A REMARK ON POWER COMPARISON THEOREM FOR HALF-LINEAR DIFFERENTIAL EQUATIONS

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Abstract. We consider the half-linear second order differential equation which is viewed as a perturbation of the so-called Riemann-Weber half-linear differential equation. We present a comparison theorem with respect to the power of the half-linearity in the equation under consideration. Our research is motivated by the recent results published by J. Sugie, N. Yamaoka, Acta Math. Hungar. *111* (2006), 165–179.

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1. Introduction

The oscillation theory of half-linear second order differential equations

$$(r(t)\Phi_\alpha(x'))' + c(t)\Phi_\alpha(x) = 0, \quad \Phi_\alpha(x) := |x|^{\alpha-1}x, \quad \alpha > 0, \tag{1}$$

has attracted considerable attention in the recent years, let us mention at least the books [1], [4] and the references given therein. It was shown that the linear Sturmian theory extends directly to (1) and hence this equation can be classified as oscillatory or nonoscillatory similarly to the linear case. Elbert and Mirzov with their papers [5] and [9] are usually regarded as pioneers of the half-linear oscillation theory.

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In the majority of oscillation criteria for (1), this equation is regarded as a perturbation of the (nonoscillatory) one term equation

$$(r(t)\Phi_\alpha(x'))' = 0$$

and (non)oscillation criteria for (1) are formulated in terms of the asymptotic behavior of the function c with respect to the function r. Roughly speaking, if the function c is "sufficiently positive" ("not too positive") with respect to the function r, equation (1) becomes oscillatory (remains nonoscillatory). A more general approach consists in regarding (1) as a perturbation of the equation of the same form

$$(r(t)\Phi_\alpha(x'))' + \tilde{c}(t)\Phi_\alpha(x) = 0 \tag{2}$$

and formulating conditions for (non)oscillation in terms of the asymptotic behavior of the function $(c - \tilde{c})h^{\alpha+1}$, where h is a certain distinguished solution of (2); we refer to [4, Sec. 5.2] and [3] for more details. This approach was used for the first time in the paper of Elbert [6], where (1) with $r(t) \equiv 1$ is viewed as a perturbation of the half-linear Euler equation with the critical coefficient

$$(\Phi_\alpha(x'))' + \Gamma_\alpha\Phi_\alpha(x) = 0, \quad \Gamma_\alpha := \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}. \tag{3}$$

The half-linear version of the classical Sturmian comparison theorem concerns a pair of equations (1) and

$$(R(t)\Phi_\alpha(x'))' + C(t)\Phi_\alpha(x) = 0 \tag{4}$$

and states that under the inequalities

$$0 < R(t) \leqslant r(t), \quad c(t) \leqslant C(t)$$

(4) oscillates faster than equation (1). More precisely, between any two consecutive zeros of a nontrivial solution of (1) there is at least one zero of any nontrivial solution of (4). In our paper we are concerned with another type of comparison theorems, namely, *with respect to the power* α in (1). The basic statement along this line is established in [11] (in a more general setting of dynamic equations on time scales). Under some additional assumptions (which are trivially satisfied for $r(t) \equiv 1$ as treated later in this paper), it states that if $\beta > \alpha$ and (1) is oscillatory, then the equation

$$(r(t)\Phi_\beta(x'))' + c(t)\Phi_\beta(x) = 0$$

is also oscillatory.

In our paper we are motivated by the results presented in the paper [13], where the equation

$$(5)\qquad (\Phi_\alpha(x'))' + \frac{1}{t^{\alpha+1}}\,[\Gamma_\alpha + \gamma_\alpha\delta(t)]\,\Phi_\alpha(x) = 0, \quad \gamma_\alpha := \Big(\frac{\alpha}{\alpha+1}\Big)^\alpha,$$

is compared with the equation of the same form, but with α replaced by β. We will recall the results of [13] in more detail in the next section. We are also motivated by the results of [2] where the equation

$$(6)\qquad (\Phi_\alpha(x'))' + \Big[\frac{\Gamma_\alpha}{t^{\alpha+1}} + \frac{\gamma_\alpha}{2t^{\alpha+1}\lg^2 t} + c(t)\Big]\Phi_\alpha(x) = 0$$

is viewed as a perturbation of the half-linear Riemann-Weber equation (sometimes called the Euler-Weber equation)

$$(7)\qquad (\Phi_\alpha(x'))' + \frac{1}{t^{\alpha+1}}\Big[\Gamma_\alpha + \frac{\gamma_\alpha}{2t^{\alpha+1}\lg^2 t}\Big]\Phi_\alpha(x) = 0.$$

As the main result of our paper, we present a comparison theorem with respect to α for equation (6).

2. Auxiliary results

First we turn our attention to Euler and Riemann-Weber half-linear differential equations which are treated in detail in the paper of Elbert and Schneider [8]. The half-linear Euler equation

$$(8)\qquad (\Phi_\alpha(x'))' + \frac{\lambda}{t^{\alpha+1}}\Phi_\alpha(x) = 0$$

is oscillatory if and only if $\lambda > \Gamma_\alpha$ and the half-linear Riemann-Weber equation (7) with a parameter λ instead of γ_α is oscillatory if and only if $\lambda > \gamma_\alpha$.

The transformation of the independent variable $x(t) = y(s)$, $s = \lg t$, transforms equation (6) to the equation

$$(9)\qquad (\Phi_\alpha(\dot y))^{\cdot} - \alpha\Phi_\alpha(\dot y) + \Big[\Gamma_\alpha + \frac{\gamma_\alpha}{2s^2} + c(\mathrm{e}^s)\Big]\Phi_\alpha(y) = 0, \quad {}^{\cdot} = \frac{\mathrm{d}}{\mathrm{d}s}$$

and the function $v = \Phi_\alpha(\dot y/y)$ is a solution of the Riccati type differential equation

$$(10)\qquad \dot v + \frac{\gamma_\alpha}{2s^2}c(\mathrm{e}^s) + H_\alpha(v) = 0,$$

where

$$H_\alpha(v) = \alpha\Big[|v|^{(\alpha+1)/\alpha} - v + \frac{1}{\alpha+1}\Big(\frac{\alpha}{\alpha+1}\Big)^\alpha\Big].$$

The following lemma plays an important role in our investigation; its proof can be found e.g. in [4, Theorem 2.2.1].

Lemma 1. *Equation* (9) *is nonoscillatory if and only if there exists a continuously differentiable function* v *which satisfies the Riccati inequality*

$$\dot{v} + \frac{\gamma_\alpha}{2s^2} c(\mathrm{e}^s) + H_\alpha(v) \leqslant 0 \tag{11}$$

for large s.

We will also need the following statement concerning asymptotics of solutions of Riccati inequalities. It is taken from [12].

Lemma 2. *Suppose that a differentiable function* ξ *satisfies the inequality*

$$\dot{\xi}(s) + H_\alpha(\xi(s)) \leqslant 0 \quad \textit{for large } t. \tag{12}$$

Then ξ *is nonincreasing and tends to* γ_α *as* $s \to \infty$.

Now we recall the concept of the principal solution of a nonoscillatory half-linear equation and the concept of the minimal solution of the associated Riccati equation

$$w' + c(t) + \alpha(r(t))^{-1/\alpha} |w|^{(\alpha+1)/\alpha} = 0 \tag{13}$$

as introduced in [10] and later independently in [7]. Nonoscillation of (1) implies that there exists a solution of (13) which is defined on some interval $[T, \infty)$ and among all such solutions there exists a *minimal* one $\tilde{w}$, minimal in the sense that any other solution of (13) satisfies $w(t) > \tilde{w}(t)$ for large t. The principal solution of (1) is then defined as the associated solution of (1), i.e., it is given by the formula

$$\tilde{x}(t) = \exp\left\{ \int^t \Phi_\alpha^{-1}\Big(\frac{\tilde{w}(s)}{r(s)}\Big) \, \mathrm{d}s \right\},$$

where Φ_α^{-1} is the inverse function of Φ_α.

The next statement is the comparison theorem for minimal solutions of a pair of Riccati equations. Its proof can be found in [7], see also [4, Theorem 4.2.2].

Lemma 3. *Consider the pair of half-linear equations* (1), (2). *Let* $\tilde{c}(t) \geqslant c(t)$ *for large* t, *suppose that* (2) *is nonoscillatory and denote by* $\tilde{w}$, $\tilde{u}$ *the minimal solutions of* (13) *and of*

$$u' + \tilde{c}(t) + \alpha(r(t))^{-1/\alpha} |u|^{(\alpha+1)/\alpha} = 0,$$

respectively. Then $\tilde{u}(t) \geqslant \tilde{w}(t)$ *for large* t.

We conclude this section with the main results of [13] which served as motivation for our paper. In these statements, $\delta(t)$ is a continuous function which is positive for large t.

Proposition 1. *Let $0 < \alpha < \beta$.*

(i) *If equation* (5) *is nonoscillatory, then the equation*

$$(\Phi_\beta(x'))' + \frac{1}{t^{\beta+1}}\left[\Gamma_\beta + \gamma_\beta \delta(t)\right]\Phi_\beta(x) = 0 \tag{14}$$

is also nonoscillatory.

(ii) *If the equation*

$$(\Phi_\beta(x'))' + \frac{1}{t^{\beta+1}}\left[\Gamma_\beta + \nu \delta(t)\right]\Phi_\beta(x) = 0 \tag{15}$$

is nonoscillatory for some $\nu > \gamma_\beta$, then (5) *is also nonoscillatory.*

3. A power comparison theorem

In this section we present the main result of our paper—a comparison theorem with respect to the power α in the perturbed Riemann-Weber half-linear differential equation.

Theorem 1. *Let $0 < \alpha < \beta$ and suppose that the equation*

$$(\Phi_\beta(x'))' + \frac{1}{t^{\beta+1}}\left[\Gamma_\beta + \frac{\gamma_\beta}{2\lg^2 t} + \nu \delta(t)\right]\Phi_\beta(x) = 0 \tag{16}$$

is nonoscillatory for some $\nu > \gamma_\beta$. Then the equation

$$(\Phi_\alpha(x'))' + \frac{1}{t^{\alpha+1}}\left[\Gamma_\alpha + \frac{\mu}{\lg^2 t} + \gamma_\alpha \delta(t)\right]\Phi_\alpha(x) = 0 \tag{17}$$

is nonoscillatory for

$$\mu < \mu^* := \frac{\gamma_\alpha(2\gamma_\beta\nu - \gamma_\beta^2)}{2\nu^2}.$$

P r o o f. The transformation of the independent variable $x(t) = y(s)$, $s = \lg t$, transforms (17) and (16) to the equations

$$(\Phi_\beta(\dot y))^{\cdot} - \beta\Phi_\beta(\dot y) + \left[\Gamma_\beta + \frac{\gamma_\beta}{2s^2} + \nu\delta(\mathrm{e}^s)\right]\Phi_\beta(y) = 0 \tag{18}$$

and

$$(\Phi_\alpha(\dot y))^{\cdot} - \alpha\Phi_\alpha(\dot y) + \left[\Gamma_\alpha + \frac{\mu}{s^2} + \gamma_\alpha\delta(\mathrm{e}^s)\right]\Phi_\alpha(y) = 0. \tag{19}$$

The Riccati equation associated with (18) is (with $\xi = \Phi_\beta(\dot y/y)$)

$$\dot\xi + H_\beta(\xi) + \frac{\gamma_\beta}{2s^2} + \nu\delta(\mathrm{e}^s) = 0, \tag{20}$$

where

$$H_\beta(\xi) = \beta\Big[\,|\xi|^{(\beta+1)/\beta} - \xi + \frac{1}{\beta+1}\Big(\frac{\beta}{\beta+1}\Big)^{\beta}\Big].$$

We will show that solvability of (20) for large s, which is equivalent to nonoscillation of (17), implies solvability of the Riccati inequality

$$\dot\eta + H_\alpha(\eta) + \frac{\mu}{s^2} + \gamma_\alpha\delta(\mathrm{e}^s) \leqslant 0 \tag{21}$$

provided $\mu < \mu^*$. Hence, suppose that $\mu < \mu^*$ and denote

$$\varepsilon = \frac{2\nu^2(\mu^* - \mu)}{\gamma_\alpha\gamma_\beta(\nu - \gamma_\beta)}.$$

There exist $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that $(1-\varepsilon_1)(1-\varepsilon_2)^2 = 1 - \varepsilon$. Denote further

$$\widehat\varepsilon = \frac{\varepsilon_1(\nu - \gamma_\beta)}{2\gamma_\alpha\gamma_\beta}, \quad \widetilde\varepsilon = \varepsilon_2\gamma_\beta, \quad c = \frac{\nu}{\gamma_\alpha}$$

and consider the function $\eta(s) = (\xi(s) + \nu - \gamma_\beta)/c$. We will show that this function satisfies inequality (21). We have $\xi = c\eta - \nu + \gamma_\beta$ and hence

$$\begin{aligned}\dot\eta &= \frac{1}{c}\dot\xi = -\frac{1}{c}\Big[\frac{\gamma_\beta}{2s^2} + H_\beta(c\eta - \nu + \gamma_\beta) + \nu\delta(\mathrm{e}^s)\Big]\\ &= -\frac{\gamma_\alpha\gamma_\beta}{2\nu s^2} - \gamma_\alpha\delta(\mathrm{e}^s) - H_\alpha(\eta) - \frac{1}{c}H_\beta(c\eta - \nu + \gamma_\beta) + H_\alpha(\eta).\end{aligned}$$

We will estimate the last two terms in the previous computation as follows. Denote

$$F(\eta) := \frac{1}{c}H_\beta(c\eta - \nu + \gamma_\beta) - H_\alpha(\eta).$$

Then by a direct computation we have (see also [13])

$$F(\gamma_\alpha) = 0 = F'(\gamma_\alpha) \quad \text{and} \quad F''(\gamma_\alpha) = \frac{\nu - \gamma_b}{\gamma_\alpha\gamma_\beta}.$$

Hence,

$$\begin{aligned}\frac{1}{c}H_\beta(c\eta - \nu + \gamma_\beta) - H_\alpha(\eta) &> \Big(\frac{\nu - \gamma_\beta}{2\gamma_\alpha\gamma_\beta} - \widehat\varepsilon\Big)(\eta - \gamma_\alpha)^2\\ &= \frac{\nu - \gamma_\beta}{2\gamma_\alpha\gamma_\beta}(1-\varepsilon_1)(\eta - \gamma_\alpha)^2\end{aligned} \tag{22}$$

for large s since $\eta(s) \to \gamma_\alpha$ as $s \to \infty$. Next we estimate the difference $\eta - \gamma_\alpha$. To this end, we estimate the difference $\xi - \gamma_\beta$. By Lemma 2 we have $\xi(s) \searrow \gamma_\beta$ as $s \to \infty$ and the comparison theorem for minimal solutions of Riccati equations (Lemma 3) yields $\xi\left(s\right) > u\left(s\right)$ for large s, where u is the minimal solution of the equation

$$\dot{u} + H_\beta(u) + \frac{\gamma_\beta}{2s^2} = 0.$$

We have $u\left(s\right) = \left(\dot{z}\left(s\right)/z\left(s\right)\right)^\beta$, where z is the principal solution of the equation

$$(\Phi_\beta(\dot{z}))^{\cdot} - \beta\Phi_\beta(\dot{z}) + \Big[\Gamma_\beta + \frac{\gamma_\beta}{2s^2}\Big]\Phi_\beta\left(y\right) = 0$$

and it is known (see [8]) that

$$u\left(s\right) = \gamma_\beta + \frac{\gamma_\beta}{s} + o\Big(\frac{1}{s}\Big) \text{ as } s \to \infty.$$

Hence

$$\xi\left(s\right) - \gamma_\beta > u\left(s\right) - \gamma_\beta > \frac{\gamma_\beta - \widetilde{\varepsilon}}{s} = \frac{\gamma_\beta}{s}\left(1 - \varepsilon_2\right)$$

for large s. Consequently,

$$\eta - \gamma_\alpha = \frac{\gamma_\alpha}{\nu}\left(\xi - \gamma_\alpha\right) > \frac{\gamma_\alpha\gamma_\beta}{\nu s}\Big(1 - \frac{\widetilde{\varepsilon}}{\gamma_\beta}\Big) = \frac{\gamma_\alpha\gamma_\beta}{\nu s}\left(1 - \varepsilon_2\right).$$

Using this estimate and (22) we obtain

$$\begin{aligned}\frac{1}{c}H_\beta(c\eta - \nu + \gamma_\beta) - H_\alpha(\eta) &> \frac{\nu - \gamma_\beta}{2\gamma_\alpha\gamma_\beta}\left(1 - \varepsilon_1\right)\frac{\gamma_\alpha^2\gamma_\beta^2}{\nu^2 s^2}\left(1 - \varepsilon_2\right)^2 \\ &= \frac{\left(\nu - \gamma_\beta\right)\gamma_\alpha\gamma_\beta}{2\nu^2 s^2}\left(1 - \varepsilon\right)\end{aligned}$$

for large s.

Therefore,

$$\begin{aligned}\dot{\eta} &= -\gamma_\alpha\delta(\mathrm{e}^s) - \frac{\gamma_\alpha\gamma_\beta}{2\nu s^2} - H_\alpha(\eta) - \frac{1}{c}H_\beta(c\eta - \nu + \gamma_\beta) + H_\alpha(\eta) \\ &< -\gamma_\alpha\delta(\mathrm{e}^s) - H_\alpha(\eta) - \frac{1}{s^2}\frac{\gamma_\alpha\gamma_\beta}{2\nu}\Big[1 + \frac{\nu - \gamma_\beta}{\nu}\left(1 - \varepsilon\right)\Big] \\ &= -\gamma_\alpha\delta(\mathrm{e}^s) - H_\alpha(\eta) - \frac{1}{s^2}\frac{\gamma_\alpha\gamma_\beta}{2\nu}\frac{2\nu - \gamma_\beta - \varepsilon\left(\nu - \gamma_\beta\right)}{\nu} \\ &= -\gamma_\alpha\delta(\mathrm{e}^s) - H_\alpha(\eta) - \frac{\gamma_\alpha\gamma_\beta}{2\nu}\frac{2\nu - \gamma_\beta}{s^2} + \frac{\varepsilon\left(\nu - \gamma_\beta\right)\gamma_\alpha\gamma_\beta}{2\nu^2 s^2} \\ &= -\gamma_\alpha\delta(\mathrm{e}^s) - H_\alpha(\eta) - \frac{\mu^* + (\mu - \mu^*)}{s^2} \\ &= -\gamma_\alpha\delta(\mathrm{e}^s) - H_\alpha(\eta) - \frac{\mu}{s^2},\end{aligned}$$

and hence η really satisfies (21) which means that (19) and so also (16) are nonoscillatory. □

4. Remarks

(i) Part (i) of Proposition 1 can be directly extended to a more general situation than that treated in Theorem 1. Consider the half-linear differential equation

$$L_n^{(\alpha)}(x) := (\Phi_\alpha(x'))' + \frac{1}{t^{\alpha+1}}\Big[\Gamma_\alpha + \frac{\gamma_\alpha}{2}\sum_{k=1}^{n}\frac{1}{\lg_1^2 t \lg_2^2 t \dots \lg_k^2 t}\Big]\Phi_\alpha(x) = 0,$$

where $\lg_1 t = \lg t$, $\lg_{k+1} t = \lg(\lg_k t)$. The Euler equation (3) and the Riemann-Weber equation (7) are special cases of (4) with $n = 0$ and $n = 1$, respectively. It is konwn that the equation $L_n^{(\alpha)}(x) = 0$ is nonoscillatory and that the constants Γ_α and $\frac{1}{2}\gamma_\alpha$ are optimal in the sense that if we repalce one of them by a greater constant, then the equation becomes oscillatory. Based on this observation, part (i) of Proposition 1 can be reformulated as follows.

Let $0 < \alpha < \beta$ and consider the pair of equations

$$L_n^{(\alpha)}(x) + \gamma_\alpha\delta(t)\Phi_\alpha(x) = 0, \tag{23}$$

$$L_n^{(\beta)}(x) + \gamma_\beta\delta(t)\Phi_\beta(x) = 0, \tag{24}$$

where δ is a nonnegative function. If equation (23) *is nonoscillatory, then* (24) *is also nonoscillatory.*

The proof of this statement easily follows from Proposition 1 (i) since (23) can be written in the form

$$(\Phi_\alpha(x'))' + \frac{1}{t^{\alpha+1}}[\Gamma_\alpha + \gamma_\alpha\tilde{\delta}(t)]\Phi_\alpha(x) = 0,$$

where

$$\tilde{\delta}(t) = \frac{1}{2}\sum_{k=1}^{n}\frac{1}{\lg_1^2 \lg_2^2 \dots \lg_k^2 t} + \delta(t),$$

and equation (24) can be rewritten in the same way.

(ii) An important part of the proof of Theorem 1 is the estimate $\xi(s) - \gamma_\beta = \gamma_\beta/s + o(s^{-2})$ which we have used for the asymptotic formula

$$\frac{1}{c}H_\beta(c\eta(s) - \nu + \gamma_\beta) - H_\alpha(\eta(s)) = \frac{\varepsilon(\nu - \gamma_\beta)\gamma_\alpha\gamma_\beta}{2\nu^2 s^2} + o(s^{-2}) \tag{25}$$

as $s \to \infty$. If we do apply this asymptotic formula and use the mere fact that the left-hand side of (25) is nonnegative (similarly as in [13]), we may reformulate Theorem 1 as follows:

Let $0 < \alpha < \beta$ *and suppose that the equation*

$$L_n^{(\beta)}(x) + \nu\delta\Phi_\beta(x) = 0$$

is nonoscillatory for some $\nu > \gamma_\beta$. *Then the equation*

$$L_n^{(\alpha)}(x) + \Big[\frac{\gamma_\alpha(\gamma_\beta - \nu)}{2\nu t^{\alpha+1}\lg^2 t} + \gamma_\alpha\delta(t)\Big]\Phi_\alpha(x) = 0$$

is also nonoscillatory.

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