130 (2005) MATHEMATICA BOHEMICA No. 1, 49–62

A SCALAR VOLTERRA DERIVATIVE FOR THE PoU-INTEGRAL

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(Received March 1, 2004)

Abstract. A weak form of the Henstock Lemma for the PoU-integrable functions is given. This allows to prove the existence of a scalar Volterra derivative for the PoU-integral. Also the PoU-integrable functions are characterized by means of Pettis integrability and a condition involving finite pseudopartitions.

Keywords: Pettis integral, McShane integral, PoU integral, Volterra derivative

MSC 2000: 28B05, 46G10

1. Introduction

In [8] and [9] J. Jarník and J. Kurzweil introduced an integration process (called PU-integral) for real valued functions on an interval of $\mathbb{R}^n$ with the use of suitably regular C^1-partitions of unity, instead of the usual partitions. The PU-integral is non-absolutely convergent and in dimension one falls properly in between the Lebesgue and the Kurzweil-Henstock integrals.

In [4], without assuming any regularity condition for the applied partition of unity, there is studied an integral (called the PoU-integral) for Banach valued functions defined on a σ-finite quasi-Radon measure space. In particular it is proved that it is equivalent to the generalized McShane integral as defined by Fremlin in [6].

Here we continue the investigation of the vector valued PoU-integrable functions started in [4]. In Section 3, using a form of the Henstock Lemma (see Proposition 1), we characterize the Pettis integrable functions which are also PoU-integrable by means of finite pseudopartitions (Theorem 1).

In Section 4, using a suitable derivation base satisfying the strong Vitali covering condition, we prove the existence of a scalar form of the Volterra derivative of the

Supported by MURST of Italy.

PoU-integral (Theorem 2). We observe that a stronger form of Volterra derivative for an operator associated to a PoU-integrable function cannot exist (see Remark 4).

2. Notations and definitions

Let $(\Omega, \mathscr{T}, \mathscr{F}, \mu)$ be a non-empty σ-finite outer regular quasi-Radon measure space, where $\mathscr{T}$ is the family of the open sets in Ω, and $\mathscr{F}$ is the family of all μ-measurable sets. Unless specified otherwise, the terms "measure", "measurable" and "almost everywhere" (briefly "a.e.") are referred to the measure μ. For a set E, we denote by χ_E the characteristic function of E. A set $E \subset \Omega$ is called *negligible* if $\mu(E) = 0$. Given a function $\theta \in L^1(\Omega, \mathbb{R})$, we set $S_\theta = \{\omega \in \Omega\colon \theta(\omega) \neq 0\}$. A *generalized McShane partition* (or simply an Mc-*partition*) (see [6] Definitions 1A) in Ω is a countable (eventually finite) set of pairs $P = \{(E_i, \omega_i)\colon i = 1, 2, \ldots\}$ where $(E_i)_i$ is a disjoint family of measurable sets of finite measure and $\omega_i \in \Omega$ for each $i = 1, 2 \ldots$. If $\mu\Big(\Omega \setminus \bigcup\limits_i E_i\Big) = 0$, we say that P is an Mc-*partition of* Ω. A *generalized pseudopartition* (or simply a *pseudopartition*) in Ω is a countable (eventually finite) set of pairs $\mathscr{Q} = \{(\theta_i, \omega_i)\colon i = 1, 2, \ldots\}$ where, for each $i = 1, 2, \ldots$, $\omega_i \in \Omega$ and $\theta_i\colon \Omega \to \mathbb{R}$ are nonnegative measurable functions such that the sets S_{θ_i} are of positive finite measure and $\sum\limits_i \theta_i \leqslant 1$ a.e. in Ω. If $\sum\limits_i \theta_i = 1$ a.e. in Ω, we say that $\mathscr{Q}$ is a *pseudopartition of* Ω.

Note that if $\mathscr{P} = \{(E_i, \omega_i)\colon i = 1, 2, \ldots\}$ is an Mc-partition in Ω, then $\mathscr{P}^* = \{(\chi_{E_i}, \omega_i)\colon i = 1, 2, \ldots\}$ is a pseudopartition in Ω, called the pseudopartition *induced* by $\mathscr{P}$. A *gauge* on Ω is a function $\Delta\colon \Omega \to \mathscr{T}$ such that $\omega \in \Delta(\omega)$ for each $\omega \in \Omega$. We say that an Mc-partition $\{(E_i, \omega_i)\colon i = 1, 2, \ldots\}$ (respectively a pseudopartition $\{(\theta_i, \omega_i)\colon i = 1, 2, \ldots\}$) is *subordinate to a gauge* Δ if $E_i \subset \Delta(\omega_i)$ (resp. $S_{\theta_i} \subset \Delta(\omega_i)$) for $i = 1, 2, \ldots$.

R e m a r k 1. If $\mathscr{P} = \{(E_i, \omega_i)\colon i = 1, 2, \ldots\}$ is an Mc-partition subordinate to a gauge Δ, then the pseudopartition P^* induced by P is also subordinate to Δ.

R e m a r k 2. It has been proved by Fremlin (see [6], Remark 1B (d)) that corresponding to each gauge Δ there is an Mc-partition of Ω subordinate to Δ. Therefore by Remark 1 the set of pseudopartitions subordinate to any gauge Δ is not empty.

From now on X is a real Banach space with dual X^* and $\mathscr{B}(X^*)$ is the closed unit ball of X^*.

Definition 1. We recall the following definitions.

a) A function $f\colon \Omega \to X$ is said to be Pettis integrable if $x^* f$ is Lebesgue integrable on Ω for each $x^* \in X^*$, and for every measurable set $E \subset \Omega$ there is a vector

$\nu(E) = \int_E f \in X$ such that $x^*(\nu(E)) = \int_E x^* f(\omega)\,\mathrm{d}\mu$ for all $x^* \in X^*$. The set function $\nu\colon \mathscr{F} \to X$ is called the indefinite Pettis integral of f.

As it is known (cf. [1]) ν is a countably additive vector measure, continuous with respect to μ (in the sense that for each $\varepsilon > 0$ there is $\eta > 0$ such that if $\mu(E) < \eta$ then $\|\nu(E)\| < \varepsilon$).

b) A function $f\colon \Omega \to X$ is said to be *McShane integrable* (see [6] Definitions 1A) (briefly Mc-integrable), with *McShane integral* $z \in X$ if for each $\varepsilon > 0$ there exists a gauge $\Delta\colon \Omega \to \mathscr{T}$ such that

$$\limsup_n \left\| z - \sum_{i \leqslant n} \mu(E_i) f(\omega_i) \right\| < \varepsilon$$

for each Mc-partition $\{(E_i, \omega_i)\colon\ i = 1, 2\ldots\}$ of Ω subordinate to Δ.

If f is an Mc-integrable function on Ω we set $z = (\mathrm{Mc})\int_\Omega f$.

Definition 2 (see [4] Definition 2)**.** A function $f\colon \Omega \to X$ is said to be PoU-*integrable* with PoU-*integral* $z \in X$ if for each $\varepsilon > 0$ there exists a gauge $\Delta\colon \Omega \to \mathscr{T}$ such that

$$\limsup_n \left\| z - \sum_{i \leqslant n} f(\omega_i) \int_\Omega \theta_i \right\| < \varepsilon$$

for each pseudopartition $\{(\theta_i, \omega_i)\colon\ i = 1, 2, \ldots\}$ of Ω subordinate to Δ.

If f is a PoU-integrable function on Ω we set $z = (\mathrm{PoU})\int_\Omega f$.

Remark 3. It has been proved that the family of Mc-integrable functions coincides with the family of PoU-integrable ones (see [4] Corollary 1).

3. PoU-integral

The following proposition is a form of the Henstock Lemma for PoU-integrable functions.

Proposition 1. *Let $f\colon \Omega \to X$ be a* PoU*-integrable function. Then for each $\varepsilon > 0$ there exists a gauge $\Delta\colon \Omega \to \mathscr{T}$ such that*

$$\left\| \sum_{i=1}^{n} \left(f(\omega_i) \int_\Omega \theta_i - (\mathrm{PoU}) \int_\Omega \theta_i f \right) \right\| < \varepsilon$$

for each finite pseudopartition $\{(\theta_i, \omega_i)\colon\ i = 1, \ldots, n\}$ in Ω subordinate to Δ.

P r o o f. By [4] Corollary 2 for each measurable, nonnegative real valued bounded function θ, also the function θf is PoU-integrable and $\int_\Omega \theta f$ is well defined. Fix $\varepsilon > 0$ and find a gauge $\Delta_1\colon \Omega \to \mathscr{T}$ such that

$$\limsup_n \left\| (\mathrm{PoU}) \int_\Omega f - \sum_{i \leqslant n} f(\omega_i) \int_\Omega \theta_i \right\| < \frac{\varepsilon}{4} \tag{1}$$

for each pseudopartition $\{(\theta_i, \omega_i)\colon\ i = 1, 2, \ldots\}$ of Ω subordinate to Δ. By Remark 3 the function f is Mc-integrable, thus according to [6] Lemma 2B we can find a gauge Δ_2 such that

$$\left\| (\mathrm{Mc}) \int_E f - \sum_{i \leqslant n} f(\omega_i) \mu(E_i) \right\| < \frac{\varepsilon}{4} \tag{2}$$

whenever $\{(E_i, \omega_i)\colon\ i = 1, \ldots, n\}$ is a finite Mc-partition in Ω subordinate to Δ_2 and $\bigcup\limits_{i=1}^{n} E_i = E$. Set $\Delta(\omega) = \Delta_1(\omega) \cap \Delta_2(\omega)$. Let $\{(\theta_i, \omega_i)\colon\ i = 1, \ldots, p\}$ be a finite pseudopartition in Ω subordinate to Δ. If $S = \bigcup\limits_{i=1}^{p} S_{\theta_i}$ then $\sum\limits_{i=1}^{p} \theta_i \leqslant \chi_S$. Set $S_1 = S_{\theta_1}$, $S_2 = S_{\theta_2} \setminus S_1, \ldots, S_p = S_{\theta_p} \setminus \bigcup\limits_{i=1}^{p-1} S_i$. Without loss of generality, we can assume that $\left(1 - \sum\limits_{i=1}^{p} \theta_i\right) > 0$. Define

$$\theta_{p+i} = \left(1 - \sum_{j=i}^{p} \theta_j\right) \chi_{S_i}.$$

Let $Q = \{(\theta_1, \omega_1), \ldots, (\theta_p, \omega_p), (\theta_{p+1}, \omega_1), \ldots, (\theta_{2p}, \omega_p)\}$. Then Q is a finite pseudopartition subordinate to Δ, indeed for $i = 1, \ldots, p$

$$S_{\theta_{p+i}} = S_i \subset \Delta(\omega_i).$$

Moreover

$$\sum_{i=1}^{2p} \theta_i = \chi_S. \tag{3}$$

As S is a measurable subset of Ω, according to [6] Theorem 1N, for each partition $\{(F_j, u_j)\colon\ j = 1, 2, \ldots\}$ of $\Omega \setminus S$ subordinate to Δ, we have

$$\limsup_m \left\| (\mathrm{Mc}) \int_{\Omega \setminus S} f - \sum_{j \leqslant m} \mu(F_j) f(u_j) \right\| < \frac{\varepsilon}{4}. \tag{4}$$

Now let $\{(F_j, u_j)\colon\ j = 1, 2, \ldots\}$ be a fixed partition satisfying (4). For $j = 1, 2, \ldots$, set $\theta_{2p+j} = \chi_{F_j}$ and $\omega_{2p+j} = u_j$, then $R = \{(\theta_i, \omega_i)\colon\ i = 1, 2, \ldots\}$ is a pseudopartition of Ω subordinate to Δ, therefore

$$\limsup_n \left\| (\mathrm{PoU})\int_\Omega f - \sum_{i\leqslant n} f(\omega_i)\int_\Omega \theta_i \right\| < \frac{\varepsilon}{4}. \tag{5}$$

Let $x^* \in \mathscr{B}(X^*)$. For $n > 2p$, by (3), we get

$$\begin{aligned}
&\left|\sum_{i=1}^{p} x^*\left[f(\omega_i)\int_\Omega \theta_i - (\mathrm{PoU})\int_\Omega f\theta_i\right]\right| \qquad (6)\\
&\leqslant \left|\sum_{i=1}^{p} x^*\left[f(\omega_i)\int_\Omega \theta_i - (\mathrm{PoU})\int_\Omega f\theta_i\right] + \sum_{i=p+1}^{2p} x^*\left[f(\omega_i)\int_\Omega \theta_i - (\mathrm{PoU})\int_\Omega f\theta_i\right]\right.\\
&\quad\left. + \sum_{i=2p+1}^{n} x^* f(\omega_i)\int_\Omega \theta_i - \int_{\Omega\setminus S} x^* f\right|\\
&\quad +\left|\sum_{i=p+1}^{2p} x^*\left[f(\omega_i)\int_\Omega \theta_i - (\mathrm{PoU})\int_\Omega f\theta_i\right]\right| + \left|\sum_{i=2p+1}^{n} x^* f(\omega_i)\int_\Omega \theta_i - \int_{\Omega\setminus S} x^* f\right|\\
&= \left|\sum_{i=1}^{2p} x^* f(\omega_i)\int_\Omega \theta_i - \int_\Omega x^* f\sum_{i=1}^{2p}\theta_i + \sum_{i=2p+1}^{n} x^* f(\omega_i)\int_\Omega \theta_i - \int_{\Omega\setminus S} x^* f\right|\\
&\quad +\left|\sum_{i=p+1}^{2p} x^*\left[f(\omega_i)\int_\Omega \theta_i - (\mathrm{PoU})\int_\Omega f\theta_i\right]\right| + \left|x^*\left(\sum_{i=2p+1}^{n} \mu(F_i)f(\omega_i) - (\mathrm{Mc})\int_{\Omega\setminus S} f\right)\right|\\
&\leqslant \left|\sum_{i=1}^{n} x^* f(\omega_i)\int_\Omega \theta_i - \int_S x^* f - \int_{\Omega\setminus S} x^* f\right| + \left|\sum_{i=p+1}^{2p} x^*\left[f(\omega_i)\int_\Omega \theta_i - (\mathrm{PoU})\int_\Omega f\theta_i\right]\right|\\
&\quad +\left\|\sum_{i=2p+1}^{n} \mu(F_i)f(\omega_i) - (\mathrm{Mc})\int_{\Omega\setminus S} f\right\| = \left|x^*\left[\sum_{i=1}^{n} f(\omega_i)\int_\Omega \theta_i - (\mathrm{PoU})\int_\Omega f\right]\right|\\
&\quad +\left|\sum_{i=p+1}^{2p} x^*\left[f(\omega_i)\int_{S_i} \theta_i - (\mathrm{PoU})\int_\Omega f\theta_i\right]\right| + \left\|\sum_{i=2p+1}^{n} \mu(F_i)f(\omega_i) - (\mathrm{Mc})\int_{\Omega\setminus S} f\right\|.
\end{aligned}$$

Now, evaluating the three terms separately, if n is sufficiently large, by (5), we have

$$\left|x^*\left[\sum_{i=1}^{n} f(\omega_i)\int_\Omega \theta_i - (\mathrm{PoU})\int_\Omega f\right]\right| \leqslant \left\|\sum_{i=1}^{n} f(\omega_i)\int_\Omega \theta_i - (\mathrm{PoU})\int_\Omega f\right\| < \frac{\varepsilon}{4} \tag{7}$$

and by (4)

$$\left\|\sum_{i=2p+1}^{n} \mu(F_i)f(\omega_i) - (\mathrm{Mc})\int_{\Omega\setminus S} f\right\| < \frac{\varepsilon}{4}. \tag{8}$$

By [4] Lemma 1 we obtain

$$\text{(9)}\quad \Big|\sum_{i=p+1}^{2p} x^*\Big[f(\omega_i)\int_{S_i}\theta_i - (\mathrm{PoU})\int_\Omega f\theta_i\Big]\Big|$$
$$\leqslant \Big|\sum_{i=p+1}^{2p}\int_{L_i'}(x^*f(\omega_i)-x^*f)\,\mathrm{d}\mu\Big| + \Big|\sum_{i=p+1}^{2p}\int_{L_i''}(x^*f(\omega_i)-x^*f)\,\mathrm{d}\mu\Big|,$$

where L_i', $i=p+1,\dots,2p$, are pairwise disjoint measurable sets with $L_i'\subset\{t\in S_i\colon x^*f(t)-x^*f(\omega_i)\geqslant 0\}$, L_i'', $i=p+1,\dots,2p$, are pairwise disjoint measurable sets with $L_i''\subset\{t\in S_i\colon x^*f(t)-x^*f(\omega_i)<0\}$, and $\bigcup\limits_{i=p+1}^{2p} S_i=\bigcup\limits_{i=p+1}^{2p}(L_i'\cup L_i'')$. Since $\{(L_i',\omega_i)\colon i=p+1,\dots 2p\}$ and $\{(L_i'',\omega_i)\colon i=p+1,\dots 2p\}$ are two finite partitions in Ω subordinate to Δ, from (2) we get

$$\text{(10)}\quad \Big|\sum_{i=p+1}^{2p}\int_{L_i'}(x^*f(\omega_i)-x^*f)\,\mathrm{d}\mu\Big| + \Big|\sum_{i=p+1}^{2p}\int_{L_i''}(x^*f(\omega_i)-x^*f)\,\mathrm{d}\mu\Big|$$
$$= \Big|\sum_{i=p+1}^{2p}\int_{L_i'}x^*f\,\mathrm{d}\mu - |L_i'|x^*f(\omega_i)\Big| + \Big|\sum_{i=p+1}^{2p}\int_{L_i''}x^*f\,\mathrm{d}\mu - |L_i''|x^*f(\omega_i)\Big|$$
$$< \frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\frac{\varepsilon}{2}.$$

By (6), (7), (8), (9) and (10) we get

$$\Big|\sum_{i=1}^{p} x^*\Big[f(\omega_i)\int_\Omega\theta_i - (\mathrm{PoU})\int_\Omega f\theta_i\Big]\Big| < \varepsilon.$$

Since this is true for each $x^*\in\mathscr{B}(X^*)$ we get

$$\Big\|\sum_{i=1}^{p}\Big[f(\omega_i)\int_\Omega\theta_i - (\mathrm{PoU})\int_\Omega f\theta_i\Big]\Big\| \leqslant \varepsilon,$$

and the assertion follows. □

Proposition 2. *Let $f\colon\Omega\to X$ be a* PoU*-integrable function. Then for each $\varepsilon>0$ there exists a gauge $\Delta\colon\Omega\to\mathscr{T}$ such that*

$$\sup_{x^*\in\mathscr{B}(X^*)}\sum_{i=1}^{n}\Big|x^*\Big[f(\omega_i)\int_\Omega\theta_i - (\mathrm{PoU})\int_\Omega\theta_i f\Big]\Big| < \varepsilon$$

for each finite pseudopartition $\{(\theta_i,\omega_i)\colon i=1,\dots,n\}$ in Ω subordinate to Δ.

P r o o f. The result follows from Proposition 1 and by the standard argument of splitting the sum of real numbers into the sum of nonnegative and negative parts to obtain the absolute value inside the summation sign (see for instance [3] Proposition 1). $\square$

Lemma 1. *Let $f\colon \Omega \to X$ be a Pettis integrable function and let θ_i, $i = 1, 2, \ldots$ be nonnegative functions in $L^1(\Omega, \mathbb{R})$ such that $\sum\limits_i \theta_i = 1$ a.e. in Ω. Then for each $\varepsilon > 0$, there exists a natural number M such that for each $n > M$,*

$$\Big\| \int_\Omega \sum_{i=n+1}^{\infty} \theta_i f \Big\| < \varepsilon.$$

P r o o f. According to the σ-finiteness of μ, $\Omega = \bigcup\limits_j \Omega_j$, where Ω_j are disjoint measurable sets of finite measure. Since f is Pettis integrable, ν is a strongly additive measure on $\mathscr{F}$. Then (see [2] Corollary 12, p. 105) the set of variations $\{|x^*\nu|\colon x^* \in \mathscr{B}(X^*)\}$ is uniformly strongly additive. So there exists a natural number K such that

$$|x^*\nu|\Big(\bigcup_{k=K+1}^{\infty} \Omega_k\Big) = \int_{\bigcup\limits_{k=K+1}^{\infty} \Omega_k} |x^* f| \,\mathrm{d}\mu < \frac{\varepsilon}{5} \tag{11}$$

for all $x^* \in \mathscr{B}(X^*)$. Moreover, for each $i = 1, 2, \ldots$, θ_i is a nonnegative real valued essentially bounded function, so by [5] Theorem 1.1.2, $\theta_i f$ is Pettis integrable. Set $f_n = \sum\limits_{i=1}^{n} \theta_i f$. Since $\sum\limits_{i=1}^{\infty} \theta_i = 1$ a.e., $\lim\limits_n f_n(\omega) = \lim\limits_n \sum\limits_{i=1}^{n} \theta_i f(\omega) = \sum\limits_{i=1}^{\infty} \theta_i f(\omega) = f(\omega)$ a.e. in Ω.

If $T = \bigcup\limits_{k=1}^{K} \Omega_k$, then T is a set of finite measure. According to the continuity of ν with respect to μ over the set T, there exists $\eta > 0$ such that if $F \subset T$, $F \in \mathscr{F}$, and $\mu(F) < \eta$, then

$$\|\nu(F)\| \leqslant \frac{\varepsilon}{5}. \tag{12}$$

Set now $T_m = \{\omega \in T\colon \|f_p(\omega) - f(\omega)\| \leqslant \frac{1}{5}\varepsilon(1 + \mu(T))^{-1}$ for all $p \geqslant m\}$. Then $\mu(T \backslash \bigcup_m T_m) = 0$, and there exists a natural number M such that $\mu^*(T_M) \geqslant \mu(T) - \eta$. Let $C \in \mathscr{F}$ be such that $T_M \subseteq C \subseteq T$ and $\mu(C) = \mu^*(T_M)$. For all $n > M$ we have

$$\begin{aligned} \Big\| \int_C f - \int_C f_n \Big\| &= \Big\| \int_{T_M} f - \int_{T_M} f_n \Big\| \\ &\leqslant \overline{\int}_{T_M} \|f - f_n\| \leqslant \frac{\varepsilon}{5(1 + \mu(T))} \mu^*(T_M) < \frac{\varepsilon}{5}. \end{aligned} \tag{13}$$

Moreover for all $x^* \in X^*$ and for all $n \in \mathbb{N}$, $|x^*(f_n)| \leqslant |x^*(f)|$, so if F is any measurable subset of T with $\mu(F) < \eta$, by (12) we deduce

$$\Big\| \int_F f_n \Big\| \leqslant \sup_{x^* \in \mathscr{B}(X^*)} \int_F |x^* f_n| \leqslant \sup_{x^* \in \mathscr{B}(X^*)} \int_F |x^* f| \leqslant \frac{2\varepsilon}{5}, \tag{14}$$

for each $n \in \mathbb{N}$. Let $n > M$; since $\mu(T \setminus C) < \eta$, by (11), (12), (13) and (14) we infer

$$\begin{aligned} \Big\| \int_\Omega f - \int_\Omega f_n \Big\| &\leqslant \Big\| \int_C f - \int_C f_n \Big\| + \Big\| \int_{T\setminus C} f \Big\| \\ &\quad + \Big\| \int_{T\setminus C} f_n \Big\| + \Big\| \int_{\bigcup\limits_{k=K+1}^{\infty} \Omega_k} \Big(\sum_{n+1}^{\infty} \theta_i \Big) f \Big\| \\ &\leqslant \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{2\varepsilon}{5} + \sup_{x^* \in \mathscr{B}(X^*)} \int_{\bigcup\limits_{k=K+1}^{\infty} \Omega_k} |x^* f| \, \mathrm{d}\mu < \varepsilon. \end{aligned} \tag{15}$$

Thus $\int_\Omega f_n$ converges to $\int_\Omega f$ strongly. Since

$$\Big\| \int_\Omega f_n - \int_\Omega f \Big\| = \Big\| \int_\Omega \sum_{i=n+1}^{\infty} \theta_i f \Big\|,$$

for all $n > M$, by (15) we deduce that

$$\Big\| \int_\Omega \sum_{i=n+1}^{\infty} \theta_i f \Big\| < \varepsilon, \tag{16}$$

and the assertion follows. $\square$

The following characterization of the PoU-integral involving only finite pseudopartitions holds.

Theorem 1. *Let $f\colon \Omega \to X$. The function f is* PoU-*integrable if and only if f is Pettis integrable and for each $\varepsilon > 0$ there exists a gauge $\Delta\colon \Omega \to \mathscr{T}$ such that for each finite pseudopartition $Q = \{(\theta_i, \omega_i)\colon i = 1, \ldots, p\}$ in Ω subordinate to Δ, satisfying the inequality $\| \int_\Omega (1 - \sum\limits_{i=1}^{p} \theta_i) f \| < \varepsilon$, we have*

$$\Big\| \sum_{i \leqslant p} f(\omega_i) \int_\Omega \theta_i - \int_\Omega f \Big\| < \varepsilon.$$

Proof. Assume first that f is PoU-integrable, then it is Pettis integrable. Let $\varepsilon > 0$ be fixed. By the previous lemma, if θ_i, $i = 1, 2, \ldots$ are nonnegative functions in $L^1(\Omega, \mathbb{R})$ such that $\sum\limits_{i=1}^{\infty} \theta_i = 1$ a.e. in Ω, then there exists a natural number M such that for each $n > M$,

$$\left\| \int_\Omega \sum_{i=n+1}^{\infty} \theta_i f \right\| < \frac{\varepsilon}{2}. \tag{17}$$

Moreover by Proposition 1 there is a gauge $\Delta\colon \Omega \to \mathscr{T}$ such that

$$\left\| \sum_{i=1}^{n} \left(f(\omega_i) \int_\Omega \theta_i - (\mathrm{PoU}) \int_\Omega \theta_i f \right) \right\| < \frac{\varepsilon}{2} \tag{18}$$

for each finite pseudopartition $\{(\theta_i, \omega_i)\colon i = 1, \ldots, n\}$ in Ω subordinate to Δ. Let now $Q = \{(\theta_i, \omega_i)\colon i = 1, \ldots, p\}$ be a finite pseudopartition in Ω subordinate to Δ and such that $\| \int_\Omega (1 - \sum\limits_{i=1}^{p} \theta_i) f \| < \frac{\varepsilon}{2}$. Then, by (17) and (18), it follows that

$$\begin{aligned} \left\| \sum_{i=1}^{p} f(\omega_i) \int_\Omega \theta_i - \int_\Omega f \right\| &\leqslant \left\| \sum_{i=1}^{p} \left(f(\omega_i) \int_\Omega \theta_i - (\mathrm{PoU}) \int_\Omega \theta_i f \right) \right\| \\ &\quad + \left\| \int_\Omega \left(1 - \sum_{i=1}^{p} \theta_i \right) f \right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Conversely, let f be a Pettis integrable function and let $\varepsilon > 0$ be fixed. By hypothesis there is Δ such that

$$\left\| \sum_{i=1}^{p} f(\omega_i) \int_\Omega \theta_i - \int_\Omega f \right\| < \varepsilon,$$

for each finite pseudopartition $Q = \{(\theta_i, \omega_i)\colon i = 1, \ldots, p\}$ of Ω subordinate to Δ and satisfying $\left\| \int_\Omega \left(1 - \sum\limits_{i=1}^{p} \theta_i\right) f \right\| < \varepsilon$. Let $R = \{(\vartheta_i, \omega_i)\colon i = 1, 2, \ldots\}$ be a pseudopartition of Ω subordinate to Δ. By Lemma 1 there is a natural number M such that for each $n > M$,

$$\left\| \int_\Omega \sum_{i=n+1}^{\infty} \vartheta_i f \right\| < \varepsilon. \tag{19}$$

Now for each $n > M$, $R' = \{(\vartheta_i, \omega_i)\colon i = 1, \ldots, n\}$ is a finite pseudopartition subordinate to Δ and satisfying (19). Therefore

$$\left\| \sum_{i=1}^{n} f(\omega_i) \int_\Omega \vartheta_i - \int_\Omega f \right\| < \varepsilon.$$

Since n is large enough it follows that the function f is PoU-integrable. $\square$

4. Scalar Volterra derivative for the PoU-integral

In this section we will define a scalar form of the Volterra derivative for the PoU-integral of Banach valued functions f.

We recall that a *derivation base* on Ω (see for example [13], Chapter 8) is a nonempty subset $\mathscr{B}$ of $\mathscr{F} \times \Omega$. For a set $E \subset \Omega$ we write

$$\mathscr{B}(E) = \{(A,\omega) \in \mathscr{B}\colon\ A \subset E\} \quad \text{and} \quad \mathscr{B}[E] = \{(A,\omega) \in \mathscr{B}\colon\ \omega \in E\}.$$

If Δ is a gauge defined on Ω we denote by

$$\mathscr{B}_\Delta = \{(A,\omega) \in \mathscr{B}\colon\ A \subset \Delta(\omega)\}.$$

We say that a base $\mathscr{B}$ is

- a *fine base* on a set $E \subset \Omega$ if for any $\omega \in E$ and for any gauge Δ the set $\mathscr{B}_\Delta[\{\omega\}]$ is nonempty;
- a *filtering base* if for each $\omega \in \Omega$, the set $\mathscr{B}[\{\omega\}]$ is a directed set.

It is known that the Vitali covering Theorem is an important tool for classical derivation theorems on functions defined on subsets of $\mathbb{R}^n$. It is perhaps worth recalling at this point that a derivation base $\mathscr{B}$ with the strong Vitali property differentiates all L^1-integrals.

Definition 3. A derivation base $\mathscr{B}$ satisfies the *strong Vitali property* if, for every $\mathscr{B}^* \subset \mathscr{B}$, fine on a set E, and every $\varepsilon > 0$, there exist finitely many couples $(A_1,\omega_1),(A_2,\omega_2),\dots,(A_n,\omega_n)$ in $\mathscr{B}^*$ such that the sets $A_1, A_2, \dots, A_n$ are pairwise disjoint and

$$\mu\biggl(E \nabla \bigcup_{i=1}^{n} A_i\biggr) < \varepsilon,$$

where the symbol ∇ denotes the symmetric difference.

As usual the symbol L^∞ denotes the family of all essentially bounded functions $\theta\colon\ \Omega \to \mathbb{R}$. If f is a PoU-integrable function we associate to f the operator $F\colon\ L^\infty \to X$, setting

$$F(\theta) = (\mathrm{PoU})\int_\Omega f\theta. \tag{20}$$

Observe that since by [4] Corollary 2, essentially bounded functions are multipliers for the PoU-integrable functions, the operator F is well defined.

In the following the symbol $\mathscr{L}$ will denote the family of all measurable functions $\theta\colon\ \Omega \to \mathbb{R}$ such that $0 \leqslant \theta \leqslant 1$ a.e. and the set S_θ is of positive finite measure. From now on we will consider the base $\mathbf{B} = \{(S_\theta, \omega)\}$, where $\theta \in \mathscr{L}$. Since by Remark 2 the set of pseudopartitions subordinate to any gauge Δ is not empty, the family $\mathbf{B}_\Delta$ is a fine base. For (S_{θ_1}, ω) and $(S_{\theta_2}, \omega) \in \mathbf{B}$, define $(S_{\theta_1}, \omega) \gg (S_{\theta_2}, \omega)$ if there exist two open sets Ω_1 and Ω_2 such that $S_{\theta_1} \subset \Omega_1$, $S_{\theta_2} \subset \Omega_2$ and $\omega \in \Omega_2 \subset \Omega_1$. Moreover, as the family of gauges on Ω is directed downward (see [6] Remark 1D), the base $\mathbf{B}$ is filtering if we consider the induced order.

In [11], considering functions of bounded variation, a type of Volterra derivative of the integral of a scalar valued L^1-integrable function f is defined, and it is proved that it coincides with the function f.

We recall that a functional $J\colon L^\infty \to \mathbb{R}$ is *Volterra* $\mathbf{B}$*-differentiable* at $\omega \in \Omega$ (see [11]) if there is a real number α such that

$$\lim \frac{J(\theta)}{\int_\Omega \theta} = \alpha$$

where the limit is taken in the directed set $\mathbf{B}[\{\omega\}]$.

We extend the definition of scalar derivative of a function F given by Pettis in [10] to the scalar Volterra derivative of an operator $J\colon L^\infty \to X$.

Definition 4. We say that a function $g\colon \Omega \to X$ is a scalar Volterra $\mathbf{B}$-derivative of the operator $J\colon L^\infty \to X$, if for each $x^* \in X^*$, the real valued functional x^*J is Volterra $\mathbf{B}$-differentiable at almost all $\omega \in \Omega$ and

$$\lim x^*\Big(\frac{J(\theta)}{\int_\Omega \theta}\Big) = x^* g(\omega) \tag{21}$$

where the limit is taken in the directed set $\mathbf{B}[\{\omega\}]$.

Theorem 2. *Let* $f\colon \Omega \to X$ *be a* PoU*-integrable function and let* F *be the associated operator defined in* (20). *Assume that the derivation base* $\mathbf{B}$ *satisfies the strong Vitali property. Then the function* f *is a scalar Volterra* $\mathbf{B}$*-derivative of* F.

Proof. Fix $x^* \in X^*$ and let N_{x^*} be the set of all $\omega \in \Omega$ for which (21) fails. Given $\omega \in N_{x^*}$ there is $\eta(\omega) > 0$ such that for each gauge Δ we can find a function $\theta \in \mathscr{L}$ subordinate to Δ, with $\int_\Omega \theta < 1/\eta(\omega)$ for which

$$\Big|x^*\Big(f(\omega)\int_\Omega \theta - (\mathrm{PoU})\int_\Omega f\theta\Big)\Big| \geqslant \eta(\omega)\int_\Omega \theta.$$

Fix an integer $n \geqslant 1$ and set $N_n = \{\omega \in N_{x^*}\colon\ \eta(\omega) > 1/n\}$. If $\varepsilon > 0$, according to Proposition 2 we find Δ_1 so that

$$\sup_{x^*\in\mathscr{B}(X^*)} \sum_{i=1}^{\infty} \Big|x^*\Big[f(\omega_i)\int_\Omega \theta_i - (\mathrm{PoU})\int_\Omega \theta_i f\Big]\Big| < \frac{\varepsilon}{2n} \tag{22}$$

for each pseudopartition $\{(\theta_i,\omega_i)\colon i=1,2,\ldots\}$ in Ω subordinate to Δ_1. Let $\mathscr{S}$ be the family of all functions $\theta\in\mathscr{L}$ such that $S_\theta\subset\Delta(\omega)$ for a $\omega\in N_n$ and for some gauge Δ, with $\Delta(\omega)\subset\Delta_1(\omega)$ and

$$\left|x^*\left(f(\omega)\int_\Omega\theta-(\mathrm{PoU})\int_\Omega f\theta\right)\right|\geqslant\frac{1}{n}\int_\Omega\theta.$$

Then the family $\mathscr{S}^*=\{(S_\theta,\omega_\theta)\}$ is a fine base of N_n. Indeed for any $\omega\in N_n$ and for any gauge Δ with $\Delta(\omega_\theta)\subset\Delta_1(\omega_\theta)$, the set $\mathscr{S}^*_\Delta[\{\omega\}]$ is not empty. Moreover we may assume that $\int_\Omega\theta\geqslant\frac12\mu(S_\theta)$. Since the strong Vitali property holds, there are couples $(S_1,\omega_1),(S_2,\omega_2),\ldots\in\mathscr{S}^*$ such that $S_1,S_2,\ldots$ are pairwise disjoint and $\mu\Bigl(N_n\nabla\bigcup\limits_{i=1}^{\infty}S_i\Bigr)=0$. If $\theta_i=\theta_{S_i}$ and $\omega_i=\omega_{S_i}$, then $Q=\{(\theta_1,\omega_1),(\theta_2,\omega_2),\ldots\}$ is a pseudopartition in Ω subordinate to Δ_1. For each p, by (22), we get

$$\sum_{i=1}^{p}\mu(S_i)\leqslant2\sum_{i=1}^{p}\int_\Omega\theta_i\leqslant2n\sum_{i=1}^{p}\left|x^*\left[\int_\Omega\theta_i f(\omega_i)-(\mathrm{PoU})\int_\Omega\theta_i f\right]\right|<\varepsilon.$$

By the arbitrariness of ε, it follows that $\mu(N_n)=0$ and as $N_{x^*}=\bigcup\limits_{n=1}^{\infty}N_n$ we see that $\mu(N_{x^*})=0$, and the assertion follows. □

R e m a r k 4. In the previous theorem we prove that a scalar derivative of the operator F defined in (20) exists, that is for each $x^*\in X^*$ and for all $\omega\notin N_{x^*}$, where N_{x^*} is a negligible set,

$$\lim x^*\Bigl(\frac{F(\theta)}{\int_\Omega\theta}\Bigr)=x^*f(\omega).$$

If the previous equality holds for all $\omega\in N$, where the negligible set N is independent of x^*, we say that the function $f(\omega)$ is the *weak Volterra* **B**-*derivative* of F. Observe that the condition concerning the existence of a scalar derivative of the operator F cannot be improved to the existence of the weak derivative as the following example shows.

E x a m p l e . Let $\{x_{ij}\}$ be the unit vector base in ℓ_2 arranged in a double sequence. For each $i\in\mathbb{N}$, define $f_i\colon[0,1]\to\ell_2$ by

$$f_i(t)=\begin{cases}2^ix_{ij} & \text{if}\quad t\in[j-1/2^i,j-1/2^i+1/4^i],\ j=1,2,\ldots2^i,\\ \varphi & \text{otherwise,}\end{cases}$$

where φ denotes the null vector in ℓ_2. Let $f(t)=\sum\limits_{i=1}^{\infty}f_i(t)$, then $f\colon[0,1]\to\ell_2$ is Pettis integrable (see [12] Example 10.9). Since f is measurable, it is McShane integrable (see [7] Theorem 17) and also PoU-integrable (see [4] Theorem 1). Let

$$F(\theta) = (\mathrm{PoU})\int_0^1 f\theta$$

be the operator associated to f.

We recall that a function $h\colon [0,1] \to X$ is the *weak derivative* of an operator H defined on the family of all subintervals of $[0,1]$ to X, if for each $x^* \in X^*$, and for all ω outside a negligible set N,

$$\lim_{|I|\to 0} x^*\Big(\frac{H(I)}{|I|}\Big) = x^* h(\omega)$$

where I is an arbitrary interval containing ω.

As showed in [12], the weak derivative of the Pettis primitive F of the function f does not exist. Now we observe that, when $\theta = \chi_{[a,b]}$ with $0 \leqslant a < b \leqslant 1$, in the subnet $\{[a,b],\omega\}$ where $\omega \in [a,b]$, the scalar **B**-Volterra derivative of the operator F coincides with the scalar derivative, with respect to the Lebesgue measure, of the indefinite Pettis integral of the function f. Therefore the weak **B**-Volterra derivative of F cannot exist.

A c k n o w l e d g m e n t. The author thanks Professor L. Di Piazza for helpful discussions and comments.

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