A Poroelastoplastic Model for Saturated Clays Incorporating the Modified Cam-Clay Model

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Outline

Setting of the model

Model

- Non-stationary isothermal saturated water flow in a deformable clay.
- The clay is composed of an *incompressible* solid matrix (index s) and a porous space completely filled by water (index w).
- The poroelastoplastic modified Cam-Clay model with non-linear elasticity is used for the solid skeleton.
- Negligible inertial effects.
- Lagrangian formulation.
- The small-strain assumption.
- Compressive-positive pressures, tensile-positive stresses.
- Based on [Cou04].

Balance Laws

Water mass balance

Under the small-strain assumption:

$$\frac{\partial(\phi\rho_w)}{\partial t} + \operatorname{div}(\rho_w \boldsymbol{q}_{rw}) = 0$$

t — the time $\qquad$ ρ_w — the water mass density

ϕ — the Lagrangian porosity (with respect to the initial configuration)

$\boldsymbol{q}_{rw} \equiv n(\boldsymbol{v}_w - \boldsymbol{v}_s)$ — the Darcy velocity

n — the Eulerian porosity (with respect to the deformed configuration)

$\boldsymbol{v}_w$ — the water velocity $\qquad$ $\boldsymbol{v}_s$ — the skeleton velocity

Equilibrium equation

Under the small-strain assumption:

$$\mathbf{div}\,\boldsymbol{\sigma} + ((1-\phi_0)\rho_s^0 + \phi\rho_w)\boldsymbol{f} = \mathbf{0}$$

$\boldsymbol{\sigma}$ — the Cauchy stress tensor

ϕ_0 — an initial Lagrangian porosity

ρ_s^0 — the initial matrix mass density

$\boldsymbol{f}$ — a body force density

Constitutive Relationships

Water density

By considering the water to be compressible:

$$\frac{d\rho_w}{\rho_w} = \frac{dp_w}{K_w}$$

d — the differential operator with respect to time

p_w — the water pressure $\qquad K_w$ — the water bulk modulus

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p_w — the water pressure K_w — the water bulk modulus

Assuming K_w constant (over some range of pressures), one obtains by integration:

$$\rho_w = \rho_w^0 e^{(p_w - p_w^0)/K_w}$$

ρ_w^0, p_w^0 — initial values of the water density and pressure

Darcy's law

Transport of water is described by:

$$\boldsymbol{q}_{rw} = \frac{\boldsymbol{k}}{\mu_w}(-\nabla p_w + \rho_w \boldsymbol{f})$$

$\boldsymbol{k}$ — the (intrinsic) permeability tensor of the porous medium

μ_w — the dynamic viscosity of water

Porosity

The solid grains forming the matrix generally undergo negligible volume changes and the matrix can be assumed to be *incompressible*. This means that the matrix volume remains unchanged during the deformation:

$$(1 - n)\,\mathrm{d}V_t = (1 - n_0)\,\mathrm{d}V_0$$

n_0 — the initial Eulerian porosity

$\mathrm{d}V_0$ — an arbitrary infinitesimal volume in the initial configuration

$\mathrm{d}V_t$ — the corresponding infinitesimal volume in the deformed configuration

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Use of transport formulae gives in the framework of small strains:

$$\phi = \phi_0 + \varepsilon_v$$

$\varepsilon_v \equiv \operatorname{tr}\boldsymbol{\varepsilon}$ — the volumetric strain

$\boldsymbol{\varepsilon} \equiv \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top\right)$ — the linear strain tensor

$\boldsymbol{u}$ — the displacement vector of the skeleton

Plastic strain

Poroplasticity is the ability of porous materials to undergo permanent strains. In the context of small strains, the strain tensor ε can be decomposed into a reversible part (elastic, superscript el) and an irreversible one (plastic, superscript p) as follows:

$$\varepsilon = \varepsilon^{el} + \varepsilon^{p}$$

Deviatoric stress and strain

Our constitutive stress:

$$\boldsymbol{\sigma}' \equiv \boldsymbol{\sigma} + p_w \boldsymbol{I} \text{ — Terzaghi's effective stress}$$

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We introduce the decompositions:

$$\boldsymbol{\sigma}' = \boldsymbol{s} - p' \boldsymbol{I}$$

$p' \equiv -\frac{1}{3} \operatorname{tr} \boldsymbol{\sigma}'$ — the effective pressure $\qquad$ $\boldsymbol{s}$ — the deviatoric stress tensor

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_d + \frac{1}{3} \varepsilon_v \boldsymbol{I}$$

$\boldsymbol{\varepsilon}_d$ — the deviatoric strain tensor

Triaxial stress conditions

When the porous material is subjected to an axial pressure $-\sigma_1$ in one direction and a uniform pressure $-\sigma_2 = -\sigma_3$ in the orthogonal directions, and the material is *isotropic*, it suffices to consider:

$$q \equiv -(\sigma_1 - \sigma_3) \text{ — the deviatoric stress}$$

$$\epsilon_q \equiv -\frac{2}{3}(\varepsilon_1 - \varepsilon_3) \text{ — the deviatoric strain}$$

$$\varepsilon_1, \varepsilon_3\,(= \varepsilon_2) \text{ — principal strains}$$

and we shall take

$$\epsilon_v := -\varepsilon_v$$

Non-linear poroelasticity

The following elastic behaviour of clays has been experimentally found:

$$d\epsilon_v^{el} = \kappa^* \frac{dp'}{p'} \qquad d\epsilon_q^{el} = \frac{dq}{3\mu}$$

$$\kappa^* := \frac{\kappa}{1 + e_0} \qquad e_0 = \frac{\phi_0}{1 - \phi_0}$$ — an initial void ratio

κ — an elastic stiffness parameter $\quad$ μ — the shear modulus

$\left(K(p') := \dfrac{p'}{\kappa^*}\right.$ — the *tangent* bulk modulus$\left.\right)$

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$$\left(K(p') := \frac{p'}{\kappa^*} \text{ — the } \textit{tangent} \text{ bulk modulus}\right)$$

By integration:

$$\epsilon_v^{el} = \kappa^* \ln \frac{p'}{p'_0} \qquad \epsilon_q^{el} = \frac{q-q_0}{3\mu}$$

p'_0, q_0 — initial values of p'_0 and q

and by inversion:

$$p' = p'_0 \exp\left(\frac{\epsilon_v^{el}}{\kappa^*}\right) \qquad q = 3\mu\epsilon_q^{el} + q_0$$

Loading function

$$f(p', q, p_{co}) = \left(p' - \frac{p_{co}}{2}\right)^2 + \frac{q^2}{M^2} - \left(\frac{p_{co}}{2}\right)^2$$

p_{co} — the effective consolidation pressure

M — a material parameter

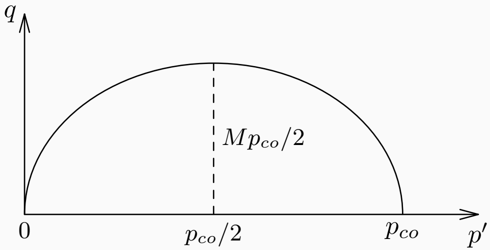

Figure 1: Yield surface $f = 0$.

Flow rule

$$d\epsilon_v^p = d\lambda \frac{\partial f}{\partial p'} = 2d\lambda\left(p' - \frac{p_{co}}{2}\right) \qquad d\epsilon_q^p = d\lambda \frac{\partial f}{\partial q} = 2d\lambda \frac{q}{M^2}$$

where the plastic multiplier $d\lambda$ satisfies the complementarity conditions:

$$d\lambda \geq 0 \qquad f \leq 0 \qquad d\lambda \cdot f = 0$$

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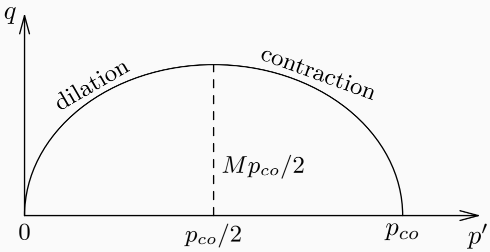

Figure 2: Yield surface.

Hardening

The incremental law:

$$\frac{dp_{co}}{p_{co}} = \frac{1}{\lambda^* - \kappa^*} d\epsilon_v^p \qquad \lambda^* := \frac{\lambda}{1 + e_0} \qquad \kappa < \lambda \text{ — a parameter}$$

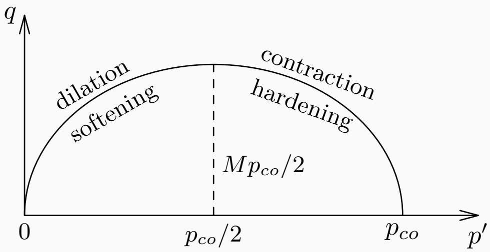

Figure 3: Yield surface.

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By integration:

$$p_{co} = p_{co}^0 \exp\left(\frac{\epsilon_v^p}{\lambda^* - \kappa^*}\right)$$

p_{co}^0 — a reference effective consolidation pressure

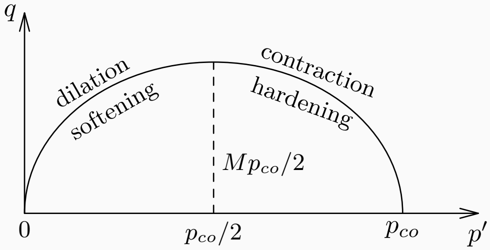

Figure 3: Yield surface.

Input values

Parameters	
Water bulk modulus	K_w
Permeability (tensor)	$\boldsymbol{k}$
Dynamic viscosity of water	μ_w
Elastic stiffness parameter	κ
Shear modulus	μ
Shear strength	M
Plastic stiffness parameter	λ

Initial (reference) values	
Water pressure	p_w^0
Water density	ρ_w^0
Porosity (or void ratio)	ϕ_0 (e_0)
Matrix density	ρ_s^0
Consolidation pressure	p_{co}^0
(Displacement	$\boldsymbol{u}_0 = \boldsymbol{0}$)

Thermodynamical Consistency

Dissipation condition

In the context of small isothermal strains, the non-negativeness of the dissipation associated with the skeleton saturated by water can be written as the following Clausius–Duhem inequality:

$$\boldsymbol{\sigma} : d\boldsymbol{\varepsilon} + p_w d\phi - d\Psi_s \geq 0$$

Ψ_s — the Helmholtz free energy of the skeleton

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By the incompressibility condition $d\phi = d\varepsilon_v$:

$$\boldsymbol{\sigma}' : d\boldsymbol{\varepsilon} - d\Psi_s \geq 0$$

and for an isotropic material under triaxial stress conditions:

$$p' d\epsilon_v + q d\epsilon_q - d\Psi_s \geq 0$$

Energy decomposition

Owing to the additive character of energy, the energy Ψ_s can be split into two parts:
(i) the elastic energy F stored in the skeleton during reversible mechanical processes;
(ii) the locked energy Z that is stored in the skeleton when irreversible (mechanical) processes take place:

$$\Psi_s = F(\epsilon_v^{el}, \epsilon_q^{el}) + Z(\chi)$$

χ — a hardening state variable

Thermodynamical relations I

Inserting the energy decomposition into the dissipation condition, one gets:

$$\left(p' - \frac{\partial F}{\partial \epsilon_v^{el}}\right) d\epsilon_v^{el} + \left(q - \frac{\partial F}{\partial \epsilon_q^{el}}\right) d\epsilon_q^{el} + p' d\epsilon_v^{p} + q d\epsilon_q^{p} + \zeta d\chi \geq 0$$

$$\zeta \equiv -\frac{\mathrm{d}Z}{\mathrm{d}\chi}$$ — the hardening force

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From here:

$$p' = \frac{\partial F}{\partial \epsilon_v^{el}} \qquad q = \frac{\partial F}{\partial \epsilon_q^{el}}$$

$$p' d\epsilon_v^{p} + q d\epsilon_q^{p} + \zeta d\chi \geq 0$$

Thermodynamical relations II

Alternatively, by introducing the energy G by the following Legendre transformation:

$$G(p', q) = p'\epsilon_v^{el} + q\epsilon_q^{el} - F(\epsilon_v^{el}, \epsilon_q^{el})$$

Thermodynamical relations II

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one obtains the state equations:

$$\epsilon_v^{el} = \frac{\partial G}{\partial p'} \qquad \epsilon_q^{el} = \frac{\partial G}{\partial q}$$

Thermodynamical consistency of the poroelasticity

The energy potential G:

$$G(p', q) = \kappa^* p' \left(\ln \frac{p'}{p'_0} - 1 \right) + \frac{(q - q_0)^2}{6\mu}$$

The energy potential F:

$$F(\epsilon_v^{el}, \epsilon_q^{el}) = \kappa^* p'_0 \exp\left(\frac{\epsilon_v^{el}}{\kappa^*} \right) + \frac{3}{2}\mu \left(\epsilon_q^{el} \right)^2 + \epsilon_q^{el} q_0$$

Thermodynamical consistency of the poroplasticity

We identify:

$\chi = \epsilon_v^p$ — the hardening variable $\qquad \zeta = -p_{co}$ — the hardening force

and we require:

$$p_{co} = \frac{\mathrm{d}Z}{\mathrm{d}\epsilon_v^p}$$

This is satisfied by taking:

$$Z(\epsilon_v^p) = (\lambda^* - \kappa^*) p_{co}^0 \exp\left(\frac{\epsilon_v^p}{\lambda^* - \kappa^*} \right)$$

The dissipation condition

It suffices to verify:

$$p' d\epsilon_v^p + q d\epsilon_q^p - p_{co} d\epsilon_v^p \geq 0$$

or by inserting the flow rule:

$$d\lambda \left[2p' \left(p' - \frac{p_{co}}{2} \right) + 2\frac{q^2}{M^2} - 2p_{co} \left(p' - \frac{p_{co}}{2} \right) \right] \geq 0$$

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In virtue of the complementarity conditions either $d\lambda = 0$ or ($d\lambda > 0$ and $f = 0$). In the latter case, one arrives at:

$$2p'\left(p' - \frac{p_{co}}{2}\right) + 2\frac{q^2}{M^2} - 2p_{co}\left(p' - \frac{p_{co}}{2}\right) = p_{co}(p_{co} - p')$$

Therefore one can conclude that the dissipated energy is non-negative over the whole range of admissible effective pressures p' ($p' \leq p_{co}$).

Extension to general stress states

One can express the deviatoric strain ϵ_q and the deviatoric stress q from triaxial stress conditions as functions of the deviatoric tensors $\boldsymbol{\varepsilon}_d$ and $\boldsymbol{s}$:

$$\epsilon_q^2 = \frac{2}{3}\boldsymbol{\varepsilon}_d : \boldsymbol{\varepsilon}_d \qquad \epsilon_q q = \boldsymbol{\varepsilon}_d : \boldsymbol{s} \qquad q^2 = \frac{3}{2}\boldsymbol{s} : \boldsymbol{s}$$

This leads to:

$$F(\epsilon_v^{el}, \boldsymbol{\varepsilon}_d^{el}) = \kappa^* p_0' \exp\left(\frac{\epsilon_v^{el}}{\kappa^*}\right) + \mu \boldsymbol{\varepsilon}_d^{el} : \boldsymbol{\varepsilon}_d^{el} + \boldsymbol{\varepsilon}_d^{el} : \boldsymbol{s}_0$$

$$f(p', \boldsymbol{s}, p_{co}) = \left(p' - \frac{p_{co}}{2}\right)^2 + \frac{2}{3M^2}\boldsymbol{s} : \boldsymbol{s} - \left(\frac{p_{co}}{2}\right)^2$$

and

$$\boldsymbol{s} = \frac{\partial F}{\partial \boldsymbol{\varepsilon}_d^{el}} = 2\mu \boldsymbol{\varepsilon}_d^{el} + \boldsymbol{s}_0$$

$$d\boldsymbol{\varepsilon}_d^{p} = d\lambda \frac{\partial f}{\partial \boldsymbol{s}} = \frac{d\lambda}{3M^2}\boldsymbol{s}$$

Field equations

Continuity equation

One obtains:

$$\frac{\partial(\phi\rho_w)}{\partial t} = \rho_w\frac{\partial\phi}{\partial t} + \phi\frac{\partial\rho_w}{\partial t} = \rho_w\frac{\partial\varepsilon_v}{\partial t} + \frac{\phi\rho_w}{K_w}\frac{\partial p_w}{\partial t}$$

$$\operatorname{div}(\rho_w\boldsymbol{q}_{rw}) = \operatorname{div}\left(\rho_w\frac{\boldsymbol{k}}{\mu_w}(-\nabla p_w + \rho_w\boldsymbol{f})\right)$$

and the water mass balance equation provides:

$$\rho_w\frac{\partial\varepsilon_v}{\partial t} + \frac{\phi\rho_w}{K_w}\frac{\partial p_w}{\partial t} = -\operatorname{div}\left(\rho_w\frac{\boldsymbol{k}}{\mu_w}(-\nabla p_w + \rho_w\boldsymbol{f})\right)$$

Equilibrium equation

By taking:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}' - p_w \boldsymbol{I} = \boldsymbol{s} - p' \boldsymbol{I} - p_w \boldsymbol{I}$$

and invoking the stress–strain relationship, one obtains:

$$-\frac{\partial p'}{\partial \varepsilon_v^{el}} \nabla \varepsilon_v^{el} + 2\mu \, \mathbf{div} \, \boldsymbol{\varepsilon}_d^{el} - \nabla p_w + \big((1-\phi_0)\rho_s^0 + \phi \rho_w\big) \boldsymbol{f} = \mathbf{0}$$

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