HIGH-RESOLUTION OBSERVING TECHNIQUES

This lecture I gave in August and May 1999 to the doctorands at the Charles University in Prague and at the Karl-Franzens Universität in Graz. It should introduce the student to theoretical background and practical solutions of problems of high-resolution observations in solar physics. The structure of the lecture is as follows:

- Preliminaries in optics
- Atmospheric turbulence
- Long-exposure images
- Short-exposure images
- Real-time image corrections
- Data acquisition and reduction

Problems of scattered light are treated in another lecture.

The lecture is mostly based on the excellent reviews by von der Lühe (1992) and by Bonet (1999):

Bonet J.A., 1999, *High spatial resolution imaging in solar physics*, in A. Hanslmeier and M. Messerotti (eds.), "Motions in the solar atmosphere", Dordrecht: Kluwer, pp. 1–34, and the references therein.

von der Lühe O., 1992, *High spatial resolution techniques*, in F. Sánchez, M. Collados, and M. Vázquez (eds.), "Solar observations: Techniques and interpretation", Cambridge University Press, pp. 3–68, and the references therein

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PRELIMINARIES IN OPTICS

Monochromatic plane wave

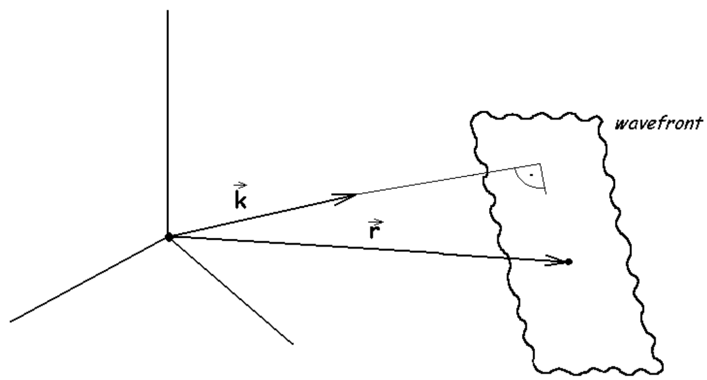
$$E(\mathbf{r}, t) = E_0 \exp(j(\omega t - \mathbf{k} \cdot \mathbf{r})) \tag{1}$$

where $|\mathbf{k}| = 2\pi/\lambda$

Only the real part of the expression has a physical meaning.

The time-independent part is a **complex amplitude**

$$\psi(\mathbf{r}) = E_0 \exp(-j\, \mathbf{k} \cdot \mathbf{r}) \tag{2}$$



The origin of the coordinate system is at the center of the entrance pupil.

Turbulence in the Earth's atmosphere →
→ random variations of the refractive index →
→ fluctuations in the optical-path length →
→ fluctuating phase delays in the wave.

Resulting **phase delay** $\phi(\mathbf{r}, t)$ describes the wavefront aberration.

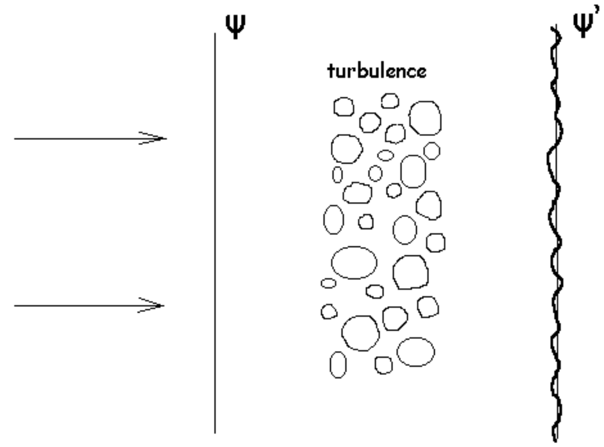
Corrugated wave:

$$E'(\mathbf{r}, t) = E_0 \exp(j(\omega t - (\mathbf{k} \cdot \mathbf{r} + \phi(\mathbf{r}, t)))) \tag{3}$$

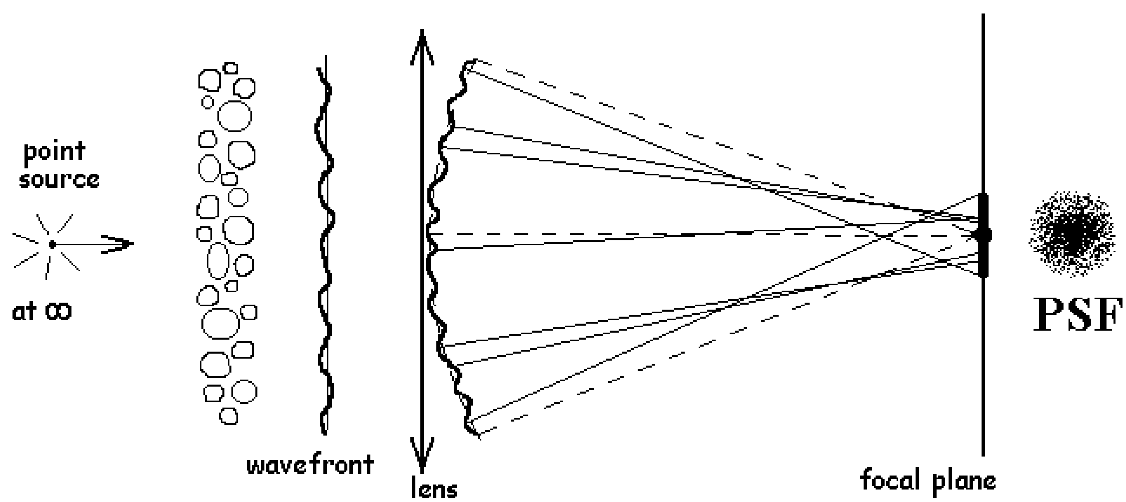
with complex amplitude

$$\psi'(\mathbf{r}, t) = \psi(\mathbf{r}) \exp(-j\, \phi(\mathbf{r}, t)) \tag{4}$$

Corrugated wavefront:



A simple image-forming system:

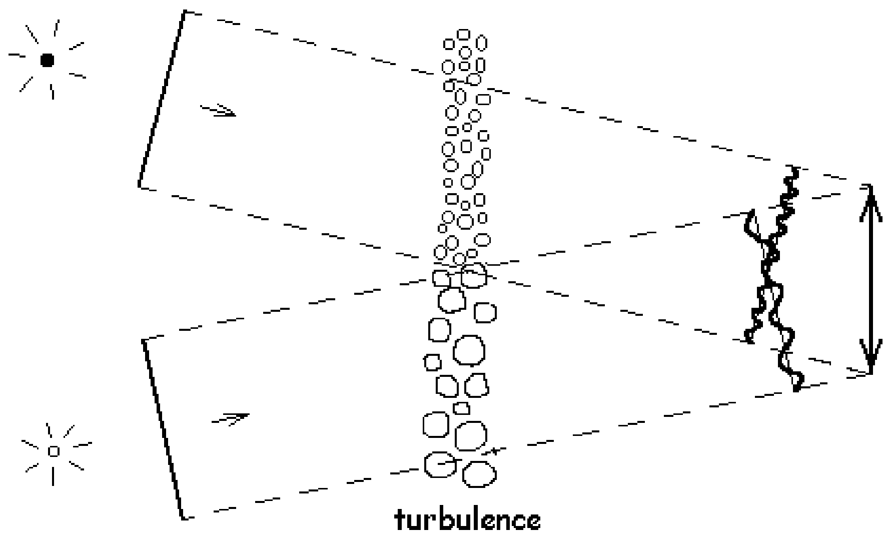


Point-spread function PSF
is a response of the telescope+atmosphere system to the point-source signal.

In general, it depends on the positions in the object and image planes and on the time.

Effects of corrugated wavefront: a seeing

- blurring
- image motion
- image distortion



The dependence of the PSF on the direction of propagation (i.e. on the position of the source) is called **anisoplanatism**.

The region (solid angle) where the PSF is space-invariant is called **isoplanatic patch**.
It is quite small, of about 5” at a good seeing.

Effects of the telescope:

- diffraction on the finite-size entrance pupil
- optical aberrations

Extended objects (the Sun) are composed of a collection of incoherent point sources.

An ideal telescope+atmosphere system (no turbulence, no aberrations, no diffraction on the entrance pupil) will form an image of the extended object in the focal plane with a "true" intensity distribution $i_0(x', y')$, where x', y' are "true" positions of images of elementary point sources.

Due to disturbing effects, characterized by the PSF $s(x, y, x', y', t)$ we obtain the observed intensity distribution

$$i(x, y, t) = \int\int i_0(x', y')\, s(x, y, x', y', t)\, \mathrm{d}x'\mathrm{d}y' \tag{5}$$

For the *isoplanatic* optical system (PSF does not change its shape with position) (5) can be written as

$$i(x, y, t) = \int\int i_0(x', y')\, s(x - x', y - y', t)\, \mathrm{d}x'\mathrm{d}y' \tag{6}$$

This is a **convolution:**

$$i(x, y, t) = i_0(x, y) * s(x, y, t) \tag{7}$$

or, in vectorial notation, where $\mathbf{q} = (x, y)$

$$i(\mathbf{q}, t) = i_0(\mathbf{q}) * s(\mathbf{q}, t) \tag{8}$$

When operating with a convolution, the **Fourier transform** is useful, because

$$I(\mathbf{u}, t) = I_0(\mathbf{u})\, S(\mathbf{u}, t) \tag{9}$$

where $\mathbf{u}$ is the spatial frequency vector (coordinates in the Fourier domain), and
$I(\mathbf{u}, t) = \mathcal{F}(i(\mathbf{q}, t)), \quad S(\mathbf{u}, t) = \mathcal{F}(s(\mathbf{q}, t)), \quad \text{etc.}$

$S(\mathbf{u}, t)$ is called **optical transfer function OTF**,
$|S(\mathbf{u}, t)|$ is the **modulation transfer function MTF**.

OTF is, in general, a complex function. If the PSF is real (always) and even (special cases), then the OTF is real and OTF = MTF.

Reconstruction of the "true" intensity distribution

– formal solution:

$$i_0(\mathbf{q}) = \mathcal{F}^{-1}\left(\frac{I(\mathbf{u},t)}{S(\mathbf{u},t)}\right) \tag{10}$$

Determination of the OTF

"Fourier transform of the intensity distribution at the focal plane is proportional to the autocorrelation of the complex amplitude of the wave at the pupil: $I \approx \mathrm{corr}(\psi' W, \psi' W)$."

In particular, the Fourier transform of the response of the system to a monochromatic (wavelength λ) distant point source (the OTF) can be computed from

$$S(\mathbf{u},t) \approx \int_{-\infty}^{\infty} \psi'(\mathbf{r},t) W(\mathbf{r})\, \psi'^{*}(\mathbf{r}-\lambda f\mathbf{u},t) W^{*}(\mathbf{r}-\lambda f\mathbf{u})\, \mathrm{d}\mathbf{r} \tag{11}$$

where $\psi'(\mathbf{r},t) = E_0 \exp(-j(\mathbf{k}\cdot\mathbf{r} + \phi(\mathbf{r},t))) = \psi(\mathbf{r}) \exp(-j\,\phi(\mathbf{r},t))$
is the incident complex amplitude,
$\mathbf{r}$ varies over the entrance pupil,
f is the focal length of the telescope,
$W(\mathbf{r})$ is the **pupil transmission function:**
$W(\mathbf{r}) = 0$ outside the pupil
$W(\mathbf{r}) \neq 0$ inside the pupil, describing telescope aberrations,
$\lambda f\mathbf{u}$ represents the autocorrelation shift in the focal plane,
$\lambda\mathbf{u}$ is an angle corresponding to the autocorrelation shift

Assuming that:
1) The source is located on the optical axis of the telescope,
2) $E_0 = const$ (no scintillations are present, valid for $D > 10$ cm),
we have $\psi(\mathbf{r}) = E_0 = const$, and the **instantaneous OTF** is

$$S(\mathbf{u},t) \approx \int_{-\infty}^{\infty} W(\mathbf{r})\, W^{*}(\mathbf{r}-\lambda f\mathbf{u})\, \exp(-j(\phi(\mathbf{r},t) - \phi(\mathbf{r}-\lambda f\mathbf{u},t)))\, \mathrm{d}\mathbf{r} \tag{12}$$

OTF of an aberration-free, diffraction-limited telescope

No atmospheric degradation: $\phi(\mathbf{r}, t) = 0$

Aberration-free telescope with circular aperture D:
$W(\mathbf{r}) = 1 \quad$ for $r \leq D/2$
$W(\mathbf{r}) = 0 \quad$ for $r > D/2$

From (12) we obtain

$$S(\mathbf{u}) \approx \int_{-\infty}^{\infty} W(\mathbf{r})\, W(\mathbf{r} - \lambda f \mathbf{u})\, \mathrm{d}\mathbf{r} \tag{13}$$

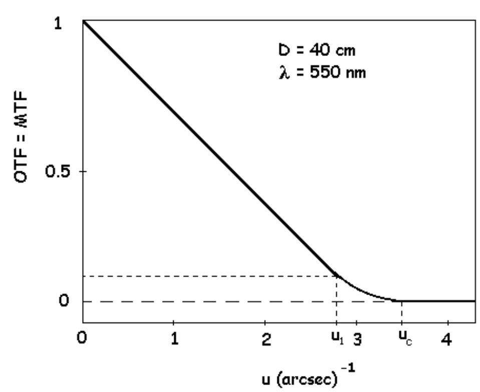
Solving the integral and normalizing it to 1 at the origin we get a rotationally symmetric function

$$S(u) = \frac{2}{\pi}\left(\arccos(\alpha u) - \alpha u\sqrt{1 - \alpha^2 u^2}\right) \tag{14}$$

$$\alpha = \lambda f / D \tag{15}$$

spatial frequency u is measured in the same units as $1/\alpha$ (e.g. cm^{-1})
In this case, $S(u) \equiv$ OTF is real, i.e. OTF = MTF.

The telescope behaves as a low-pass frequency filter with cut-off at $\alpha u_c = 1$. $\quad u_c = D/(\lambda f) \quad (\mathrm{cm}^{-1}) \qquad u_c = D/\lambda \quad (\mathrm{radian}^{-1})$



The cutoff u_c corresponds to an angular distance on the sky

$$\theta_c = \lambda / D \quad \text{(radian)}$$

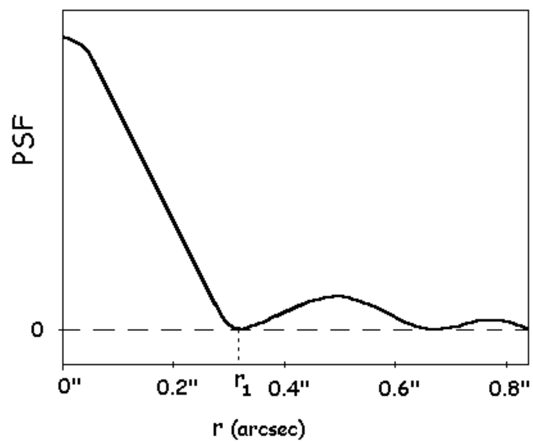
We cannot get any information about objects smaller than θ_c.

PSF of an aberration-free, diffraction-limited telescope

$$\mathrm{PSF} \equiv s(r) = \mathcal{F}^{-1}(S(u))$$
$$s(r) = \frac{1}{\pi}\left(\frac{J_1(\frac{\pi D}{\lambda f}r)}{r}\right)^2 \quad (16)$$

This is so-called **Airy function**.
J_1 is the Bessel function of order 1.

The first minimum (0) of $s(r)$ is at $r_1 = 3.83\,\lambda f/(\pi D)$ (cm)



r_1 corresponds to angular distance $\theta_1 = r_1/f = 3.83\,\lambda/(\pi D)$,

$$\theta_1 = 1.22\,\lambda/D = 1.22\,\theta_c \quad \text{(radian)}$$

Rayleigh resolution limit:
"Two points on the sky are resolved when a maximum of the Airy's pattern of the first one falls into the minimum of the second one" – that is, when they are at the angular distance θ_1.

θ_1 corresponds to spatial frequency $u_1 < u_c$. At Rayleigh resolution limit the value of OTF is 0.0894, so that at this frequency the telescope still transmits about 9 % of the original signal.

ATMOSPHERIC TURBULENCE

Atmospheric turbulence $\rightarrow$ changes of temperature $T \rightarrow$
$\rightarrow$ changes of refractive index n

$$\Delta n = 7.6 \cdot 10^{-5} \frac{p}{T^2} \Delta T \tag{17}$$

p (pressure) in mb, T in K
Good sites are those with small ΔT in the atmosphere. The statistics of the temperature field is determined by the turbulent motions:

- Heating of the terrestrial surface $\rightarrow$
  $\rightarrow$ large-scale thermal convection. This corresponds to the **outer scale of turbulence,** $L_0 \sim 30\,\mathrm{m}$
- **Turbulent regime:** fragmentation of the large-scale convection pattern into smaller cells down to a certain limit, l_0.
- **Inner scale of turbulence,** $l_0 \sim 3\,\mathrm{mm}$
  At this scale, the kinetic energy is dissipated into heat by viscous friction. The turbulent regime takes place between L_0 and l_0.

Statistical description of the temperature field:
Temperature structure function D_T
is defined as a statistical variance (σ^2) of temperature differences between two points at a distance ρ:

$$D_T(\rho) = \langle\, |T(\mathbf{r}, t) - T(\mathbf{r} + \vec{\rho}, t)|^2 \,\rangle \tag{18}$$

$\langle ... \rangle$ means *averaging in time.*

In the turbulent regime, for $l_0 \ll \rho \ll L_0$, the **Obukhov's law** is valid:

$$D_T(\rho) = C_T^2\, \rho^{2/3} \tag{19}$$

where C_T is called **temperature structure constant**.

Statistical description of the refractive index fluctuations:
Index structure function D_n
is derived from (17) and (19):

$$D_n(\rho) = (7.6 \cdot 10^{-5} \frac{p}{T^2})^2 \, C_T^2 \, \rho^{2/3} \tag{20}$$

$$D_n(\rho) = C_n^2 \, \rho^{2/3} \tag{21}$$

where C_n is the **index structure "constant"**.
C_n depends on p, T, i.e. it varies with the height in the atmosphere.

Temporal fluctuations of atmospheric inhomogeneities are in the frequency range of 1 – 100 Hz.

- Short exposure time of about 10 ms or less can freeze the image but the instantaneous OTF is complicated.
- Long exposure time > 0.1 s makes time-averaging of the instantaneous OTF and the resulting OTF is more simple.

LONG-EXPOSURE IMAGES

Long-exposure OTF

Long exposure time > 0.1 s produces time-averaging $\langle ... \rangle$ of the instantaneous OTF. In the Fourier domain:

$$\langle I(\mathbf{u},t)\rangle = I_0(\mathbf{u})\,\langle S(\mathbf{u},t)\rangle \tag{22}$$

$$\langle S(\mathbf{u},t)\rangle \approx \int_{-\infty}^{\infty} W(\mathbf{r})\,W^*(\mathbf{r}-\lambda f\mathbf{u})\,\langle \exp(-j(\phi(\mathbf{r},t)-\phi(\mathbf{r}-\lambda f\mathbf{u},t)))\rangle\,\mathrm{d}\mathbf{r} \tag{23}$$

$\phi(\mathbf{r},t)-\phi(\mathbf{r}-\lambda f\mathbf{u},t)$ is a random variable for which a Gaussian distribution can be assumed, so that

$$\langle \exp(-j(\phi(\mathbf{r},t)-\phi(\mathbf{r}-\lambda f\mathbf{u},t)))\rangle = \exp\left(-\frac{1}{2}\langle\,|\phi(\mathbf{r},t)-\phi(\mathbf{r}-\lambda f\mathbf{u},t)|^2\,\rangle\right) \tag{24}$$

The term (a real number) $\langle\,|\phi(\mathbf{r},t)-\phi(\mathbf{r}-\lambda f\mathbf{u},t)|^2\,\rangle$
is the mean phase-delay difference between two points at the telescope pupil, separated by

$$\vec{\rho} = \lambda\, f\,\mathbf{u}$$

and, analogously to D_T, we can call it **wave structure function**

$$D_\phi(\rho) = \langle\,|\phi(\mathbf{r},t)-\phi(\mathbf{r}+\vec{\rho},t)|^2\,\rangle \tag{25}$$

Since ϕ is a function of the refractive index n, D_ϕ can be related to the statistics of the fluctuations of n along the ray path in the atmosphere, and (25) turns into

$$D_\phi(\rho) = 2.91\,k^2\,(\cos\gamma)^{-1}\,\rho^{5/3}\int C_n^2(h)\,\mathrm{d}h \tag{26}$$

$k = 2\pi/\lambda$,
γ is the zenith distance, and
h is the height in the atmosphere.
We see that C_n determines the quality of image.

Finally, using (25) in (23), the long-exposure OTF can be written as

$$\langle S(\mathbf{u},t)\rangle \approx \int_{-\infty}^{\infty} W(\mathbf{r})\,W^*(\mathbf{r}-\lambda f\mathbf{u})\,\mathrm{d}\mathbf{r}\;\cdot\;\exp\left(-\frac{1}{2}\,D_\phi(\rho)\right) \tag{27}$$

Fried parameter r_0

$$r_0 = \left(0.423\, k^2\, (\cos\gamma)^{-1} \int C_n^2(h)\, \mathrm{d}h\right)^{-3/5} \tag{28}$$

$$D_\phi(\rho) = 2.91\, k^2\, (\cos\gamma)^{-1}\, \rho^{5/3} \int C_n^2(h)\, \mathrm{d}h$$

$$D_\phi(\rho) = 6.88\, \left(\frac{\rho}{r_0}\right)^{5/3} \tag{29}$$

r_0 has a dimension of length and can be used as a unique parameter to characterize the seeing.

Let us come back to (27):

$$\langle S(\mathbf{u},t)\rangle \approx \int_{-\infty}^{\infty} W(\mathbf{r})\, W^*(\mathbf{r}-\lambda f \mathbf{u})\, \mathrm{d}\mathbf{r}\, \cdot\, \exp\left(-\frac{1}{2}\, D_\phi(\lambda f u)\right)$$

In the long-exposure OTF, the telescopic and atmospheric parts can be *separated*: $\langle S(\mathbf{u},t)\rangle = S_{tel}(\mathbf{u}) \cdot \langle S_{atm}(u,t)\rangle$

OTF of the aberration-free telescope:

$$S_{tel}(u) = \frac{2}{\pi} \left(\arccos(\alpha u) - \alpha u \sqrt{1-\alpha^2 u^2}\right)$$

where $\alpha = \lambda f / D$.

The long-exposure atmospheric OTF: (a real and even function)

$$\langle S_{atm}(u,t)\rangle = \exp\left(-\frac{1}{2}\, D_\phi(\lambda f u)\right)$$

$$\langle S_{atm}(u,t)\rangle = \exp\left(-3.44\, \left(\frac{\lambda f u}{r_0}\right)^{5/3}\right)$$

$$\langle S_{atm}(u,t)\rangle = \exp(-3.44\, (\beta u)^{5/3}) \tag{30}$$

where $\beta = \lambda f / r_0$.

We see that r_0 plays a similar role in the long-exposure atmospheric OTF like D in the telescopic OTF. Also, in both cases, OTF = MTF.

The Fried parameter can be interpreted as the diameter of a diffraction-limited telescope located outside the atmosphere, that gives the same resolution as an infinitely large (non-diffracting) perfect telescope observing through the atmosphere.

The larger r_0, the better the seeing. At very good sites (Canary Islands), 20 cm $< r_0 <$ 30 cm, occasionally $r_0 \simeq 50$ cm.

Practical resolution of the system telescope+atmosphere for long exposures:

- $D < r_0$: *The resolution is limited by the telescope.*
  The cut-off frequency is $u_c = D/(\lambda f)$, corresponding to the minimum angular distance $\theta_c = \lambda/D$.
- $D > r_0$: *The resolution is limited by the atmosphere.*
  The "cut-off frequency" is $u_c = r_0/(\lambda f)$, corresponding to the minimum angular distance $\theta_c = \lambda/r_0$.
  Since $r_0 \approx (k^2)^{-3/5}$ – see (28), $r_0 \approx \lambda^{6/5}$, and $\theta_c \approx \lambda^{-1/5}$.
  Therefore, seeing slowly improves with increasing wavelength.

SHORT-EXPOSURE IMAGES

Short exposures (< 10 ms) can "freeze" the image, giving higher spatial resolution than the long exposures.

The short-exposure PSF

- has an irregular shape
- varies with time
- varies with position

The short-exposure OTF = the instantaneous OTF
(a complex function):

$$S(\mathbf{u},t) \approx \int_{-\infty}^{\infty} W(\mathbf{r})\, W^*(\mathbf{r}-\lambda f \mathbf{u})\, \exp(-j(\phi(\mathbf{r},t) - \phi(\mathbf{r}-\lambda f \mathbf{u},t)))\, \mathrm{d}\mathbf{r}$$

We can get some information about this OTF taking a *series* of short-exposure images (within a time interval shorter than evolutionary changes in the object), and performing a *suitable averaging in time.*

This method is called **speckle interferometry**.

A series of N short-exposure images $i_i(\mathbf{q})$:

$$I_i(\mathbf{u}) = \mathcal{F}(i_i(\mathbf{q}))$$

$$I_i(\mathbf{u}) = I_0(\mathbf{u})\, S_i(\mathbf{u})$$

where $S_i(\mathbf{u})$ is an instantaneous OTF of the i-th image.

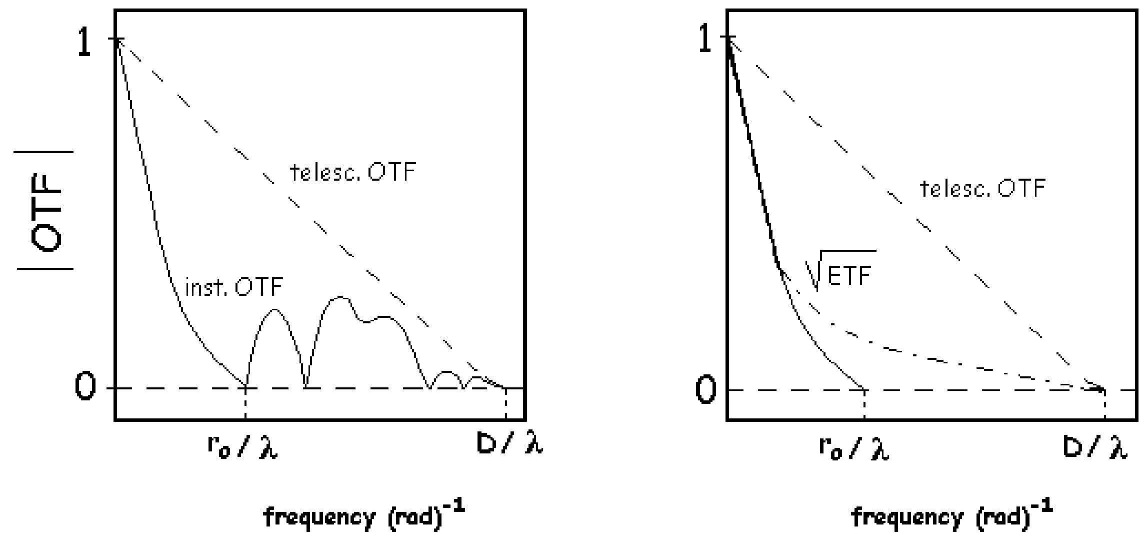
With the ordinary time-averaging

$$\frac{1}{N}\sum_{i=1}^{N} I_i(\mathbf{u}) = I_0(\mathbf{u}) \underbrace{\frac{1}{N}\sum_{i=1}^{N} S_i(\mathbf{u})}_{long-exposure\ OTF} \tag{31}$$

we loose information at high frequencies, since the summation $\sum S_i(\mathbf{u})$ produces cancellations in the complex Fourier components, mainly in the high-frequency range.

Labeyrie's method: suitable time-averaging of absolute values which avoids cancellations of positive and negative values

$$\underbrace{\frac{1}{N}\sum_{i=1}^{N}|I_i(\mathbf{u})|^2}_{\textit{mean power spectrum}} = |I_0(\mathbf{u})|^2 \underbrace{\frac{1}{N}\sum_{i=1}^{N}|S_i(\mathbf{u})|^2}_{\textit{ETF}} \qquad (32)$$



Energy transfer function ETF

$$\mathrm{ETF} = \frac{1}{N}\sum_{i=1}^{N}|S_i(\mathbf{u})|^2$$

preserves averaged high-frequency components up to D/λ.

In the low-frequency range, $\sqrt{\mathrm{ETF}} \simeq$ long-exposure OTF.

Knowing the ETF, we can restore $|I_0(\mathbf{u})|$, i.e. the *amplitude spectrum* of the object, but not the *phase spectrum*. That means that the restored intensities i_0 are correct but they are not necessarily located at correct places.
There are also methods to restore phase spectra (*Knox & Thompson*).

How to obtain the ETF ?

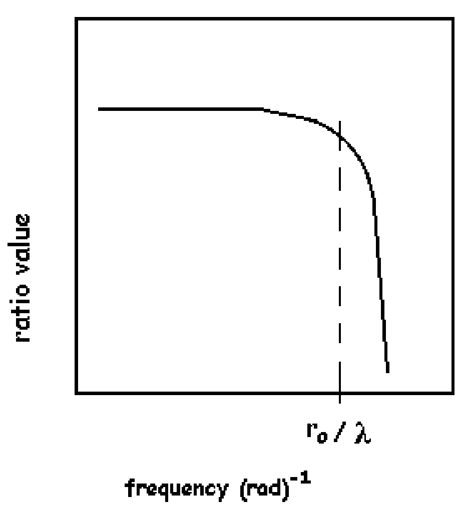
Stellar observations: from time-averaged power spectra of unresolved bright stars (point sources).

Solar observations: reference point sources are not available.
The ETF has to be obtained from **numerical models** parametrized by the Fried parameter r_0 and the telescope aperture D (*Korff*).

And how to find r_0 ?
Comparing (31) and (32) we get

$$\frac{|\sum_{i=1}^{N} I_i(\mathbf{u})|^2}{\sum_{i=1}^{N} |I_i(\mathbf{u})|^2} = \frac{|\sum_{i=1}^{N} S_i(\mathbf{u})|^2}{\sum_{i=1}^{N} |S_i(\mathbf{u})|^2} = \frac{|\, long - exp.\ OTF\,|^2}{ETF} \tag{33}$$

Both ETF and long-exposure OTF depend on r_0 – see the figure of OTF and ETF. The ratio (33) is nearly constant at low frequencies, shows a strong bend at $u \simeq r_0/\lambda$, and then decreases. Fitting model curves to observed ratio $|\sum I_i(\mathbf{u})|^2 / \sum |I_i(\mathbf{u})|^2$ we obtain r_0, the model, and, finally, the ETF.



Polynomial description of wavefront aberrations:

Zernike polynomials

After passing the atmosphere+telescope system, at the exit pupil (radius R) of the telescope we receive the corrugated wavefront whose aberrations $\phi(r,\theta)$ (r,θ are polar coordinates) can be expanded in a series of orthogonal Zernike polynomials $Z_j(\rho,\theta)$, $\rho = r/R$,

$$\phi(r,\theta) = \sum_i a_i \, Z_i(\rho,\theta) \tag{34}$$

where a_i are weighting factors.

The first few Zernike polynomials correspond to the classical optical aberrations:

Z_1 : piston (constant offset, no effect)

$Z_{2,3}$: tilt (image motion)

Z_4 : defocus (change of wavefront curvature)

$Z_{5,6}$: astigmatism

Z_{7-10} : coma

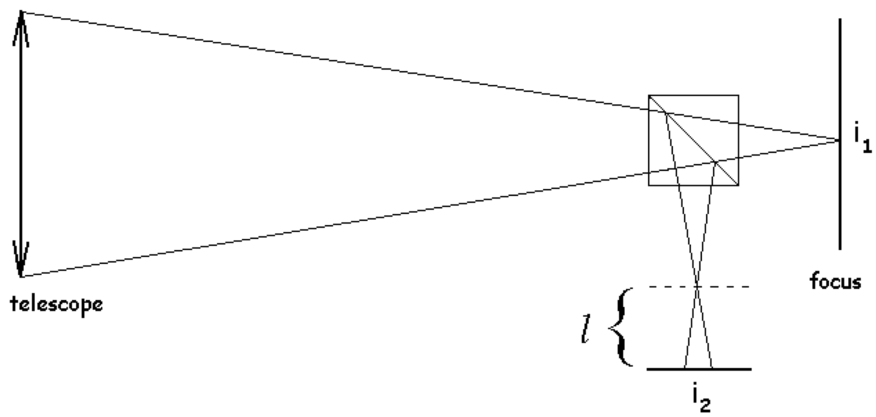
Z_{11} : spherical aberration

...

The polynomial expansion of $\phi(r, \theta)$ is a utilized in the

Phase-diversity technique for image reconstruction.

Two images, i_1, i_2, are observed simultaneously:
i_1 in the focal plane
i_2 out of focus, by a known distance l



Known amount of defocus → known phase error (phase diversity) $\Delta\phi \approx Z_4$

$$i_1(\mathbf{q}) = i_0(\mathbf{q}) * s_1(\mathbf{q})$$
$$i_2(\mathbf{q}) = i_0(\mathbf{q}) * s_2(\mathbf{q})$$

We must find a combination of object i_0 and PSFs s_1, s_2 that minimizes a functional

$$\int ((i_1(\mathbf{q}) - i_0(\mathbf{q}) * s_1(\mathbf{q}))^2 + (i_2(\mathbf{q}) - i_0(\mathbf{q}) * s_2(\mathbf{q}))^2) \, \mathrm{d}\mathbf{q}$$

or, equivalently, in the Fourier domain

$$\int (|I_1(\mathbf{u}) - I_0(\mathbf{u}) \, S_1(\mathbf{u})|^2 + |I_2(\mathbf{u}) - I_0(\mathbf{u}) \, S_2(\mathbf{u})|^2) \, \mathrm{d}\mathbf{u}$$

The difference between S_1, S_2, due to defocus, is known. I_1, I_2 come from observations. From the minimization, using the expansion of S_1, S_2 into Zernike polynomials, we obtain I_0.

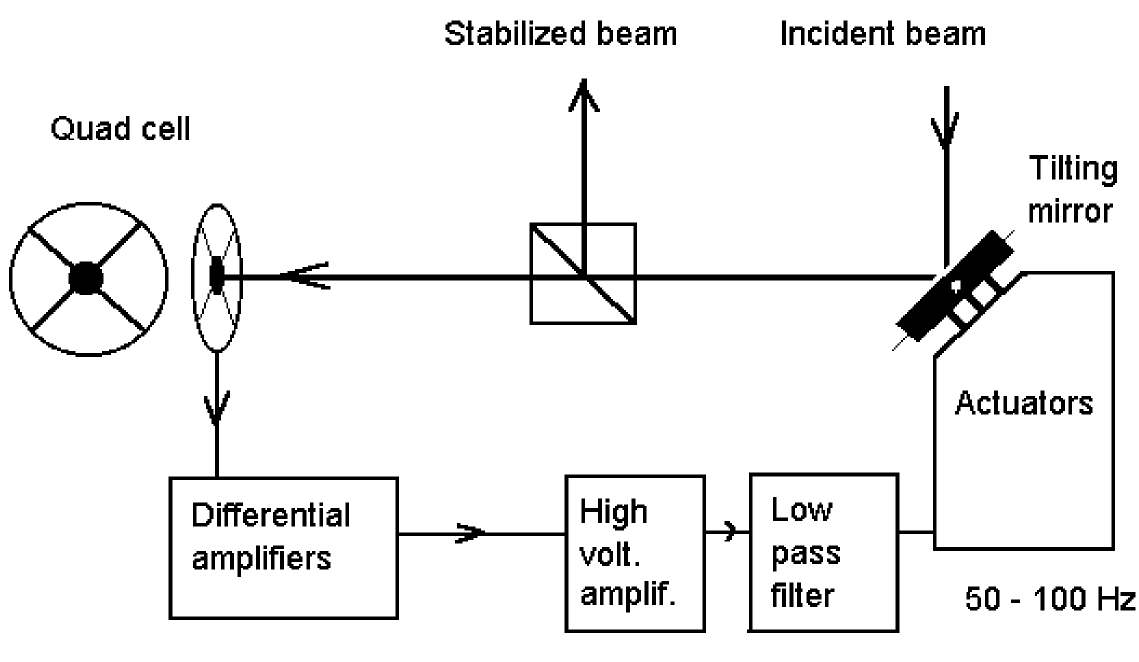
REAL-TIME IMAGE CORRECTION

Tilts of the wavefront (Z_2, Z_3) $\rightarrow$ image motion
– *sunspot tracker, correlation tracker*

Higher-order aberrations $\rightarrow$ image motion, defocus, astigmatism, coma, image distortion, ...
– *adaptive optics systems*

Sunspot tracker

A significant, high-contrast feature in the image (sunspot, pore).

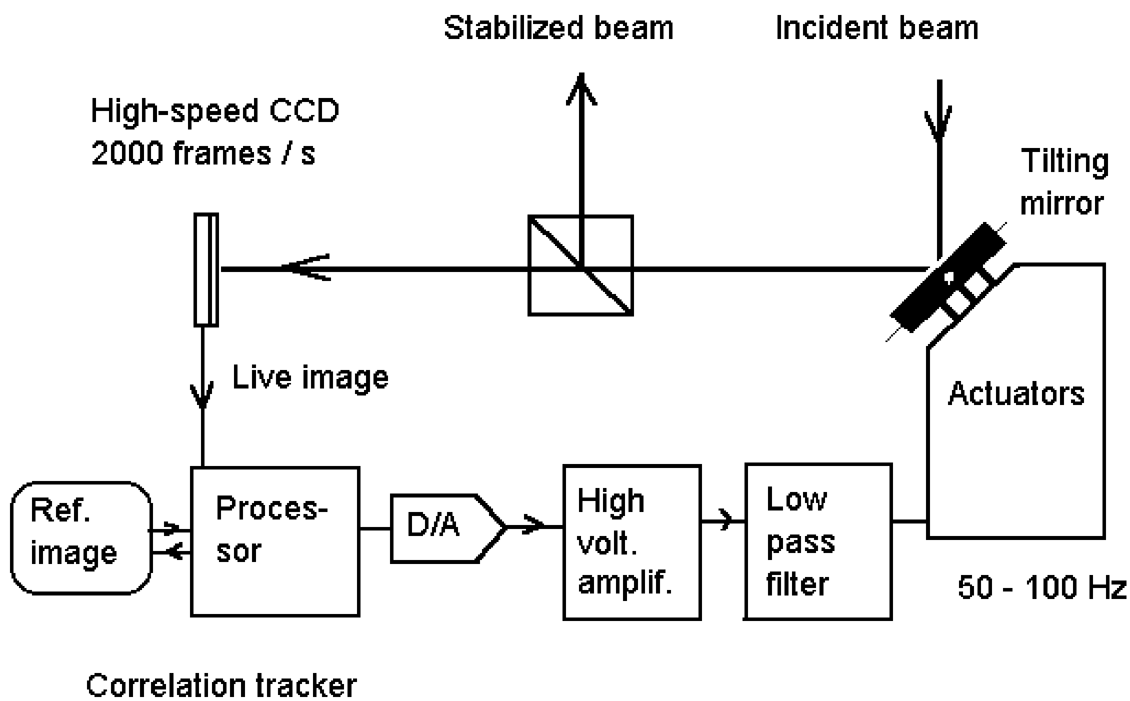


Sunspot tracker

Analog system, 2 degrees of freedom.
Simple, reliable, often used.

Correlation tracker

Low-contrast, homogeneous structures (solar granulation).



Digital system, 2 degrees of freedom.
In the processor, the *live image* is correlated with a *reference image* (updated every 10 s), and the shift of image corresponding to the best correlation is computed.

Fast *absolute differences* algorithm used for correlation:

$$D(k,l) = \sum_i \sum_j |i_{\rm ref}(i+k, j+l) - i_{\rm live}(i,j)| = minimum$$

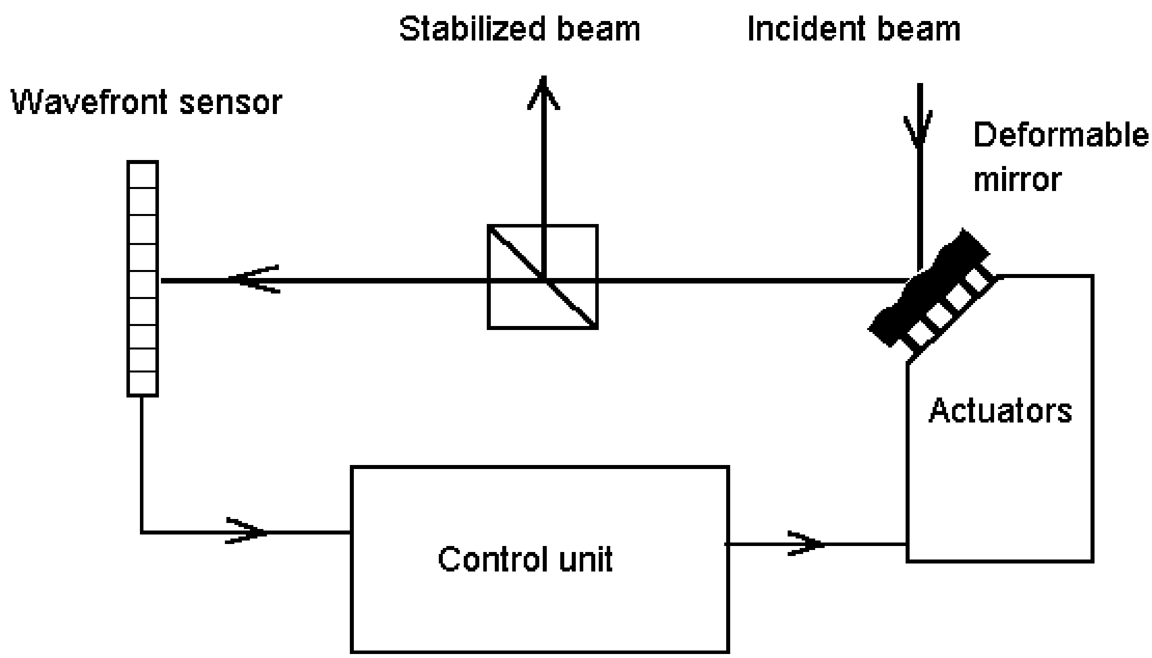
where (i,j) is the pixel position at the detector, and (k,l) is the correlation shift.

Adaptive optics systems

Deformable mirrors (segmented or elastic) driven by digital systems.

- *Slow systems* (several Hz) compensate time-dependent aberrations and the telescope, e.g. slow mechanical deformations (large thin mirrors).
- *Fast systems* (500–1000 Hz) for seeing corrections (in development).

N degrees of freedom = N channels
To compensate the effects of atmospheric turbulence and aberrations of the telescope, i.e. to reach the diffraction-limited resolution,
$N \geq (D/r_0)^2$ (D – aperture, r_0 – Fried parameter).



Adaptive optics system

The wavefront sensor determines the wavefront deformation ϕ.
Hartmann-Shack wavefront sensor:
2-D array of detectors, each detector has its own microlens.
Each local ϕ produces a shift of image on the corresponding detector.

DATA ACQUISITION AND REDUCTION

Image acquisition

2-D solid-state detectors (CCD cameras) $\rightarrow$ images in the form of 2-D digital arrays of image elements – pixels.

To achieve high spatial resolution:

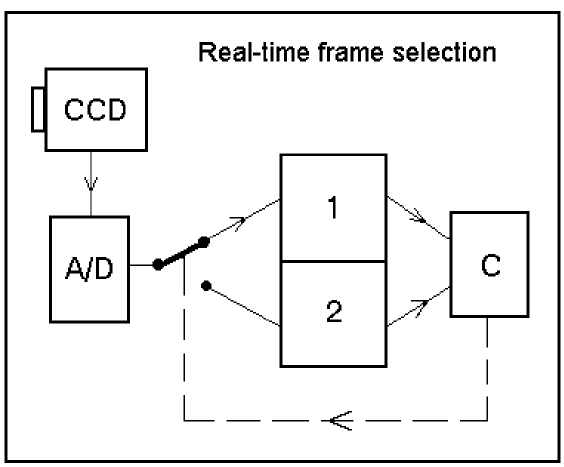
- short exposures
- moments of good seeing
- selection of recorded frames

Real-time frame selection

Indicator of image quality: *rms contrast* of granular field.

$$\Delta i_{\rm rms} = \frac{1}{\bar{i}} \sqrt{\frac{\sum_{n=1}^{N} \sum_{m=1}^{M} (i(n,m) - \bar{i})^2}{N\,M}} = \frac{\mathrm{stdev}(i)}{\bar{i}} \qquad (35)$$

$N \times M$ – size of the granular field (in pixels)
$\bar{i}$ – mean intensity in the granular field



Better frame remains in the buffer, worse frame is overwritten by the actual one. At the end of selection period remains the best frame.

Image pre-treatment

Each pixel of the detector slightly differs in:

- dark current dc – response to zero illumination
- gain – responsitivity (sensitivity) to illumination i

Assuming the detector to be linear:

$$response = i \cdot gain + dc$$
$$i = \frac{response - dc}{gain} \tag{36}$$

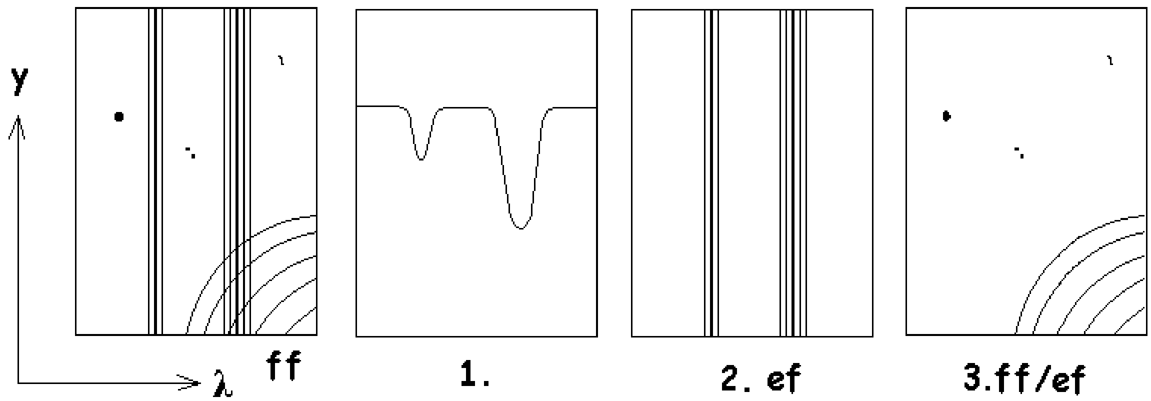
We need a **map of dark current** – an image taken with obstructed light path. To remove the noise we average many (~ 50) such images.

We need a **map of gain**, which can also include other disturbing effects:
– dust particles on optical surfaces,
– gradients of illumination,
– interference fringes.
This map is called **flatfield** ff. It is obtained as an image of a uniform source under identical observing conditions.
In practice, it can be obtained by defocussing the telescope, and taking many frames while the telescope moves quickly to blur the structures in the image. About 50–100 frames are necessary to average.

Then, the first step in data reduction is the *dc + flatfield correction.* From (36) we have for each pixel

$$i = \frac{i_{\mathrm{raw}} - dc}{ff - dc} \tag{37}$$

What to do with **images of spectra** where also spectral lines are present in the flatfield?
Spectrograph settings and wavelength must be preserved.



1. Compress the flatfield ff in y-direction, to obtain one average row (in the flatfield, spectral lines must be exactly in the y-direction).
2. Expand the average row by replicating to the size of the original 2-D flatfield. This expanded image ef contains information only about the spectral lines.
3. Divide $ff\,/\,ef$. The result is a flatfield without spectral lines.

Normalization

To compensate for changes in transparency of the sky and changes of exposure times, each frame is usually normalized to the mean intensity $\overline{i_q}$ of a quiet (undisturbed) region (in spectra, to the mean continuum intensity of a quiet region): $i_{\rm norm} = i/\,\overline{i_q}$.

Image deconvolution

How to compute the "true" intensity distribution i_0 from the observed i ? At a given instant (fixed time)

$$i(\mathbf{q}) = i_0(\mathbf{q}) * s(\mathbf{q})$$

and in the Fourier domain

$$I(\mathbf{u}) = I_0(\mathbf{u})\, S(\mathbf{u})$$

We can determine the OTF in the following cases:
– diffraction-limited telescope,
– long exposures,
– short exposures.

The formal solution is given by (10):

$$i_0(\mathbf{q}) = \mathcal{F}^{-1}\left(\frac{I(\mathbf{u})}{S(\mathbf{u})}\right)$$

Problems:

- Information at frequencies beyond the cutoff cannot be restored.
- **NOISE** $n(\mathbf{q})$ – mostly thermal noise of the CCD detector.

Assumption:
No correlation between signal and noise. → The noise can be considered as an additive contribution to the signal.

The noisy observed image can be expressed as:

$$i_N(\mathbf{q}) = i_0(\mathbf{q}) * s(\mathbf{q}) + n(\mathbf{q}) = i(\mathbf{q}) + n(\mathbf{q})$$

and in the Fourier domain

$$I_N(\mathbf{u}) = I_0(\mathbf{u})\, S(\mathbf{u}) + N(\mathbf{u}) = I(\mathbf{u}) + N(\mathbf{u}) \tag{38}$$

The formal solution of the deconvolution then gives:

$$\frac{I_N(\mathbf{u})}{S(\mathbf{u})} = \underbrace{\frac{I(\mathbf{u})}{S(\mathbf{u})}}_{I_0(\mathbf{u})} + \frac{N(\mathbf{u})}{S(\mathbf{u})} \tag{39}$$

This means that the noise is also restored (amplified). At high frequencies, where the amplification is strong and the signal-to-noise ratio (SNR) is low, the noise will be enhanced too much.

Prior to the restoration, the noise has to be *filtered*:

$$I_F(\mathbf{u}) = (I(\mathbf{u}) + N(\mathbf{u}))\,\Phi(\mathbf{u}) \tag{40}$$

where $\Phi(\mathbf{u})$ is a filter.

Optimum filter $\Phi(\mathbf{u})$

$\Phi(\mathbf{u})$ (a real function) weights Fourier components according to the level of noise at each frequency $\mathbf{u}$.

A condition to define the optimum filter:

$$\delta = \int_{image} (i(\mathbf{q}) - i_F(\mathbf{q}))^2 \, \mathrm{d}\mathbf{q} = minimum$$

In the Fourier domain (according to the Parseval's theorem)

$$\delta = \int |I(\mathbf{u}) - I_F(\mathbf{u})|^2 \, \mathrm{d}\mathbf{u} = \int \epsilon^2 \, \mathrm{d}\mathbf{u} = minimum \tag{41}$$

where ϵ is a residual error. From (40) and (41) we get

$$\delta = \int |\underbrace{I(\mathbf{u})(1 - \Phi(\mathbf{u}))}_{signal} - \underbrace{N(\mathbf{u})\,\Phi(\mathbf{u})}_{noise}|^2 \, \mathrm{d}\mathbf{u} = minimum \tag{42}$$

Since we assumed that the noise is uncorrelated with the signal, the cross products in the development of the squared modulus will be zero and (42) can be re-written as

$$\delta = \int (|I(\mathbf{u})|^2 (1 - \Phi(\mathbf{u}))^2 + |N(\mathbf{u})|^2 \Phi^2(\mathbf{u})) \, \mathrm{d}\mathbf{u} = \int \epsilon^2 \mathrm{d}\mathbf{u} = minimum \tag{43}$$

Finding the minimum of δ,

$$\frac{\partial \epsilon^2}{\partial \Phi} = -2 \, |I(\mathbf{u})|^2 (1 - \Phi(\mathbf{u})) + 2 \, |N(\mathbf{u})|^2 \Phi(\mathbf{u}) = 0$$

we obtain the optimum filter

$$\Phi(\mathbf{u}) = \frac{|I(\mathbf{u})|^2}{|I(\mathbf{u})|^2 + |N(\mathbf{u})|^2} \tag{44}$$

The optimum filter is determined by the power of observed signal and by the power of noise.

Combining the optimum filter (44) with the restoration process (10) we obtain the **optimum restoration filter** $\Phi_R(\mathbf{u})$

$$\Phi_R(\mathbf{u}) = \frac{|I(\mathbf{u})|^2}{|I(\mathbf{u})|^2 + |N(\mathbf{u})|^2} \cdot \frac{1}{S(\mathbf{u})} \tag{45}$$

and the image deconvolution has to be done following to the formula

$$i_0(\mathbf{q}) = \mathcal{F}^{-1}(I_N(\mathbf{u}) \cdot \Phi_R(\mathbf{u})) \tag{46}$$

Practical realization of $\Phi_R(\mathbf{u})$ for $S(\mathbf{u}) \in \mathcal{R}$, i.e. OTF = MTF (diffraction-limited telescope, long exposures, amplitude restoration for short exposures – $S(\mathbf{u}) = \sqrt{\mathrm{ETF}}$).

Substituting $I(\mathbf{u}) = I_0(\mathbf{u})\, S(\mathbf{u})$ in (45) we have

$$\Phi_R(\mathbf{u}) = \frac{|I_0(\mathbf{u})|^2\, S^2(\mathbf{u})}{|I_0(\mathbf{u})|^2\, S^2(\mathbf{u}) + |N(\mathbf{u})|^2} \cdot \frac{1}{S(\mathbf{u})}$$

that means

$$\Phi_R(\mathbf{u}) = \frac{S(\mathbf{u})}{S^2(\mathbf{u}) + |N(\mathbf{u})/I_0(\mathbf{u})|^2} \tag{47}$$

The restoration filter depends on the MTF and on the ratio of powers of the noise and signal, i.e. on the power of 1/SNR.

Variations of SNR with $\mathbf{u}$ can be modelled:

- *By a constant*:

  $$|N(\mathbf{u})/I_0(\mathbf{u})|^2 = 1/C$$

  This is a rough approximation but it works in many cases when the noise is not too strong. C must be determined empirically from the shape of restored images; $C \simeq 10$ to 50.

  $$\Phi_R(\mathbf{u}) = \frac{C+1}{C} \cdot \frac{S(\mathbf{u})}{S^2(\mathbf{u}) + 1/C} \tag{48}$$

- *By a linear relation*:

  $$|N(\mathbf{u})/I_0(\mathbf{u})|^2 = 0 \qquad \text{for} \quad |\mathbf{u}| \leq |\mathbf{u}_0|$$

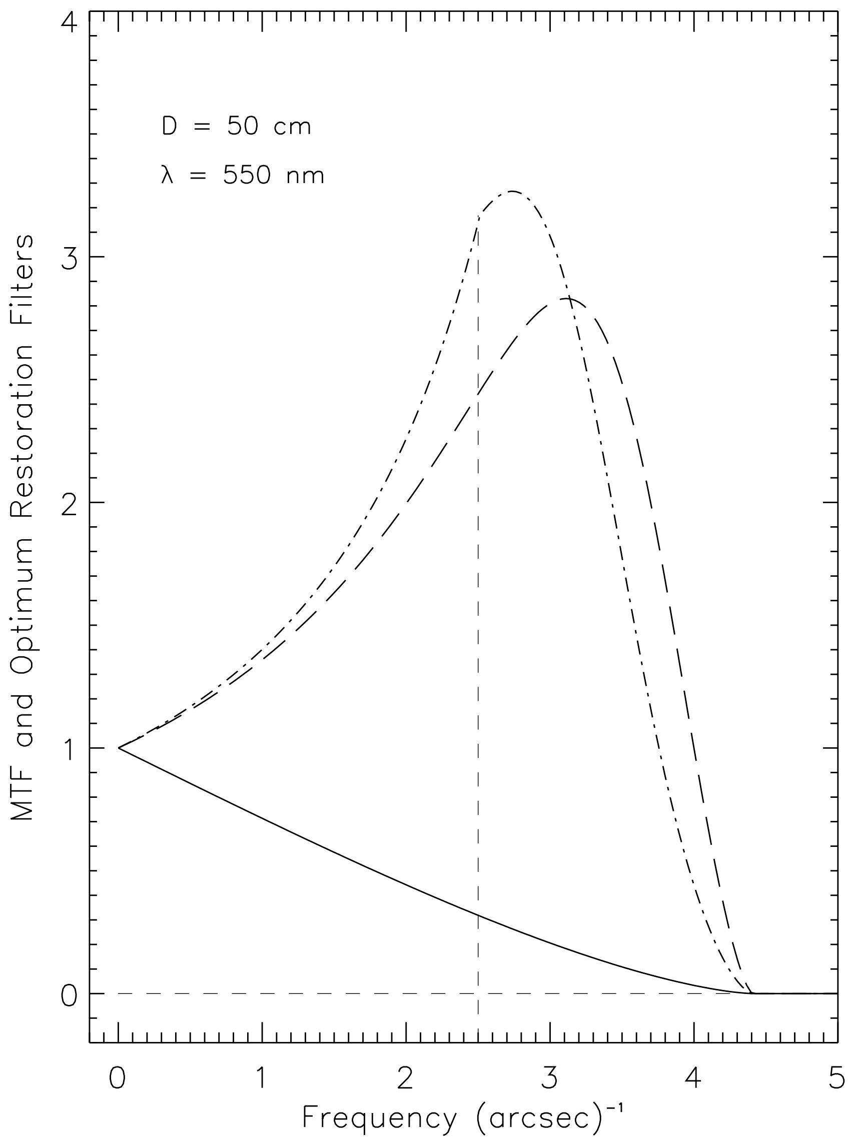
  $$|N(\mathbf{u})/I_0(\mathbf{u})|^2 = K(\mathbf{u} - \mathbf{u}_0) \qquad \text{for} \quad |\mathbf{u}| > |\mathbf{u}_0|$$

  So that,

  $$\Phi_R(\mathbf{u}) = 1/S(\mathbf{u}) \qquad \text{for} \quad |\mathbf{u}| \leq |\mathbf{u}_0|$$

  $$\Phi_R(\mathbf{u}) = \frac{S(\mathbf{u})}{S^2(\mathbf{u}) + K(\mathbf{u} - \mathbf{u}_0)} \qquad \text{for} \quad |\mathbf{u}| > |\mathbf{u}_0| \tag{49}$$

  Here the filtering takes place only for $|\mathbf{u}| > |\mathbf{u}_0|$ and we can better determine the shape of the filter, but two free parameters, $K, \mathbf{u}_0$, have to be found by trial and error.



An example of the telescope MTF for $D = 50$ cm and $\lambda = 550$ nm (*solid line*) together with optimum restoration filters:

1. Constant SNR (*long dash*), $C = 30$.
2. Linear increase of SNR (*dash-dot*), $K = 0.05$, $u_0 = 2.5$ (arcsec)$^{-1}$ – see the vertical dashed line.

The cutoff frequency u_c is 4.4 (arcsec)$^{-1}$.

Time series
– corrections for image motion and distortion

Time series are necessary to study the dynamics of solar features, e.g. lifetimes, photometric and morphological evolution, proper motions, oscillations, etc. Also necessary for speckle interferometry.

Image motion

Global displacements of images in the series – wavefront tilts and mechanical vibrations of the telescope.

Correction:
Realignment of each frame with a *reference image* i_R
by cross-correlation techniques.
Correlation:

$$C(\mathbf{d}) = \int i_j(\mathbf{q})\, i_R(\mathbf{q}+\mathbf{d})\, \mathrm{d}\mathbf{q} \tag{50}$$

We look for a shift $\mathbf{\Delta}$, where $C(\mathbf{\Delta}) = maximum$,
and shifting $i_j(\mathbf{q})$ by $-\mathbf{\Delta}$ we compensate the displacement.

Reference image:

- If the observed structures do not evolve significantly during the series, we can take any frame as a reference (e.g. first, middle, average image...)
- If the observed structures exhibit significant evolutionary changes, we use the *method of cumulative displacements*:
  1. We compute a set of relative displacements $\mathbf{\Delta}_{j,j-1}$ of couples of consecutive frames i_{j-1}, i_j.
  2. The vectorial summation of relative displacements gives the cumulative displacement of j-th frame with respect to the first one in the series:

$$\mathbf{\Delta}_j = \sum_{k \le j} \mathbf{\Delta}_{k,k-1}$$

Image distortion

Image distortion is due to anisoplanatic image motion. It is seen in a movie as relative random displacements between different parts of the image.

Correction: **destretching**

To correct for image distortion we must determine a map of local distortions $\mathbf{\Delta}(\mathbf{q}) = (\Delta_x(\mathbf{q}), \Delta_y(\mathbf{q}))$ for each frame with respect to a reference image.

Reference image:
Temporal running average of a few images preceding and following the current frame.

$\mathbf{\Delta}(\mathbf{q})$ is determined from *local correlation*:

$$C(\mathbf{d}, \mathbf{q}) = \int i(\mathbf{h})\, i_R(\mathbf{h} + \mathbf{d})\, w(\mathbf{q} - \mathbf{h})\, \mathrm{d}\mathbf{h} \tag{51}$$

$w(\mathbf{q} - \mathbf{h})$ – correlation window centered on a variable position $\mathbf{q}$; usually a Gaussian bell with FWHM comparable with the isoplanatic patch (2"–4"). w moves along the whole image.

$i(\mathbf{h})\, w(\mathbf{q} - \mathbf{h})$ – a part of the image centered at $\mathbf{q}$ that will be correlated with i_R.

$C(\mathbf{d}, \mathbf{q})$ represents a correlation (for a shift $\mathbf{d}$) around the position $\mathbf{q}$. Since the displacements are small, we need to compute $C(\mathbf{d}, \mathbf{q})$ only for a few values of $\mathbf{d}$ to find

$$C(\mathbf{\Delta}, \mathbf{q}) = maximum.$$

Having $\mathbf{\Delta}$ for each $\mathbf{q}$ we determined $\mathbf{\Delta}(\mathbf{q})$ – a map of local distortions. Then, the rectified (destretched) image will be

$$i_{rect}(\mathbf{q}) = i(\mathbf{q} - \mathbf{\Delta}(\mathbf{q}))$$

Since $\mathbf{\Delta}(\mathbf{q})$ is not necessarily integer, an interpolation has to be used.

END