Micro-Macro Continuum Approach to Porous Media

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Constituent (or **phase**) — a part of the porous medium that is separated from other such parts by sharp interfaces (e.g., a solid, water, oil, air).

Micro- and macro-scales of a porous medium

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- Impractical for modelling due to possibly complex configurations of particular constituents.

Macro-scale

- The inner constitution of the porous medium is ignored. The variables and quantities are defined at *every point* in the porous medium domain.

Two major approaches for passing from the micro- to the macro-scale:

- the **volume fraction concept**,
- **homogenisation** (requires a *periodic* structure).
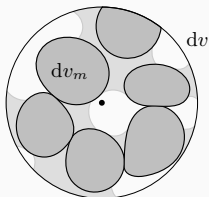
Representative elementary volumes

In the *volume fraction concept*, each point of a control space of a porous medium is considered to be a centroid of a so-called **representative elementary volume** or **average volume element** dv:

- small enough ("infinitesimal"),
- composed representatively of microscopic volume elements dv_m of the constituents.
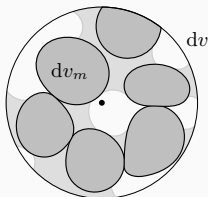
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The **partial volume element** dv^π
$\equiv$ the volume of a constituent π within dv.

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$$S_f \equiv \frac{\mathrm{d}v^f}{\mathrm{d}v - \mathrm{d}v^s} = \frac{\eta_f}{\phi} \text{ — the \textbf{saturation} of fluid } f.$$

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$$\frac{\int_{\mathrm{d}v^\pi} \rho \xi \, \mathrm{d}v_m}{\int_{\mathrm{d}v^\pi} \rho \, \mathrm{d}v_m}, \qquad \rho \text{ — the microscopic mass density,}$$

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$$\dots,$$

area averages, *ensemble* averages over the number of particles,...

Associating the average volumes $\mathrm{d}v$ with each point of the control space in the previous definitions, one obtains macroscopic quantities which are defined in the *total* control space, and which can be interpreted as *local statistical averages* of values at the underlying microscale.

Overlapping partial continua

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Within this context, the porous material is theoretically substituted by a model where each constituent is "smeared" over the control space, and it occupies the total volume *simultaneously* with the other constituents. One then speaks of **overlapping partial continua**.

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- In **averaging theories**, averaging is used for introducing the macroscopic description (typically by equations) from a *microscopic* one.

Example: Darcy's law

- obtained empirically at the macroscopic level first,

- can also be derived from the microscopic momentum balance equation for a fluid in the void space.

No matter how the macroscopic description is obtained, it should be chosen such that it is *relevant* for all the physical phenomena involved in the intended applications!

Three scales of a porous medium can be distinguished in the *continuum-fracture* approach to EDZ (excavation damage zones):

- **micro**-**scale** — continuum location of individual constituents,

- **macro**-**scale** — overlapping partial continua over the total volume (only *main* discrete fractures excluded in case of potential preferential pathways).

Multi-scale approach to EDZ

Three scales of a porous medium can be distinguished in the *continuum-fracture* approach to EDZ (excavation damage zones):

- **micro-scale** — continuum location of individual constituents,
- **meso-scale** — overlapping partial continua + a fine discrete fracture network,
- **macro-scale** — overlapping partial continua over the total volume (only *main* discrete fractures excluded in case of potential preferential pathways).

Eulerian versus Lagrangian Approach to Poroelasticity under Small Strains

T. Ligurský

The model

- non-stationary isothermal saturated water flow in a deformable porous medium
- isotropic elastic skeleton
- negligible inertial effects
- the assumption of small perturbations
- compressive-positive pore pressures, tensile-positive stresses
- [Cou04], [LS98]

Continuum approach, continuity assumption

The porous medium is treated as the superimposition of continua of its constituents where each point is simultaneously occupied by points of all constituents. It is assumed that there exists a macroscopic scale at which the inner constitution of the medium can be ignored but which is relevant for all the physical phenomena involved in the intended applications. The physics is supposed to vary continuously at this scale.

The skeleton deformation

When subjected to external forces and to variations in pressure of the water, the skeleton deforms. We introduce:

$\boldsymbol{X}$ — the position vector of a skeleton particle in an initial configuration, i.e., at time $t = 0$

$\boldsymbol{x}$ — the position vector in the current (deformed) configuration:

$$\boldsymbol{x} = \boldsymbol{X} + \boldsymbol{u} \tag{1}$$

$\boldsymbol{u}$ — the displacement vector

and

$\boldsymbol{F} \equiv \boldsymbol{I} + \boldsymbol{\nabla u}$ — the deformation gradient

$\boldsymbol{E} \equiv \dfrac{1}{2}(\boldsymbol{F}^\top \boldsymbol{F} - \boldsymbol{I}) = \dfrac{1}{2}\big(\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^\top + (\boldsymbol{\nabla u})^\top \boldsymbol{\nabla u}\big)$ — the Green-Lagrange strain tensor

$J \equiv \det \boldsymbol{F}$ — the Jacobian of the deformation

Any initial infinitesimal volume $\mathrm{d}V_0$ transforms into a current infinitesimal volume $\mathrm{d}V_t$ through the relation:

$$\mathrm{d}V_t = J \, \mathrm{d}V_0 \tag{2}$$

Eulerian and Lagrangian porosities

n — the Eulerian porosity: $n \, \mathrm{d}V_t$ is the volume of the void space in the current volume $\mathrm{d}V_t$

ϕ — the Lagrangian porosity, referring the *current porous* volume to the *initial* volume $\mathrm{d}V_0$ according to:

$$\phi\,\mathrm{d}V_0 = n\,\mathrm{d}V_t \stackrel{(2)}{=} nJ\,\mathrm{d}V_0$$
$$\phi = Jn \tag{3}$$

Balance equations

Let Ω_0 be an initial total volume which deforms into a current total volume Ω_t, and let $V_0 \subset \Omega_0$ be an arbitrary initial material volume which transforms into a current volume $V_t \subset \Omega_t$.

Water mass balance
The *Eulerian* form in the *current* configuration:

$$\frac{\mathrm{D}_w}{\mathrm{D}t} \int_{V_t} \rho_w n \, \mathrm{d}V_t = 0$$

$\dfrac{\mathrm{D}_w}{\mathrm{D}t}$ — the total (or material or particle) time derivative with respect to water

ρ_w — the water mass density

or equivalently:

$$\int_{V_t} \left(\frac{\partial(\rho_w n)}{\partial t} + \mathrm{div}(\rho_w n \boldsymbol{v}_w) \right) \mathrm{d}V_t = 0 \qquad \boldsymbol{v}_w \text{ — the water velocity}$$

$$\int_{V_t} \left(\frac{\mathrm{D}_s(\rho_w n)}{\mathrm{D}t} + \rho_w n \, \mathrm{div}\,\boldsymbol{v}_s + \mathrm{div}(\rho_w \boldsymbol{q}_{rw}) \right) \mathrm{d}V_t = 0$$

$\boldsymbol{v}_s$ — the solid velocity

$\dfrac{\mathrm{D}_s}{\mathrm{D}t}$ — the total time derivative with respect to the solid

$\left(\text{for a field } \mathcal{G}\text{: } \dfrac{\mathrm{D}_s\mathcal{G}}{\mathrm{D}t} = \dfrac{\partial\mathcal{G}}{\partial t} + \boldsymbol{v}_s \cdot \nabla\mathcal{G} \right)$

$\boldsymbol{q}_{rw} \equiv n(\boldsymbol{v}_w - \boldsymbol{v}_s)$ — the water specific discharge relative to the solid (also called Darcy velocity or filtration vector)

and in the local form:

$$\boxed{\frac{\mathrm{D}_s(\rho_w n)}{\mathrm{D}t} + \rho_w n \, \mathrm{div}\,\boldsymbol{v}_s + \mathrm{div}(\rho_w \boldsymbol{q}_{rw}) = 0 \quad \text{in } \Omega_t} \tag{4}$$

Use of the transformations

$$\int_{V_t} \left(\frac{\mathrm{D}_s(\rho_w n)}{\mathrm{D}t} + \rho_w n \, \mathrm{div}\,\boldsymbol{v}_s \right) \mathrm{d}V_t = \int_{V_0} \left(\frac{\partial(\rho_w n)}{\partial t} + \mathrm{div}(\rho_w n \boldsymbol{v}_s) \right) J \, \mathrm{d}V_0$$

$$= \int_{V_0} \frac{\mathrm{d}(\rho_w n J)}{\mathrm{d}t} \, \mathrm{d}V_0 \stackrel{(3)}{=} \int_{V_0} \frac{\mathrm{d}(\rho_w \phi)}{\mathrm{d}t} \, \mathrm{d}V_0$$

$$\int_{V_t} \mathrm{div}(\rho_w \boldsymbol{q}_{rw}) \, \mathrm{d}V_t = \int_{V_0} \mathrm{div}\,\boldsymbol{M} \, \mathrm{d}V_0$$

$\boldsymbol{M} \equiv J\boldsymbol{F}^{-1}(\rho_w \boldsymbol{q}_{rw})$ — the Lagrangian relative flow vector of water mass

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provides the *Lagrangian* continuity equations in the *initial* (reference) configuration:

$$\int_{V_0} \left(\frac{\mathrm{d}(\rho_w \phi)}{\mathrm{d}t} + \operatorname{div} \boldsymbol{M} \right) \mathrm{d}V_0 = 0$$

$$\boxed{\frac{\mathrm{d}(\rho_w \phi)}{\mathrm{d}t} + \operatorname{div} \boldsymbol{M} = 0 \quad \text{in } \Omega_0} \tag{5}$$

Solid mass balance

The *Eulerian* form:

$$\frac{\mathrm{D}_s}{\mathrm{D}t} \int_{V_t} \rho_s (1 - n) \, \mathrm{d}V_t = 0 \tag{6}$$

$$\rho_s \text{ — the solid mass density}$$

or equivalently

$$\int_{V_t} \left(\frac{\mathrm{D}_s(\rho_s(1-n))}{\mathrm{D}t} + \rho_s(1-n) \operatorname{div} \boldsymbol{v}_s \right) \mathrm{d}V_t = 0$$

$$\boxed{\frac{\mathrm{D}_s(\rho_s(1-n))}{\mathrm{D}t} + \rho_s(1-n) \operatorname{div} \boldsymbol{v}_s = 0 \quad \text{in } \Omega_t} \tag{7}$$

Integration of (6) gives the *Lagrangian* alternative:

$$\frac{\mathrm{D}_s}{\mathrm{D}t} \int_{V_t} \rho_s(1-n) \, \mathrm{d}V_t = \frac{\mathrm{d}}{\mathrm{d}t} \int_{V_0} \rho_s(1-n) J \, \mathrm{d}V_0 = 0$$

$$\int_{V_0} \rho_s(1-n) J \, \mathrm{d}V_0 = \int_{V_0} \rho_{s0}(1 - n_0) \, \mathrm{d}V_0 = \int_{V_0} \rho_{s0}(1 - \phi_0) \, \mathrm{d}V_0 \tag{8}$$

$$\rho_{s0} \text{ — the initial solid mass density} \qquad n_0 = \phi_0 \text{ — the initial porosity}$$

$$\boxed{\rho_s(1-n) J = \rho_{s0}(1 - \phi_0) \quad \text{in } \Omega_0} \tag{9}$$

Equilibrium equation

The *Eulerian* form:

$$\boxed{\operatorname{\mathbf{div}} \boldsymbol{\sigma} + (\rho_s(1-n) + \rho_w n)\boldsymbol{f} = \boldsymbol{0} \quad \text{in } \Omega_t} \tag{10}$$

$$\boldsymbol{\sigma} \text{ — the Cauchy stress tensor} \qquad \boldsymbol{f} \text{ — a body force density}$$

Using the transport formulae

$$\int_{V_t} \operatorname{\mathbf{div}} \boldsymbol{\sigma} \, \mathrm{d}V_t = \int_{V_0} \operatorname{\mathbf{div}}(\boldsymbol{F}\boldsymbol{\Pi}) \, \mathrm{d}V_0 \qquad \boldsymbol{\Pi} \equiv J \boldsymbol{F}^{-1} \boldsymbol{\sigma} \boldsymbol{F}^{-\top} \text{ — the Piola-Kirchhoff stress tensor}$$

$$\int_{V_t} \rho_s(1-n)\boldsymbol{f} \, \mathrm{d}V_t = \int_{V_0} \rho_s(1-n)\boldsymbol{f} J \, \mathrm{d}V_0 \overset{(9)}{=} \int_{V_0} \rho_{s0}(1-\phi_0)\boldsymbol{f} \, \mathrm{d}V_0$$

$$\int_{V_t} \rho_w n \boldsymbol{f} \, \mathrm{d}V_t = \int_{V_0} \rho_w n \boldsymbol{f} J \, \mathrm{d}V_0 \overset{(3)}{=} \int_{V_0} \rho_w \phi \boldsymbol{f} \, \mathrm{d}V_0$$

one derives the *Lagrangian* form:

$$\boxed{\operatorname{\mathbf{div}}(\boldsymbol{F}\boldsymbol{\Pi}) + (\rho_{s0}(1-\phi_0) + \rho_w \phi)\boldsymbol{f} = \boldsymbol{0} \quad \text{in } \Omega_0} \tag{11}$$

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Constitutive relationships

Water density

$$\frac{\mathrm{d}\rho_w}{\rho_w} = \frac{\mathrm{d}p_w}{K_w} \tag{12}$$

p_w — the water pressure K_w — the water bulk modulus

Considering K_w constant (over some range of pressures), one can integrate (12) into the form:

$$\rho_w = \rho_{w0}e^{(p_w - p_{w0})/K_w} \tag{13}$$

ρ_{w0}, p_{w0} — initial values of the water density and pressure

Darcy's law

$$\boldsymbol{q}_{rw} = \frac{\boldsymbol{k}}{\mu_w}(-\nabla p_w + \rho_w \boldsymbol{f}) \tag{14}$$

$\boldsymbol{k}$ — the (intrinsic) permeability tensor of the porous medium
μ_w — the dynamic viscosity of water

Assumption 1 (Small transformations). The displacement gradient is small:

$$\|\boldsymbol{\nabla u}\| \ll 1$$

− Under this assumption:

$$\boldsymbol{E} \approx \boldsymbol{\varepsilon} \equiv \frac{1}{2}\big(\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^\top\big) \text{ — the linear strain tensor}$$

$$J \approx 1 + \operatorname{div}\boldsymbol{u} = 1 + \varepsilon_v \tag{15}$$

$$\varepsilon_v \equiv \operatorname{tr}\boldsymbol{\varepsilon} = \operatorname{div}\boldsymbol{u} \text{ — the volumetric strain}$$

$$\boldsymbol{\Pi} \approx \boldsymbol{\sigma}$$

Stress tensor

$$\mathrm{d}\boldsymbol{\sigma} + \alpha\mathrm{d}p_w\boldsymbol{I} = \boldsymbol{D}\mathrm{d}\boldsymbol{\varepsilon} \tag{16}$$

α — Biot's coefficient $\boldsymbol{D}$ — a tangent elastic stiffness tensor of the solid skeleton

Solid mass content
− With regard to the solid mass balance equations, a constitutive equation either for the porosity or for the solid density is needed

Porosity

$$\mathrm{d}\phi = \alpha\mathrm{d}\varepsilon_v + \frac{\mathrm{d}p_w}{N} \tag{17}$$

N — Biot's modulus

Considering α and N constant (over some range of strains and pressures), one can integrate (17) into the form:

$$\phi = \phi_0 + \alpha\varepsilon_v + \frac{p_w - p_{w0}}{N} \tag{18}$$

Moreover, assuming that the skeleton bulk modulus K is constant as well and that the solid grains (matrix) are linearly elastic, one can derive:

$$\alpha = 1 - \frac{K}{K_s} \qquad \frac{1}{N} = \frac{\alpha - \phi_0}{K_s} \tag{19}$$

K_s — the bulk modulus of the solid grains

Solid density

By assuming $\rho_s = \rho_s(p_w, \operatorname{tr} \boldsymbol{\sigma}')$ for Terzaghi's effective stress $\boldsymbol{\sigma}' \equiv \boldsymbol{\sigma} + p_w \boldsymbol{I}$ and using $\alpha = 1 - K/K_s$ from (19), one can obtain

$$
\begin{aligned}
\frac{1}{\rho_s} \frac{\mathrm{D}_s \rho_s}{\mathrm{D}t} &= \frac{1}{\rho_s} \frac{\partial \rho_s}{\partial p_w} \frac{\mathrm{D}_s p_w}{\mathrm{D}t} + \frac{1}{\rho_s} \frac{\partial \rho_s}{\partial (\operatorname{tr} \boldsymbol{\sigma}')} \frac{\mathrm{D}_s (\operatorname{tr} \boldsymbol{\sigma}')}{\mathrm{D}t} \\
&= \frac{1}{1-n} \left(\frac{\alpha - n}{K_s} \frac{\mathrm{D}_s p_w}{\mathrm{D}t} - (1-\alpha) \operatorname{div} \boldsymbol{v}_s \right)
\end{aligned}
\tag{20}
$$

Complete equations

The small perturbation assumption

$=$ Assumptions 1–4.

Assumption 2 (Small displacements). The displacements of the skeleton particles are small:

$$\|\boldsymbol{u}/L\| \ll 1$$

L — the length scaling the dimensions of the porous structure

– This assumption allows us to merge the initial configuration and the current one:

$$\boldsymbol{x} \approx \boldsymbol{X} \qquad \Omega_t \approx \Omega_0 =: \Omega$$

Assumptions 1 and 2 constitute together the **small strain assumption**.

Assumption 3. Small variations of the porosity:

$$\left| \frac{\phi - \phi_0}{\phi_0} \right| \ll 1 \quad \text{or} \quad \left| \frac{n - n_0}{n_0} \right| \ll 1$$

Assumption 4. Small variations of the water mass density:

$$\left| \frac{\rho_w - \rho_{w0}}{\rho_{w0}} \right| \ll 1$$

– Assumptions 3 and 4 allow us to take:

$$\phi \approx \phi_0 \qquad n \approx n_0 \qquad \rho_w \approx \rho_{w0}$$

and together with Assumption 1:

$$\rho_s \overset{(8)}{=} \frac{\rho_{s0}(1 - n_0)}{(1 - n)J} \overset{(15)}{\approx} \frac{\rho_{s0}(1 - n_0)}{(1 - n_0)(1 + \varepsilon_v)} \approx \rho_{s0}$$

Lagrangian approach [Cou04]

When adopting the small perturbation assumption,

$$
\begin{aligned}
\frac{\partial(\rho_w \phi)}{\partial t} &= \rho_w \frac{\partial \phi}{\partial t} + \phi \frac{\partial \rho_w}{\partial t} \overset{(17),(12)}{=} \rho_w \left(\alpha \frac{\partial \varepsilon_v}{\partial t} + \frac{1}{N} \frac{\partial p_w}{\partial t} \right) + \phi \frac{\rho_w}{K_w} \frac{\partial p_w}{\partial t} \\
&\approx \rho_{w0} \left(\frac{1}{N} + \frac{\phi_0}{K_w} \right) \frac{\partial p_w}{\partial t} + \rho_{w0} \alpha \frac{\partial \varepsilon_v}{\partial t}
\end{aligned}
$$

$$
\operatorname{div} \boldsymbol{M} \approx \operatorname{div}(\rho_w \boldsymbol{q}_{rw}) \overset{(14)}{=} \operatorname{div}\left(\rho_w \frac{\boldsymbol{k}}{\mu_w} (-\nabla p_w + \rho_w \boldsymbol{f}) \right) \approx \operatorname{div}\left(\rho_{w0} \frac{\boldsymbol{k}}{\mu_w} (-\nabla p_w + \rho_{w0} \boldsymbol{f}) \right)
$$

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and the Lagrangian water mass balance equation (5) leads to:

$$\rho_{w0}\left(\frac{1}{N} + \frac{\phi_0}{K_w}\right)\frac{\partial p_w}{\partial t} + \rho_{w0}\alpha\frac{\partial \varepsilon_v}{\partial t} = -\,\mathrm{div}\left(\rho_{w0}\frac{\boldsymbol{k}}{\mu_w}(-\nabla p_w + \rho_{w0}\boldsymbol{f})\right) \quad \text{in } \Omega \tag{21}$$

Furthermore

$$\mathbf{div}(\boldsymbol{F}\boldsymbol{\Pi}) \approx \mathbf{div}\,\boldsymbol{\sigma} \qquad \rho_w\phi \approx \rho_{w0}\phi_0$$

and the Lagrangian equilibrium equation (11) becomes:

$$\mathbf{div}\,\boldsymbol{\sigma} + \big(\rho_{s0}(1-\phi_0) + \rho_{w0}\phi_0\big)\boldsymbol{f} = \mathbf{0} \quad \text{in } \Omega \tag{22}$$

Time discretisation
With regard to the approximations made above, when the system (21)&(22) is discretised in time, one can update the values of ρ_{w0} and ϕ_0 according to (13) and (18):

$$\rho_{w0} := \rho_{w0}e^{(p_w - p_{w0})/K_w} \qquad \phi_0 := \phi_0 + \alpha\varepsilon_v + \frac{p_w - p_{w0}}{N}$$

and potentially also the coefficients K_w, α and N in the discretised (21)&(22) at the end of each time step. However, the term $\rho_{s0}(1-\phi_0)$ in (22) and the computational domain Ω (with a mesh from space discretisation) remain fixed to their initial states at time $t = 0$.

Eventually the Eulerian porosity n, which quantifies appropriately the actual porosity, can be obtained from the Lagrangian porosity ϕ by (3) and (15).

Eulerian approach [LS98]

By developing the time derivatives in the Eulerian mass balance equations (4) and (7) one obtains

$$\rho_w\frac{\mathrm{D}_s n}{\mathrm{D}t} + n\frac{\mathrm{D}_s\rho_w}{\mathrm{D}t} + \rho_w n\,\mathrm{div}\,\boldsymbol{v}_s = -\,\mathrm{div}(\rho_w\boldsymbol{q}_{rw}) \tag{23}$$

$$\frac{\mathrm{D}_s(1-n)}{\mathrm{D}t} + \frac{1-n}{\rho_s}\frac{\mathrm{D}_s\rho_s}{\mathrm{D}t} + (1-n)\,\mathrm{div}\,\boldsymbol{v}_s = 0 \tag{24}$$

Elimination of $\mathrm{D}_s\rho_s/\mathrm{D}t$ from (24) by (20) gives

$$\frac{\mathrm{D}_s n}{\mathrm{D}t} = \frac{\alpha - n}{K_s}\frac{\mathrm{D}_s p_w}{\mathrm{D}t} + (\alpha - n)\,\mathrm{div}\,\boldsymbol{v}_s \tag{25}$$

which inserted together with (12) and (14) into (23) yields

$$\rho_w\left(\frac{\alpha - n}{K_s} + \frac{n}{K_w}\right)\frac{\mathrm{D}_s p_w}{\mathrm{D}t} + \rho_w\alpha\,\mathrm{div}\,\boldsymbol{v}_s = -\,\mathrm{div}\left(\rho_w\frac{\boldsymbol{k}}{\mu_w}(-\nabla p_w + \rho_w\boldsymbol{f})\right) \tag{26}$$

When adopting the small perturbation assumption,

$$\Omega_t \approx \Omega \qquad \rho_w \approx \rho_{w0} \qquad n \approx n_0$$

Assumption 5. The deformation velocity of the solid skeleton is small:

$$\|\boldsymbol{v}_s\| \ll 1$$

– Under Assumption 5 in addition to the small perturbation assumption,

$$\frac{\mathrm{D}_s}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{v}_s \cdot \nabla \approx \frac{\partial}{\partial t}$$

$$\operatorname{div} \boldsymbol{v}_s = \operatorname{div} \frac{\mathrm{D}_s \boldsymbol{u}}{\mathrm{D}t} = \operatorname{div} \frac{\partial \boldsymbol{u}}{\partial t} + \operatorname{div}((\boldsymbol{\nabla}\boldsymbol{u})\boldsymbol{v}_s) \stackrel{\text{Assumptions 1,5}}{\approx} \operatorname{div} \frac{\partial \boldsymbol{u}}{\partial t} = \frac{\partial \varepsilon_v}{\partial t}$$

Hence (26) can be rewritten as:

$$\boxed{\rho_{w0}\left(\frac{\alpha - n_0}{K_s} + \frac{n_0}{K_w}\right)\frac{\partial p_w}{\partial t} + \rho_{w0}\alpha\frac{\partial \varepsilon_v}{\partial t} = -\operatorname{div}\left(\rho_{w0}\frac{\boldsymbol{k}}{\mu_w}(-\nabla p_w + \rho_{w0}\boldsymbol{f})\right) \quad \text{in } \Omega} \tag{27}$$

Further

$$\rho_s(1 - n) \approx \rho_{s0}(1 - n_0) \qquad \rho_w n \approx \rho_{w0}n_0$$

and the Eulerian equilibrium equation (10) leads to:

$$\boxed{\mathbf{div}\,\boldsymbol{\sigma} + (\rho_{s0}(1 - n_0) + \rho_{w0}n_0)\boldsymbol{f} = \mathbf{0} \quad \text{in } \Omega} \tag{28}$$

In addition, the evolution equations (20) and (25) for ρ_s and n can be approximated by

$$\frac{1}{\rho_s}\frac{\partial \rho_s}{\partial t} = \frac{1}{1-n}\left(\frac{\alpha - n}{K_s}\frac{\partial p_w}{\partial t} - (1-\alpha)\frac{\partial \varepsilon_v}{\partial t}\right) \tag{29}$$

$$\frac{\partial n}{\partial t} = (\alpha - n)\left(\frac{1}{K_s}\frac{\partial p_w}{\partial t} + \frac{\partial \varepsilon_v}{\partial t}\right) \tag{30}$$

Taking α and K_s constant, one can integrate (30) into

$$n = \alpha + (n_0 - \alpha)\exp\left(-\frac{p_w - p_{w0}}{K_s} - \varepsilon_v\right)$$

with the first-order Taylor approximation

$$n \approx \alpha + (n_0 - \alpha)\left(1 - \frac{p_w - p_{w0}}{K_s} - \varepsilon_v\right) = n_0 + (\alpha - n_0)\left(\frac{p_w - p_{w0}}{K_s} + \varepsilon_v\right) \tag{31}$$

Similarly from (29):

$$\begin{aligned}
\rho_s &= \rho_{s0}\exp\left(\int_0^t \frac{1}{1-n}\left(\frac{\alpha - n}{K_s}\frac{\partial p_w}{\partial t} - (1-\alpha)\frac{\partial \varepsilon_v}{\partial t}\right)\mathrm{d}\tau\right) \\
&\approx \rho_{s0}\exp\left(\frac{1}{1-n_0}\left(\frac{\alpha - n_0}{K_s}(p_w - p_{w0}) - (1-\alpha)\varepsilon_v\right)\right) \\
&\approx \rho_{s0}\left(1 + \frac{1}{1-n_0}\left(\frac{\alpha - n_0}{K_s}(p_w - p_{w0}) - (1-\alpha)\varepsilon_v\right)\right) \tag{32}
\end{aligned}$$

Time discretisation
When the system (27)&(28) is discretised in time, one can update the computational domain Ω (with a mesh from space discretisation) according to (1):

$$\Omega := \Omega + \boldsymbol{u}(\Omega)$$

the values of ρ_{w0}, ρ_{s0} and n_0 according to (13), (32) and (31):

$$\rho_{w0} := \rho_{w0}e^{(p_w - p_{w0})/K_w} \qquad \rho_{s0} := \rho_{s0}\left(1 + \frac{1}{1-n_0}\left(\frac{\alpha - n_0}{K_s}(p_w - p_{w0}) - (1-\alpha)\varepsilon_v\right)\right)$$

$$n_0 := n_0 + (\alpha - n_0)\left(\frac{p_w - p_{w0}}{K_s} + \varepsilon_v\right)$$

and potentially also the coefficients K_w, K_s and α in the discretised (27)&(28) at the end of each time step.

Summary

- Since $\phi_0 = n_0$ in the initial configuration at time $t = 0$, the systems (21)&(22) and (27)&(28) are the same whenever the equality $1/N = (\alpha - \phi_0)/K_s$ in (19) holds, despite different forms of the Eulerian water mass balance equation (4) and the Lagrangian one (5), and different sorts of approximations made in the Eulerian and Lagrangian equations.
- Differences occur only in the potential update of the initial states in successive time steps in the case of time discretisation, mainly due to differences between the Lagrangian porosity ϕ and the Eulerian one n. Note that then the computational domain Ω (with its mesh) has to be updated together with the porosity n_0 in the Eulerian approach so that n_0 quantifies properly the initial porosity at the beginning of the time steps!
- One can easily verify that the equations (18) and (31) for the increments of the porosities ϕ and n are mutually related by (3) and (15) for small ε_v and small $p_w - p_{w0}$ although these two equations have been introduced independently of one another.

References

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