Saturated Thermoporoelasticity

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The model

- non-stationary non-isothermal saturated water flow in a deformable porous medium
- isotropic elastic skeleton
- negligible inertial effects
- continuum approach, continuity assumption
- the assumption of thermal equilibrium
- the assumption of small perturbations (small transformations, small displacements, small variations of the porosity, water mass density and temperature) + the assumption of small deformation velocity
- compressive-positive pore pressures, tensile-positive stresses
- an extract from [Cou04] with the Lagrangian approach + a connection and adaptation of the Eulerian approach from [LS98]

Notation

t — the time

$\boldsymbol{u}$ — the displacement vector of the skeleton

$\boldsymbol{id} + \boldsymbol{u}$ — the deformation of the skeleton

$\boldsymbol{F} = \boldsymbol{I} + \boldsymbol{\nabla u}$ — the deformation gradient

$\boldsymbol{\varepsilon} \equiv \frac{1}{2}\big(\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^\top\big)$ — the linear strain tensor

$\varepsilon_v \equiv \operatorname{tr} \boldsymbol{\varepsilon} = \operatorname{div} \boldsymbol{u}$ — the volumetric strain

$J = \det(\boldsymbol{I} + \boldsymbol{\nabla u})$ — the Jacobian of the deformation
($\approx 1 + \varepsilon_v$ under the assumption of small transformations)

n — the Eulerian porosity

$\phi = Jn$ — the Lagrangian porosity

Balance laws

Skeleton mass balance

The Eulerian form (in the current configuration):

$$\frac{\partial(\rho_s(1-n))}{\partial t} + \operatorname{div}\big(\rho_s(1-n)\boldsymbol{v}_s\big) = 0 \tag{1}$$

ρ_s — the matrix mass density $\qquad$ $\boldsymbol{v}_s$ — the skeleton velocity

or equivalently:

$$\frac{\mathrm{D}_s(\rho_s(1-n))}{\mathrm{D}t} + \rho_s(1-n)\operatorname{div}\boldsymbol{v}_s = 0 \tag{2}$$

$\frac{\mathrm{D}_s}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{v}_s \cdot \nabla$ — the total time derivative with respect to the skeleton

The Lagrangian form (in the initial configuration):

$$\rho_s(1-n)J = \rho_{s0}(1-\phi_0) \tag{3}$$

ρ_{s0} — the initial skeleton mass density
$\phi_0(= n_0)$ — the initial Lagrangian (= initial Eulerian) porosity

Water mass balance

The Eulerian form:

$$\frac{\partial(\rho_w n)}{\partial t} + \operatorname{div}(\rho_w n \boldsymbol{v}_w) = 0 \tag{4}$$

ρ_w — the water mass density
$\boldsymbol{v}_w$ — the water velocity

or equivalently, referring to the skeleton motion:

$$\frac{\mathrm{D}_s(\rho_w n)}{\mathrm{D}t} + \rho_w n \operatorname{div} \boldsymbol{v}_s + \operatorname{div}(\rho_w \boldsymbol{q}_{rw}) = 0 \tag{5}$$

$\boldsymbol{q}_{rw} \equiv n(\boldsymbol{v}_w - \boldsymbol{v}_s)$ — the water specific discharge relative to the skeleton (or Darcy velocity or filtration vector)

The Lagrangian alternative:

$$\frac{\mathrm{d}(\rho_w \phi)}{\mathrm{d}t} + \operatorname{div} \boldsymbol{M} = 0 \tag{6}$$

$$\boldsymbol{M} \equiv J\boldsymbol{F}^{-1}(\rho_w \boldsymbol{q}_{rw}) \text{ — the Lagrangian relative flow vector of water mass} \tag{7}$$

Balance of momentum

For any current material domain V_t:

$$\frac{\mathrm{D}_s}{\mathrm{D}t}\int_{V_t} \rho_s(1-n)\boldsymbol{v}_s \,\mathrm{d}V_t + \frac{\mathrm{D}_w}{\mathrm{D}t}\int_{V_t} \rho_w n \boldsymbol{v}_w \,\mathrm{d}V_t = \int_{V_t} \rho \boldsymbol{f} \,\mathrm{d}V_t + \int_{\partial V_t} \boldsymbol{T} \,\mathrm{d}a \tag{8}$$

$\dfrac{\mathrm{D}_w}{\mathrm{D}t}$ — the total time derivative with respect to the water
$\rho \equiv \rho_s(1-n) + \rho_w n$ — the mass density of the porous medium (including both the skeleton and the water)
$\boldsymbol{f}$ — a body force density $\quad$ $\boldsymbol{T}$ — a surface force density

Applying the transport theorem to each component of the left-hand side of (8) one gets:

$$\begin{aligned}
&\frac{\mathrm{D}_s}{\mathrm{D}t}\int_{V_t} \rho_s(1-n)v_{si} \,\mathrm{d}V_t + \frac{\mathrm{D}_w}{\mathrm{D}t}\int_{V_t} \rho_w n v_{wi} \,\mathrm{d}V_t \\
&\quad = \int_{V_t} \left(\frac{\partial(\rho_s(1-n)v_{si})}{\partial t} + \operatorname{div}\big(\rho_s(1-n)v_{si}\boldsymbol{v}_s\big) + \frac{\partial(\rho_w n v_{wi})}{\partial t} + \operatorname{div}(\rho_w n v_{wi} \boldsymbol{v}_w) \right) \mathrm{d}V_t \quad \forall i
\end{aligned}$$

which together with the mass balance equations (1) and (4) yields:

$$\frac{\mathrm{D}_s}{\mathrm{D}t}\int_{V_t} \rho_s(1-n)\boldsymbol{v}_s \,\mathrm{d}V_t + \frac{\mathrm{D}_w}{\mathrm{D}t}\int_{V_t} \rho_w n \boldsymbol{v}_w \,\mathrm{d}V_t = \int_{V_t} \big(\rho_s(1-n)\boldsymbol{a}_s + \rho_w n \boldsymbol{a}_w\big) \,\mathrm{d}V_t$$

$\boldsymbol{a}_s \equiv \dfrac{\mathrm{D}_s \boldsymbol{v}_s}{\mathrm{D}t} = \dfrac{\partial \boldsymbol{v}_s}{\partial t} + (\boldsymbol{\nabla} \boldsymbol{v}_s)\boldsymbol{v}_s$ — the skeleton acceleration
$\boldsymbol{a}_w \equiv \dfrac{\mathrm{D}_w \boldsymbol{v}_w}{\mathrm{D}t} = \dfrac{\partial \boldsymbol{v}_w}{\partial t} + (\boldsymbol{\nabla} \boldsymbol{v}_w)\boldsymbol{v}_w$ — the water acceleration

Use of the Cauchy stress tensor in the surface integral of (8) in connection with the divergence theorem gives:

$$\int_{\partial V_t} \boldsymbol{T}\,\mathrm{d}a = \int_{\partial V_t} \boldsymbol{\sigma}\boldsymbol{n}\,\mathrm{d}a = \int_{V_t} \mathbf{div}\,\boldsymbol{\sigma}\,\mathrm{d}V_t$$

$\boldsymbol{\sigma}$ — the Cauchy stress tensor $\qquad$ $\boldsymbol{n}$ — the outward unit normal to V_t

Hence one can rewrite (8) in the form:

$$\int_{V_t} \big(\mathbf{div}\,\boldsymbol{\sigma} + \rho\boldsymbol{f} - \rho_s(1-n)\boldsymbol{a}_s - \rho_w n\boldsymbol{a}_w\big)\,\mathrm{d}V_t = \mathbf{0}$$

which leads to the local equation of motion:

$$\mathbf{div}\,\boldsymbol{\sigma} + \rho\boldsymbol{f} - \rho_s(1-n)\boldsymbol{a}_s - \rho_w n\boldsymbol{a}_w = \mathbf{0} \tag{9}$$

By neglecting the inertial forces one arrives at:

Equilibrium equation
The Eulerian form:

$$\mathbf{div}\,\boldsymbol{\sigma} + (\rho_s(1-n) + \rho_w n)\boldsymbol{f} = \mathbf{0} \tag{10}$$

The Lagrangian counterpart:

$$\mathbf{div}(\boldsymbol{F}\boldsymbol{\Pi}) + (\rho_{s0}(1-\phi_0) + \rho_w\phi)\boldsymbol{f} = \mathbf{0} \tag{11}$$

$$\boldsymbol{\Pi} \equiv J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-\top} \text{ — the Piola-Kirchhoff stress tensor} \tag{12}$$

Balance of moment of momentum

$\Longrightarrow$ symmetry of the stress tensor $\boldsymbol{\sigma}$

Energy conservation

Partial stress tensors
The stress tensor $\boldsymbol{\sigma}$ can be decomposed as:

$$\boldsymbol{\sigma} = (1-n)\boldsymbol{\sigma}_s + n\boldsymbol{\sigma}_w \tag{13}$$

$\boldsymbol{\sigma}_s, \boldsymbol{\sigma}_w$ — the partial stress tensors related to the solid matrix and the water, respectively

so that:

$$\boldsymbol{T} = \boldsymbol{T}_s + \boldsymbol{T}_w \qquad \boldsymbol{T}_s = (1-n)\boldsymbol{\sigma}_s\boldsymbol{n}, \quad \boldsymbol{T}_w = n\boldsymbol{\sigma}_w\boldsymbol{n} \tag{14}$$

Here one can take approximately

$$\boldsymbol{\sigma}_w = -p_w\boldsymbol{I} \tag{15}$$

p_w — the water pressure

because the water shear stress plays the main role in the interaction force between the water and the skeleton (which does not explicitly appear here).

Mechanical energy equation

The work rate $\mathcal{P}_{V_t}(\boldsymbol{f},\boldsymbol{T})$ supplied by the external body and surface forces to a material domain V_t is given by:

$$\begin{aligned}\mathcal{P}_{V_t}(\boldsymbol{f},\boldsymbol{T}) &\equiv \int_{V_t}\big(\rho_s(1-n)\boldsymbol{f}\cdot\boldsymbol{v}_s+\rho_w n\boldsymbol{f}\cdot\boldsymbol{v}_w\big)\,\mathrm{d}V_t+\int_{\partial V_t}\big(\boldsymbol{T}_s\cdot\boldsymbol{v}_s+\boldsymbol{T}_w\cdot\boldsymbol{v}_w\big)\,\mathrm{d}a\\ &=\int_{V_t}\big(\rho\boldsymbol{f}\cdot\boldsymbol{v}_s+\rho_w\boldsymbol{f}\cdot\boldsymbol{q}_{rw}\big)\,\mathrm{d}V_t+\int_{\partial V_t}\big(\boldsymbol{T}\cdot\boldsymbol{v}_s+\boldsymbol{T}_w\cdot(\boldsymbol{v}_w-\boldsymbol{v}_s)\big)\,\mathrm{d}a\end{aligned}$$

The divergence theorem and the symmetry of the stress tensor $\boldsymbol{\sigma}$ yield:

$$\int_{\partial V_t}\boldsymbol{T}\cdot\boldsymbol{v}_s\,\mathrm{d}a=\int_{\partial V_t}(\boldsymbol{\sigma}\boldsymbol{n})\cdot\boldsymbol{v}_s\,\mathrm{d}a=\int_{V_t}\big((\mathbf{div}\,\boldsymbol{\sigma})\cdot\boldsymbol{v}_s+\boldsymbol{\sigma}:\boldsymbol{d}_s\big)\,\mathrm{d}a$$

$$\boldsymbol{d}_s\equiv\frac{1}{2}\big(\boldsymbol{\nabla}\boldsymbol{v}_s+(\boldsymbol{\nabla}\boldsymbol{v}_s)^\top\big)\text{ — the Eulerian strain rate tensor}\tag{16}$$

Owing to (14) and (15) one gets:

$$\int_{\partial V_t}\boldsymbol{T}_w\cdot(\boldsymbol{v}_w-\boldsymbol{v}_s)\,\mathrm{d}a=\int_{\partial V_t}(\boldsymbol{\sigma}_w\boldsymbol{n})\cdot\boldsymbol{q}_{rw}\,\mathrm{d}a=\int_{\partial V_t}(-p_w\boldsymbol{q}_{rw})\cdot\boldsymbol{n}\,\mathrm{d}a=\int_{V_t}\big(-\operatorname{div}(p_w\boldsymbol{q}_{rw})\big)\,\mathrm{d}V_t$$

The total derivative of the kinetic energy associated with the skeleton and water particles in V_t reads:

$$\begin{aligned}&\frac{\mathrm{D}_s}{\mathrm{D}t}\int_{V_t}\frac{1}{2}\rho_s(1-n)\boldsymbol{v}_s\cdot\boldsymbol{v}_s\,\mathrm{d}V_t+\frac{\mathrm{D}_w}{\mathrm{D}t}\int_{V_t}\frac{1}{2}\rho_w n\boldsymbol{v}_w\cdot\boldsymbol{v}_w\,\mathrm{d}V_t\\ &\quad=\int_{V_t}\frac{1}{2}\bigg(\frac{\partial(\rho_s(1-n)\boldsymbol{v}_s\cdot\boldsymbol{v}_s)}{\partial t}+\operatorname{div}\big(\rho_s(1-n)(\boldsymbol{v}_s\cdot\boldsymbol{v}_s)\boldsymbol{v}_s\big)\\ &\qquad\qquad+\frac{\partial(\rho_w n\boldsymbol{v}_w\cdot\boldsymbol{v}_w)}{\partial t}+\operatorname{div}(\rho_w n(\boldsymbol{v}_w\cdot\boldsymbol{v}_w)\boldsymbol{v}_w)\bigg)\,\mathrm{d}V_t\\ &\quad\overset{(1),(4)}{=}\int_{V_t}\frac{1}{2}\bigg(\rho_s(1-n)\frac{\mathrm{D}_s(\boldsymbol{v}_s\cdot\boldsymbol{v}_s)}{\mathrm{D}t}+\rho_w n\frac{\mathrm{D}_w(\boldsymbol{v}_w\cdot\boldsymbol{v}_w)}{\mathrm{D}t}\bigg)\,\mathrm{d}V_t\\ &\quad=\int_{V_t}\big(\rho_s(1-n)\boldsymbol{a}_s\cdot\boldsymbol{v}_s+\rho_w n\boldsymbol{a}_w\cdot\boldsymbol{v}_w\big)\,\mathrm{d}V_t\\ &\quad=\int_{V_t}\Big(\big(\rho_s(1-n)\boldsymbol{a}_s+\rho_w n\boldsymbol{a}_w\big)\cdot\boldsymbol{v}_s+\rho_w\boldsymbol{a}_w\cdot\boldsymbol{q}_{rw}\Big)\,\mathrm{d}V_t\end{aligned}$$

Use of the equation of motion (9) finally leads to the mechanical energy equation in the form:

$$\begin{aligned}&\mathcal{P}_{V_t}(\boldsymbol{f},\boldsymbol{T})-\frac{\mathrm{D}_s}{\mathrm{D}t}\int_{V_t}\frac{1}{2}\rho_s(1-n)\boldsymbol{v}_s\cdot\boldsymbol{v}_s\,\mathrm{d}V_t-\frac{\mathrm{D}_w}{\mathrm{D}t}\int_{V_t}\frac{1}{2}\rho_w n\boldsymbol{v}_w\cdot\boldsymbol{v}_w\,\mathrm{d}V_t\\ &\quad=\int_{V_t}\Big(\big(\mathbf{div}\,\boldsymbol{\sigma}+\rho\boldsymbol{f}-\rho_s(1-n)\boldsymbol{a}_s-\rho_w n\boldsymbol{a}_w\big)\cdot\boldsymbol{v}_s\\ &\qquad\qquad+\boldsymbol{\sigma}:\boldsymbol{d}_s-\operatorname{div}(p_w\boldsymbol{q}_{rw})+\rho_w(\boldsymbol{f}-\boldsymbol{a}_w)\cdot\boldsymbol{q}_{rw}\Big)\,\mathrm{d}V_t\\ &\quad=\int_{V_t}\big(\boldsymbol{\sigma}:\boldsymbol{d}_s-\operatorname{div}(p_w\boldsymbol{q}_{rw})+\rho_w(\boldsymbol{f}-\boldsymbol{a}_w)\cdot\boldsymbol{q}_{rw}\big)\,\mathrm{d}V_t\end{aligned}\tag{17}$$

where the right-hand side can be interpreted as the strain work rate related to the porous medium contained in V_t.

By neglecting the inertia effects one obtains:

$$\mathcal{P}_{V_t}(\boldsymbol{f},\boldsymbol{T})=\int_{V_t}\big(\boldsymbol{\sigma}:\boldsymbol{d}_s-\operatorname{div}(p_w\boldsymbol{q}_{rw})+\rho_w\boldsymbol{f}\cdot\boldsymbol{q}_{rw}\big)\,\mathrm{d}V_t$$

(which can be alternatively derived by applying the equilibrium equation (10) solely to $\mathcal{P}_{V_t}(\boldsymbol{f},\boldsymbol{T})$ in the procedure above).

Energy balance

The conservation of energy in all its possible forms currently contained in a material volume V_t is expressed by:

$$\frac{\mathrm{D}_s}{\mathrm{D}t}\int_{V_t}\rho_s(1-n)\left(e_s+\frac{1}{2}\boldsymbol{v}_s\cdot\boldsymbol{v}_s\right)\mathrm{d}V_t+\frac{\mathrm{D}_w}{\mathrm{D}t}\int_{V_t}\rho_w n\left(e_w+\frac{1}{2}\boldsymbol{v}_w\cdot\boldsymbol{v}_w\right)\mathrm{d}V_t$$
$$=\mathcal{P}_{V_t}(\boldsymbol{f},\boldsymbol{T})-\int_{\partial V_t}\boldsymbol{q}\cdot\boldsymbol{n}\,\mathrm{d}a \quad (18)$$

e_s, e_w — the specific internal energies of the solid matrix and the water, respectively
$\boldsymbol{q}$ — the heat flux vector

Use of the transport theorem furnishes:

$$\begin{aligned}
&\frac{\mathrm{D}_s}{\mathrm{D}t}\int_{V_t}\rho_s(1-n)e_s\,\mathrm{d}V_t+\frac{\mathrm{D}_w}{\mathrm{D}t}\int_{V_t}\rho_w n e_w\,\mathrm{d}V_t\\
&=\int_{V_t}\left(\frac{\partial(\rho_s(1-n)e_s)}{\partial t}+\operatorname{div}\big(\rho_s(1-n)e_s\boldsymbol{v}_s\big)+\frac{\partial(\rho_w n e_w)}{\partial t}+\operatorname{div}(\rho_w n e_w\boldsymbol{v}_w)\right)\mathrm{d}V_t\\
&=\int_{V_t}\left(\frac{\mathrm{D}_s(\rho_s(1-n)e_s)}{\mathrm{D}t}+\rho_s(1-n)e_s\operatorname{div}\boldsymbol{v}_s+\frac{\mathrm{D}_s(\rho_w n e_w)}{\mathrm{D}t}+\rho_w n e_w\operatorname{div}\boldsymbol{v}_s+\operatorname{div}(\rho_w e_w\boldsymbol{q}_{rw})\right)\mathrm{d}V_t
\end{aligned}$$

which together with the mechanical energy equation (17) and the divergence theorem allows us to rewrite (18) as:

$$\int_{V_t}\Bigg(\frac{\mathrm{D}_s(\rho_s(1-n)e_s)}{\mathrm{D}t}+\rho_s(1-n)e_s\operatorname{div}\boldsymbol{v}_s+\frac{\mathrm{D}_s(\rho_w n e_w)}{\mathrm{D}t}+\rho_w n e_w\operatorname{div}\boldsymbol{v}_s+\operatorname{div}(\rho_w e_w\boldsymbol{q}_{rw})$$
$$-\boldsymbol{\sigma}:\boldsymbol{d}_s+\operatorname{div}(p_w\boldsymbol{q}_{rw})-\rho_w(\boldsymbol{f}-\boldsymbol{a}_w)\cdot\boldsymbol{q}_{rw}+\operatorname{div}\boldsymbol{q}\Bigg)\mathrm{d}V_t=0 \quad (19)$$

Neglecting the inertia force provides the local form:

$$\frac{\mathrm{D}_s(\rho_s(1-n)e_s)}{\mathrm{D}t}+\rho_s(1-n)e_s\operatorname{div}\boldsymbol{v}_s+\frac{\mathrm{D}_s(\rho_w n e_w)}{\mathrm{D}t}+\rho_w n e_w\operatorname{div}\boldsymbol{v}_s+\operatorname{div}(\rho_w e_w\boldsymbol{q}_{rw})$$
$$-\boldsymbol{\sigma}:\boldsymbol{d}_s+\operatorname{div}(p_w\boldsymbol{q}_{rw})-\rho_w\boldsymbol{f}\cdot\boldsymbol{q}_{rw}+\operatorname{div}\boldsymbol{q}=0$$

Use of (2) and (5) then yields the Euler energy equation:

$$\boxed{\rho_s(1-n)\frac{\mathrm{D}_s e_s}{\mathrm{D}t}+\rho_w n\frac{\mathrm{D}_s e_w}{\mathrm{D}t}=\boldsymbol{\sigma}:\boldsymbol{d}_s-\operatorname{div}(p_w\boldsymbol{q}_{rw}+\boldsymbol{q})-\rho_w\boldsymbol{q}_{rw}\cdot\nabla e_w+\rho_w\boldsymbol{f}\cdot\boldsymbol{q}_{rw}} \quad (20)$$

With the aim of transporting equation (19) from the current volume V_t to the corresponding volume V_0 in the skeleton initial configuration, we firstly introduce:

$e\equiv\rho_s(1-n)e_s+\rho_w n e_w$ — the overall density of internal energy per unit volume of porous medium

$h_w\equiv e_w+\dfrac{p_w}{\rho_w}$ — the water specific enthalpy (21)

and express (19) in the form:

$$\int_{V_t}\left(\frac{\mathrm{D}_s e}{\mathrm{D}t}+e\operatorname{div}\boldsymbol{v}_s-\boldsymbol{\sigma}:\boldsymbol{d}_s+\operatorname{div}(\rho_w h_w\boldsymbol{q}_{rw}+\boldsymbol{q})-\rho_w(\boldsymbol{f}-\boldsymbol{a}_w)\cdot\boldsymbol{q}_{rw}\right)\mathrm{d}V_t=0$$

Now use of standard transport formulae provides:

$$\int_{V_t}\left(\frac{\mathrm{D}_s e}{\mathrm{D}t}+e\,\mathrm{div}\,\boldsymbol{v}_s\right)\mathrm{d}V_t=\int_{V_0}\left(\frac{\partial e}{\partial t}+\boldsymbol{v}_s\cdot\nabla e+e\,\mathrm{div}\,\boldsymbol{v}_s\right)J\,\mathrm{d}V_0=\int_{V_0}\frac{\mathrm{d}(eJ)}{\mathrm{d}t}\,\mathrm{d}V_0=\int_{V_0}\frac{\mathrm{d}E}{\mathrm{d}t}\,\mathrm{d}V_0$$

$E\equiv eJ$ — the overall Lagrangian density of internal energy per unit of initial porous medium volume

$$\int_{V_t}\mathrm{div}\,\boldsymbol{q}\,\mathrm{d}V_t=\int_{V_0}\mathrm{div}\,\boldsymbol{Q}\,\mathrm{d}V_0$$

$\boldsymbol{Q}\equiv J\boldsymbol{F}^{-1}\boldsymbol{q}$ — the Lagrangian heat flow vector

$$\int_{V_t}\rho_w(\boldsymbol{f}-\boldsymbol{a}_w)\cdot\boldsymbol{q}_{rw}\,\mathrm{d}V_t\overset{(7)}{=}\int_{V_0}(\boldsymbol{f}-\boldsymbol{a}_w)\cdot(\boldsymbol{F}\boldsymbol{M})\,\mathrm{d}V_0 \tag{22}$$

Moreover, one has for a particle which was initially located by a position vector $\boldsymbol{X}$ and is currently located by a position vector $\boldsymbol{x}=\boldsymbol{X}+\boldsymbol{u}$:

$$\nabla h_w(\boldsymbol{X})=(\boldsymbol{F}(\boldsymbol{X}))^\top\nabla h_w(\boldsymbol{x})$$

$$\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}t}(\boldsymbol{X})=\frac{\mathrm{d}(\boldsymbol{\nabla}\boldsymbol{u})}{\mathrm{d}t}(\boldsymbol{X})=\left(\boldsymbol{\nabla}\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\right)(\boldsymbol{X})=\boldsymbol{\nabla}\boldsymbol{v}_s(\boldsymbol{x})\boldsymbol{F}(\boldsymbol{X})$$

From here:

$$\begin{aligned}\int_{V_t}\mathrm{div}(\rho_w h_w\boldsymbol{q}_{rw})\,\mathrm{d}V_t&=\int_{V_t}\big(h_w\,\mathrm{div}(\rho_w\boldsymbol{q}_{rw})+\rho_w\boldsymbol{q}_{rw}\cdot\nabla h_w\big)\,\mathrm{d}V_t\\&=\int_{V_0}\big(h_w\,\mathrm{div}\,\boldsymbol{M}+\rho_w\boldsymbol{q}_{rw}\cdot(\boldsymbol{F}^{-\top}\nabla h_w)J\big)\,\mathrm{d}V_0\\&=\int_{V_0}\big(h_w\,\mathrm{div}\,\boldsymbol{M}+(J\boldsymbol{F}^{-1}(\rho_w\boldsymbol{q}_{rw}))\cdot\nabla h_w\big)\,\mathrm{d}V_0=\int_{V_0}\mathrm{div}(h_w\boldsymbol{M})\,\mathrm{d}V_0\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{d}_s\overset{(16)}{=}\frac{1}{2}\big(\boldsymbol{\nabla}\boldsymbol{v}_s+(\boldsymbol{\nabla}\boldsymbol{v}_s)^\top\big)&=\frac{1}{2}\left(\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}t}\boldsymbol{F}^{-1}+\boldsymbol{F}^{-\top}\left(\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}t}\right)^\top\right)\\&=\frac{1}{2}\boldsymbol{F}^{-\top}\left(\boldsymbol{F}^\top\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}t}+\left(\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}t}\right)^\top\boldsymbol{F}\right)\boldsymbol{F}^{-1}=\boldsymbol{F}^{-\top}\frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t}\boldsymbol{F}^{-1}\end{aligned}$$

$\boldsymbol{E}\equiv\frac{1}{2}(\boldsymbol{F}^\top\boldsymbol{F}-\boldsymbol{I})$ — the Green-Lagrange strain tensor

$$\begin{aligned}\int_{V_t}\boldsymbol{\sigma}:\boldsymbol{d}_s\,\mathrm{d}V_t&=\int_{V_0}\boldsymbol{\sigma}:\left(\boldsymbol{F}^{-\top}\frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t}\boldsymbol{F}^{-1}\right)J\,\mathrm{d}V_0\\&=\int_{V_0}\left(J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-\top}\right):\frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t}\,\mathrm{d}V_0\overset{(12)}{=}\int_{V_0}\boldsymbol{\Pi}:\frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t}\,\mathrm{d}V_0\end{aligned}$$

Altogether, this furnishes:

$$\int_{V_0}\left(\frac{\mathrm{d}E}{\mathrm{d}t}-\boldsymbol{\Pi}:\frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t}+\mathrm{div}(h_w\boldsymbol{M}+\boldsymbol{Q})-(\boldsymbol{f}-\boldsymbol{a}_w)\cdot(\boldsymbol{F}\boldsymbol{M})\right)\mathrm{d}V_0=0$$

Neglecting the inertia force delivers the Lagrangian local energy equation:

$$\frac{\mathrm{d}E}{\mathrm{d}t}=\boldsymbol{\Pi}:\frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t}-\mathrm{div}(h_w\boldsymbol{M}+\boldsymbol{Q})+\boldsymbol{f}\cdot(\boldsymbol{F}\boldsymbol{M}) \tag{23}$$

Finally, we express the energy balance in terms of entropies.

Assumption. The water and the matrix are locally in *thermal equilibrium*, that is, at the same absolute temperature T (the timescale of the modelled phenomena is substantially larger than the relaxation time required to reach thermal equilibrium locally).

Let:

$$s_s \text{ — the Eulerian specific entropy of the matrix}$$
$$s_w \text{ — the Eulerian water specific entropy}$$
$$S \equiv (\rho_s(1-n)s_s + \rho_w n s_w)J \text{ — the overall Lagrangian density of entropy per unit of initial porous medium volume} \tag{24}$$
$$\Psi \equiv E - TS \text{— the overall Lagrangian density of Helmholtz free energy} \tag{25}$$

Then

$$\begin{aligned} T\frac{\mathrm{d}S}{\mathrm{d}t} &= \frac{\mathrm{d}E}{\mathrm{d}t} - \frac{\mathrm{d}\Psi}{\mathrm{d}t} - S\frac{\mathrm{d}T}{\mathrm{d}t} \\ &\overset{(23)}{=} \boldsymbol{\Pi} : \frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t} - (h_w - Ts_w)\,\mathrm{div}\,\boldsymbol{M} - S\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{\mathrm{d}\Psi}{\mathrm{d}t} - (\nabla h_w - T\nabla s_w)\cdot\boldsymbol{M} + \boldsymbol{f}\cdot(\boldsymbol{F}\boldsymbol{M}) \\ &\quad - \mathrm{div}\,\boldsymbol{Q} - T\,\mathrm{div}(s_w\boldsymbol{M}) \end{aligned}$$

which provides the entropy balance in the form of the Lagrangian thermal equation:

$$\boxed{T\left(\frac{\mathrm{d}S}{\mathrm{d}t} + \mathrm{div}(s_w\boldsymbol{M})\right) = -\,\mathrm{div}\,\boldsymbol{Q} + \Phi_s + \Phi_w} \tag{26}$$
$$\Phi_s \equiv \boldsymbol{\Pi} : \frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t} - (h_w - Ts_w)\,\mathrm{div}\,\boldsymbol{M} - S\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{\mathrm{d}\Psi}{\mathrm{d}t} \text{ — the Lagrangian density of skeleton dissipation} \tag{27}$$
$$\Phi_w \equiv -(\nabla h_w - T\nabla s_w)\cdot\boldsymbol{M} + \boldsymbol{f}\cdot(\boldsymbol{F}\boldsymbol{M}) \text{ — the Lagrangian density of water dissipation} \tag{28}$$

The identification of Φ_s and Φ_w as the dissipation terms related respectively to the skeleton and water will be done later.

Constitutive relationships

Lagrangian approach

Water

One can obtain the following water state equations by applying the first two laws of thermostatics:

$$h_w = h_w(p_w, s_w) \qquad \frac{1}{\rho_w} = \frac{\partial h_w}{\partial p_w} \qquad T = \frac{\partial h_w}{\partial s_w} \tag{29}$$

These state equations can be inverted with respect to the conjugate sets of thermodynamical state variables (p_w, s_w) and $(1/\rho_w, T)$. Indeed, by introducing:

$$\psi_w \equiv e_w - Ts_w \overset{(21)}{=} h_w - \frac{p_w}{\rho_w} - Ts_w \text{ — the water specific Helmholtz free energy} \tag{30}$$

one gets:

$$\mathrm{d}\psi_w = \frac{\partial h_w}{\partial p_w}\mathrm{d}p_w + \frac{\partial h_w}{\partial s_w}\mathrm{d}s_w - \frac{\mathrm{d}p_w}{\rho_w} - p_w\mathrm{d}\left(\frac{1}{\rho_w}\right) - T\mathrm{d}s_w - s_w\mathrm{d}T \overset{(29)}{=} -p_w\mathrm{d}\left(\frac{1}{\rho_w}\right) - s_w\mathrm{d}T$$

and the state equations take the alternative form:

$$\psi_w = \psi_w\left(\frac{1}{\rho_w}, T\right) \qquad p_w = -\frac{\partial \psi_w}{\partial(1/\rho_w)} \qquad s_w = -\frac{\partial \psi_w}{\partial T} \tag{31}$$

Equations (29) can also be only partially inverted with respect to the couple of conjugate variables (s_w, T) by introducing:

$$g_w \equiv h_w - Ts_w = \psi_w + \frac{p_w}{\rho_w} \text{ — the water specific free enthalpy (also called the Gibbs potential)}$$

so that:

$$\mathrm{d}g_w = \frac{\partial h_w}{\partial p_w}\mathrm{d}p_w + \frac{\partial h_w}{\partial s_w}\mathrm{d}s_w - T\mathrm{d}s_w - s_w\mathrm{d}T \overset{(29)}{=} \frac{1}{\rho_w}\mathrm{d}p_w - s_w\mathrm{d}T$$

and one obtains:

$$g_w = g_w(p_w, T) \qquad \frac{1}{\rho_w} = \frac{\partial g_w}{\partial p_w} \qquad s_w = -\frac{\partial g_w}{\partial T} \tag{32}$$

By differentiating (32) one finally arrives at the constitutive equations:

$$\frac{\mathrm{d}\rho_w}{\rho_w} = \frac{\mathrm{d}p_w}{K_w} - \beta_w \mathrm{d}T \tag{33}$$

K_w — the water bulk modulus
β_w — the water volumetric thermal expansion coefficient

$$\mathrm{d}s_w = -\beta_w \frac{\mathrm{d}p_w}{\rho_w} + c_{p_w}\frac{\mathrm{d}T}{T} \tag{34}$$

c_{p_w} — the water specific heat capacity at constant pressure

Note that considering K_w and β_w constant (over some ranges of pressures and temperatures), one can integrate (33) into the form:

$$\rho_w = \rho_{w0} e^{(p_w - p_{w0})/K_w - \beta_w (T - T_0)}$$

ρ_{w0}, p_{w0}, T_0 — initial values of the water density, pressure and temperature

Darcy's law
(for negligible inertial forces)

$$\boldsymbol{q}_{rw} = \frac{\boldsymbol{k}}{\mu_w}(-\nabla p_w + \rho_w \boldsymbol{f}) \tag{35}$$

$\boldsymbol{k}$ — the (intrinsic) permeability tensor of the porous medium (in a general anisotropic case)
μ_w — the dynamic viscosity of water

Let us look at the water dissipation term Φ_w (28) at this point. Employing (29) one has:

$$\nabla h_w - T\nabla s_w = \frac{\partial h_w}{\partial p_w}\nabla p_w + \frac{\partial h_w}{\partial s_w}\nabla s_w - T\nabla s_w = \frac{1}{\rho_w}\nabla p_w$$

so we can rewrite Φ_w as:

$$\Phi_w = -\frac{1}{\rho_w}(\nabla p_w) \cdot \boldsymbol{M} + \boldsymbol{f} \cdot (\boldsymbol{F}\boldsymbol{M})$$

This can be expressed more conveniently in the Eulerian form. Let:

$$\varphi_w \equiv \Phi_w J^{-1} \text{ — the Eulerian density of water dissipation}$$

From

$$\begin{aligned}\int_{V_0} \frac{1}{\rho_w}(\nabla p_w) \cdot \boldsymbol{M}\,\mathrm{d}V_0 &= \int_{V_t} \frac{1}{\rho_w}\left(\boldsymbol{F}^\top \nabla p_w\right) \cdot \boldsymbol{M} J^{-1}\,\mathrm{d}V_t \\ &= \int_{V_t} (\nabla p_w) \cdot \left(\frac{1}{\rho_w}\boldsymbol{F}\boldsymbol{M}J^{-1}\right)\mathrm{d}V_t \overset{(7)}{=} \int_{V_t} (\nabla p_w) \cdot \boldsymbol{q}_{rw}\,\mathrm{d}V_t\end{aligned}$$

and (22) one gets:

$$\int_{V_t} \varphi_w \,\mathrm{d}V_t = \int_{V_0} \Phi_w \,\mathrm{d}V_0 = \int_{V_0} \left(-\frac{1}{\rho_w} (\nabla p_w) \cdot \boldsymbol{M} + \boldsymbol{f} \cdot (\boldsymbol{F}\boldsymbol{M}) \right) \mathrm{d}V_0 = \int_{V_t} (-\nabla p_w + \rho_w \boldsymbol{f}) \cdot \boldsymbol{q}_{rw} \,\mathrm{d}V_t$$

$$\varphi_w = (-\nabla p_w + \rho_w \boldsymbol{f}) \cdot \boldsymbol{q}_{rw} \tag{36}$$

The non-negativeness of the dissipation associated with the water flow $\varphi_w \geq 0$ in combination with Darcy's law (35) then requires:

$$(-\nabla p_w + \rho_w \boldsymbol{f}) \cdot \left(\frac{\boldsymbol{k}}{\mu_w} (-\nabla p_w + \rho_w \boldsymbol{f}) \right) \geq 0$$

which implies that $\boldsymbol{k}/\mu_w$ has to be positive semidefinite.

Fourier's law

$$\boldsymbol{q} = -\boldsymbol{\kappa} \nabla T \tag{37}$$

$\boldsymbol{\kappa}$ — a tensor of thermal conductivities (in a general anisotropic case)

Skeleton

To derive the constitutive equations for the skeleton, we identify Φ_s (27) with the dissipation related to the sole skeleton first. For this purpose, the use of water mass balance equation (6) allows us to rewrite Φ_s in the form:

$$\Phi_s = \boldsymbol{\Pi} : \frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t} + (h_w - T s_w) \frac{\mathrm{d}(\rho_w \phi)}{\mathrm{d}t} - S \frac{\mathrm{d}T}{\mathrm{d}t} - \frac{\mathrm{d}\Psi}{\mathrm{d}t}$$

Due to (24) and the additive character of entropy one has:

$$S_s \equiv \rho_s (1 - n) s_s J = S - \rho_w n s_w J = S - \rho_w \phi s_w \tag{38}$$

— the skeleton Lagrangian density of entropy per unit of initial volume

and accordingly for the free energy (25):

$\Psi_s = \Psi - \rho_w \phi \psi_w$ — the skeleton Lagrangian density of free energy per unit of initial volume

These definitions in combination with (30) and (31) give:

$$\begin{aligned}
\Phi_s &= \boldsymbol{\Pi} : \frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t} + \left(\psi_w + \frac{p_w}{\rho_w} \right) \frac{\mathrm{d}(\rho_w \phi)}{\mathrm{d}t} - (S_s + \rho_w \phi s_w) \frac{\mathrm{d}T}{\mathrm{d}t} - \frac{\mathrm{d}(\Psi_s + \rho_w \phi \psi_w)}{\mathrm{d}t} \\
&= \boldsymbol{\Pi} : \frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t} + \psi_w \frac{\mathrm{d}(\rho_w \phi)}{\mathrm{d}t} + p_w \frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{p_w \phi}{\rho_w} \frac{\mathrm{d}\rho_w}{\mathrm{d}t} - S_s \frac{\mathrm{d}T}{\mathrm{d}t} - \rho_w \phi s_w \frac{\mathrm{d}T}{\mathrm{d}t} \\
&\quad - \frac{\mathrm{d}\Psi_s}{\mathrm{d}t} - \psi_w \frac{\mathrm{d}(\rho_w \phi)}{\mathrm{d}t} - (\rho_w \phi) \left(\frac{\partial \psi_w}{\partial (1/\rho_w)} \frac{\mathrm{d}(1/\rho_w)}{\mathrm{d}t} + \frac{\partial \psi_w}{\partial T} \frac{\mathrm{d}T}{\mathrm{d}t} \right) \\
&= \boldsymbol{\Pi} : \frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t} + p_w \frac{\mathrm{d}\phi}{\mathrm{d}t} - S_s \frac{\mathrm{d}T}{\mathrm{d}t} - \frac{\mathrm{d}\Psi_s}{\mathrm{d}t}
\end{aligned}$$

Eventually, use of energy G_s defined by:

$$G_s \equiv \Psi_s - p_w \phi$$

leads to:

$$\begin{aligned}
\Phi_s &= \boldsymbol{\Pi} : \frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t} + p_w \frac{\mathrm{d}\phi}{\mathrm{d}t} - S_s \frac{\mathrm{d}T}{\mathrm{d}t} - \frac{\mathrm{d}G_s}{\mathrm{d}t} - p_w \frac{\mathrm{d}\phi}{\mathrm{d}t} - \phi \frac{\mathrm{d}p_w}{\mathrm{d}t} \\
&= \boldsymbol{\Pi} : \frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t} - \phi \frac{\mathrm{d}p_w}{\mathrm{d}t} - S_s \frac{\mathrm{d}T}{\mathrm{d}t} - \frac{\mathrm{d}G_s}{\mathrm{d}t}
\end{aligned}$$

When the assumption of small transformations is fulfilled, $\boldsymbol{\Pi}$ and $\boldsymbol{E}$ can be replaced by $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, and one can write:

$$\Phi_s = \boldsymbol{\sigma} : \frac{\mathrm{d}\boldsymbol{\varepsilon}}{\mathrm{d}t} - \phi\frac{\mathrm{d}p_w}{\mathrm{d}t} - S_s\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{\mathrm{d}G_s}{\mathrm{d}t}$$

The skeleton energy G_s admits $\boldsymbol{\varepsilon}$, p_w and T as natural arguments since their rates explicitly appear in the above expression for the dissipation related to the skeleton. In thermoporoelasticity, G_s is therefore considered in the form:

$$G_s = G_s(\boldsymbol{\varepsilon}, p_w, T) \tag{39}$$

In addition, the dissipation related to the skeleton is zero, that is:

$$\Phi_s = \boldsymbol{\sigma} : \frac{\mathrm{d}\boldsymbol{\varepsilon}}{\mathrm{d}t} - \phi\frac{\mathrm{d}p_w}{\mathrm{d}t} - S_s\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{\mathrm{d}G_s}{\mathrm{d}t} = 0 \tag{40}$$

$$\left(\boldsymbol{\sigma} - \frac{\partial G_s}{\partial \boldsymbol{\varepsilon}}\right) : \frac{\mathrm{d}\boldsymbol{\varepsilon}}{\mathrm{d}t} - \left(\phi + \frac{\partial G_s}{\partial p_w}\right)\frac{\mathrm{d}p_w}{\mathrm{d}t} - \left(S_s + \frac{\partial G_s}{\partial T}\right)\frac{\mathrm{d}T}{\mathrm{d}t} = 0$$

Since variations of any variable among the set of state variables for the skeleton $\boldsymbol{\varepsilon}$, p_w and T can occur irrespective of the variations of the other variables, this produces the state equations:

$$\boldsymbol{\sigma} = \frac{\partial G_s}{\partial \boldsymbol{\varepsilon}} \qquad \phi = -\frac{\partial G_s}{\partial p_w} \qquad S_s = -\frac{\partial G_s}{\partial T}$$

By differentiating these equations while taking into account (39) and Maxwell's symmetry relations (the symmetry of partial derivatives) and restricting ourselves to an isotropic material, we finally obtain the constitutive equations:

Are there any requirements on the range of $\boldsymbol{D}$ so that the differentiation admits the form of the 1st equation? – Probably yes, but should it not have a full rank due to invertibility of the stress-strain relation?

$$\begin{aligned}\mathrm{d}\boldsymbol{\sigma} &= \boldsymbol{D}\left(\mathrm{d}\boldsymbol{\varepsilon} - \frac{\beta}{3}\mathrm{d}T\boldsymbol{I}\right) - \alpha\mathrm{d}p_w\boldsymbol{I} \\ &= \lambda\mathrm{d}\varepsilon_v\boldsymbol{I} + 2\mu\mathrm{d}\boldsymbol{\varepsilon} - \alpha\mathrm{d}p_w\boldsymbol{I} - \beta K\mathrm{d}T\boldsymbol{I}\end{aligned} \tag{41}$$

$$\mathrm{d}\phi = \alpha\mathrm{d}\varepsilon_v + \frac{\mathrm{d}p_w}{N} - \beta_\phi\mathrm{d}T \tag{42}$$

$$\mathrm{d}S_s = \beta K\mathrm{d}\varepsilon_v - \beta_\phi\mathrm{d}p_w + C\frac{\mathrm{d}T}{T} \tag{43}$$

$\boldsymbol{D}$ — a tangent elastic stiffness tensor of the skeleton
λ, μ — the Lamé coefficients of the skeleton
$K = (3\lambda + 2\mu)/3$ — the skeleton bulk modulus (44)
β — the skeleton volumetric thermal expansion coefficient
α — Biot's coefficient $\qquad N$ — Biot's modulus
β_ϕ — a volumetric thermal expansion coefficient related to the porosity
C — the skeleton heat capacity at constant strain and pressure

Considering α, N and β_ϕ constant (over some ranges of strains, pressures and temperatures), one can integrate (42) into the form:

$$\phi - \phi_0 = \alpha\varepsilon_v + \frac{p_w - p_{w0}}{N} - \beta_\phi(T - T_0) \tag{45}$$

By introducing:

$$\sigma_v \equiv \frac{1}{3}\operatorname{tr}\boldsymbol{\sigma} \text{ — the hydrostatic part of the stress tensor } \boldsymbol{\sigma}$$

and combining (43) and (41), one can express the entropy variation as a function of the volumetric stress σ_v instead of the volumetric strain ε_v to see that:

$$C = C_{\sigma_v} - \beta^2 KT \qquad C_{\sigma_v} \text{ — the skeleton heat capacity at constant stress and pressure} \tag{46}$$

Furthermore, let:

$$K_s \text{ — the matrix bulk modulus}$$

Under the assumptions of the constancy of α, N, K and K_s, and small variations of the porosity (and small transformations), one can derive:

$$\alpha = 1 - \frac{K}{K_s} \qquad \frac{1}{N} = \frac{\alpha - n_0}{K_s} \tag{47}$$

Using $n_0 = \phi_0$ one can also show:

$$\beta = \beta_s \qquad \beta_\phi = \beta_s(\alpha - \phi_0) \qquad C_{\sigma_v} = (1 - \phi_0)C_{\sigma_{sv}} \tag{48}$$

$$\beta_s \text{ — the matrix volumetric thermal expansion coefficient}$$
$$C_{\sigma_{sv}} \text{ — the solid matrix heat capacity at constant stress}$$

One takes $\alpha = 1$ and $1/N = 0$ ($1/K_s = 0$) for an incompressible solid matrix (not $\beta_\phi = 0$ in contrast to [Cou04, §4.1.3], where incompressibility with respect to pressure as well as temperature changes is considered).

Eulerian approach

The constitutive equations (33) and (41) for ρ_w and $\boldsymbol{\sigma}$ and Darcy's and Fourier's laws (35) and (37) are considered in the same form as in the Lagrangian approach.

Matrix density

– an alternative to (42) for the Lagrangian porosity ϕ.
Introducing:

$$\sigma_{sv} \equiv \frac{1}{3} \operatorname{tr} \boldsymbol{\sigma}_s \text{ — the hydrostatic part of the partial stress tensor } \boldsymbol{\sigma}_s$$

and considering $\rho_s = \rho_s(\sigma_{sv}, T)$ in analogy with the constitutive relationship for the water density ρ_w (33) furnishes:

$$\frac{\mathrm{d}\rho_s}{\rho_s} = -\frac{\mathrm{d}\sigma_{sv}}{K_s} - \beta_s \mathrm{d}T \tag{49}$$

The stress partition (13) with (15) yields:

$$\sigma_v = (1-n)\sigma_{sv} - np_w$$
$$(1-n)\sigma_{sv} = \sigma_v + np_w = \frac{1}{3} \operatorname{tr} \boldsymbol{\sigma'} - (1-n)p_w$$
$$\boldsymbol{\sigma'} \equiv \boldsymbol{\sigma} + p_w \boldsymbol{I} \text{ — Terzaghi's effective stress}$$

which under the assumption of small variations of the porosity leads to:

$$\mathrm{d}\sigma_{sv} = \frac{\mathrm{d}(\operatorname{tr} \boldsymbol{\sigma'})}{3(1-n)} - \mathrm{d}p_w$$
$$\frac{\mathrm{d}\rho_s}{\rho_s} \overset{(49)}{=} \frac{\mathrm{d}p_w}{K_s} - \frac{\mathrm{d}(\operatorname{tr} \boldsymbol{\sigma'})}{3(1-n)K_s} - \beta_s \mathrm{d}T$$

Further, one gets from (41) and the expression for α in (47):

$$\mathrm{d}\boldsymbol{\sigma'} = \lambda \mathrm{d}\varepsilon_v \boldsymbol{I} + 2\mu \mathrm{d}\boldsymbol{\varepsilon} + \frac{K}{K_s}\mathrm{d}p_w \boldsymbol{I} - \beta K \mathrm{d}T \boldsymbol{I}$$
$$\mathrm{d}(\operatorname{tr} \boldsymbol{\sigma'}) = \operatorname{tr}\Big(\lambda \mathrm{d}\varepsilon_v \boldsymbol{I} + 2\mu \mathrm{d}\boldsymbol{\varepsilon} + \frac{K}{K_s}\mathrm{d}p_w \boldsymbol{I} - \beta K \mathrm{d}T \boldsymbol{I}\Big) \overset{(44)}{=} 3K\Big(\mathrm{d}\varepsilon_v + \frac{\mathrm{d}p_w}{K_s} - \beta \mathrm{d}T\Big)$$

and using α from (47) again, one finally arrives at:

$$\frac{\mathrm{d}\rho_s}{\rho_s} = \frac{1}{1-n}\Big(\frac{\alpha - n}{K_s}\mathrm{d}p_w - (1-\alpha)\mathrm{d}\varepsilon_v - \big(\beta_s(1-n) - \beta(1-\alpha)\big)\mathrm{d}T\Big) \tag{50}$$

Enthalpies

The Euler energy equation will be expressed by means of enthalpies instead of entropies s_w and S_s from the Lagrangian approach. Taking into account our sign convention of pressures and stresses, we define in analogy (correctly?) with the water specific enthalpy h_w (21):

$h_s \equiv ?$

$$h_s \equiv e_s - \frac{\sigma_{sv}}{\rho_s} \text{ — the specific enthalpy of the solid matrix} \tag{51}$$

In view of (29) and (32):

$$\mathrm{d}h_w = \frac{\partial h_w}{\partial p_w}\mathrm{d}p_w + \frac{\partial h_w}{\partial s_w}\mathrm{d}s_w = \frac{1}{\rho_w}\mathrm{d}p_w + T\left[\left(\frac{\partial s_w}{\partial p_w}\right)_T \mathrm{d}p_w + \left(\frac{\partial s_w}{\partial T}\right)_{p_w}\mathrm{d}T\right]$$

where T or p_w in the subindices mean that the partial derivatives are taken at T or p_w held constant, respectively. Now one can see from (34) and (33) that:

$$\left(\frac{\partial s_w}{\partial p_w}\right)_T = -\frac{\beta_w}{\rho_w} = -\left(\frac{\partial(1/\rho_w)}{\partial T}\right)_{p_w} \qquad \left(\frac{\partial s_w}{\partial T}\right)_{p_w} = \frac{c_{p_w}}{T}$$

which yields:

$$\mathrm{d}h_w = c_{p_w}\mathrm{d}T + \left[\frac{1}{\rho_w} - T\left(\frac{\partial(1/\rho_w)}{\partial T}\right)_{p_w}\right]\mathrm{d}p_w = c_{p_w}\mathrm{d}T + \frac{1}{\rho_w}\left[1 + \frac{T}{\rho_w}\left(\frac{\partial \rho_w}{\partial T}\right)_{p_w}\right]\mathrm{d}p_w \tag{52}$$

Accordingly (?)

$\mathrm{d}h_s = ?$

$$\mathrm{d}h_s = c_{\sigma_{sv}}\mathrm{d}T - \left[\frac{1}{\rho_s} - T\left(\frac{\partial(1/\rho_s)}{\partial T}\right)_{\sigma_{sv}}\right]\mathrm{d}\sigma_{sv} = c_{\sigma_{sv}}\mathrm{d}T - \frac{1}{\rho_s}\left[1 + \frac{T}{\rho_s}\left(\frac{\partial \rho_s}{\partial T}\right)_{\sigma_{sv}}\right]\mathrm{d}\sigma_{sv} \tag{53}$$

$$c_{\sigma_{sv}} = T\left(\frac{\partial s_s}{\partial T}\right)_{\sigma_{sv}} \text{ — the solid matrix specific heat capacity at constant stress}$$

Since entropy is an extensive quantity, one has:

$$(1-\phi_0)\mathfrak{S}_s = S_s \overset{(24)}{=} \rho_s(1-n)s_s J \overset{(3)}{=} \rho_{s0}(1-\phi_0)s_s$$

$$\mathfrak{S}_s \text{ — the Lagrangian solid matrix entropy}$$

$$\rho_{s0}c_{\sigma_{sv}} = \rho_{s0}T\left(\frac{\partial s_s}{\partial T}\right)_{\sigma_{sv}} = T\left(\frac{\partial \mathfrak{S}_s}{\partial T}\right)_{\sigma_{sv}} = C_{\sigma_{sv}} \tag{54}$$

Complete equations

Lagrangian approach

Continuity equation

When adopting the small perturbation assumption,

$$\frac{\partial(\rho_w\phi)}{\partial t} = \rho_w\frac{\partial\phi}{\partial t} + \phi\frac{\partial\rho_w}{\partial t} \overset{(42),(33)}{=} \rho_w\left(\alpha\frac{\partial\varepsilon_v}{\partial t} + \frac{1}{N}\frac{\partial p_w}{\partial t} - \beta_\phi\frac{\partial T}{\partial t}\right) + \phi\left(\frac{\rho_w}{K_w}\frac{\partial p_w}{\partial t} - \beta_w\rho_w\frac{\partial T}{\partial t}\right)$$

$$\approx \rho_{w0}\alpha\frac{\partial\varepsilon_v}{\partial t} + \rho_{w0}\left(\frac{1}{N} + \frac{\phi_0}{K_w}\right)\frac{\partial p_w}{\partial t} - \rho_{w0}(\beta_\phi + \phi_0\beta_w)\frac{\partial T}{\partial t}$$

$$\operatorname{div}\boldsymbol{M} \approx \operatorname{div}(\rho_w\boldsymbol{q}_{rw}) \overset{(35)}{=} \operatorname{div}\left(\rho_w\frac{\boldsymbol{k}}{\mu_w}(-\nabla p_w + \rho_w\boldsymbol{f})\right) \approx \operatorname{div}\left(\rho_{w0}\frac{\boldsymbol{k}}{\mu_w}(-\nabla p_w + \rho_{w0}\boldsymbol{f})\right)$$

and the Lagrangian water mass balance equation (6) yields:

$$\boxed{\begin{aligned}&\rho_{w0}\alpha\frac{\partial\varepsilon_v}{\partial t}+\rho_{w0}\left(\frac{1}{N}+\frac{\phi_0}{K_w}\right)\frac{\partial p_w}{\partial t}-\rho_{w0}(\beta_\phi+\phi_0\beta_w)\frac{\partial T}{\partial t}\\&\qquad=-\operatorname{div}\left(\rho_{w0}\frac{\boldsymbol{k}}{\mu_w}(-\nabla p_w+\rho_{w0}\boldsymbol{f})\right)\end{aligned}}\tag{55}$$

Equilibrium equation

In the Lagrangian equilibrium equation (11) one can take:

$$\mathbf{div}(\boldsymbol{F\Pi})\approx\mathbf{div}\,\boldsymbol{\sigma}\qquad\rho_w\phi\approx\rho_{w0}\phi_0$$

which furnishes:

$$\boxed{\mathbf{div}\,\boldsymbol{\sigma}+\big(\rho_{s0}(1-\phi_0)+\rho_{w0}\phi_0\big)\boldsymbol{f}=\mathbf{0}}\tag{56}$$

Thermal equation

The small perturbation assumption is now extended by the following:

Assumption. Small variations of the temperature:

$$\left|\frac{T-T_0}{T_0}\right|\ll 1$$

Under this assumption one can take $T\approx T_0$, which together with the remaining small perturbation assumptions allows us to write:

$$\frac{\mathrm{d}S}{\mathrm{d}t}\overset{(38)}{=}\frac{\mathrm{d}S_s}{\mathrm{d}t}+\rho_w\phi\frac{\mathrm{d}s_w}{\mathrm{d}t}+s_w\frac{\mathrm{d}(\rho_w\phi)}{\mathrm{d}t}\qquad\operatorname{div}(s_w\boldsymbol{M})\overset{(6)}{=}\boldsymbol{M}\cdot\nabla s_w-s_w\frac{\mathrm{d}(\rho_w\phi)}{\mathrm{d}t}$$

$$\begin{aligned}T\left(\frac{\mathrm{d}S}{\mathrm{d}t}+\operatorname{div}(s_w\boldsymbol{M})\right)&\approx T\left(\frac{\partial S_s}{\partial t}+\rho_w\phi\frac{\partial s_w}{\partial t}+(\rho_w\boldsymbol{q}_{rw})\cdot\nabla s_w\right)\\&\overset{(43),(34)}{\approx}\beta KT_0\frac{\partial\varepsilon_v}{\partial t}-\beta_\phi T_0\frac{\partial p_w}{\partial t}+C\frac{\partial T}{\partial t}-\phi_0\beta_wT_0\frac{\partial p_w}{\partial t}+\rho_{w0}\phi_0c_{p_w}\frac{\partial T}{\partial t}\\&\qquad+\boldsymbol{q}_{rw}\cdot(-\beta_wT_0\nabla p_w+\rho_{w0}c_{p_w}\nabla T)\end{aligned}$$

$$-\operatorname{div}\boldsymbol{Q}\approx-\operatorname{div}\boldsymbol{q}\overset{(37)}{=}\operatorname{div}(\boldsymbol{\kappa}\nabla T)$$

$$\Phi_s\overset{(40)}{=}0\qquad\Phi_w\approx\varphi_w\overset{(36)}{\approx}(-\nabla p_w+\rho_{w0}\boldsymbol{f})\cdot\boldsymbol{q}_{rw}$$

and the Lagrangian thermal equation (26) becomes:

$$\boxed{\begin{aligned}&\beta KT_0\frac{\partial\varepsilon_v}{\partial t}-T_0(\beta_\phi+\phi_0\beta_w)\frac{\partial p_w}{\partial t}+(C+\rho_{w0}\phi_0c_{p_w})\frac{\partial T}{\partial t}\\&\qquad=\operatorname{div}(\boldsymbol{\kappa}\nabla T)+\big(-\rho_{w0}c_{p_w}\nabla T-(1-\beta_wT_0)\nabla p_w+\rho_{w0}\boldsymbol{f}\big)\cdot\boldsymbol{q}_{rw}\end{aligned}}\tag{57}$$

where $\boldsymbol{q}_{rw}$ is given by Darcy's law (35).

Eulerian approach

Continuity equation

When adopting the assumptions of small perturbations and small deformation velocity,

$$\rho_w\approx\rho_{w0}\qquad n\approx n_0\qquad\rho_s\approx\rho_{s0}$$

$$\frac{\mathrm{D}_s}{\mathrm{D}t}=\frac{\partial}{\partial t}+\boldsymbol{v}_s\cdot\nabla\approx\frac{\partial}{\partial t}\qquad\operatorname{div}\boldsymbol{v}_s=\operatorname{div}\frac{\mathrm{D}_s\boldsymbol{u}}{\mathrm{D}t}\approx\operatorname{div}\frac{\partial\boldsymbol{u}}{\partial t}=\frac{\partial\varepsilon_v}{\partial t}$$

and one can rewrite the Eulerian mass balance equations (2) and (5) and the constitutive equation for ρ_s (50) as:

$$\frac{\partial(1-n)}{\partial t} + \frac{1-n_0}{\rho_{s0}}\frac{\partial \rho_s}{\partial t} + (1-n_0)\frac{\partial \varepsilon_v}{\partial t} = 0 \tag{58}$$

$$\rho_{w0}\frac{\partial n}{\partial t} + n_0\frac{\partial \rho_w}{\partial t} + \rho_{w0}n_0\frac{\partial \varepsilon_v}{\partial t} = -\operatorname{div}(\rho_{w0}\boldsymbol{q}_{rw}) \tag{59}$$

$$\frac{1-n_0}{\rho_{s0}}\frac{\partial \rho_s}{\partial t} = \frac{\alpha-n_0}{K_s}\frac{\partial p_w}{\partial t} - (1-\alpha)\frac{\partial \varepsilon_v}{\partial t} - \big(\beta_s(1-n_0)-\beta(1-\alpha)\big)\frac{\partial T}{\partial t} \tag{60}$$

Elimination of $\partial\rho_s/\partial t$ from (58) by (60) gives:

$$\frac{\partial n}{\partial t} = \frac{\alpha-n_0}{K_s}\frac{\partial p_w}{\partial t} + (\alpha-n_0)\frac{\partial \varepsilon_v}{\partial t} - \big(\beta_s(1-n_0)-\beta(1-\alpha)\big)\frac{\partial T}{\partial t} \tag{61}$$

which inserted together with (33) and (35) into (59) yields:

$$\boxed{\begin{aligned}\rho_{w0}\alpha\frac{\partial \varepsilon_v}{\partial t} + \rho_{w0}\left(\frac{\alpha-n_0}{K_s}+\frac{n_0}{K_w}\right)\frac{\partial p_w}{\partial t} - \rho_{w0}\big(\beta_s(1-n_0)-\beta(1-\alpha)+n_0\beta_w\big)\frac{\partial T}{\partial t}&\\ = -\operatorname{div}\left(\rho_{w0}\frac{\boldsymbol{k}}{\mu_w}(-\nabla p_w+\rho_{w0}\boldsymbol{f})\right)&\end{aligned}} \tag{62}$$

Expressions for ρ_s and n

Taking α, K_s, β_s and β constant, one can integrate (60) and (61) into:

$$\begin{aligned}\rho_s &= \rho_{s0}\left(1+\frac{1}{1-n_0}\left(\frac{\alpha-n_0}{K_s}(p_w-p_{w0}) - (1-\alpha)\varepsilon_v - \big(\beta_s(1-n_0)-\beta(1-\alpha)\big)(T-T_0)\right)\right)\\ n &= n_0 + \frac{\alpha-n_0}{K_s}(p_w-p_{w0}) + (\alpha-n_0)\varepsilon_v - \big(\beta_s(1-n_0)-\beta(1-\alpha)\big)(T-T_0)\end{aligned} \tag{63}$$

Equilibrium equation

In the Eulerian equilibrium equation (10) one can take:

$$\rho_s(1-n) \approx \rho_{s0}(1-n_0) \qquad \rho_w n \approx \rho_{w0}n_0$$

which leads to:

$$\boxed{\operatorname{\mathbf{div}}\boldsymbol{\sigma} + (\rho_{s0}(1-n_0)+\rho_{w0}n_0)\boldsymbol{f} = \boldsymbol{0}} \tag{64}$$

Thermal equation

By (21), (51), (52) and (53), we switch from internal energies to enthalpies to get:

$$\begin{aligned}\rho_s\frac{\mathrm{D}_s e_s}{\mathrm{D}t} &= \rho_s\frac{\mathrm{D}_s h_s}{\mathrm{D}t} - \frac{\sigma_{sv}}{\rho_s}\frac{\mathrm{D}_s\rho_s}{\mathrm{D}t} + \frac{\mathrm{D}_s\sigma_{sv}}{\mathrm{D}t} = \rho_s c_{\sigma_{sv}}\frac{\mathrm{D}_s T}{\mathrm{D}t} - \frac{T}{\rho_s}\left(\frac{\partial\rho_s}{\partial T}\right)_{\sigma_{sv}}\frac{\mathrm{D}_s\sigma_{sv}}{\mathrm{D}t} - \frac{\sigma_{sv}}{\rho_s}\frac{\mathrm{D}_s\rho_s}{\mathrm{D}t}\\ \rho_w\frac{\mathrm{D}_s e_w}{\mathrm{D}t} &= \rho_w\frac{\mathrm{D}_s h_w}{\mathrm{D}t} + \frac{p_w}{\rho_w}\frac{\mathrm{D}_s\rho_w}{\mathrm{D}t} - \frac{\mathrm{D}_s p_w}{\mathrm{D}t} = \rho_w c_{p_w}\frac{\mathrm{D}_s T}{\mathrm{D}t} + \frac{T}{\rho_w}\left(\frac{\partial\rho_w}{\partial T}\right)_{p_w}\frac{\mathrm{D}_s p_w}{\mathrm{D}t} + \frac{p_w}{\rho_w}\frac{\mathrm{D}_s\rho_w}{\mathrm{D}t}\end{aligned}$$

and the energy equation (20) becomes:

$$\begin{aligned}&\big(\rho_s(1-n)c_{\sigma_{sv}}+\rho_w n c_{p_w}\big)\frac{\mathrm{D}_s T}{\mathrm{D}t} - \frac{(1-n)T}{\rho_s}\left(\frac{\partial\rho_s}{\partial T}\right)_{\sigma_{sv}}\frac{\mathrm{D}_s\sigma_{sv}}{\mathrm{D}t} + \frac{nT}{\rho_w}\left(\frac{\partial\rho_w}{\partial T}\right)_{p_w}\frac{\mathrm{D}_s p_w}{\mathrm{D}t}\\ &- \frac{(1-n)\sigma_{sv}}{\rho_s}\frac{\mathrm{D}_s\rho_s}{\mathrm{D}t} + \frac{np_w}{\rho_w}\frac{\mathrm{D}_s\rho_w}{\mathrm{D}t} = \boldsymbol{\sigma}:\boldsymbol{d}_s - \operatorname{div}(p_w\boldsymbol{q}_{rw}+\boldsymbol{q}) - \rho_w\boldsymbol{q}_{rw}\cdot\nabla e_w + \rho_w\boldsymbol{f}\cdot\boldsymbol{q}_{rw}\end{aligned} \tag{65}$$

Now one obtains from (2) and (5):

$$\frac{1-n}{\rho_s}\frac{\mathrm{D}_s\rho_s}{\mathrm{D}t} = \frac{\mathrm{D}_s n}{\mathrm{D}t} - (1-n)\operatorname{div}\boldsymbol{v}_s, \qquad \frac{n}{\rho_w}\frac{\mathrm{D}_s\rho_w}{\mathrm{D}t} = -\frac{\mathrm{D}_s n}{\mathrm{D}t} - n\operatorname{div}\boldsymbol{v}_s - \frac{1}{\rho_w}\operatorname{div}(\rho_w\boldsymbol{q}_{rw})$$

The stress partition (13) with (15) yields:

$$\sigma_v = (1-n)\sigma_{sv} - np_w$$
$$(1-n)\frac{\mathrm{D}_s\sigma_{sv}}{\mathrm{D}t} = \frac{\mathrm{D}_s\sigma_v}{\mathrm{D}t} + n\frac{\mathrm{D}_s p_w}{\mathrm{D}t} + (\sigma_{sv}+p_w)\frac{\mathrm{D}_s n}{\mathrm{D}t}$$

Besides, taking into account also the symmetry of $\boldsymbol{\sigma}$ and the decomposition:

$$\boldsymbol{\sigma}_s = \boldsymbol{s}_s + \sigma_{sv}\boldsymbol{I} \qquad \boldsymbol{s}_s \text{ — the deviatoric part of } \boldsymbol{\sigma}_s \tag{66}$$

one may write:

$$\boldsymbol{\sigma} : \boldsymbol{d}_s = \big((1-n)\boldsymbol{\sigma}_s + n\boldsymbol{\sigma}_w\big) : \boldsymbol{\nabla}\boldsymbol{v}_s = (1-n)\boldsymbol{s}_s : \boldsymbol{\nabla}\boldsymbol{v}_s + (1-n)\sigma_{sv}\operatorname{div}\boldsymbol{v}_s - np_w\operatorname{div}\boldsymbol{v}_s$$

Hence, with the aid of the identity:

$$-\operatorname{div}\left(\frac{p_w}{\rho_w}\rho_w\boldsymbol{q}_{rw}\right) + \frac{p_w}{\rho_w}\operatorname{div}(\rho_w\boldsymbol{q}_{rw}) - \rho_w\boldsymbol{q}_{rw}\cdot\nabla e_w = -\rho_w\boldsymbol{q}_{rw}\cdot\nabla h_w$$

equation (65) takes the form:

$$\begin{aligned}\big(\rho_s(1-n)c_{\sigma_{sv}} + \rho_w n c_{p_w}\big)\frac{\mathrm{D}_s T}{\mathrm{D}t} - \frac{T}{\rho_s}\left(\frac{\partial\rho_s}{\partial T}\right)_{\sigma_{sv}}\frac{\mathrm{D}_s\sigma_v}{\mathrm{D}t} + \left[\frac{nT}{\rho_w}\left(\frac{\partial\rho_w}{\partial T}\right)_{p_w} - \frac{nT}{\rho_s}\left(\frac{\partial\rho_s}{\partial T}\right)_{\sigma_{sv}}\right]\frac{\mathrm{D}_s p_w}{\mathrm{D}t}\\ = (\sigma_{sv}+p_w)\left[1 + \frac{T}{\rho_s}\left(\frac{\partial\rho_s}{\partial T}\right)_{\sigma_{sv}}\right]\frac{\mathrm{D}_s n}{\mathrm{D}t} + (1-n)\boldsymbol{s}_s : \boldsymbol{\nabla}\boldsymbol{v}_s - \operatorname{div}\boldsymbol{q} + \rho_w(-\nabla h_w + \boldsymbol{f})\cdot\boldsymbol{q}_{rw}\end{aligned}$$

Under the extended small perturbation assumption and the assumption of small deformation velocity, this can be rewritten in the form:

$$\begin{aligned}\big(\rho_{s0}(1-n_0)c_{\sigma_{sv}} + \rho_{w0}n_0c_{p_w}\big)\frac{\partial T}{\partial t} - \frac{T_0}{\rho_s}\left(\frac{\partial\rho_s}{\partial T}\right)_{\sigma_{sv}}\frac{\partial\sigma_v}{\partial t} + \left[\frac{n_0T_0}{\rho_w}\left(\frac{\partial\rho_w}{\partial T}\right)_{p_w} - \frac{n_0T_0}{\rho_s}\left(\frac{\partial\rho_s}{\partial T}\right)_{\sigma_{sv}}\right]\frac{\partial p_w}{\partial t}\\ = (\sigma_{sv}+p_w)\left[1 + \frac{T_0}{\rho_s}\left(\frac{\partial\rho_s}{\partial T}\right)_{\sigma_{sv}}\right]\frac{\partial n}{\partial t} + (1-n_0)\boldsymbol{s}_s : \boldsymbol{\nabla}\frac{\partial\boldsymbol{u}}{\partial t} - \operatorname{div}\boldsymbol{q} + \rho_{w0}(-\nabla h_w + \boldsymbol{f})\cdot\boldsymbol{q}_{rw}\end{aligned} \tag{67}$$

According to (33), (49), (41) and (52):

$$\frac{1}{\rho_w}\left(\frac{\partial\rho_w}{\partial T}\right)_{p_w} = -\beta_w \qquad \frac{1}{\rho_s}\left(\frac{\partial\rho_s}{\partial T}\right)_{\sigma_{sv}} = -\beta_s$$

$$\frac{\partial\sigma_v}{\partial t} = \frac{1}{3}\operatorname{tr}\frac{\partial\boldsymbol{\sigma}}{\partial t} = \frac{1}{3}\operatorname{tr}\left(\lambda\frac{\partial\varepsilon_v}{\partial t}\boldsymbol{I} + 2\mu\frac{\partial\boldsymbol{\varepsilon}}{\partial t} - \alpha\frac{\partial p_w}{\partial t}\boldsymbol{I} - \beta K\frac{\partial T}{\partial t}\boldsymbol{I}\right) \overset{(44)}{=} K\frac{\partial\varepsilon_v}{\partial t} - \alpha\frac{\partial p_w}{\partial t} - \beta K\frac{\partial T}{\partial t}$$

$$\rho_{w0}\nabla h_w = \rho_{w0}c_{p_w}\nabla T + \frac{\rho_{w0}}{\rho_w}\left[1 + \frac{T}{\rho_w}\left(\frac{\partial\rho_w}{\partial T}\right)_{p_w}\right]\nabla p_w \approx \rho_{w0}c_{p_w}\nabla T + (1-\beta_wT_0)\nabla p_w$$

which together with (37) substituted into (67) yields:

$$\boxed{\begin{aligned}\beta_sKT_0\frac{\partial\varepsilon_v}{\partial t} - T_0\big(\beta_s(\alpha-n_0) + n_0\beta_w\big)\frac{\partial p_w}{\partial t} + \big(\rho_{s0}(1-n_0)c_{\sigma_{sv}} - \beta_s\beta KT_0 + \rho_{w0}n_0c_{p_w}\big)\frac{\partial T}{\partial t}\\ = (1-\beta_sT_0)(\sigma_{sv}+p_w)\frac{\partial n}{\partial t} + (1-n_0)\boldsymbol{s}_s : \boldsymbol{\nabla}\frac{\partial\boldsymbol{u}}{\partial t} + \operatorname{div}(\boldsymbol{\kappa}\nabla T)\\ + \big(-\rho_{w0}c_{p_w}\nabla T - (1-\beta_wT_0)\nabla p_w + \rho_{w0}\boldsymbol{f}\big)\cdot\boldsymbol{q}_{rw}\end{aligned}} \tag{68}$$

where $\boldsymbol{q}_{rw}$ is given by Darcy's law (35).

- Eventually, $\partial n/\partial t$ can be substituted from (61). (The term related to mechanical work caused by porosity changes can usually be neglected according to [LS98].)
- An expression for $\boldsymbol{s}_s$ can be obtained from $\boldsymbol{\sigma}_s$ by (66), and the stress partition (13) with (15) and the stress-strain relationship (41). (The term related to viscous dissipation of the skeleton is usually neglected in literature.)

Summary

One can see that the Lagrangian system (55)–(57) coincides with the Eulerian one (62)&(64)&(68) if:

- $\phi_0 = n_0$ (they are equal to their original values from the initial configuration);
- the expressions (46) and (54) for C and $C_{\sigma_{sv}}$ are taken into account (with $T \approx T_0$);
- the expressions for α, $1/N$, β, β_ϕ and C_{σ_v} from (47) and (48) are valid;
- one omits

$$(1 - \beta_s T_0)(\sigma_{sv} + p_w)\frac{\partial n}{\partial t} + (1 - n_0)\boldsymbol{s}_s : \boldsymbol{\nabla}\frac{\partial \boldsymbol{u}}{\partial t}$$

  from the Eulerian thermal equation. (The first term seems to be negligible under the assumption of small variations of the porosity, but is the second term also negligible under our assumptions?)

Neglection of $(1 - n_0)\boldsymbol{s}_s : \boldsymbol{\nabla}(\partial \boldsymbol{u}/\partial t)$?

One can also show that equations (45) and (63) for current values of the Lagrangian porosity ϕ and Eulerian porosity n are then approximately related by:

$$\phi = Jn \approx (1 + \varepsilon_v)n$$

when ε_v, $p_w - p_{w0}$ and $T - T_0$ are small.

References

[Cou04] O. Coussy. *Poromechanics.* John Wiley & Sons, 2004.

[LS98] R. W. Lewis and B. A. Schrefler. *The Finite Element Method in the Static and Dynamic Deformation and Consolidation of Porous Media.* John Wiley, 2nd edition, 1998.