



Modelling Processes in Porous Media IV: From Saturated Thermoporoelasticity to Unsaturated One

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The Model

Balance Laws

Constitutive Equations

Simplifications of the Two-phase Flow Model

The Model

- Non-stationary non-isothermal unsaturated water flow in a deformable porous medium.
- Isotropic elastic skeleton (index s).
- The porous space filled by water (index w) and a gas phase (index g) formed by a mixture of water vapour (index v) and 'dry' air (index a).
- Possible phase changes between the water and its vapour.
- Negligible inertial effects.
- The assumption of thermal equilibrium (the fluids and the matrix are locally at the same temperature).
- The assumption of small deformations.
- An extract from [Cou04] + an adaptation of the Eulerian approach from [LS98].

\mathbf{u} — the displacement vector of the skeleton

$\mathbf{id} + \mathbf{u}$ — the deformation of the skeleton

$\mathbf{F} = \mathbf{I} + \nabla \mathbf{u}$ — the deformation gradient

$J = \det(\mathbf{I} + \nabla \mathbf{u})$ — the Jacobian of the deformation

n, ϕ — the Eulerian and Lagrangian porosity: $n \, dV_t = \phi \, dV_0$ is the current porous space in a current volume dV_t or in the corresponding initial volume dV_0 ($nJ = \phi$)

S_f — the degree of saturation relative to fluid f : $S_f n \, dV_t = S_f \phi \, dV_0$ is the volume occupied by fluid f in current volume dV_t or initial volume dV_0

Both the vapour and the air occupy the whole part of the porous space filled by the gas phase:

$$S_v = S_a = S_g \quad S_w + S_g = 1$$

Balance Laws

$$\frac{D_s}{Dt} \int_{V_t} \rho_s(1 - n) dV_t = 0$$

$\frac{D_s}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}_s \cdot \nabla$ — the total time derivative with respect to the skeleton

t — the time

\mathbf{v}_s — the skeleton velocity

ρ_s — the solid mass density

The Eulerian local form:

$$\frac{D_s(\rho_s(1 - n))}{Dt} + \rho_s(1 - n) \operatorname{div} \mathbf{v}_s = 0$$

The Lagrangian alternative:

$$(1 - n)\rho_s J = (1 - \phi_0)\rho_{s0}$$

$\phi_0 (= n_0)$ — the initial Lagrangian (= initial Eulerian) porosity

ρ_{s0} — the initial skeleton mass density

$$\frac{D_w}{Dt} \int_{V_t} n S_w \rho_w dV_t = - \int_{V_t} \dot{r}_{w \rightarrow v} dV_t$$

$\frac{D_w}{Dt}$ — the total time derivative with respect to the water

ρ_w — the water mass density

$\dot{r}_{w \rightarrow v}$ — the rate of water mass changing into vapour per unit of current volume

The local Eulerian continuity equation (referring to the skeleton motion):

$$\frac{D_s(nS_w\rho_w)}{Dt} + nS_w\rho_w \operatorname{div} \mathbf{v}_s + \operatorname{div}(\rho_w \mathbf{q}_{rw}) = -\dot{r}_{w \rightarrow v}$$

$\mathbf{q}_{rw} \equiv nS_w(\mathbf{v}_w - \mathbf{v}_s)$ — the water specific discharge relative to the skeleton
(or Darcy velocity or filtration vector)

\mathbf{v}_w — the water velocity

The Lagrangian counterpart:

$$\frac{d(\phi S_w \rho_w)}{dt} + \operatorname{div} \mathbf{M}_w = -\dot{m}_{w \rightarrow v}$$

$\mathbf{M}_w \equiv J\mathbf{F}^{-1}(\rho_w \mathbf{q}_{rw})$ — the Lagrangian relative flow vector of water mass

$\dot{m}_{w \rightarrow v} \equiv J\dot{r}_{w \rightarrow v}$ — the rate of water mass changing into vapour per unit
of initial volume

$$\frac{D_v}{Dt} \int_{V_t} nS_g \rho_v dV_t = \int_{V_t} \dot{r}_{w \rightarrow v} dV_t$$

$\frac{D_v}{Dt}$ — the total time derivative with respect to the vapour

ρ_v — the vapour (partial) mass density

(= the mass of vapour per unit volume of the gas phase)

The local Eulerian form:

$$\frac{D_s(nS_g \rho_v)}{Dt} + nS_g \rho_v \operatorname{div} \mathbf{v}_s + \operatorname{div}(\rho_v \mathbf{q}_{rv}) = \dot{r}_{w \rightarrow v}$$

$\mathbf{q}_{rv} \equiv nS_g(\mathbf{v}_v - \mathbf{v}_s)$ — the vapour specific discharge relative to the skeleton

\mathbf{v}_v — the vapour velocity

The Lagrangian form:

$$\frac{d(\phi S_g \rho_v)}{dt} + \operatorname{div} \mathbf{M}_v = \dot{m}_{w \rightarrow v}$$

$\mathbf{M}_v \equiv J\mathbf{F}^{-1}(\rho_v \mathbf{q}_{rv})$ — the Lagrangian relative flow vector of vapour mass

$$\frac{D_a}{Dt} \int_{V_t} nS_g \rho_a dV_t = 0$$

$\frac{D_a}{Dt}$ — the total time derivative with respect to the air

ρ_a — the dry air (partial) mass density

The local Eulerian form:

$$\frac{D_s(nS_g \rho_a)}{Dt} + nS_g \rho_a \operatorname{div} \mathbf{v}_s + \operatorname{div}(\rho_a \mathbf{q}_{ra}) = 0$$

$\mathbf{q}_{ra} \equiv nS_g(\mathbf{v}_a - \mathbf{v}_s)$ — the air specific discharge relative to the skeleton

\mathbf{v}_a — the dry air velocity

The Lagrangian form:

$$\frac{d(\phi S_g \rho_a)}{dt} + \operatorname{div} \mathbf{M}_a = 0$$

$\mathbf{M}_a \equiv J\mathbf{F}^{-1}(\rho_a \mathbf{q}_{ra})$ — the Lagrangian relative flow vector of air mass

$$\frac{D_s}{Dt} \int_{V_t} (1-n)\rho_s \mathbf{v}_s dV_t + \sum_{f=w,v,a} \frac{D_f}{Dt} \int_{V_t} nS_f \rho_f \mathbf{v}_f dV_t = \int_{V_t} \rho \mathbf{f} dV_t + \int_{\partial V_t} \mathbf{T} da$$

$\rho \equiv (1-n)\rho_s + \sum_{f=w,v,a} nS_f \rho_f$ — the apparent mass density of the total porous medium

\mathbf{f} — a body force density \mathbf{T} — a surface force density

The local equation of motion:

$$\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{f} - (1-n)\rho_s \mathbf{a}_s - \sum_{f=w,v,a} nS_f \rho_f \mathbf{a}_f - \dot{r}_{w \rightarrow v} (\mathbf{v}_v - \mathbf{v}_w) = \mathbf{0}$$

$\boldsymbol{\sigma}$ — the Cauchy stress tensor

$\mathbf{a}_s \equiv \frac{D_s \mathbf{v}_s}{Dt} = \frac{\partial \mathbf{v}_s}{\partial t} + (\nabla \mathbf{v}_s) \mathbf{v}_s$ — the skeleton acceleration

$\mathbf{a}_f \equiv \frac{D_f \mathbf{v}_f}{Dt} = \frac{\partial \mathbf{v}_f}{\partial t} + (\nabla \mathbf{v}_f) \mathbf{v}_f$ — the acceleration of fluid f

By disregarding the dynamic effects (quasistatic approximation):

$$\operatorname{div} \boldsymbol{\sigma} + \left((1 - n)\rho_s + \sum_{f=w,v,a} nS_f \rho_f \right) \mathbf{f} = \mathbf{0}$$

The Lagrangian counterpart:

$$\operatorname{div}(\mathbf{F}\boldsymbol{\Pi}) + \left((1 - \phi_0)\rho_{s0} + \sum_{f=w,v,a} \phi S_f \rho_f \right) \mathbf{f} = \mathbf{0}$$

$$\boldsymbol{\Pi} \equiv \mathbf{J}\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\top} \text{ — the Piola-Kirchhoff stress tensor}$$

$$\begin{aligned} \frac{D_s}{Dt} \int_{V_t} \mathbf{x} \times (1-n)\rho_s \mathbf{v}_s dV_t + \sum_{f=w,v,a} \frac{D_f}{Dt} \int_{V_t} \mathbf{x} \times n S_f \rho_f \mathbf{v}_f dV_t \\ = \int_{V_t} \mathbf{x} \times \rho \mathbf{f} dV_t + \int_{\partial V_t} \mathbf{x} \times \mathbf{T} da \end{aligned}$$

\mathbf{x} — the position vector

\implies the symmetry of the stress tensor $\boldsymbol{\sigma}$

$$\begin{aligned}
 & \frac{D_s}{Dt} \int_{V_t} (1-n)\rho_s \left(e_s + \frac{1}{2} \mathbf{v}_s \cdot \mathbf{v}_s \right) dV_t \\
 & + \sum_{f=w,v,a} \frac{D_f}{Dt} \int_{V_t} n S_f \rho_f \left(e_f + \frac{1}{2} \mathbf{v}_f \cdot \mathbf{v}_f \right) dV_t \\
 & = \int_{V_t} \left((1-n)\rho_s \mathbf{f} \cdot \mathbf{v}_s + \sum_{f=w,v,a} n S_f \rho_f \mathbf{f} \cdot \mathbf{v}_f \right) dV_t \\
 & + \int_{\partial V_t} \left(\mathbf{T}_s \cdot \mathbf{v}_s + \sum_{f=w,v,a} \mathbf{T}_f \cdot \mathbf{v}_f + \mathbf{T}_{int} \cdot \mathbf{v}_s \right) da - \int_{\partial V_t} \mathbf{q} \cdot \mathbf{n} da
 \end{aligned}$$

e_s — the specific (i.e., per mass unit) internal energy of the skeleton

e_f — the specific internal energy of fluid f

$\mathbf{T}_s, \mathbf{T}_f$ — the surface forces related to the porous solid and fluid f

\mathbf{T}_{int} — the overall surface tension exerted along the solid-fluid and the fluid-fluid (water-gas) interfaces

\mathbf{q} — the heat flux vector \mathbf{n} — the outward unit normal to V_t

$$\frac{D_s}{Dt} \int_{V_t} (1-n) \rho_s s_s dV_t + \sum_{f=w,v,a} \frac{D_f}{Dt} \int_{V_t} n S_f \rho_f s_f dV_t \geq - \int_{\partial V_t} \frac{\mathbf{q} \cdot \mathbf{n}}{T} da$$

s_s — the specific entropy of the skeleton

s_f — the fluid specific entropy

T — the absolute temperature

$$\Phi \equiv \Phi_s + \Phi_{\rightarrow} + \sum_{f=w,v,a} \Phi_f + \Phi_{th} \geq 0$$

$$\Phi_s \approx \boldsymbol{\sigma} : \frac{d\boldsymbol{\varepsilon}}{dt} + \sum_{f=w,v,a} p_f \frac{d(\phi S_f)}{dt} - S_s \frac{dT}{dt} - \frac{d\Psi_s}{dt}$$

$$\boldsymbol{\varepsilon} \equiv \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \text{ — the linear strain tensor}$$

p_f — the pressure of fluid f

S_s, Ψ_s — the skeleton Lagrangian densities of entropy and free energy per unit of initial volume

$$\Phi_{\rightarrow} = (g_w - g_v) \dot{m}_{w \rightarrow v}$$

g_w, g_v — the Gibbs potentials of water and vapour

$$\Phi_f J^{-1} = (-\nabla p_f + \rho_f \mathbf{f}) \cdot \mathbf{q}_{rf}$$

$$\Phi_{th} J^{-1} = -\frac{\mathbf{q}}{T} \cdot \nabla T$$

Owing to the very distinct nature of the dissipations, one can substitute the unique inequality $\Phi \geq 0$ by four *separate* inequalities:

$$\Phi_s \geq 0 \quad \Phi_{\rightarrow} \geq 0 \quad \sum_{f=w,v,a} \Phi_f \geq 0 \quad \Phi_{th} \geq 0$$

Constitutive Equations

By differentiating classical fluid state equations one obtains for fluid f :

$$\frac{d\rho_f}{\rho_f} = \frac{dp_f}{K_f} - \beta_f dT$$

$$ds_f = -\beta_f \frac{dp_f}{\rho_f} + c_{pf} \frac{dT}{T}$$

K_f — the fluid bulk modulus

β_f — the fluid volumetric thermal expansion coefficient

c_{pf} — the fluid specific heat capacity at constant pressure

In particular, we assume that the gas phase is an *ideal mixture of vapour and dry air*, which behave as *ideal gases*:

$p_g = p_v + p_a$ — the pressure of the gaseous vapour–air mixture

$$p_f = \frac{RT}{\mathcal{M}_f} \rho_f \quad f = v, a$$

R — the ideal gas constant

\mathcal{M}_f — the molar mass of fluid f

The local thermodynamic equilibrium between the water and its vapour requires:

$$\begin{aligned}\Phi_{\rightarrow} &= (g_w - g_v)\dot{m}_{w\rightarrow v} = 0 \\ g_w(p_w, T) &= g_v(p_v, T)\end{aligned}\quad (1)$$

When assuming sufficiently slow changes so that the water–vapour equilibrium is maintained throughout the evolution, and considering the water density ρ_w as constant (in comparison with the vapour density ρ_v), one can derive **Kelvin's law** by differentiation of (1):

$$p_w - p_{atm} = \frac{\rho_w RT}{\mathcal{M}_v} \ln \frac{p_v}{p_{vs}(T)}$$

$p_{vs}(T)$ — the pressure of the saturating vapour at temperature T when the water pressure p_w is equal to the atmospheric pressure p_{atm}

$$p_{vs}(T) = p_{vs}(T_0) \exp \left\{ \frac{M_v}{RT} \left[\frac{\mathcal{L}_0}{T_0} (T - T_0) + (c_{p_w} - c_{p_v}) \left(T - T_0 - T \ln \frac{T}{T_0} \right) \right] \right\}$$

$\mathcal{L}_0 = T_0(s_{v0} - s_{w0})$ — the latent heat of vaporisation at temperature T_0
and $p_w = p_{atm}$

s_{v0}, s_{w0} — the specific entropies of the vapour and the water at
temperature T_0

One can decompose the transport \mathbf{q}_{rf} of a component $f = v, a$ of the vapour–air gaseous mixture into an **advective** part \mathbf{q}_{rg} and a **diffusive** part ($\mathbf{q}_{rf} - \mathbf{q}_{rg}$) by introducing \mathbf{q}_{rg} as follows:

c_v, c_a — the molar concentrations of the vapour and the air

$c \equiv c_v + c_a$ — the total gas molar concentration

$x_f \equiv \frac{c_f}{c}$ — the mole fraction of fluid $f = v, a$

$\mathbf{q}_{rg} \equiv x_v \mathbf{q}_{rv} + x_a \mathbf{q}_{ra}$ — the specific discharge of the gas phase relative to the skeleton (*molar average*)

Transport of water and the gas phase as a whole:

$$\mathbf{q}_{rf} = \frac{\mathbf{k} k_{rf}}{\mu_f} (-\nabla p_f + \rho_f \mathbf{f}), \quad f = w, g$$

\mathbf{k} — the intrinsic permeability tensor of the porous medium

$k_{rf} = k_{rf}(S_f) \in [0, 1]$ — the relative permeability to fluid f

μ_f — the dynamic viscosity of fluid f

Diffusion of the vapour and the air through the gas:

$$\mathbf{q}_{rv} - \mathbf{q}_{rg} = -\frac{\rho_g}{\rho_v} \mathbf{D} \nabla \left(\frac{\rho_v}{\rho_g} \right)$$

$$\mathbf{q}_{ra} - \mathbf{q}_{rg} = -\frac{\rho_g}{\rho_a} \mathbf{D} \nabla \left(\frac{\rho_a}{\rho_g} \right)$$

\mathbf{D} — a diffusion tensor

It is worth to mention that the non-negativeness of the dissipation associated with the fluid transport $\sum_{f=w,v,a} \Phi_f \geq 0$ is satisfied if \mathbf{k}/μ_w , \mathbf{k}/μ_g and \mathbf{D} are all positive semidefinite. But this does not hold if \mathbf{q}_{rg} is chosen as the *mass average* instead of the *molar* one!

Heat conduction:

$$\mathbf{q} = -\kappa \nabla T$$

κ — a tensor of thermal conductivities

The non-negativeness of the dissipation associated with heat conduction $\Phi_{th} \geq 0$ requires κ to be positive semidefinite.

In thermoporoelasticity, the mechanical dissipation Φ_s associated with the skeleton is zero:

$$\Phi_s = \boldsymbol{\sigma} : \frac{d\boldsymbol{\varepsilon}}{dt} + \sum_{f=w,v,a} p_f \frac{d(\phi S_f)}{dt} - S_s \frac{dT}{dt} - \frac{d\Psi_s}{dt} = 0$$

Owing to the additive character of energy, one can decompose the skeleton free energy Ψ_s as:

$$\Psi_s(\boldsymbol{\varepsilon}, \phi, S_w, T) = \Psi_s^*(\boldsymbol{\varepsilon}, \phi, T) + \phi U(\phi, S_w, T)$$

Ψ_s^* — the free energy of the porous solid alone per unit of initial volume

ϕU — the overall interfacial free energy per unit of initial volume

$$U(\phi, S_w, T) = \phi^{-1/3} \Gamma(S_w, T)$$

The decomposition of Ψ_s leads to:

$$p_c = -\frac{\partial U}{\partial S_w}$$
$$\sigma : \frac{d\varepsilon}{dt} - \phi \frac{d\pi}{dt} - S_s^* \frac{dT}{dt} - \frac{dG_s^*}{dt} = 0 \quad (2)$$
$$\pi = S_w p_w + S_g p_g - \frac{2}{3} U \text{ — the equivalent pore pressure}$$
$$S_s^* = S_s + \phi \frac{\partial U}{\partial T} \text{ — the entropy of the porous solid alone}$$
$$G_s^* \equiv \Psi_s^* - \pi \phi$$

Further, (2) provides the skeleton state equations:

$$G_s^* = G_s^*(\varepsilon, \pi, T) \quad \sigma = \frac{\partial G_s^*}{\partial \varepsilon} \quad \phi = -\frac{\partial G_s^*}{\partial \pi} \quad S_s^* = -\frac{\partial G_s^*}{\partial T}$$

In view of the equality

$$p_c = - \frac{\partial U}{\partial S_w}$$

there is a relationship between the capillary pressure p_c and the saturation S_w :

$$p_c = p_c(\phi, S_w, T)$$

or

$$S_w = S_w(\phi, p_c, T)$$

Moreover, the interfacial energy U can be expressed as:

$$U(\phi, S_w, T) = \int_{S_w}^1 p_c(\phi, S, T) dS$$

By differentiating the skeleton state equations while restricting oneself to an isotropic material one obtains:

$$d\boldsymbol{\sigma} = \left(K - \frac{2}{3}\mu \right) d\varepsilon_v \mathbf{I} + 2\mu d\boldsymbol{\varepsilon} - \alpha d\pi \mathbf{I} - \beta K dT \mathbf{I}$$

$$d\phi = \alpha d\varepsilon_v + \frac{d\pi}{N} - \beta_\phi dT$$

$$dS_s^* = \beta K d\varepsilon_v - \beta_\phi d\pi + \frac{C}{T} dT$$

K, μ — the bulk and shear moduli of the porous solid

$\varepsilon_v \equiv \text{tr } \boldsymbol{\varepsilon} = \text{div } \mathbf{u}$ — the volumetric strain

β — the porous solid volumetric thermal expansion coefficient

α — Biot's coefficient N — Biot's modulus

β_ϕ — a volumetric thermal expansion coefficient related to the porosity

C — the porous solid heat capacity at constant strain and pressure

The Model

Balance Laws

Constitutive Equations

Alternatives for an Eulerian formulation

Simplifications of the Two-phase Flow Model

By introducing the hydrostatic part of the partial stress tensor σ_s related to the porous solid alone:

$$\sigma_{sv} \equiv \frac{1}{3} \text{tr} \sigma_s$$

and considering $\rho_s = \rho_s(\sigma_{sv}, T)$ in analogy with the state equations for fluids, one can take (?):

$$\frac{d\rho_s}{\rho_s} = -\frac{d\sigma_{sv}}{K_s} - \beta_s dT$$

K_s — the matrix bulk modulus

β_s — the matrix volumetric thermal expansion coefficient

This together with the constitutive equation for σ yields:

$$\frac{d\rho_s}{\rho_s} = \frac{1}{1-n} \left(\frac{\alpha-n}{K_s} d\pi - (1-\alpha) d\varepsilon_v - (\beta_s(1-n) - \beta(1-\alpha)) dT \right)$$

According to the classical thermodynamics of fluids:

$$h_f = e_f + \frac{p_f}{\rho_f} \text{ — the specific enthalpy of fluid } f$$

$$dh_f = c_{p_f} dT + \frac{1}{\rho_f} (1 - \beta_f T) dp_f$$

By introducing the decomposition:

$$e_s = e_s^* + e_{int}$$

e_s^* — the specific internal energy of the solid matrix alone

e_{int} — the specific overall interfacial internal energy

and the specific solid matrix enthalpy by:

$$h_s^* \equiv e_s^* - \frac{1}{(1 - \phi_0)\rho_{s0}} \boldsymbol{\sigma} : \boldsymbol{\varepsilon} - \frac{1}{(1 - \phi_0)\rho_{s0}} \pi \phi$$

one can derive from the dissipation condition $\Phi_s = 0$:

$$(1 - \phi_0)\rho_{s0} dh_s^* = -\boldsymbol{\varepsilon} : d\boldsymbol{\sigma} - \phi d\pi + \beta T d\sigma_v + (\alpha\beta - \beta_\phi) T d\pi + C_\sigma dT$$

$$\sigma_v \equiv \frac{1}{3} \text{tr} \boldsymbol{\sigma} \text{ — the hydrostatic part of the stress tensor } \boldsymbol{\sigma}$$

C_σ — the skeleton heat capacity at constant stress and pressure

$$(1 - \phi_0)\rho_{s0} de_{int} = \frac{2}{3} U d\phi - \phi p_c dS_w - T d\left(\phi \frac{\partial U}{\partial T}\right)$$

Simplifications of the Two-phase Flow Model

Simplifications that make it possible to eliminate the unknown gas pressure p_g from the previous equations and to reduce the general unsaturated model introduced before, which treats a complete two-phase flow.

The simplifications are based on the assumptions, which seem to be relevant for clays under moderate conditions as clays have a high intrinsic permeability to gas.

(i) The gas pressure is not much greater than several times atmospheric pressure whereas the capillary pressure reaches values much greater than atmospheric pressure. Hence:

$$|p_c| \gg |p_g| \quad |S_w p_c| \gg |p_g|$$

(ii) The changes of the gas pressure are so small that (a) they are negligible with respect to the changes of the capillary pressure, (b) the Darcian advective transport of the vapour–air mixture considered as a whole is much smaller than the Fickian molecular diffusion, (c) the changes of the gas pressure in the Fick law are negligible when compared with the changes of the vapour pressure:

$$\begin{aligned} |dp_c| \gg |dp_g| \quad |S_w dp_c| \gg |dp_g| \\ \left\| \frac{k k_{rg}}{\mu_g} (-\nabla p_g + \rho_g \mathbf{f}) \right\| \ll \left\| -\frac{p_g}{p_v} \mathbf{D} \nabla \left(\frac{p_v}{p_g} \right) \right\| \\ -\frac{p_g}{p_v} \mathbf{D} \nabla \left(\frac{p_v}{p_g} \right) \approx -\frac{1}{p_v} \mathbf{D} \nabla p_v \end{aligned}$$

(iii) The body force exerted on the gaseous vapour–air mixture can be neglected with respect to the forces exerted on the skeleton and water:

$$\left((1 - n)\rho_s + \sum_{f=w,v,a} nS_f\rho_f \right) \mathbf{f} \approx ((1 - n)\rho_s + nS_w\rho_w) \mathbf{f}$$

(iv) The thermal capacity of the dry air as well as the heat supplied by its motion can be neglected with respect to the other components of the porous medium.

Under the previous assumptions:

$$p_c \approx -p_w$$

$$dp_c \approx -dp_w$$

$$S_w = S_w(\phi, p_w, T) \quad dS_w = \frac{\partial S_w}{\partial \phi} d\phi + \frac{\partial S_w}{\partial p_w} dp_w + \frac{\partial S_w}{\partial T} dT$$

$$\pi \approx S_w p_w - \frac{2}{3} U$$

$$d\pi \approx S_w dp_w + \frac{1}{3} p_w dS_w - \frac{2}{3} \left(\frac{\partial U}{\partial \phi} d\phi + \frac{\partial U}{\partial T} dT \right)$$

$$\mathbf{q}_{rv} = \mathbf{q}_{rg} + (\mathbf{q}_{rv} - \mathbf{q}_{rg}) = \frac{\mathbf{k}k_{rg}}{\mu_g} (-\nabla p_g + \rho_g \mathbf{f}) - \frac{\rho_g}{\rho_v} \mathbf{D} \nabla \left(\frac{p_v}{\rho_g} \right) \approx -\frac{1}{\rho_v} \mathbf{D} \nabla p_v$$

Alternatively, when the gas phase is approximately at a constant (hydrostatic atmospheric) pressure, but this value is not much smaller than the capillary pressure as assumed before, one can take the value of the gas pressure as the reference (zero) one. Then one also gets:

$$p_c \approx -p_w \quad S_w p_w + S_g p_g \approx S_w p_w$$

However, one has to be aware that the water pressure is diminished by the value of the gas pressure in this case.

Field equations are then derived by inserting the constitutive equations into:

- the sum of the water and vapour mass balance equations:

$$\frac{d(\phi S_w \rho_w + \phi(1 - S_w) \rho_v)}{dt} + \text{div}(\mathbf{M}_w + \mathbf{M}_v) = 0$$

or

$$\frac{D_s(n(S_w \rho_w + (1 - S_w) \rho_v))}{Dt} + n(S_w \rho_w + (1 - S_w) \rho_v) \text{div} \mathbf{v}_s + \text{div}(\rho_w \mathbf{q}_{rw} + \rho_v \mathbf{q}_{rv}) = 0$$

- the equilibrium equation
- the energy equation



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