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Comment on the linear mirror instability near the threshold

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Linear threshold condition of the mirror instability in a homogeneous, multi-species collisionless plasma with a general class of distribution functions is obtained in the low-frequency, long-wavelength limit of the Vlasov-Maxwell equation. In the case of one cold species, the conditions of the validity of the threshold condition and the behavior of the instability near threshold are also presented. It is confirmed that finite Larmor radius effects do not change the threshold condition. The linear threshold condition is extended to the case of hot species with a general class of distribution functions. In this case the conditions of the validity of the threshold condition or the behavior of the instability near threshold are hard to get analytically. Previous analytical and numerical results are discussed.

I. INTRODUCTION

Mirror instability[1–3] is one of many different electromagnetic instabilities driven by the particle temperature anisotropies and is relevant in collisionless, laboratory,[4] space[5] and astrophysical[6] plasmas. A general form of the threshold condition of the mirror instability for multi-species, bi-Maxwellian particles in the low-frequency, long-wavelength limit of the Vlasov-Maxwell equation and may be given in the form[7,8]

$$\sum_s \beta_{s\perp}\left(\frac{T_{s\perp}}{T_{s\|}}-1\right) > 1+\frac{\left(\sum_s \rho_s \frac{T_{s\perp}}{T_{s\|}}\right)^2}{2\sum_s \frac{\rho_s^2}{\beta_{s\|}}} \tag{1}$$

For symbol definitions see Appendix. In the approximation of one cold species (with $\beta_{s\perp},\ \beta_{s\|}\rightarrow 0$) the last term at the right hand side of (1) disappears (which corresponds to the vanishing parallel electric field) and the condition reads:[9]

$$\sum_s \beta_{s\perp}\left(\frac{T_{s\perp}}{T_{s\|}}-1\right) > 1 \tag{2}$$

It is noteworthy, that all the species contribute to the condition (1) so that it covers all the special cases such as the proton mirror and the electron mirror (or the field swelling[10]) instabilities.

Shapiro & Shevchenko[11] generalized the mirror threshold condition for one ion species s and cold electrons as

$$-\frac{m_s}{p_B}\int \frac{v_\perp^4}{4}\frac{\partial f_s}{\partial v_\|^2}d\boldsymbol{v}-\beta_{\perp s} > 1. \tag{3}$$

The same threshold condition can be obtained from an energetic principle.[12,13]

Using an adiabatic linear response of the ion distribution function (and cold electrons) and the plasma neutrality, Ref. 14 showed an importance of Landau resonance for the mirror instability and stressed the necessity of the kinetic treatment for the mirror instability. The same quasi-hydrodynamic approach was used to include hot electrons in the case of bi-Maxwellian particle distribution functions and the threshold condition (1) was recovered.[15,16] However, the predicted behaviors of the instability near the threshold in Refs. 15 and 16 were different. Pokhotelov et al.[17] generalized the mirror threshold for general ion and electron distribution functions using the same quasi-hydrodynamic approach.

Hasegawa[9] considered Finite Larmor Radius (FLR) effects (in the approximation one cold species) and showed that FLR effects stabilize modes with sufficiently short wavelengths but do not change the threshold condition. Similar results were obtained by Hall[8] in the case of hot species. On the other hand, Pokhotelov et al.[18,19] revisited the linear theory of the mirror instability and suggested that FLR effects importantly modify the mirror threshold condition.

In this paper we reexamine the work on the role of FLR effects[8,9] in the case of bi-Maxwellian particle distribution functions and extend this analysis to a general class of particle distribution functions.

II. LINEAR THEORY

We assume a neutral multi-species plasma

$$\sum_s \rho_s = 0 \tag{4}$$

with bi-Maxwellian distribution functions

$$f_s=\frac{n_s}{(2\pi)^{3/2}v_{s\|}v_{s\perp}^2}\exp\left(-\frac{v_\|^2}{2v_{s\|}^2}-\frac{v_\perp^2}{2v_{s\perp}^2}\right) \tag{5}$$

We suppose the background magnetic field is in the z direction, and the wave fields vary as $e^{i(k_\perp x+k_\| z-\omega t)}$. The general form of the dispersion relation is

$$\det\left(\boldsymbol{K}-k^2\boldsymbol{1}+\boldsymbol{k}\boldsymbol{k}\right)=0 \tag{6}$$

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where $\boldsymbol{K} = \omega^2/c^2\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}$ is the dielectric tensor;[20] $\mathbf{1}$ denotes the identity tensor.

In the low-frequency and long-wavelength limit,

$$\frac{|\omega|}{\omega_{cs}} \ll 1, \qquad \frac{k_\perp v_{s\perp}}{\omega_{cs}} \ll 1, \quad \text{and} \quad \frac{k_\| v_{s\|}}{\omega_{cs}} \ll 1, \tag{7}$$

the dispersion relation (6) can be factored (at least for the threshold condition[8]) to

$$K_{xx} - k_\|^2 = 0 \tag{8}$$

and

$$K_{yy} + \frac{K_{yz}^2}{\tilde{K}_{zz}} - k^2 = 0 \tag{9}$$

where

$$\tilde{K}_{zz} = K_{zz} + k_\perp^2 \frac{\beta_\perp - \beta_\|}{2}. \tag{10}$$

A. One Cold Species

In the case of one cold species the term $K_{yz}^2/\tilde{K}_{zz}$ is negligible and the dispersion relation reads[9]

$$K_{yy} - k^2 = 0. \tag{11}$$

In the limit of (7) and for

$$\frac{|\omega|}{\sqrt{2}k_\| v_{s\|}} \ll 1 \quad \text{and} \quad \frac{|\omega|^2}{k^2 v_A^2} \ll 1 \tag{12}$$

one can obtain from (11) the relation

$$P k_\|^2 + Q k_\perp^2 = 0 \tag{13}$$

with

$$P = 1 + \sum_s \left(\frac{1}{2} + \frac{13}{16}\lambda_s\right)(\beta_{s\perp} - \beta_{s\|}) \tag{14}$$

$$Q = 1 + \sum_s \left\{\beta_{\perp s}\left(1 - \frac{3}{2}\lambda_s\right)\left[1 + \frac{A_s}{2}Z'(\zeta_s)\right]\right\} \tag{15}$$

where $\zeta_s = \omega/(\sqrt{2}k_\| v_{s\|})$ and $\lambda_s = k_\perp^2 r_{gs}^2$.

The threshold for a plasma with one ion population s and with cold electrons is

$$\Gamma_s = \beta_{\perp s}(A_s - 1) - 1 > 0 \tag{16}$$

for $k_\|/k_\perp \to 0$ and $k_\perp \to 0$

Near the threshold, $0 < \Gamma_s \ll 1$, the maximum growth rate $\gamma_{\rm m}$ is given as

$$\gamma_{\rm m} = \frac{\omega_{cs}}{4\sqrt{3\pi}} \frac{\Gamma_s^2}{\beta_{s\perp} A_s^{3/2} \Pi_s^{1/2}} \tag{17}$$

where $\Pi_s = 1 + (\beta_{s\perp} - \beta_{s\|})/2$. The maximum growth rate appears at $k_{\perp\rm m}$ and $k_{\|\rm m}$ which are given as

$$k_{\perp\rm m} r_{gs} = \sqrt{\frac{\Gamma_s}{6}} \quad \text{and} \quad k_{\|\rm m} r_{gs} = \frac{1}{2\sqrt{6}} \frac{\Gamma_s}{\Pi_s^{1/2}} \tag{18}$$

Note that these expressions, equations (17) and (18) are slightly modified when compared to the original results of Ref. 9 where it is assumed $P \sim 1$.

The previous results can be easily generalized to a multicomponent plasma with one cold species. In this case the threshold (2) is given as

$$\Gamma = \left[\sum_s \beta_{s\perp}(A_s - 1)\right] - 1 > 0 \tag{19}$$

Near the threshold $0 < \Gamma \ll 1$ the mirror dispersion is given by

$$\gamma = \sqrt{\frac{2}{\pi}} k_\| \tilde{v} \left(\Gamma - \frac{3}{2}\tilde{r}^2 k_\perp^2 - \frac{k_\|^2}{k_\perp^2}\Pi\right) \tag{20}$$

where

$$\Pi = 1 + \frac{1}{2}\sum_s (\beta_{s\perp} - \beta_{s\|}) \tag{21}$$

$$\tilde{r}^2 = \sum_s \beta_{s\perp} r_{gs}^2 (A_s - 1) \tag{22}$$

$$\tilde{v}^{-1} = \sum_s \frac{\beta_{s\perp} A_s}{v_{s\|}} \tag{23}$$

Under the conditions $\Pi > 0$ and $\tilde{r}^2 > 0$ the maximum growth rate is

$$\gamma_{\rm m} = \frac{1}{4\sqrt{3\pi}} \frac{\tilde{v}}{\tilde{r}} \frac{\Gamma^2}{\Pi^{1/2}}, \tag{24}$$

and appears at $k_{\perp\rm m}$ and $k_{\|\rm m}$ given as

$$k_{\perp\rm m}\tilde{r} = \sqrt{\frac{\Gamma}{6}} \quad \text{and} \quad k_{\|\rm m}\tilde{r} = \frac{1}{2\sqrt{6}} \frac{\Gamma}{\Pi^{1/2}} \tag{25}$$

Relations (24,25) are a simple generalization of (17,18) and exhibit similar behavior: near the threshold the maximum growth rate appears at long wavelengths with respect to species gyroradii.

The factorization (8,9)[21] in the long-wavelength and low-frequency limit is valid (at least for the threshold condition[8]) for any distribution function in the form

$$f_s = f_s(v_\|^2, v_\perp) \tag{26}$$

In this case for the approximation of one cold species one gets a condition from (11) in the limit $k_\|/k_\perp \to 0$, $\gamma/k_\| \to 0$ in the form[11]

$$\Gamma = \sum_s \frac{m_s}{p_B} \int \frac{v_\perp^4}{4} \tilde{f}_s d\boldsymbol{v} - \sum_s \beta_{\perp s} - 1 > 0, \tag{27}$$

where $\tilde{f}_s$ is defined as

$$\tilde{f}_s = -\frac{\partial f_s}{\partial v_\parallel^2}. \tag{28}$$

If we define

$$\tilde{v}^{-1} = \sqrt{2\pi}\sum_s \frac{m_s}{p_B}\int \frac{v_\perp^4}{4}\delta(v_\parallel)\tilde{f}_s d\boldsymbol{v} \tag{29}$$

and

$$\tilde{r}^2 = \sum_s \frac{m_s}{24 p_B}\frac{1}{\omega_{cs}^2}\int \left(v_\perp^6 \tilde{f}_s - 3v_\perp^4 f_s\right) d\boldsymbol{v} \tag{30}$$

one recovers the dispersion (20) and for the conditions

$$\Pi > 0, \quad \tilde{v} > 0 \quad \text{and} \quad \tilde{r}^2 > 0 \tag{31}$$

the relations (24,25) with Π given by (21).

B. Hot species

In the general case of hot species the factorization (8,9) is only applicable for the threshold condition[8] and one cannot neglect in the dispersion relation (9) the term $K_{yz}^2/\tilde{K}_{zz}$ which corresponds to the existence of the parallel electric field. The threshold condition may be obtained as

$$\sum_s \beta_{s\perp}\left(A_s - 1\right) > 1 + \frac{\left(\sum_s \rho_s A_s\right)^2}{2\sum_s \frac{\rho_s^2}{\beta_{s\parallel}}} \tag{32}$$

assuming that

$$k_\parallel \propto \Gamma, \quad k_\perp \propto \Gamma^{1/2} \text{ and } \gamma \propto \Gamma^2 \tag{33}$$

where Γ is a small parameter denoting a distance from the threshold in analogy with the case of one cold species (24,25). In order to investigate the behavior of the mirror instability near the threshold the full dispersion (6) is necessary;[8] however, this relation leads to a cubic equation in γ. Consequently, contrary to the case of one cold species, we were not able to obtain simple relations for the maximum growth rate and its position as well as additional conditions for the validity of the threshold (31).

Finally, for the distribution function $f_s = f_s(v_\parallel^2, v_\perp)$ assuming (33) one can get the threshold condition

$$\sum_s \frac{m_s}{p_B}\int \frac{v_\perp^4}{4}\tilde{f}_s d\boldsymbol{v} - \beta_\perp > 1 + \frac{\left(\sum_s q_s \int v_\perp^2 \tilde{f}_s d\boldsymbol{v}\right)^2}{4p_B \sum_s \frac{q_s^2}{m_s}\int \tilde{f}_s d\boldsymbol{v}} \tag{34}$$

which is equivalent to the general mirror threshold condition derived in Ref. 17. As in the case of bi-Maxwellian particles, relation (6) leads to a cubic equation in γ so that we were not able to obtain simple relations for the maximum growth rate and its position as well as additional conditions for the validity of the threshold (34).

III. DISCUSSION

For the case of one cold species the relations (24,25) and the mirror threshold condition (27) are only valid for (31): $\Pi < 0$ gives the threshold for the fluid fire hose instabilities[3] (however, the full kinetic treatment predicts two different fire hose instabilities[5] which have generally lower thresholds); $\tilde{v} < 0$ may lead to another type of instability[17,19] and $\tilde{r}^2 < 0$ may destabilize the mirror mode even for $\Gamma < 0$ below (27).

Recently Refs. 18,19 suggested that the mirror maximum growth rate appears for $k_\perp r_{gs} \sim 1$ and that the threshold conditions is largely modified by the FLR effects. However, the approximation used in Ref. 18,19 (as well as in this paper) is only valid at the low-frequency, long-wavelength limit $k_\perp r_{gs} \ll 1$. It is straightforward to show[22] that the threshold calculated from equation (25) in Ref. 18 is identical to (16) and that near the threshold the behavior of the maximum grow rate and its position is identical to (17,18).

In the case of hot species, contrary to our results, Hall[8] obtained analytically the behavior of the mirror instability near the threshold. However, his results were derived under some simplifying assumptions and moreover he assumed $k_\parallel \propto \Gamma$, $k_\perp \propto \Gamma$ and $\gamma \propto \Gamma$ which gives a different term ordering when compared to our results where (33) is used in analogy with the case of one cold species. On the other hand, the numerical solution of the full kinetic dispersion of the mirror instability[23,24] gives results qualitatively similar to the analytical results.[8,9] We were not able to obtain analytically additional conditions for the validity of the threshold conditions (32,34). On the other hand, in some cases the results of the full kinetic treatment (6) are in a good agreement with the analytical threshold condition (32).[25]

The quasi-magnetohydrodynamic approach[15–17] is largely compatible with the factorization (9) so that its predictions of the behavior of the mirror instability in the hot species case may be questionable.

IV. CONCLUSION

We have presented the linear threshold condition (27) for the mirror instability in the homogeneous, multi-species plasma with a general class of distribution functions (26) in the case of one cold species from the low-frequency, long-wave length limit of the linear Vlasov-Maxwell equations. We have presented the conditions (31) of the validity of the threshold condition (27) as well as the behavior of the maximum growth rate and its position near the threshold in this case (24,25) as a generalization of the previous results.[9] The linear threshold condition is not modified by FLR effects in agreement with Hasegawa[9] as well as with the reexamined results of Pokhotelov et al.[18]

Furthermore, we have derived the threshold condition (34) for the mirror instability in the homogeneous, multi-species with a general class of distribution functions (26) in the case hot species using the same approach. This condition is in agreement with the previous results.[7,8,17] In this case we were not able to derive analytically the conditions of validity of the threshold condition or the behavior of the instability near the threshold. Although such analytic analysis could be done under some simplifying assumptions,[8] we conclude that in a general case of a plasma with hot species it is advisable to use the full dispersion relation (6) of Vlasov-Maxwell equation.[23,24] We expect that these results are also relevant for the (drift) mirror instability in inhomogeneous plasmas.

APPENDIX: DEFINITIONS

We use the subscripts $\perp$ and $\|$ to denote the directions with respect to the ambient magnetic field $\boldsymbol{B}_0$ with $B_0 = |\boldsymbol{B}_0|$ denoting its magnitude; the subscript s denotes different species. Here f_s denotes the distribution functions, $n_s = \int f_s d\boldsymbol{v}$ is the number density and $T_{s\perp} = m_s \int v_\perp^2 f_s d\boldsymbol{v}/(2n_s k_B)$ and $T_{s\|} = m_s \int v_\|^2 f_s d\boldsymbol{v}/(n_s k_B)$ are the (effective) perpendicular and parallel temperatures, respectively, and we define $A_s = T_{s\perp}/T_{s\|}$. Here $p_B = B_0^2/2\mu_0$ denotes the magnetic pressure and we define the particle betas as $\beta_{s\|} = n_s k_B T_{s\|}/p_B$, $\beta_{s\perp} = n_s k_B T_{s\perp}/p_B$, and the total betas as $\beta_\perp = \sum_s \beta_{s\perp}$, $\beta_\| = \sum_s \beta_{s\|}$. Here the thermal velocities are defined as $v_{s\|} = (k_B T_{s\|}/m_s)^{1/2}$ and $v_{s\perp} = (k_B T_{s\perp}/m_s)^{1/2}$. The cyclotron frequency are and $\omega_{cs} = q_s B_0/m_s$, respectively, the electron plasma frequency is $\omega_{ps} = (n_s q_s^2/m_s\epsilon_0)^{1/2}$. The gyroradius is given as $r_{gs} = v_{\perp s}/\omega_{cs}$. In these expressions: m_s denote the mass, q_s denotes the charge, and $\rho_s = q_s n_s$ is the charge density. Here μ_0 and ϵ_0 stand for the vacuum magnetic permeability and electric permittivity, respectively, and k_B is Boltzmann constant. ω denotes the (complex) wave frequency, γ denotes the growth/damping rate, $\boldsymbol{k}$ denotes the wave vector, $\boldsymbol{k} = (k_\perp, 0, k_\|)$ whereas γ_m denotes the maximum growth rate and $k_{\perp\mathrm{m}}$ and $k_{\|\mathrm{m}}$ denote the corresponding wave vector components. Here $\boldsymbol{\epsilon}$ denotes the dispersion tensor, $\boldsymbol{K} = \omega^2/c^2\boldsymbol{\epsilon}$, and $\mathbf{1}$ denotes the identity tensor; Z and Z' denote the plasma dispersion function and its derivative, respectively.

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[1] A. A. Vedenov and R. Z. Sagdeev, in *Plasma Physics and the Problem of Controlled Thermonuclear Reactions* (Pergamon, New York, 1958), vol. 3, pp. 332–339.
[2] L. I. Rudakov and R. Z. Sagdeev, in *Plasma Physics and the Problem of Controlled Thermonuclear Reactions* (Pergamon, New York, 1958), vol. 3, pp. 321–331.
[3] M. Tajiri, J. Phys. Soc. Japan **22**, 1482 (1967).
[4] S. A. Chandrasekhar, A. N. Kaufman, and K. M. Watson, Proc. R. Soc. London, Ser. A **245**, 435 (1958).
[5] P. Hellinger, P. Trávníček, J. C. Kasper, and A. J. Lazarus, Geophys. Res. Lett. **33**, L09101 (2006).
[6] P. Sharma, G. H. Hammett, E. Quataert, and J. M. Stone, Astrophys. J. **637**, 952 (2006).
[7] T. H. Stix, *The Theory of Plasma Waves* (McGraw-Hill, New York, 1962).
[8] A. N. Hall, J. Plasma Phys. **21**, 431 (1979).
[9] A. Hasegawa, Phys. Fluids **12**, 2642 (1969).
[10] S. Migliuolo, J. Geophys. Res. **91**, 7981 (1986).
[11] V. D. Shapiro and V. I. Shevchenko, Sov. Phys. JETP, Engl. Transl. **18**, 1109 (1964).
[12] T. G. Northrop and A. W. Schardt, J. Geophys. Res. **85**, 25 (1980).
[13] M. P. Leubner and N. Schupfer, J. Geophys. Res. **106**, 12993 (2001).
[14] D. J. Southwood and M. G. Kivelson, J. Geophys. Res. **98**, 9181 (1993).
[15] F. G. E. Pantellini and S. J. Schwartz, J. Geophys. Res. **100**, 3539 (1995).
[16] O. A. Pokhotelov, M. A. Balikhin, H. S.-C. K. Alleyne, and O. G. Onishchenko, J. Geophys. Res. **105**, 2393 (2000).
[17] O. A. Pokhotelov, R. A. Treumann, R. Z. Sagdeev, M. A. Balikhin, O. G. Onishchenko, V. P. Pavlenko, and I. Sandberg, J. Geophys. Res. **107**, 1312 (2002).
[18] O. A. Pokhotelov, R. Z. Sagdeev, M. A. Balikhin, and R. A. Treumann, J. Geophys. Res. **109**, A09213 (2004).
[19] O. A. Pokhotelov, M. A. Balikhin, R. Z. Sagdeev, and R. A. Treumann, J. Geophys. Res. **110**, A10206 (2005).
[20] M. L. Goldstein, H. K. Wong, and A. F. Viñas, J. Geophys. Res. **90**, 302 (1985).
[21] M. Gedalin, Y. E. Lyubarsky, M. Balikhin, and C. T. Russell, Phys. Plasmas **8**, 2934 (2001).
[22] E. A. Kuznetsov, T. Passot, and P.-L. Sulem, Phys. Rev. Lett. **98**, 235003 (2007).
[23] C. P. Price, D. W. Swift, and L.-C. Lee, J. Geophys. Res. **91**, 101 (1986).
[24] V. Génot, S. J. Schwartz, C. Mazelle, M. Balikhin, M. Dunlop, and T. M. Bauer, J. Geophys. Res. **106**, 21611 (2001).
[25] P. Trávníček, P. Hellinger, M. G. G. T. Taylor, C. P. Escoubet, I. Dandouras, and E. Lucek, Geophys. Res. Lett. **34**, L15104 (2007).