STABILITY PROCESSES OF MOVING INVARIANT MANIFOLDS IN UNCERTAIN IMPULSIVE DIFFERENTIAL-DIFFERENCE EQUATIONS

Gani Tr. Stamov, Sliven

(Received September 6, 2007)

Abstract. We present a result on the stability of moving invariant manifolds of nonlinear uncertain impulsive differential-difference equations. The result is obtained by means of piecewise continuous Lyapunov functions and a comparison principle.

Keywords: moving invariant set, stability theory, uncertain impulsive differential-difference system

MSC 2000: 34A37, 34K45

1. Introduction

The necessity of studying uncertain differential equations is due to the fact that these equations are a useful mathematical tool in modelling many processes of the real world [7], [8].

A natural generalization of these equations are the uncertain impulsive ordinary differential equations or uncertain impulsive differential-difference equations. These equations may be used for mathematical simulation of processes and phenomena which are subject to short-term perturbations during their evolution. The duration of the perturbations is negligible in comparison with the duration of the process considered, therefore it can be considered instantaneous. In [1], [2], [3], [12], for instance, the reader can find some fundamental results on the theory of impulsive systems.

The applications of uncertain impulsive differential equations to mathematical simulation request finding some criteria for stability of their solutions. One of the most important parts of the qualitative theory of differential equations is the theory of stability of the invariant manifolds. The main results related to the study of the existence and stability of invariant manifolds for uncertain impulsive differential and integro-differential equations can be found in [9], [10], [11].

The main purpose of this paper is to derive "easily verifiable" sufficient conditions for the stability of moving invariant manifolds for a class of uncertain impulsive differential-difference equations. The paper is organized as follows. In Section 2 we give some preliminaries and main definitions. In Section 3 we investigate the stability of moving invariant manifolds. By means of piecewise continuous auxiliary functions which are analogues of the classical Lyapunov's functions sufficient conditions are obtained. The main idea comes from the works of Lakshmikantham, Leela and Martynyuk for stability of uncertain differential systems and moving invariant sets as the parametric changes [4], [5] and from the works [1], [7], [8]. The investigations are carried out also by using a comparison principle which permits us to reduce the study of impulsive differential-difference equations to the study of a scalar differential equation. The results we obtain generalize those in [6], [13].

2. Statement of the problem. Preliminary notes

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with elements $x = \operatorname{col}(x_1, x_2, \dots, x_n)$ and norm $|\cdot| = \Big(\sum\limits_{k=1}^{n} x_k^2\Big)^{1/2}$, let Ω be a domain in $\mathbb{R}^n$ containing the origin, $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$, $h > 0$, $\varphi_0 \in C[[t_0 - h, t_0], \mathbb{R}^n]$, $S_\varrho = \{x \in \mathbb{R}^n \colon |x| = \varrho\}$, $B_\varrho = \{x \in \mathbb{R}^n \colon |x| < \varrho\}$, $\varrho > 0$.

We will consider the system of uncertain impulsive differential-difference equations

$$
\begin{cases}
\dot{x}(t) = f(t, x(t), x(t-h), \lambda), \ t \neq \tau_k, \ t > t_0, \\
x(t) = \varphi_0(t), \ t \in [t_0 - h, t_0], \\
\Delta x(\tau_k) = I_k(x(\tau_k), \lambda), \ \tau_k > t_0, \ k = 1, 2, \dots,
\end{cases}
\tag{1}
$$

where

i) $t_0 \in \mathbb{R}$, $f \in C[(t_0, \infty) \times \Omega \times \Omega \times \mathbb{R}^d, \mathbb{R}^n]$ and $\lambda \in \mathbb{R}^d$ is an uncertain parameter;

ii) $t_0 = \tau_0 < \tau_1 < \ldots < \tau_k < \ldots$, $\lim\limits_{k \to \infty} \tau_k = \infty$;

iii) $\Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k)$, $k = 1, 2, \ldots$;

iv) $I_k \in C[\Omega \times \mathbb{R}^d, \mathbb{R}^n]$, $k = 1, 2, \ldots$.

We introduce the following notation: $x(t) = x(t; t_0, \varphi_0)$ is the solution of the problem (1); $J^+(t_0, \varphi_0)$ is the maximal interval of type $[t_0, \beta)$ in which the solution $x(t) = x(t; t_0, \varphi_0)$ is defined; $\|\varphi\| = \sup\limits_{t \in [t_0 - h, t_0]} |\varphi(t)|$ is the norm of a function $\varphi \in C[[t_0 - h, t_0], \mathbb{R}^n]$.

The solutions $x(t)$ of the problems in the form (1) are piecewise continuous functions with discontinuities of the first kind at the points $\tau_k > t_0$, $k = 1, 2, \ldots$. At these points the solutions $x(t)$ are continuous from the left, that is, at the moments of impulse effects $\tau_k' s$ the following relations are valid:

$$x(\tau_k - 0) = x(\tau_k), \quad x(\tau_k + 0) = x(\tau_k) + I_k(x(\tau_k), \lambda), \quad k = 1, 2, \ldots.$$

If for some positive integer j we have $\tau_k < \tau_j + h < \tau_{k+1}$, $k = 0, 1, 2, \ldots$, then in the interval $[\tau_j + h, \tau_{k+1}]$ the solution of problem (1) coincides with the solution of the problem

$$\begin{cases} \dot{y}(t) = f(t, y(t), x(t - h + 0), \lambda), \\ y(\tau_j + h) = x(\tau_j + h), \end{cases}$$

and if $\tau_j + h = \tau_k$ for $j = 0, 1, 2, \ldots$, $k = 1, 2, \ldots$, then in the interval $[\tau_j + h, \tau_{k+1}]$ the solution $x(t)$ coincides with the solution of the problem

$$\begin{cases} \dot{y}(t) = f(t, y(t), x(t - h + 0), \lambda), \\ y(\tau_j + h) = x(\tau_j + h) + I_k(x(\tau_j + h), \lambda). \end{cases}$$

If the point $x(\tau_k) + I_k(x(\tau_k), \lambda) \notin \Omega$, then the solution of problem (1) is not defined for $t > \tau_k$.

For existence and uniqueness results of (1) see [2], [3].

Consider the following sets:

$$G_k = \{(t, x) \in [t_0, \infty) \times \Omega\colon\ \tau_k < t < \tau_{k+1}\},\ k = 0, 1, 2, \ldots,\ G = \bigcup_{k=0}^{\infty} G_k;$$
$$W_k = \{(t, u) \in \mathbb{R}_+^2\colon\ \tau_k < t < \tau_{k+1}\},\ k = 0, 1, 2, \ldots;$$
$$K = \{a \in C[\mathbb{R}_+, \mathbb{R}_+]\colon\ a \text{ is strictly increasing in } \mathbb{R}_+, \text{ and } a(0) = 0\};$$

$PC[\mathbb{R}_+, \mathbb{R}^n] = \{x\colon\ \mathbb{R}_+ \to \mathbb{R}^n$, x is a piecewise continuous function with discontinuities of the first kind at τ_k, $k = 1, 2, \ldots$ and $x(\tau_k - 0) = x(\tau_k)\}$;

$V_0 = \{V \in C[G, \mathbb{R}_+],\ V(t, 0) = 0,\ t \in [t_0, \infty)$ is locally Lipschitz in $x \in \mathbb{R}^n$ on each of the sets G_k, and $\displaystyle\lim_{\substack{(t,x)\to(\tau_k, x_0)\\ (t,x)\in G_{k+1}}} V(t, x) = V(\tau_k + 0, x_0)\}$.

Let $V \in V_0$. For $x \in PC(\mathbb{R}_+, \Omega)$ and $t \in [t_0, \infty),\ t \neq \tau_k,\ k = 1, 2, \ldots$ we define

$$D_- V(t, x(t)) = \lim_{\delta \to 0^-} \inf \delta^{-1}\{V(t + \delta, x(t) + \delta f(t, x(t), x(t - h), \lambda)) - V(t, x(t))\}.$$

Our aim is to reduce the study of the system (1) to the study of a simple scalar impulsive differential equation with impulses at fixed moments and an uncertain parameter.

For convenience, let us state the following hypotheses.

(A_0) $w\colon \mathbb{R}^3_+ \to \mathbb{R}_+$ is continuous on $(\tau_k, \tau_{k+1}] \times \mathbb{R}^2_+$, $\tau_k < \tau_{k+1}$, $\lim\limits_{k\to\infty} \tau_k = \infty$, $w(t,0,0) = 0$, the limits

$$w(\tau_k + 0, u_0, \mu) = \lim_{\substack{(t,u,\mu)\to(\tau_k,u_0,\mu)\\ (t,u)\in W_{k+1}}} w(t,u,\mu)$$

exist and are finite, $\psi_k \in C[\mathbb{R}^2_+, \mathbb{R}]$, $\psi_k(u,\mu)$, $k = 0,1,2,\ldots$ are nondecreasing in u for $\mu \in \mathbb{R}_+$, and $r(t;t_0,u_0)$ is the maximal solution of the impulsive differential equation

$$\text{(2)} \qquad \begin{cases} \dot u = w(t,u,\mu),\ t \neq \tau_k,\ t > t_0, \\ \Delta u(\tau_k) = u(\tau_k + 0) - u(\tau_k - 0) = \psi_k(u(\tau_k),\mu),\ k = 1,2,\ldots, \\ u(t_0 + 0) = u_0,\ t_0 \in \mathbb{R}_+, \end{cases}$$

existing on $[t_0, \infty)$.

(A_1) $V \in V_0$ and for $t > t_0$, $x \in E_0$, we have

$$D_- V(t,x(t)) \leqslant w(t, V(t,x(t)),\mu),\ t \neq \tau_k,\ k = 0,1,2,\ldots$$
$$(D_- V(t,x(t)) \geqslant w(t, V(t,x(t)),\mu),\ t \neq \tau_k,\ k = 0,1,2,\ldots)$$

where

$$E_0 = \{x \in PC[\mathbb{R}_+, \Omega]\colon\ V(s,x(s)) \leqslant V(t,x(t)),\ t - h \leqslant s \leqslant t,\ t \in [t_0,\infty)\}$$

and

$$V(t, x(t) + I_k(x(t),\lambda)) \leqslant \psi_k(V(t,x(t)),\mu), \quad t = \tau_k,\ k = 1,2,\ldots,$$
$$(V(t, x(t) + I_k(x(t),\lambda)) \geqslant \psi_k(V(t,x(t)),\mu), \quad t = \tau_k,\ k = 1,2,\ldots).$$

Theorem 1. *Assume that conditions (A_0) and (A_1) are satisfied.*
Then if $x(t) = x(t;t_0,\varphi_0)$ is any solution of (1) existing on $[t_0,\infty)$, we have

$$V(t,x(t)) \leqslant r(t;t_0,u_0),\ t \geqslant t_0 \quad \textit{provided } V(t_0 + 0, \varphi_0) \leqslant u_0$$

or

$$V(t,x(t)) \geqslant r(t;t_0,u_0),\ t \geqslant t_0 \quad \textit{provided } V(t_0 + 0, \varphi_0) \geqslant u_0.$$

Proof. The proof of Theorem 1 is analogous to the proof of Lemma 2 in [12].

Now we present definitions which are borrowed from [10] and concern the invariance and stability of moving invariant manifolds of systems (1) and (2).

Definition 1 [10]**.** Let $r_k = r_k(\lambda) > 0$, $k = 0, 1, 2, \ldots$. Then we say that the manifold M, where

$$M = \bigcup_{k=1}^{\infty} M_k, \quad M_k = \{x \in \mathbb{R}^n\colon\ (t, x) \in G_k,\ |x| = r_k\},\ k = 0, 1, 2, \ldots,$$

is:

1. *invariant and uniformly stable* (US) *with respect to* (1) if
   i) $\|\varphi_0\| = r_0 \Rightarrow |x(t)| = r_k$, $t \in (\tau_k, \tau_{k+1}]$, $k = 0, 1, 2, \ldots$,
   ii) given $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$r_0 - \delta < \|\varphi_0\| < r_0 + \delta \Rightarrow r_k - \varepsilon < |x(t)| < r_k + \varepsilon,\ t \in (\tau_k, \tau_{k+1}],\ k = 0, 1, 2, \ldots;$$

2. *invariant and uniformly asymptotically stable* (UAS) *with respect to* (1) if M is (US) and there exist $\delta_0 > 0$ and $T = T(\varepsilon) > 0$ such that if $t_0 + T \in (\tau_l, \tau_{l+1}]$ for some $l = 0, 1, 2, \ldots$, then

$$r_0 - \delta < \|\varphi_0\| < r_0 + \delta \Rightarrow r_l - \varepsilon < |x(t)| < r_l + \varepsilon,\ t \in (t_0 + T, \tau_{l+1}].$$

   If

$$r_k - \varepsilon < |x(t)| < r_k + \varepsilon,\ t \in (\tau_k, \tau_{k+1}],\ k \geqslant l + 1,$$

   and $t_0 + T = \tau_p + 0$ for some $p = 1, 2, \ldots$, then

$$r_0 - \delta < \|\varphi_0\| < r_0 + \delta \Rightarrow r_k - \varepsilon < |x(t)| < r_k + \varepsilon,\ t \in (\tau_k, \tau_{k+1}],\ k \geqslant p,$$

   where $x(t) = x(t; t_0, \varphi_0)$ is a solution of (1) on $J^+(t_0, \varphi_0)$.

Definition 2 [10]**.** Let $R_k = R_k(\mu) > 0$, $k = 0, 1, 2, \ldots$. Then we say that the manifold

$$u = \bigcup_{k=1}^{\infty} u_k,\ u_k = \{u \in \mathbb{R}_+\colon\ (t, u) \in W_k,\ u = R_k\},\ k = 0, 1, 2, \ldots,$$

is:

1. *invariant and uniformly stable* (US) *with respect to* (2) if
   i) $u_0 = R_0 \Rightarrow R_k = u(t)$, $t \in (\tau_k, \tau_{k+1}]$, $k = 0, 1, 2, \ldots$,
   ii) given $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$R_0 - \delta < u_0 < R_0 + \delta \Rightarrow R_k - \varepsilon < u(t) < R_k + \varepsilon,\ t \in (\tau_k, \tau_{k+1}],\ k = 0, 1, 2, \ldots;$$

2. *invariant and uniformly asymptotically stable* (UAS) *with respect to* (2) if there exist $\delta_0 > 0$ and $T = T(\varepsilon) > 0$ such that if $t_0 + T \in (\tau_l, \tau_{l+1}]$ for some $l = 0, 1, 2, \dots$ then

$$R_0 - \delta < u_0 < R_0 + \delta \Rightarrow R_l - \varepsilon < u(t) < R_l + \varepsilon,\ t \in (t_0 + T, \tau_{l+1}].$$

If

$$R_k - \varepsilon < u(t) < R_k + \varepsilon,\ t \in (\tau_k, \tau_{k+1}],\ k \geqslant l + 1$$

and $t_0 + T = \tau_p + 0$ for some $p = 1, 2, \dots$, then

$$R_0 - \delta < u_0 < R_0 + \delta \Rightarrow R_k - \varepsilon < u(t) < R_k + \varepsilon,\ t \in (\tau_k, \tau_{k+1}],\ k \geqslant p,$$

where $u(t) = u(t; t_0, u_0)$ is a solution of (2).

We define, for simplicity, the sets

$$E_1^{(k)} = \{x;\ x \in E_0,\ x(t) \in \Omega \setminus B_{r_k}\},\ k = 0, 1, 2, \dots,$$
$$E_2^{(k)} = \{x;\ x \in E_0,\ x(t) \in B_{r_k} \cup S_{r_k}\},\ k = 0, 1, 2, \dots.$$

3. Main result

Theorem 2. *Assume:*

(H_0) *For each $\lambda \in \mathbb{R}^d$ there exist a sequence $\{r_k\}_{k=1}^{\infty}$, $r_k = r_k(\lambda)$ such that $r_k(\lambda) > 0$ and $r_k(\lambda) \to 0$ as $|\lambda| \to 0$, and $r_k(\lambda) \to \infty$ as $|\lambda| \to \infty$ for each $k = 0, 1, 2, \dots$.*

(H_1) *There exist functions $V \in V_0$ and $a, b \in K$ such that*

$$b(|x|) \leqslant V(t, x) \quad \text{for } t \neq \tau_k,\ x \in E_1^{(k)}$$

and

$$V(t, x) \leqslant a(|x|) \quad \text{for } t \neq \tau_k,\ x \in E_2^{(k)},\ k = 0, 1, 2, \dots.$$

(H_2)

$$D_{-}V(t, x) \leqslant w(t, V(t, x), r_k) \quad \text{for } t \neq \tau_k,\ x \in E_1^{(k)}$$

and

$$D_{-}V(t, x) \geqslant w(t, V(t, x), r_k) \quad \text{for } t \neq \tau_k,\ x \in E_2^{(k)},\ k = 0, 1, 2, \dots.$$

(H_3)

$$V(\tau_k + 0, x(\tau_k) + I_k(x(\tau_k), \lambda)) \leqslant \psi_k(V(\tau_k, x(\tau_k)), \mu) \quad \text{for } x \in E_1^{(k)}$$

and

$$V(\tau_k+0, x(\tau_k)+I_k(x(\tau_k),\lambda)) \geqslant \psi_k(V(\tau_k, x(\tau_k)), \mu) \quad \text{for } x\in E_2^{(k)},\ k=0,1,2,\ldots.$$

(H_4) *For each sequence* $\{r_k\}_{k=0}^{\infty}$, $r_k=r_k(\lambda)>0$, *there exists a sequence* $\{R_k\}_{k=0}^{\infty}$ *such that* $R_k = R(r_k)\geqslant 0$ *with* $R_k\to 0$ *as* $r_k\to 0$ *and* $R_k\to\infty$ *as* $r_k\to\infty$, $k=0,1,2,\ldots$, *and* $u=R$, $R=\bigcup\limits_{k=0}^{\infty} R_k$ *is invariant and* (UAS) *relative to* (2).

Then if for any $r_k>0$, $a(r_k)=b(r_k)=R(r_k)$, *the manifold* $M=\bigcup\limits_{k=1}^{\infty} M_k$ *is invariant and* (UAS) *relative to* (1).

P r o o f. Assume that condition (H_4) is fulfilled for some $\{r_k\}_{k=0}^{\infty}$, $r_k=r_k(\lambda)>0$. First we shall prove that the manifold M is invariant with respect to (1).

If not, there would exists a solution of (1) with $\|\varphi_0\|=r_0$ and $t_2>t_1\geqslant t_0$ such that the following two cases may occur:

C a s e 1. If $t_1\in(\tau_k,\tau_{k+1}]$ and $t_2\in(\tau_l,\tau_{l+1}]$, $k\geqslant l$, then $|x(t_1)|=r_k$, $|x(t_2)|>r_l$, $x\in E_0$ is such that $x(t)\in\Omega\setminus B_{r_\sigma}$, $t\in[t_1,t_2]$, where $\sigma=k$ if $l=k$, or $\sigma=k,k+1,\ldots,l$ if $l>k$.

From (H_1) and (H_2) for $V(t,x(t))$ it follows that

$$\begin{aligned}
&D_-V(t,x(t))\leqslant w(t,u(t;t_1,V(t_1,x(t_1))),r_\sigma) \quad \text{if } t\in[t_1,t_2]\setminus\{\tau_\sigma\in[t_1,t_2]\},\\
&V(\tau_\sigma+0,x(\tau_\sigma)+I_\sigma(x(\tau_\sigma),\lambda))\leqslant\psi_\sigma(V(\tau_\sigma,x(\tau_\sigma)),r_\sigma) \quad \text{for } \tau_\sigma\in[t_1,t_2].
\end{aligned}$$

Using the comparison Theorem 1 we have

$$V(t,x(t))\leqslant u(t;t_1,V(t_1,x(t_1))),\ t_1\leqslant t\leqslant t_2,$$

hence

$$\begin{aligned}
b(r_\sigma)<b(|x(t_2)|)&\leqslant V(t_2,x(t_2))\leqslant u(t_2;t_1,a(|x(t_1)|))\\
&=u(t_2;t_1,a(r_\sigma))=b(r_\sigma)=a(r_\sigma)=R_\sigma, \quad \sigma=k,k+1,\ldots,l,
\end{aligned}$$

which is a contradiction.

C a s e 2. If $t_1\in(\tau_k,\tau_{k+1}]$ and $t_2\in(\tau_l,\tau_{l+1}]$, $k\geqslant l$, then $|x(t_1)|=r_k$, $|x(t_2)|<r_l$, $x\in E_0$ is such that $x(t)\in B_{r_\sigma}\cup S_{r_\sigma}$, $t\in[t_1,t_2]$, where $\sigma=k$ if $l=k$, and $\sigma=k,k+1,\ldots,l$ if $l>k$.

From (H_1) and (H_2) it follows that

$$\begin{aligned}
&D_-V(t,x(t))\geqslant w(t,u(t\colon\ t_1,V(t_1,x(t_1))),r_\sigma) \quad \text{if } t\in[t_1,t_2]\setminus\{\tau_\sigma\in[t_1,t_2]\},\\
&V(\tau_\sigma+0,x(\tau_\sigma)+I_\sigma(x(\tau_\sigma),\lambda))\geqslant\psi_\sigma(V(\tau_\sigma,x(\tau_\sigma)),r_\sigma) \quad \text{for } \tau_\sigma\in[t_1,t_2],
\end{aligned}$$

where $u(t, t_1, V(t_1, x(t_1)))$ is the solution of (2) through $(t_1, V(t_1, x(t_1))$ or

$$V(t, x(t)) \geqslant u(t; t_1, V(t_1, x(t_1))), \quad t_1 \leqslant t \leqslant t_2.$$

Similarly we obtain

$$\begin{aligned} a(r_\sigma) > a(|x(t_2)|) \geqslant V(t_2, x(t_2)) \geqslant u(t_2, t_1, b(|x(t_1)|)) \\ = u(t_2, t_1, b(r_\sigma)) = b(r_\sigma) = a(r_\sigma) = R_\sigma,\ \sigma = k, k+1, \dots, l, \end{aligned}$$

which also is a contradiction.

Let us fix $\varepsilon > 0$ and let $t_0 \in \mathbb{R}_+$ be given. Suppose that $u = R$ is (US). Then since $a(r_k) = b(r_k) = R_k$, $k = 1, 2, \dots$, given $a(r_k - \varepsilon)$, $b(r_k + \varepsilon)$, there exist $\varepsilon_1 > 0$, $\delta_1 > 0$, $\delta > 0$ such that

$$R_k + \delta_1 = a(r_k + \delta) < b(r_k + \varepsilon) = R_k + \varepsilon_1, \quad k = 0, 1, 2, \dots$$

and

$$R_k - \varepsilon_1 = a(r_k - \varepsilon) < b(r_k - \delta) = R_k - \delta_1, \quad k = 0, 1, 2, \dots.$$

If $R_0 - \delta_1 < u_0 < R_0 + \delta_1$ then $R_k - \varepsilon_1 < u(t) < R_k + \varepsilon_1$, $t \geqslant t_0$, $k = 0, 1, 2, \dots$ where $u(t)$ is a solution of (2). We claim that with this $\delta > 0$ the manifold M is (US), that is

$$r_0 - \delta < \|\varphi_0\| < r_0 + \delta \Rightarrow r_k - \varepsilon < |x(t)| < r_k + \varepsilon, \quad t \geqslant t_0,\ k = 1, 2, \dots.$$

If this were not true, there would exist a solution $x(t)$ of (1) with $r_0 - \delta < |x_0| < r_0 + \delta$ and $t_2 > t > t_1$ such that either

(a) $|x(t_2)| = r_l + \varepsilon$, $|x(t_1)| = r_k + \delta$ and $x \in E_0$ is such that $x(t) \in \Omega \setminus B_{r_\sigma} \cup S_{r_\sigma}$, $t \in [t_1, t_2]$, $t_1 \in (\tau_k, \tau_{k+1}]$, $t_2 \in (\tau_l, \tau_{l+1}]$, $l \geqslant k$, $\sigma = k, k+1, \dots, l$, or

(b) $|x(t_2)| = r_l - \varepsilon$, $|x(t_1)| = r_k - \delta$ and $x \in E_0$ is such that $x(t) \in B_{r_\sigma}$, $t \in [t_1, t_2]$, $t_1 \in (\tau_k, \tau_{k+1}]$, $t_2 \in (\tau_l, \tau_{l+1}]$, $l \geqslant k$, $\sigma = k, k+1, \dots, l$.

Consider (a). As before, we have

$$V(t, x(t)) \leqslant u(t; t_1, V(t_1, x(t_1))),\ t \in [t_1, t_2]$$

and therefore, we arrive at the contradiction

$$b(r_\sigma + \varepsilon) = b(|x(t_2)|) \leqslant V(t_2, x(t_2)) \leqslant u(t_2; t_1, a(r_\sigma + \delta)) < b(r_\sigma + \varepsilon),$$

$\sigma = k, k+1, \dots, l.$

Similarly, in case (b) we first get

$$V(t, x(t)) \geqslant u(t, t_1, V(t_1, x(t_1))),\ t \in [t_1, t_2],$$

and then it follows that

$$a(r_\sigma - \varepsilon) = a(|x(t_2)|) \geqslant V(t_2, x(t_2)) \geqslant u(t_2; t_1, a(r_\sigma - \delta)) > a(\varrho_\sigma - \varepsilon),$$

$\sigma = k, k+1, \ldots, l$, which is a contradiction. Hence M is (US).

To prove that the set M is (UAS) with respect to (1), let us first fix $\varepsilon_k = r_k$, $k = 1, 2, \ldots$, and let $\delta_k = \delta(r_k)$ so that we obtain

$$b(r_k - \delta_k) < u_0 < a(r_k + \delta) \Rightarrow 0 < u(t) < b(2r_k), \quad t \geqslant t_0,\ k = 0, 1, 2, \ldots,$$

and

$$r_0 - \delta_0 < \|\varphi_0\| < r_0 + \delta_0 \Rightarrow 0 < |x(t)| < 2r_k, \quad t \geqslant t_0,\ k = 0, 1, 2, \ldots.$$

Assume that $u = R$ is (UAS) and let $\delta = \delta(\varepsilon)$ be the same number corresponding to ε when u is (US) with respect to (2). Then for given $b(r_k + \delta)$, $a(r_k - \delta)$ there exists $T = T(\varepsilon) > 0$ such that

i) if $t_0 + T \in (\tau_l, \tau_{l+1}]$ for some $l = 1, 2, \ldots$, then

$$b(r_0 - \delta_0) < u_0 < a(r_0 + \delta_0) \Rightarrow a(r_l - \delta) < u(t) < b(r_l + \delta), \quad t \in (t_0 + T, \tau_{l+1}]$$

and

$$a(r_k - \delta) < u(t) < b(r_k + \delta), \quad t \in (\tau_k, \tau_{k+1}],\ k \geqslant l + 1,$$

ii) if $t_0 + T = \tau_p$ for some $p = 1, 2, \ldots$, then

$$b(r_0 - \delta_0) < u_0 < a(r_0 + \delta_0) \Rightarrow a(r_k - \delta) < u(t) < b(r_k + \delta), \quad t \in (\tau_k, \tau_{k+1}],\ k \geqslant p.$$

Since M is (US), it is enough to show that there exists $t^* \in (\tau_q, \tau_{q+1}] \subset (t_0, t_0 + T)$ satisfying $r_q - \delta < |x(t^*)| < r_q + \delta$. If t^* does not exists, then for $t_0 + T \in (\tau_l, \tau_{l+1}]$ we have either

(c) $x \in E_0$ is such that $x(t) \in \Omega \setminus B_{r_\sigma + \delta}$ for all $t \in [t_0, t_0 + T] \setminus \{\tau_\sigma \in (t_0, t_0 + T]$, $\sigma = 1, 2, \ldots, l\}$ or

(d) $x \in E_0$ is such that $x(t) \in B_{r_\sigma + \delta} \cup S_{r_\sigma + \delta}$ for all $t \in [t_0, t_0 + T] \setminus \{\tau_\sigma \in (t_0, t_0 + T]$, $\sigma = 1, 2, \ldots, l\}$.

In case (c), we have

$$b(r_\sigma + \delta) \leqslant V(t_0 + T, x(t_0 + T)) \leqslant u(t_0 + T; t_0, a(r_\sigma + \delta_0)) < b(r_\sigma + \delta)$$

for $\sigma = 0, 1, 2, \ldots, l$, which is a contraction. Similarly, in case (d), it follows that

$$a(r_\sigma - \delta) \geqslant V(t_0 + T, x(t_0 + T)) \geqslant u(t_0 + T; t_0, b(r_\sigma - \delta_0)) > a(r_\sigma - \delta)$$

for $\sigma = 1, 2, \ldots, l$, which is again a contraction. Hence there exists $t^* \in [t_0, t_0 + T]$ satisfying $r_q - \delta < |x(t^*)| < r_q + \delta$ and the proof of Theorem 2 is complete.

Remark 1. Note that the main result of the paper follows from an estimate of Lyapunov functions on the minimal class E_0 in assumption (A_1). This class depends on the choice of the functions $w_0(t, v, \mu)$, and $\psi_k^0(v, \mu)$, $k = 1, 2, \ldots$. Special cases of these choices are considered in [4], [5].

References

[1] *Bainov D., Kostadinov S., Nguyen Van Min*: Dichotomies and Integral Manifolds of Impulsive Differential Equations. SCT Publishing, Singapore, 1994.

[2] *Bainov D., Stamova I. M.*: Vector Lyapunov functions and conditional stability for systems of impulsive differential-difference equations. ANZIAM J. *42* (2001), 341–353.

[3] *Bainov D., Dishliev A. B., Stamova I. M.*: Continuous dependence of solutions of impulsive systems of differential-diference equations on initial data and on parameter. Bol. Soc. Parana. Mat. *18* (1998), 21–34.

[4] *Lakshmikantham, V., Leela, S., Martynyuk, A. A.*: Stability Analysis of Nonlinear Systems. Marcel Dekker, New York, 1989.

[5] *Lakshmikantham, V., Leela, S., Martynyuk, A. A.*: Practical Stability of Nonlinear Systems. World Scientific Publishing, Singapore, 1990.

[6] *Lakshmikantham, V., Vatsala, A. S.*: Stability of moving invariant sets (Sivasundaram, S., ed.). Advances in Nonlinear Dynamics. Langhorne, PA: Cordon and Breach. Stab. Control Theory Methods Appl. *5*, 1997, pp. 79–83.

[7] *Shendge, G. R.*: A new approach to the stability theory of functional differential equations. J. Math. Anal. Appl. *95* (1983), 319–334.

[8] *Siljak, D. D., Ikeda, M., Ohta, Y.*: Parametic stability. Proccedings Universita di Genova-Ohio State University Joint Conference: Birkhauser, 1991, pp. 1–20.

[9] *Stamov G.*: Stability of moving invariant maniolds for impulsive differential equations. J. Tech. Univ. Plovdiv Fundam. Sci. Appl., Ser. A Pure Appl. Math. *7* (1999), 99–107.

[10] *Stamov G.*: Stability of moving conditionally manifolds for impulsive differential equations. Adv. Stud. Contemp. Math. (Kyungshang) *9* (2004), 99–107.

[11] *Stamov G.*: Impulsive integro-differential equations and stability of moving invariant maniolds. Methods Appl. Anal. *14* (2007), 69–76.

[12] *Stamova, I., Stamov, G.*: Lyapunov-Razumikhin method for impulsive functional differential equations and applications to the population dynamics. J. Comput. Appl. Math. *130* (2001), 163–171.

[13] *Vatsala, A. S., Deo, S. G.*: Stability of moving invariant sets for functional differential systems. Int. J. Nonlin. Diff. Eq.: Theory, Methods and Appl. *3* (1997), 179–186.

Author's address: *Gani Tr. Stamov*, Technical University-Sofia, 8800 Sliven, Bulgaria, e-mail: `gstamov@abv.bg`.