On Coulomb collisions in bi-Maxwellian plasmas

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Collisional momentum and energy transport in bi-Maxwellian plasmas with a drift velocity along the ambient magnetic field are calculated from both the Fokker-Planck and Boltzmann integral approximations. The transport coefficients obtained from the two approaches are identical to the leading order (proportional to the Coulomb logarithm) and are presented here in a closed form involving generalized double hypergeometric functions.

I. INTRODUCTION

Transport in weakly collisional plasmas may strongly deviate from theoretical predictions obtained for a collision-dominated plasma.[1] Weak collisions are not generally able to keep particle distribution functions near the thermal equilibrium, a phenomenon clearly evidenced in the solar wind.[2,3] For anisotropic (bi-Maxwellian) distribution functions transport coefficients have been calculated starting from the Fokker-Planck approximation.[4,5] However, the Fokker-Planck approximation fails far from the thermal equilibrium[6] and the Boltzmann integral has to be used. This had been done for drifting bi-Maxwellian gases in a general case of an inverse-power force[7] but those results were not given in a closed form and included integrals which were calculated only in some cases/limits. In this paper we calculate the collisional momentum and energy transport in bi-Maxwellian plasmas with a drift velocity along the ambient magnetic field from both the Fokker-Planck and Boltzmann integral approximations. The transport coefficients obtained from the two approaches are identical to the leading order (proportional to the Coulomb logarithm) and are presented here in a closed form involving generalized double hypergeometric functions.

II. TRANSPORT COEFFICIENTS

Coulomb collision scattering may be approximated by a two particle interaction term via the Boltzmann collision integral. Concerning notation, here we use SI, ϵ_0 denotes the electric permittivity, index s (and t) denotes different species, q_s and m_s are the species charge and mass, respectively; $m_{st} = m_s m_t/(m_s + m_t)$ denotes an effective mass, $f_s = f_s(\boldsymbol{v})$ is the species velocity distribution function. The collisional variation of the distribution function of species s is given by a sum of terms giving the scattering on all species t in the form[8]

$$\left(\frac{\partial f_s}{\partial t}\right)_c = \sum_t \int_{\mathbb{R}^3 \times \Omega} [f_s(\boldsymbol{v}')f_t(\boldsymbol{u}') - f_s(\boldsymbol{v})f_t(\boldsymbol{u})] \, g I_{st} \mathrm{d}\Omega \mathrm{d}^3 u \tag{1}$$

where $g = |\boldsymbol{g}| = |\boldsymbol{v}-\boldsymbol{u}|$, $\boldsymbol{v}'$ and $\boldsymbol{u}'$ are post-collision velocities with respect to $\boldsymbol{v}$ and $\boldsymbol{u}$, respectively, the Rutherford cross section is given by

$$I_{st} = \frac{q_s^2 q_t^2}{64\pi^2 \epsilon_0^2 m_{st}^2} \frac{1}{g^4 \sin^4(\chi/2)} \tag{2}$$

and in the integration with respect to the solid angle Ω the standard cut-off is used

$$\int_\Omega (\;) \mathrm{d}\Omega = \int_0^{2\pi} \int_{\chi_{\min}}^{\pi} (\;) \sin\chi \mathrm{d}\chi \mathrm{d}\phi \tag{3}$$

where χ the deflection angle between $\boldsymbol{v}$ and $\boldsymbol{v}'$ and ϕ gives the rotation around $\boldsymbol{g}$. This cut-off (3) leads to the Coulomb logarithm $\ln \Lambda_{st}$.

Assuming a dominance of small angle deflections, expanding the Boltzmann integral (1) in $\boldsymbol{u}' - \boldsymbol{u}$ and $\boldsymbol{v}' - \boldsymbol{v}$ and taking first terms in the Taylor series one gets the Fokker-Planck equation, which may be given in the Landau conservative form

$$\left(\frac{\partial f_s}{\partial t}\right)_c = -\sum_t \boldsymbol{\nabla_v} \cdot \boldsymbol{j}_{st} \tag{4}$$

where the collisional current in the velocity space is

$$\boldsymbol{j}_{st} = \frac{q_s^2 q_t^2 \ln\Lambda_{st}}{8\pi\epsilon_0^2 m_s} \int_{\mathbb{R}^3} \frac{\boldsymbol{1}g^2 - \boldsymbol{gg}}{g^3} \cdot \left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial\boldsymbol{u}} - \frac{f_t}{m_s}\frac{\partial f_s}{\partial\boldsymbol{v}}\right) \mathrm{d}^3 u \tag{5}$$

where $\boldsymbol{1}$ is the unity tensor.

From (4) one may get basic transport coefficients by taking the appropriate moments of $(\partial f_s/\partial t)_c$ assuming a specific form of the distribution function. Here we assume that all considered species have bi-Maxwellian velocity distribution functions with a mean velocity parallel to the ambient magnetic field (assumed in the z direction)

$$f_s = \frac{n_s}{(2\pi)^{3/2}v_{s\|}v_{s\perp}^2} e^{-\frac{v_\perp^2}{2v_{s\perp}^2} - \frac{(v_\| - v_s)^2}{2v_{s\|}^2}} \tag{6}$$

where n_s is the particle number density,

$$v_{s\|} = \sqrt{\frac{k_B T_{s\|}}{m_s}} \quad \text{and} \quad v_{s\perp} = \sqrt{\frac{k_B T_{s\perp}}{m_s}} \tag{7}$$

are parallel and perpendicular thermal velocities, respectively (k_B being the Boltzmann constant), and v_s are parallel drift velocities.

The calculation of these moments leads to integrals in the form

$$\int\limits_0^\infty \int\limits_0^\pi \mathrm{e}^{-v^2}\mathrm{e}^{-Av^2\cos^2\theta}\mathrm{e}^{-Vv\cos\theta}P(v,\sin\theta,\cos\theta)d\theta dv \quad (8)$$

where P is a low-degree, trivariate polynomial. The integral may be evaluated by expanding two exponential terms with $\cos\theta$ into infinite sums and integrating resulting terms (assuming the double infinite sum converges). The double infinite sums are in the form of double hypergeometric functions and one arrives at the transport coefficients

$$\left(\frac{dv_s}{dt}\right)_c = \sum_t \nu_{st}\frac{v_t-v_s}{2}F^{(st)}_{1\frac{3}{2}\frac{5}{2}} \quad (9)$$

$$\left(\frac{dT_{s\parallel}}{dt}\right)_c = T_{s\parallel}\sum_t \nu_{st}\left[\frac{m_{st}}{m_t}\left(\frac{T_{t\parallel}}{T_{s\parallel}}-1\right)F^{(st)}_{1\frac{1}{2}\frac{5}{2}}\right.$$
$$\left. + \frac{(v_t-v_s)^2}{2v^2_{st\parallel}}F^{(st)}_{1\frac{3}{2}\frac{5}{2}} - 2\left(F^{(st)}_{2\frac{1}{2}\frac{5}{2}}-F^{(st)}_{1\frac{1}{2}\frac{5}{2}}\right)\right] \quad (10)$$

$$\left(\frac{dT_{s\perp}}{dt}\right)_c = T_{s\perp}\sum_t \frac{\nu_{st}}{A_{st}}\left[\frac{m_{st}}{m_t}\left(\frac{T_{t\perp}}{T_{s\perp}}-1\right)F^{(st)}_{2\frac{1}{2}\frac{5}{2}}\right.$$
$$\left. + F^{(st)}_{2\frac{1}{2}\frac{5}{2}} - F^{(st)}_{1\frac{1}{2}\frac{5}{2}}\right] \quad (11)$$

where

$$v_{st\parallel}=\sqrt{\frac{v^2_{s\parallel}+v^2_{t\parallel}}{2}},\quad \text{and}\quad v_{st\perp}=\sqrt{\frac{v^2_{s\perp}+v^2_{t\perp}}{2}} \quad (12)$$

are combined effective parallel and perpendicular velocities, respectively,

$$A_{st}=\frac{v^2_{st\perp}}{v^2_{st\parallel}}=\frac{m_tT_{s\perp}+m_sT_{t\perp}}{m_tT_{s\parallel}+m_sT_{t\parallel}} \quad (13)$$

is an effective temperature anisotropy and

$$\nu_{st}=\frac{q^2_sq^2_tn_t}{12\pi^{3/2}\epsilon^2_0m_sm_{st}v^3_{st\parallel}}\ln\Lambda_{st} \quad (14)$$

is a collision frequency of species s on species t. Here $F^{(st)}_{abc}$ are defined through generalized double hypergeometric or Kampé de Fériet functions[9]

$$F^{(st)}_{abc}=\mathrm{e}^{-\frac{(v_t-v_s)^2}{4v^2_{st\parallel}}}F^{2\cdot\cdot}_{1\cdot1}\left(\begin{array}{c}a,b\\c;b\end{array},1-A_{st},A_{st}\frac{(v_t-v_s)^2}{4v^2_{st\parallel}}\right) \quad (15)$$

For the derivation and simplification of transport coefficients (9–11) we have also used the recursive formulas (B3–B7). Another expression for $F^{(st)}_{abc}$ could be obtained from (B9)

$$F^{(st)}_{abc}=\frac{\mathrm{e}^{-\frac{(v_t-v_s)^2}{4v^2_{st\parallel}}}}{A^a_{st}}\Phi_1\left(\begin{array}{c}a,c-b\\c\end{array},1-\frac{1}{A_{st}},\frac{(v_t-v_s)^2}{4v^2_{st\parallel}}\right) \quad (16)$$

which may be derived by going through $\sin^2\theta$ rather than through $\cos^2\theta$ in (8). The transport coefficients (9–11) clearly conserve the total energy and momentum.

For inter-species collisions we have

$$F^{(ss)}_{abc}= {}_2F_1\left(\begin{array}{c}a,b\\c\end{array},1-A_{ss}\right) \quad (17)$$

and one recovers the results of Ref. 4 considering that the Kogan function[4,5,7] φ defined as

$$\varphi(x)=\begin{cases}\frac{\arctan\sqrt{x}}{\sqrt{x}} & \text{for } x>0\\ 1 & \text{for } x=0\\ \frac{\tanh^{-1}\sqrt{-x}}{\sqrt{-x}} & \text{for } x<0\end{cases} \quad (18)$$

is related to the standard hypergeometric function

$$\varphi(x)= {}_2F_1\left(\begin{array}{c}1,\frac{1}{2}\\ \frac{3}{2}\end{array},-x\right) \quad (19)$$

and using relations (A5–A7). Similarly for $v_s=v_t$ one recovers the results of Ref. 5.

The transport coefficients can be also calculated directly from the Boltzmann collision integral. This calculation also leads to integrals in the form of (8), cf., Ref. 7, Eq. (24–26). It can be easily shown that for bi-Maxwellian distribution functions with velocities parallel to the ambient magnetic field one gets the same transport coefficients (9–11) as obtained from the Fokker-Planck to the leading order $\propto \ln\Lambda$. Note that the Coulomb logarithm used in Ref. 7 is twice the standard one (cf., Refs. 8,10). The agreement between the (leading order) momentum and energy transport coefficients obtained from the Boltzmann collision integral and the Fokker-Planck approximations is in agreement with the results of Ref. 6 which indicate that large-angle collisions impact higher order moments.

III. CONCLUSIONS

We have presented a closed form of collisional transport coefficients in bi-Maxwellian plasmas drifting along the ambient magnetic field. These coefficients can be expressed in the form of double hypergeometric functions. These results can be further generalized to an inverse-power force interaction and to include a drift velocity perpendicular with respect to the ambient magnetic field; a presence of the perpendicular drift velocity leads to triple hypergeometric functions.

APPENDIX A: HYPERGEOMETRIC FUNCTION

The standard Gauss hypergeometric function can be defined as

$$
{}_2F_1 \left(\begin{array}{c} a,b \\ c \end{array}, x \right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \tag{A1}
$$

where $(a)_n$ is a Pochhammer symbol

$$
(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\dots(a+n-1). \tag{A2}
$$

The infinite sum in (A1) is absolutely convergent for $|x| < 1$. For other values an analytic continuation is to be used,[11] e.g., using the linear transformation property

$$
{}_2F_1 \left(\begin{array}{c} a,b \\ c \end{array}, x \right) = (1-x)^{-a}\, {}_2F_1 \left(\begin{array}{c} a, c-b \\ c \end{array}, \frac{x}{x-1} \right). \tag{A3}
$$

or the integral representation[12]

$$
{}_2F_1 \left(\begin{array}{c} a,b \\ c \end{array}, x \right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{t^{a-1}(1-t)^{c-a-1}}{(1-tx)^b} \mathrm{d}t. \tag{A4}
$$

One can easily check the following recurrence formulas

$$
\left(\frac{c}{b} - x \right) {}_2F_1 \left(\begin{array}{c} 1,b \\ c \end{array}, x \right) - \frac{c}{b} = \frac{c-b}{c(c+1)}\, {}_2F_1 \left(\begin{array}{c} 2, b+1 \\ c+2 \end{array}, x \right) \tag{A5}
$$

$$
{}_2F_1 \left(\begin{array}{c} 2,b \\ c \end{array}, x \right) - {}_2F_1 \left(\begin{array}{c} 1,b \\ c \end{array}, x \right) = \frac{bx}{c}\, {}_2F_1 \left(\begin{array}{c} 2, b+1 \\ c+1 \end{array}, x \right) \tag{A6}
$$

$$
(x+1)\, {}_2F_1 \left(\begin{array}{c} 1, 2-c \\ c \end{array}, x \right) - 1 = \frac{2x}{c}\, {}_2F_1 \left(\begin{array}{c} 2, 2-c \\ c+1 \end{array}, x \right) \tag{A7}
$$

APPENDIX B: MULTIPLE HYPERGEOMETRIC FUNCTIONS

A special class of double hypergeometric functions or Kampé de Fériet functions[9] is considered here. These functions can be represented as double series

$$
F_{1\cdot1}^{2\cdot\cdot} \left(\begin{array}{c} a,b \\ c;d \end{array}, x, y \right) = \sum_{n,k=0}^{\infty} \frac{(a)_{n+k}(b)_{n+k}}{(c)_{n+k}(d)_k} \frac{x^n}{n!} \frac{y^k}{k!}. \tag{B1}
$$

The double infinite series in (B1) is absolutely convergent for any y and for $|x| < 1$. Outside this region an analytic continuation is needed. Expression (B1) may be expressed in the following form

$$
F_{1\cdot1}^{2\cdot\cdot} \left(\begin{array}{c} a,b \\ c;d \end{array}, x, y \right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (d)_k} \frac{y^k}{k!}\, {}_2F_1 \left(\begin{array}{c} a+k, b+k \\ c+k \end{array}, x \right). \tag{B2}
$$

One can easily check the following recurrence relations for the double hypergeometric functions:

$$
F_{1\cdot1}^{2\cdot\cdot} \left(\begin{array}{c} a,b \\ c;d \end{array}, x, y \right) + \frac{ax}{c} F_{1\cdot1}^{2\cdot\cdot} \left(\begin{array}{c} a+1, b+1 \\ c+1; d+1 \end{array}, x, y \right)
$$
$$
= F_{1\cdot1}^{2\cdot\cdot} \left(\begin{array}{c} a, b+1 \\ c; d+1 \end{array}, x, y \right), \tag{B3}
$$

for $a \neq 0$

$$
x F_{1\cdot1}^{2\cdot\cdot} \left(\begin{array}{c} a+1, b+1 \\ c+1; d \end{array}, x, y \right) + \frac{y}{d} F_{1\cdot1}^{2\cdot\cdot} \left(\begin{array}{c} a+1, b+1 \\ c+1; d+1 \end{array}, x, y \right)
$$
$$
= \frac{c}{b} \left[F_{1\cdot1}^{2\cdot\cdot} \left(\begin{array}{c} a+1, b \\ c; d \end{array}, x, y \right) - F_{1\cdot1}^{2\cdot\cdot} \left(\begin{array}{c} a, b \\ c; d \end{array}, x, y \right) \right], \tag{B4}
$$

whereas for $a = 0$

$$
x F_{1\cdot1}^{2\cdot\cdot} \left(\begin{array}{c} 1, b+1 \\ c+1; d \end{array}, x, y \right) + \frac{y}{d} F_{1\cdot1}^{2\cdot\cdot} \left(\begin{array}{c} 1, b+1 \\ c+1; d+1 \end{array}, x, y \right)
$$
$$
= \frac{c}{b} F_{1\cdot1}^{2\cdot\cdot} \left(\begin{array}{c} 1, b \\ c; d \end{array}, x, y \right), \tag{B5}
$$

$$
b F_{1\cdot1}^{2\cdot\cdot} \left(\begin{array}{c} a, b+1 \\ c+1; d \end{array}, x, y \right) - c F_{1\cdot1}^{2\cdot\cdot} \left(\begin{array}{c} a, b \\ c; d \end{array}, x, y \right)
$$
$$
= (b-c) F_{1\cdot1}^{2\cdot\cdot} \left(\begin{array}{c} a, b \\ c+1; d \end{array}, x, y \right), \tag{B6}
$$

$$
F_{1\cdot1}^{2\cdot\cdot} \left(\begin{array}{c} 1, b+1 \\ c; d \end{array}, x, y \right) - F_{1\cdot1}^{2\cdot\cdot} \left(\begin{array}{c} 1, b \\ c; d \end{array}, x, y \right)
$$
$$
= \frac{1}{b} \left[F_{1\cdot1}^{2\cdot\cdot} \left(\begin{array}{c} 2, b \\ c; d \end{array}, x, y \right) - F_{1\cdot1}^{2\cdot\cdot} \left(\begin{array}{c} 1, b \\ c; d \end{array}, x, y \right) \right]. \tag{B7}
$$

The double hypergeometric function $F_{1\cdot1}^{2\cdot\cdot}$ in case of $b = d$ are related to Humbert generalized double cofluent hypergeometric function

$$
\Phi_1 \left(\begin{array}{c} a, b \\ c \end{array}, x, y \right) = \sum_{n,k=0}^{\infty} \frac{(a)_{n+k}(b)_n}{(c)_{n+k}} \frac{x^n}{n!} \frac{y^k}{k!} \tag{B8}
$$

$$
F_{1\cdot1}^{2\cdot\cdot} \left(\begin{array}{c} a, b \\ c; b \end{array}, x, y \right) = (1-x)^{-a} \Phi_1 \left(\begin{array}{c} a, c-b \\ c \end{array}, \frac{x}{x-1}, \frac{y}{1-x} \right) \tag{B9}
$$

Finally, for $b = d$ one can get the simple integral representation

$$
F_{1\cdot1}^{2\cdot\cdot} \left(\begin{array}{c} a, b \\ c; b \end{array}, x, y \right) \tag{B10}
$$

$$
= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{t^{a-1}(1-t)^{c-a-1}}{(1-tx)^b} \exp\left(\frac{ty}{1-tx} \right) \mathrm{d}t.
$$

The derivation of (B10) is analogic to (A4) for the standard hypergeometric function.[12]

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