, $[\![H \equiv H(F/G)]\!] = W$ and $[\![F \equiv G]\!] = \emptyset$. □

Let us denote the set of formulas valid in each strongly compositional model as **HIS**.

Theorem 6.5. $F \in$ **HIS** *iff* F *is a theorem of the proof system* SHIP *that results from adding (S) as an axiom schema to* HIP*.*

Proof. Similar to the proof of Theorem 6.3. $\Box$

Proposition 6.6. HI $\subset$ HIW $\subset$ HIS.

Proof. The non-strict inclusions are clearly true (each strongly compositional model is weakly compositional). To prove **HI** $\subset$ **HIW**, it is sufficient to show that an instance of (W) is not valid in some hyperintensional model. Take a state-based hyperintensional model where $W = \{w\}$ and $S = \{w, s_1, s_2\}$. Define $O[At] = \{\{w, s_1\}\}$, $O(r \wedge p) = \{w, s_1\}$ and $O(r \wedge q) = \{w, s_2\}$ (s_1 is an 'impossible world' where $r \wedge q$ holds but q does not). Let O be defined on the rest of the arguments in an arbitrary way that is consistent with the definition of a state-based hyperintensional model (obviously there is at least one such way). Now $p \equiv q$ is true in w, but $(r \wedge p) \equiv (r \wedge q)$ is not.

To prove **HIW** $\subset$ **HIS**, it is sufficient to prove that an instance of the right-to-left implication of (S) is not valid in some weakly compositional hyperintensional model. Consider a MS model $\mathcal{M} = \langle W, N, [\![\cdot]\!]\rangle$ with the following property: $[\![p]\!] \neq [\![q]\!]$, but $[\![p]\!] \in N(w)$ iff $[\![q]\!] \in N(w)$ for all $w \in W$. Define a structure $\mathcal{M}^* = \langle W, \mathcal{P}(W), [\![\cdot]\!], N, id\rangle$ where id is the identity function on $\mathcal{P}(W)$. It can be easily checked that $\mathcal{M}^*$ is a weakly compositional hyperintensional model (where $F \equiv G$ is equivalent to $\Box_U(F \leftrightarrow G)$) such that $\Box p \equiv \Box q$ is valid in $\mathcal{M}^*$ and $p \equiv q$ is not valid. $\Box$

Let P be any of the proof systems discussed in this section. Derivations of F from a set of assumptions Γ in P are defined in the standard way. As expected, F is derivable from Γ in P iff it is derivable from a finite $\{G_1, \ldots, G_n\} \subseteq \Gamma$ ('fin-derivable' from Γ) iff $(G_1 \wedge \ldots \wedge G_n) \to F$ is a theorem of P. We say that Γ is P-consistent iff $p \wedge \neg p$ is not derivable from Γ in P. We say that Γ is satisfiable in a class of models iff there is a model $\mathfrak{M}$ in the class such that $\bigcap_{G \in \Gamma} [\![G]\!]_{\mathfrak{M}} \neq \emptyset$. It is easily seen that the proofs of Theorems 6.1, 6.3 and 6.5 establish the following fact.

Theorem 6.7. *Γ is HIP-consistent (WHIP-consistent, SHIP-consistent) iff it is satisfiable in the class of all hyperintensional models (weakly compositional models, strongly compositional models).*

Proof. This is a version of the standard modal completeness argument. To establish the left-to-right implication, it is sufficient to note that, in each case, the canonical model belongs to the 'right' class of models. For the right-to-left implication, use the fact that derivability is equivalent to fin-derivability and then apply Theorems 6.1, 6.3 and 6.5. $\Box$

We say that F follows from Γ in **HI**, $\Gamma \vDash_{\mathbf{HI}} F$, iff $\Gamma \cup \{\neg F\}$ is not satisfiable in any hyperintensional model; and similarly for **HIW** and **HIS** on the one hand and the class of weakly compositional and strongly compositional models, respectively, on the other hand. Theorem 6.7 yields a form of *strong completeness* of our proof systems: F follows from Γ in a class of models iff it is derivable from Γ in the corresponding proof system.

Proposition 6.8. *$F \equiv G \in$ **HIS** only if $F = G$.*

Proof. We will show that $F \equiv G$ is a theorem of SHIP only if $F = G$. The rest follows from Theorem 6.1. Let us define a function $e : Fm \to \{0, 1\}$ as follows:

- $e(p) = 1$ for all propositional atoms p
- $e(\neg F) = 1$ iff $e(F) = 0$
- $e(F \wedge G) = 1$ iff $e(F) = e(G) = 1$
- $e(F \vee G) = 0$ iff $e(F) = e(G) = 0$
- $e(F \to G) = 0$ iff $e(F) = 1$ and $e(G) = 0$
- $e(F \leftrightarrow G) = 1$ iff $e(F) = e(G)$
- $e(\Box F) = 1$ for all F
- $e(\Diamond F) = 0$ for all F
- $e(F \equiv G) = 1$ iff $F = G$
- $e(\Box_U F) = 1$ iff $e(F) = 1$

It is easy to check that the set $\{F ; e(F) = 1\}$ contains all theorems of SHIP, but no formula $F \equiv G$ where $F \neq G$. □

Proposition 6.8 may be seen as a form of a triviality result; it entails that no 'interesting' formulas of the form $F \equiv G$ are valid in either class of models studied in this article. This is indeed not very surprising given the fact that we do not rule out models where O is injective. On the other hand our logics can be used to study arbitrary sets of assumptions. For instance, the question whether some F is entailed by a set of 'meaning postulates' (formulas of the form $F \equiv G$) together with additional 'factual assumptions' (formulas of other forms) boils down questions of satisfiability in classes of hyperintensional models and, via Theorem 6.7, to the question of existence of derivations in the corresponding proof systems.

We conclude by considering a yet more general version of our semantics. An *equivalence model* is $\mathfrak{M} = \langle W, C, \approx, O, N_C, I \rangle$ where $\langle W, C, O, N_C, I \rangle$ is a hyperintensional model and $\approx$ is an equivalence relation on C. Moreover, we require that

- $[\![F \equiv G]\!] = \begin{cases} W & \text{if } O(F) \approx O(G); \\ \emptyset & \text{otherwise.} \end{cases}$
- $O(F) \approx O(G)$ only if $[\![F]\!] = [\![G]\!]$
- every $N_C(w)$ is closed under $\approx$ (i.e. if $c \approx c'$, then $c \in N_C(w)$ iff $c' \in N_C(w)$).

It is clear that equivalence models are a generalisation of hyperintensional models. Informally, we may see $\approx$ as a relation of 'strong equivalence' on contents; equivalent contents determine the same intension and have the same modal properties expressed by $\Box$.[21] Nevertheless, such a generalization does not yield a different logic.

Theorem 6.9. *The set of formulas valid in all equivalence models is* **HI**.

Proof. It is sufficient to show that for every equivalence model there is a hyperintensional model that validates precisely the same formulas. Thus take an equivalence model $\mathfrak{M}$ and define $\mathfrak{M}'$ by stipulating $C' = \{[c]_\approx \,;\, c \in C\}$ ($[c]_\approx$ is the equivalence class of c under $\approx$); $O'(F) = [O(F)]_\approx$; $N'_C(w) = \{[c]_\approx \,;\, c \in N_C(w)\}$ and $I'([c]_\approx) = I(c)$. Note that N'_C and I' are well-defined since they do not depend on the choice of a representative of $[c]_\approx$. Now $w \in [\![F]\!]$ iff $w \in I(O(F))$ iff $w \in I'([O(F)]_\approx)$ iff $w \in I'(O'(F))$. Hence, $[\![F]\!] = W$ iff $[\![F]\!]' = W$. $\Box$

Nevertheless, $\approx$ adds some complexity in that Propositions 6.2 and 6.4 are not valid if $\mathfrak{M}$ is assumed to be an equivalence model (then $\approx$ expresses equivalence of content, not identity of content; but the latter is involved in the compositionality principles). To avoid these issues, one would need to extend $\mathcal{L}_\equiv$ by another binary operator that would differentiate between $\approx$ and $=$ on C.

7 Conclusion

The aim of this article was to introduce a general semantic framework for hyperintensional modal logics that subsumes the major approaches to hyperintensional modality known from the literature as special cases. To this

[21] As an example of such a relation, consider various conversion relations between the denotations of λ-terms in Transparent Intensional Logic [17]. The relation of procedural isomorphism on the set of constructions was defined to express a similar notion, see [18]. Another example, if $\Box$ is assumed to correspond to epistemic attitudes of an agent, is intensional equivalence 'recognized' by the agent.

end, we generalised the Montague–Scott semantics for modal logics. The generalisation builds on an explicit representation of fine-grained sentential contents (to which modalities pertain) without assuming any specific features of these contents. In fact, arbitrary objects may play the role of contents in our semantics. This approach enables to develop hyperintensional modal logic without taking sides in the philosophical debates about the nature of sentential content. As such, the framework also provides a common logical ground for the proponents of rival semantic theories and, potentially, is suitable for introducing logics motivated by specific semantic approaches that are comprehensible to proponents of rival theories of content.

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