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# Global bifurcation for quasivariational inequalities of reaction-diffusion type

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**Abstract**

We consider a reaction-diffusion system with implicit unilateral boundary conditions introduced by Mosco. Under the assumptions considered, diffusion driven instability occurs for the same system but with linear mixed boundary conditions. We show that global continua of stationary spatially nonhomogeneous solutions bifurcate in the domain of stability of the trivial solution of the problem with classical boundary conditions mentioned, where bifurcation for this classical problem is excluded. In the case when a bifurcation parameter is a size of the domain, the result says that bifurcation for the unilateral problem occurs for some diffusion coefficients for which bifurcation for the classical boundary conditions is excluded, and for some diffusion coefficients it occurs for the unilateral problem in smaller domains than for the classical boundary conditions. The problem is formulated as a quasivariational inequality and the proof is based on the Leray-Schauder degree.

*Key words:* Global bifurcation, quasivariational inequality, unilateral implicit boundary conditions, Leray-Schauder degree, reaction-diffusion system.
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## 1 Introduction.

Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^{\mathbb{N}}$, $\mathbb{N} \geq 2$, with a Lipschitzian boundary $\partial\Omega$ and $\Gamma_D$, $\Gamma_N$, $\Gamma_U$ are pairwise disjoint open parts of $\partial\Omega$ such that

$$\text{meas}\ \Gamma_D > 0, \quad \text{meas}\ \Gamma_U > 0, \quad \text{meas}\ \{\partial\Omega \setminus (\Gamma_D \cup \Gamma_N \cup \Gamma_U)\} = 0. \tag{1}$$

Our goal is to prove the existence and describe a location of a global bifurcation of stationary solutions for the reaction-diffusion system

$$\begin{cases} u_t = d_1\Delta u + b_{11}u + b_{12}v + n_1(d_1, d_2, u, v) \\ v_t = d_2\Delta v + b_{21}u + b_{22}v + n_2(d_1, d_2, u, v) \end{cases} \quad \text{in} \quad (0, +\infty) \times \Omega, \tag{2}$$

with the unilateral implicit boundary conditions

$$\begin{cases} u = 0, \quad v = 0 \quad \text{on} \quad \Gamma_D, \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \setminus \Gamma_D, \quad \dfrac{\partial v}{\partial n} = 0 \quad \text{on} \quad \Gamma_N, \\ v \geq -\displaystyle\int_{\Gamma_U} \Phi(x,y)\frac{\partial v}{\partial n}(y)d\Gamma(y), \quad \dfrac{\partial v}{\partial n} \geq 0, \\ \left(v + \displaystyle\int_{\Gamma_U} \Phi(x,y)\frac{\partial v}{\partial n}(y)d\Gamma(y)\right)\dfrac{\partial v}{\partial n} = 0 \quad \text{on} \quad \Gamma_U. \end{cases} \tag{3}$$

Here $d_1$, $d_2$ are positive real parameters (diffusion coefficients), $n_1$, $n_2$ are small perturbations, $b_{ij}$ $(i, j = 1, 2)$ are real coefficients and $\Phi \in C^2(\overline{\Omega} \times \overline{\Omega})$ is a given function. Of course, if $\Phi(x, y) \equiv 0$ in $\overline{\Omega} \times \overline{\Omega}$ then we get the Signorini boundary condition on $\Gamma_U$ for $v$. Clearly $[0, 0]$ is a solution of (2) with (3) as well as with the classical mixed boundary conditions

$$u = 0, \quad v = 0 \quad \text{on} \quad \Gamma_D, \qquad \frac{\partial u}{\partial n} = 0, \quad \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \Gamma_N \cup \Gamma_U. \tag{4}$$

The bifurcation point of (2) , (4) will be obtained in the domain of parameters where the trivial solution of the classical problem (2) , (4) is stable and where a bifurcation for (2) , (4) is excluded. To explain these relations it is necessary to start ou exposition with the evolution system (2) , (4) , but in fact we will consider the stationary problem corresponding to (2), (3) with $d$ changing along a curve $\sigma$ in $\mathbb{R}^2_+$. More precisely, we will consider a continuous mapping $\sigma = [\sigma_1, \sigma_2] : \mathbb{R}_+ \to \mathbb{R}^2_+$ and the problem

$$\begin{cases} \sigma_1(s)\Delta u + b_{11}u + b_{12}v + n_1(\sigma_1(s), \sigma_2(s), u, v) = 0 \\ \sigma_2(s)\Delta v + b_{12}u + b_{22}v + n_2(\sigma_1(s), \sigma_2(s), u, v) = 0 \end{cases} \quad \text{in} \quad \Omega, \tag{5}$$

with the unilateral boundary conditions (3) and with the real bifurcation parameter $s \in \mathbb{R}_+$.

We always suppose that

$$\begin{gathered} b_{11} > 0, \qquad b_{12} < 0, \qquad b_{21} > 0, \qquad b_{22} < 0, \\ b_{11} + b_{22} < 0, \qquad b_{11}b_{22} - b_{12}b_{21} > 0. \end{gathered} \tag{6}$$

If our system is related to a chemical reaction, $u$, $v$ describe the concentrations of reactants, then the assumptions (6) mean that $u$ and $v$ is an activator and inhibitor, respectively. It is well-known that under the assumptions (6), the trivial solution is stable as the solution of the system without any diffusion (i.e. ODE's obtained for $d_1 = d_2 = 0$), but as a solution of (2) with the classical boundary conditions (4) it is stable only for $d = [d_1, d_2]$ from some subdomain $D_S \subset \mathbb{R}^2_+$ and unstable for $d = [d_1,\ d_2]$ from $D_U = \mathbb{R}^2_+ \setminus \overline{D}_S$ (see Proposition 2.2, Figure 1). Standard methods of bifurcation theory guarantee that stationary spatially inhomogeneous solutions of the classical problem (2), (4) bifurcate from the trivial solution on the border $C_E$ between the domain of stability and instability and also in some points of the domain of instability $D_U$, and that there are no bifurcation points in the domain of stability. For the unilateral problem (2), (3), (even with $\Phi = 0$) classical approaches based on linearization fail because this problem has no real linearization.

Our goal is to show that there are global bifurcations of stationary solutions of the problem (2), (3) even in $D_S$ (domain of stability corresponding to the classical problem), where bifurcation for (2), (4), is excluded. Hence, the unilateral boundary conditions (4) have a certain destabilizing effect (in the sense of arising of bifurcation).

We will prove that if the curve $\sigma$ comes from the domain $D_S$ to a neighborhood of a suitable part of $C_E$ then there is a global bifurcation point $s_B$ for (2), (3) $s_B$ with $\sigma(s_B) \in D_S$. The problem will be formulated in terms of a quasi-variational inequality which can be written also as a strongly nonlinear operator equation with a projection onto a closed convex set depending implicitly on a solution. The Leray-Schauder degree will be used to this equation. The first result of this type (in a very simple form) for variational, not quasivariational, inequalities was proved in [1] by a method from [7]. It was based on a certain nonstandard combination of a penalty method with Dancer's global bifurcation result for equations. An other proof of a result of this type, based on a direct use of the Leray-Schauder degree (jump of the degree implies bifurcation) was given in [12]. In both cases, only particular curves $\sigma_1(s) = s$, $\sigma_2(s) = 1$ were considered and local bifurcation was described. Generalizations to general curves and various types of unilateral boundary conditions were proved in a series of papers (see e.g. references in [2]), which gives the most complex and abstract result in this direction for the case of unilateral multivalued conditions. (The conditions (3) are not included neither with $\Phi = 0$). The problem with implicit boundary conditions (3) was studied in [8] for the case of general curves intersecting the border $C_E$ in suitable places. We generalize here the result from [8] in the following directions. First, the assumptions concerning positivity on $\Gamma_U$ of some eigenfunctions of the Laplacian are weakened Further, we describe here a global behavior of the bifurcation branch, while only local bifurcation was shown in [8]. In contrast to [8], we also consider the perturbations $n_1, n_2$ dependent on diffusion coefficients and $\Phi$ depends not only on $x$. Finally, our curve coming from the domain $D_S$ need not to intersect the border $C_E$ but only its suitable neighborhood, and the bifurcation point obtained in $D_S$ is localized in more details. The last generalization mentioned has the following consequence. For the particular case $\sigma_1(s) = s^{-2}d_1$, $\sigma_2(s) = s^{-2}d_2$ $(d_1, d_2$ fixed), the parameter $s$ describes a growth of the domain, and our result says not only (as in [8]) that for some diffusion coefficients $d_1, d_2$ a bifurcation for (2), (3) occurs "sooner" (for smaller domains) than for the classical problem (2), (4), but (in addition) even for some $d_1, d_2$ for which the whole curve $\sigma$ lies in $D_S$ and (2), (4) has no bifurcation points at all.

Let us recall that implicit unilateral boundary condition considered were introduced (in a different situation) by U. Mosco [10], [3]. In our case it can describe a unilateral membrain allowing a flux of the inhibitor $v$ only in one direction (inwards $\Omega$, not outwards) on the surface $\Gamma_U$ which is surrounded by a reservoir with a basic concentration (whose value is shifted to zero in our model) which is lowered by the amount of the material just flowing through $\Gamma_U$.

## 2 Notation and general remarks.

In the whole paper, we assume that

$$
\begin{aligned}
&(i) \text{ if } \ y \in \Gamma_U \ \text{ then } \ \Phi(x,y) \geq 0 \quad \text{a.a. for all} \quad x \in \partial\Omega; \\
&(ii) \text{ if } \ x \in \Gamma_D \ \text{ then } \ \Phi(x,y) = 0 \quad \text{a.a. for all} \quad y \in \partial\Omega.
\end{aligned}
\tag{7}
$$

Suppose that $n_1$, $n_2$ are real differentiable functions on $\mathbb{R}^4$ and

$$n_j(d_1,d_2,0,0)=0, \quad \frac{\partial n_j}{\partial u}(d_1,d_2,0,0)=\frac{\partial n_j}{\partial v}(d_1,d_2,0,0)=0,\ j=1,2, \tag{8}$$

for all $d=[d_1,d_2]\in\mathbb{R}^2_+$. Assume that for each $d=[d_1,d_2]\in\mathbb{R}^2_+$ there is $c(d_1,d_2)$ such that the functions $n_j$, $j=1,2$ satisfy the growths conditions

$$|n_j(d_1,d_2,u,v)|\le c(d_1,d_2)(1+|u|^{q-1}+|v|^{q-1}),\quad j=1,2 \tag{9}$$

with some $q\ge 1$ for $\mathbb{N}=2$ and $1\le q<\dfrac{2\mathbb{N}}{\mathbb{N}-2}$ for $\mathbb{N}>2$. Furthermore, assume that

$$|n_j(d_1,d_2,u,v)-n_j(d_1^0,d_2^0,u,v)|\le c_0(d_1,d_2)(1+|u|^{q-1}+|v|^{q-1}) \tag{10}$$

for $j=1,2$, $[d_1,d_2]$, $[d_1^0,d_2^0]\in\mathbb{R}^2_+$, and where $c_0(d_1,d_2)\ge 0, c_0(d_1,d_2)\to 0$ as $[d_1,d_2]\to[d_1^0,d_2^0]$.

We denote by $\mathbb{R}_+$ the set of all positive reals, $\mathbb{R}^2_+=\mathbb{R}_+\times\mathbb{R}_+$, $\to$ and $\rightharpoonup$ will denote the strong convergence and the weak convergence, respectively.

Let us denote by $k_j$ $(j=1,2,...)$ the eigenvalues of the boundary value problem

$$\begin{cases} -\Delta u = ku & \text{in} \quad \Omega,\\ u=0 & \text{on} \quad \Gamma_D,\\ \partial u/\partial n = 0 & \text{on} \quad \partial\Omega\setminus\Gamma_D. \end{cases} \tag{11}$$

Further, let $e_j$ $(j=1,2,...)$ be the system of eigenfunctions of (11) which is complete and orthogonal in $L_2(\Omega)$. By $m_j$ we denote the multiplicity of the eigenvalue $k_j$.

**Remark 2.1** Let us consider the eigenvalue problem

$$\begin{cases} d_1\Delta u + b_{11}u + b_{12}v = \lambda u \\ d_1\Delta v + b_{21}u + b_{22}v = \lambda v \end{cases} \quad \text{in} \quad \Omega, \tag{12}$$

with the boundary conditions (4). Let us recall that if $Re\ \lambda\le -\varepsilon<0$ for all eigenvalues of the problem (12), (4) then the trivial solution of (2), (4) is linearly stable (see e.g. [14]) and if there is at last one eigenvalue of (12), (4) satisfying $Re\ \lambda > 0$ then the trivial solution of (2), (4) is linearly unstable.

Remark 2.1 together with Proposition 2.2 below justify the following notation:

$C_j := \left\{ d = [d_1, d_2] \in \mathbb{R}^2_+;\ d_2 = \dfrac{b_{12}b_{21}/k_j^2}{d_1 - b_{11}/k_j} + \dfrac{b_{22}}{k_j} \right\},\ j = 1, 2, \dots$ ($C_j$ are in $\mathbb{R}^2_+$ lying parts of certain hyperbolas – see Figure 1),

$C_E-$ the envelope of the hyperbolas $C_j,\ j = 1, 2, \dots$ (see Figure 1),

$D_U-$ the set of all $d \in \mathbb{R}^2_+$ lying to the left from $C_E$, i.e. from at least one of $C_j$ (domain of instability of the trivial solution of (2), (4) – see Remark 2.1, Proposition 2.2 and Figure 1),

$D_S := \mathbb{R}^2_+ \backslash (C_E \cup D_U)-$ the set of all $d \in \mathbb{R}^2_+$ lying to the right from $C_E$, i.e. from all $C_j$ (domain of stability of the trivial solution of (2), (4)– see Remark 2.1, Proposition 2.2 and Figure 1),

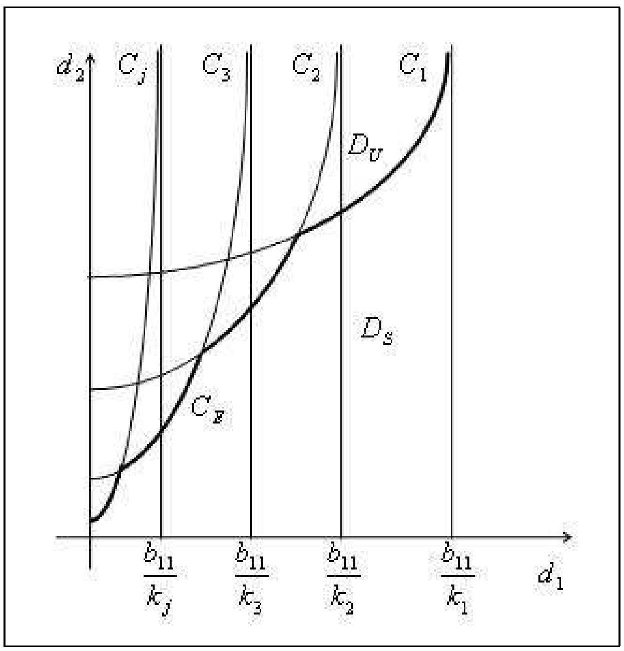


Fig. 1.

Finally, we will consider the linearized system

$$\begin{cases} d_1 \Delta u + b_{11} u + b_{12} v = 0 \\ d_2 \Delta v + b_{21} u + b_{22} v = 0 \end{cases} \quad \text{in} \quad \Omega, \tag{13}$$

and denote $E(d_1, d_2)-$ the set of all solutions $u, v$ of the problem (13), (4).

**Proposition 2.2** *Let (1), (6) hold.*

*Then* $\bigcup_{j=1}^{+\infty} C_j$ *is the set of all* $d = [d_1, d_2] \in \mathbb{R}^2_+$ *for which* $\lambda = 0$ *is an eigenvalue of the problem* (12), (4). *If* $d \in D_S$ *then there is* $\varepsilon > 0$ *such that* $Re\lambda < -\varepsilon$ *for all eigenvalues of* (12), (4) *and if* $d \in D_U$ *then there exists at least one positive eigenvalue of* (12), (4).

*If* $d$ *lies only on one hyperbola* $C_p$, *i.e.* $d \in C_p = ... = C_{p+m_p-1}$, $d \notin C_q$, $q \neq p, ..., p + m_p - 1$, $m_p$ *is the multiplicity of* $k_p$ *then*

$$E(d_1, d_2) = Lin\{[e_j, \frac{b_{21}}{d_2 k_j - b_{22}} e_j]\}_{j=p}^{p+m_p-1}.$$

*If* $d \in C_p \cap C_q$, $C_p \neq C_q$ *(the intersection of two different hyperbolas) then*

$$E(d_1, d_2) = Lin\{[e_j, \frac{b_{21}}{d_2 k_j - b_{22}} e_j]\}_{j=p}^{p+m_p-1} \cup Lin\{[e_j, \frac{b_{21}}{d_2 k_j - b_{22}} e_j]\}_{j=q}^{q+m_q-1}.$$

## 3 Problem formulation. Main result

Let us introduce the space

$$H := \{\varphi \in W^{12}(\Omega);\ \varphi = 0 \ \text{ on } \ \Gamma_D \text{ in the sense of traces }\}. \tag{14}$$

Then $H$ is a real Hilbert space with the scalar product

$$\langle u, \varphi \rangle = \int_\Omega \sum_{i=1}^{n} u_{x_i} \varphi_{x_i} dx \quad \text{for all } \ u, \ \varphi \in H,$$

and the corresponding norm $\|\varphi\|^2 = \langle \varphi, \varphi \rangle$ is equivalent to the usual Sobolev norm under the assumption (1) (see e.g. [9]). We denote $\widetilde{H} = H \times H$, $\langle U, W \rangle = \langle u, v \rangle + \langle w, z \rangle$ and $\|U\|^2 = \|u\|^2 + \|v\|^2$ for $U = [u, v]$, $W = [w, z]$, $U, \ W \in \widetilde{H}$ . Furthermore, we will introduce the Banach space

$$V := \{\, \varphi \in H :\ \Delta\varphi \in L^2(\Omega)\}, \tag{15}$$

with the norm

$$|||\varphi||| = \left( \int_\Omega |\nabla \varphi|^2 \, dx \right)^{\frac{1}{2}} + \left( \int_\Omega |\Delta \varphi|^2 \, dx \right)^{\frac{1}{2}}.$$

See e.g. [9].

**Remark 3.1** *If* $v \in V$ *then the normal derivative* $\dfrac{\partial v}{\partial n}$ *can be defined as a linear bounded functional on the space* $H$ *by*

$$\left[\frac{\partial v}{\partial n}, \varphi\right] = \int_\Omega (\Delta v \varphi + \nabla v \nabla \varphi) dx \quad \text{for all} \quad \varphi \in H.$$

If $v$ is sufficiently smooth up to the boundary then, of course,

$$\left[\frac{\partial v}{\partial n}, \varphi\right] = \int_{\partial\Omega} \frac{\partial v}{\partial n} \varphi d\Gamma \quad \text{for all} \quad \varphi \in H,$$

where $\dfrac{\partial v}{\partial n}$ in the integral on the right hand side is the classical derivative of $v$ with respect to the outer normal to $\partial\Omega$ (that means the classical Green formula holds). As usual, given a relatively open subset $\Gamma_U$ of $\partial\Omega$, we will write that $\dfrac{\partial v}{\partial n} = 0$ on $\Gamma_U$ if $\left[\dfrac{\partial v}{\partial n}, \varphi\right] = 0$ for all $\varphi \in H$ with $\varphi = 0$ on $\partial\Omega \setminus \Gamma_U$ and similarly $\dfrac{\partial v}{\partial n} \geq 0$ on $\Gamma_U$ if $\left[\dfrac{\partial v}{\partial n}, \varphi\right] \geq 0$ for all $\varphi \in H$ with $\varphi = 0$ on $\partial\Omega \setminus \Gamma_U$ and $\varphi \geq 0$ on $\Gamma_U$. Here all equalities and inequalities for $\varphi$ on parts of the boundary $\partial\Omega$ are understood in the sense of traces. Let us recall that for any $x \in \overline{\Omega}$, $\Phi(x, \cdot) \in C^2(\overline{\Omega})$, i.e. also $\Phi(x, \cdot) \in H$. Hence, for $v \in V$ and given $x$, the integrals in (3) can be understood as $\left[\dfrac{\partial v}{\partial n}, \Phi(x, \cdot)\right]$.

For any $v \in V$, define the closed convex set $K_v$ in $H$ by

$$K_v := \left\{ \varphi \in H; \quad \varphi(x) \geq -\left[\frac{\partial v}{\partial n}, \Phi(x, \cdot)\right] \quad \text{a.e. on} \quad \Gamma_U \right\}, \tag{16}$$

where $\dfrac{\partial v}{\partial n} : H \to \mathbb{R}$ is the functional from Remark 3.1.

Now, we introduce a weak solution of the problem (5), (3) as a couple $[u, v]$ satisfying the quasivariational inequality

$$\begin{cases} u, v \in V, \ v \in K_v, \\ \int_\Omega \sigma_1(s) \nabla u \nabla \varphi - (b_{11} u + b_{12} v + n_1(\sigma(s), u, v)) \varphi \, dx = 0, \\ \int_\Omega \sigma_2(s) \nabla u \nabla (\psi - v) - (b_{21} u + b_{22} v + n_2(\sigma(s), u, v))(\psi - v) \, dx \geq 0 \end{cases} \tag{17}$$

for all $\psi \in K_v$. Similarly, a weak solution of the problem (13), (3), is a couple $[u,\ v]$ satisfying the quasivariational inequality

$$\begin{cases} u, v \in V,\ v \in K_v, \\ \int\limits_\Omega d_1 \nabla u \nabla \varphi - (b_{11} u + b_{12} v)\varphi \, dx = 0 \quad \text{for all} \quad \varphi \in H, \\ \int\limits_\Omega d_2 \nabla v \nabla (\psi - v) - (b_{21} u + b_{22} v)(\psi - v) \, dx \geq 0 \ \text{ for all } \ \psi \in K_v. \end{cases} \tag{18}$$

Speaking about a solution of a boundary value problem we will have always in mind a weak solution.

**Lemma 3.2** *A couple* $U = [u, v] \in H$ *is a weak solution of the problem (5), (3) or (13), (4) if and only if* $\triangle u,\ \triangle v \in L_2(\Omega)$, *(5) or (13) hold a.e. in* $\Omega$ *and the boundary conditions (3) or (4) are fulfilled in the sense of Remark 3.1.*

**Proof** follows by standard considerations concerning weak solutions (choice of suitable test functions, Green's formula, etc.). $\square$

**Definition 3.3** A *bifurcation point* of the problem (17) is a parameter $s_B \in \mathbb{R}_+$ such that in any neighborhood of $[s_B; 0, 0] \in \mathbb{R}_+ \times V \times V$ there exists $[s; u, v] \in \mathbb{R}_+ \times V \times V$, $|||u||| + |||v||| \neq 0$ satisfying (17). By a bifurcation point of the problem (5), (3) we mean a bifurcation point of the weak formulation (17).

In the sequel, we will consider a hyperbola $C_p$ $(C_{p-1} \neq C_p = \cdots = C_{p+m_p-1} \neq C_{p+m_p})$ or two consequent hyperbolas $C_p$, $C_q$ (i.e. $C_{p-1} \neq C_p = \cdots = C_{p+m_p-1} \neq C_q = \cdots = C_{q+m_q-1} \neq C_{q+m_q}$, $q = p + m_p$) such that the following assumption (19) (i) or (19) (ii), respectively, is fulfilled:

$$\left[ \begin{array}{l} \text{(i)} \ \sum\limits_{i=p}^{p+m_p-1} \alpha_i e_i > 0 \quad \text{on} \ \ \Gamma_U \ \ \text{for some} \quad \{\alpha_i\}_{i=p}^{p+m_p-1}; \\ \text{(ii)} \ \sum\limits_{i=p}^{q+m_q-1} \widetilde{\alpha}_i e_i > 0 \quad \text{on} \ \ \Gamma_U \ \ \text{for some} \quad \{\widetilde{\alpha}_i\}_{i=p}^{q+m_q-1}. \end{array} \right. \tag{19}$$

Denote by $S$ the set of all nontrivial weak solutions of (5) , (3), i.e.

$$S = \{[s; u, v] \in \mathbb{R}_+ \times V \times V : [s; u, v] \text{ satisfies (17) with } |||u||| + |||v||| \neq 0\}.$$

**Theorem 3.4** (Main Theorem). *Assume that* $\Gamma_U \subset \partial\Omega$ *is a smooth manifold in* $\mathbb{R}^{\mathbb{N}-1}$ *with the smooth boundary. Let (6) (8), (9), (10) be fulfilled. If (19)* (i) *or (19)* (ii) *hold with some* $p$ *then there is an open set* $\mathcal{U} \subset$

$\mathbb{R}^2_+$ *which contains the arc* $C_E \cap C_p$ *or the point* $d = C_p \cap C_q$, *respectively, and has the following properties. If* $\sigma = [\sigma_1, \sigma_2] : \mathbb{R}_+ \to \mathbb{R}^2_+$ *is a curve such that* $\sigma(\widehat{s}) \in \mathcal{U} \cap D_S$ *for some* $\widehat{s}$ *and* $\sigma_1(s^0) > \dfrac{b_{11}}{k_1}$ *for some* $s^0$, $s^0 < \widehat{s}$, *then there exists a bifurcation point* $s_B \in (s^0, \widehat{s})$ *of the problem (5) , (3) with* $\sigma_1(s_B) \le \dfrac{b_{11}}{k_1}$. *Moreover, the bifurcation is global in the following sense.*
*There is a connected set* $\mathcal{F} \subseteq S$ *such that* $\overline{\mathcal{F}}$ *(the closure in* $V \times V$ *) contains a point* $[s_B; 0, 0]$ *and* $\mathcal{F}$ *has at least one of the following properties:*
*1)* $\mathcal{F}$ *is unbounded in* $\mathbb{R}_+ \times V \times V$,
*2)* $\overline{\mathcal{F}}$ *contains a point of the type* $[0; u, v]$, $|||u||| + |||v||| \neq 0$ *or* $[s; 0, 0]$ *with* $s \in \overline{\mathbb{R}}_+ \setminus [s^0, \widehat{s}]$.

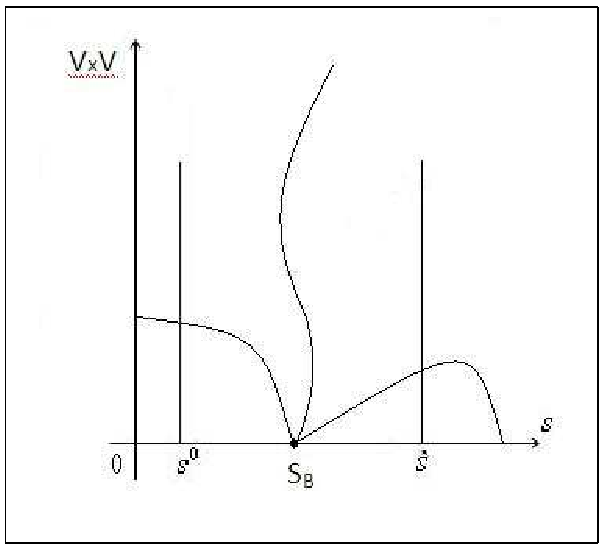


Fig. 2.

**Remark 3.5** The assertion of Theorem 3.4 remains valid if we ommit the assumption that $\Gamma_U$ is a smooth manifold but replace (19)(i) or (ii) by the stronger assumption (33) (i) or (ii) from Remark 4.6 below. This will be seen from the proof of Theorem 3.4 because none of these assumptions is used directly, only their consequence (31) is essential - see Lemma 4.5 and Remark 4.6.

**Remark 3.6** If the curve $\sigma$ in Theorem 3.4 satisfies in addition $\sigma_1(s) > \frac{b_{11}}{k_1}$ for all $s < s_0$ then we can replace $s \in R^+ \setminus [s^0, \hat{s}]$ in the case 2) by $s > \hat{s}$. It follows from the fact that there are no bifurcation points of (17) with $\sigma_1(s) > \frac{b_{11}}{k_1}$ (see Lemmas 4.4, 4.9).

## 4 Proof of the Main Result

Introduce the operators $A : H \to H$ , $N_j : \mathbb{R}^2_+ \times H \times H \to H$ $(j = 1, 2)$ by

$$\langle Au, \varphi\rangle = \int_\Omega u\varphi\, dx \quad \text{for all} \quad u,\ \varphi \in H, \tag{20}$$

$$\langle N_j(d, U), \varphi\rangle = \int_\Omega n_j(d_1, d_2, u, v)\varphi\, dx, \quad U = [u, v] \in H \times H,\ \varphi \in H. \tag{21}$$

Then

$$A \quad \text{is linear, continuous, symmetric, positive and compact,} \tag{22}$$

$$N_j : \mathbb{R}^2_+ \times H \times H \quad \text{are nonlinear, continuous, compact operators} \tag{23}$$

under the assumptions (9), (10). (This folows from the properties of the Nemyckij operators and compact embedding theorems). Moreover, using the conditions (8) and (9) it is possible to prove that

$$\lim_{\|U\|\to 0} \frac{\|N_j(d, U)\|}{\|U\|} = 0, \quad \begin{array}{l}\text{uniformly for } d \text{ from compact}\\ \text{subsets of } \mathbb{R}^2_+, \quad j = 1, 2.\end{array} \tag{24}$$

See [2], Proposition 3.2.

The weak formulation of the problem (5), (3) (i.e. (17)) can be written in the form

$$\begin{cases} u, v \in V,\ v \in K_v, \\ \sigma_1(s)u - b_{11}Au - b_{12}Av - N_1(\sigma(s), u, v) = 0, \\ \langle \sigma_2(s)v - b_{21}Au - b_{22}Av - N_2(\sigma(s), u, v), \psi - v\rangle \geq 0 \ \text{ for all } \ \psi \in K_v. \end{cases} \tag{25}$$

The weak formulation of the problem (13), (3) (i.e. (18)) the form

$$\begin{cases} u, v \in V,\ v \in K_v, \\ d_1u - b_{11}Au - b_{12}Av = 0, \\ \langle d_2v - b_{21}Au - b_{22}Av, \psi - v\rangle \geq 0 \ \text{ for all } \ \psi \in K_v. \end{cases} \tag{26}$$

For a given $d = [d_1, d_2] \in \mathbb{R}^2_+$, we denote

$$E_I(d_1,d_2)=\{[u,v]\in V\times V;\quad (26) \text{ is fulfilled }\}\quad (26).$$

Let us note that the problem (26) is positively homogenous, i.e. $[u,v]\in E_I(d_1,d_2)$ if and only if $[\tau u,\tau v]\in E_I(d_1,d_2)$ for all $\tau\in\mathbb{R}_+$.

**Remark 4.1** For any $v\in V$ we define the operator $P_v : H\to K_v$ (the projection of $H$ onto the closed convex set $K_v$) by

$$P_v(z)\in K_v,\quad \|z-P_v(z)\|=\min_{\varphi\in K_v}\|z-\varphi\|\quad \text{for all}\quad z\in H.$$

For any $v\in V$ and $z\in H$ the element $P_v(z)$ is uniquely defined by the conditions

$$P_v(z)\in K_v\quad\text{and}\quad \langle P_v(z)-z,\varphi-P_v(z)\rangle\ge 0\quad\text{for all}\quad \varphi\in K_v.$$

(See e.g. [4]). It can easily be checked that $P_{tv}(tz)=tP_v(z)$ for all $v\in V,\ z\in H,\ t>0$.

**Lemma 4.2** *Let (1), (7) hold. The operator* $A$ *and* $N_j,\ (j=1,2)$ *defined by (20), (21) maps* $V$ *and* $\mathbb{R}^2_+\times V\times V$*, respectively, into* $V$*. Further,* $A:V\to V,\ N_j:\mathbb{R}^2_+\times V\times V\to V$ *are continuous, compact operators. Moreover, if* $U_n=[u_n,v_n]\to 0,\quad W_n=[w_n,z_n]=\dfrac{U_n}{|||U_n|||}\rightharpoonup W=[w,z]$ *in* $V,\quad d^n=[d_1^n,d_2^n]\to d=[d_1,d_2],\ d_2>0$ *then*

$$b_{11}Aw_n+b_{12}Az_n-\frac{N_1(d,u_n,v_n)}{|||U_n|||}\to b_{11}Aw+b_{12}Az\quad in\quad V,$$

$$P_{z_n}\left[(d_2^n)^{-1}\left(b_{21}Aw_n+b_{22}Az_n-\frac{N_2(d^n,u_n,v_n)}{|||U_n|||}\right)\right]\to P_z[d_2^{-1}(b_{21}Aw+b_{22}Az)]\quad in\quad V.$$

**Proof** is the same as that of Lemma 2.1 in [5], where the quasivariational inequality for $u$ and equation for $v$ is considered. Cf. also [8]. □

Using Remark 4.1 we can write the problem (25) in the form

$$\begin{cases} u,v\in V,\ v\in K_v,\\ \sigma_1(s)u-b_{11}Au-b_{12}Av-N_1(\sigma(s),u,v)=0,\\ \sigma_2(s)v-P_v(b_{21}Au+b_{22}Av+N_2(\sigma(s),u,v))=0.\end{cases}\qquad (27)$$

**Remark 4.3** Let us define

$$T(d,U) = \begin{pmatrix} \dfrac{1}{d_1}(b_{11}Au + b_{12}Av + N_1(d,u,v)) \\ \dfrac{1}{d_2}P_v(b_{21}Au + b_{22}Av + N_2(d,u,v)) \end{pmatrix}, \tag{28}$$

$$T_0(d,U) = \begin{pmatrix} \dfrac{1}{d_1}(b_{11}Au + b_{12}Av) \\ \dfrac{1}{d_2}P_v(b_{21}Au + b_{22}Av) \end{pmatrix}. \tag{29}$$

Due to Lemma 4.2, $T,\ T_0 : \mathbb{R}^2_+ \times V \times V \to V \times V$ are nonlinear, continuous and compact operators. The problem (26) and (27) is equivalent to

$$U - T_0(d,U) = 0$$

and

$$U - T(\sigma(s),U) = 0, \tag{30}$$

respectively.

The Leray-Schauder degree of $I - T(\sigma(s),\cdot)$ at the point $s \in \mathbb{R}_+$ with respect to the ball $B_r = \{U \in V \times V;\ \|U\| < r\}$ and at the point zero, will be denoted by

$$\deg(I - T(\sigma(s),\cdot), B_r, 0).$$

**Lemma 4.4** (See [5], Lemma 1.2). *Let (8), (9) be fulfilled. Then for any bifurcation point $s_B$ of the problem (27), we have $E_I(\sigma_1(s_B),\sigma_2(s_B)) \neq \{0\}$.*

The following lemma is a modification of Lemma 3.1 in [6], where a variational inequality with a fixed cone $K_0$ is studied.

**Lemma 4.5** *Assume that $\Gamma_U \subset \partial\Omega$ is a smooth manifold in $\mathbb{R}^{\mathbb{N}-1}$ with the smooth boundary. Let (19)* (i) *or (19)* (ii) *hold with some $p$. Then*

$$E_I(d) \subset E(d) \tag{31}$$

*holds for all $d = [d_1, d_2] \in C_p$ or for $d = d_0 \in C_p \cap C_q$.*

**Proof.** Let $d = [d_1, d_2] \in C_p$ with some $p$ from (19) (i). By the assumption (19) (i) there is $\{\alpha_i\}_{i=p}^{p+m_p-1}$ such that $\sum\limits_{i=p}^{p+m_p-1} \alpha_i e_i >$ on $\Gamma_U$. According to Proposition 2.2, for any fixed $d = [d_1, d_2] \in C_p$ the couple $[u_0, v_0] := [\sum_{i=p}^{p+m_p-1} \alpha_i e_i, \sum_{i=p}^{p+m_p-1} \frac{b_{21}}{d_2 k_p - b_{22}} \alpha_i e_i]$ is a solution of the problem (13), (4), i.e. we have

$$\begin{aligned} &u_0 \in H, \int_\Omega d_1 \nabla u_0 \nabla \varphi - (b_{11} u_0 + b_{12} v_0)\varphi \, dx = 0 \text{ for all } \varphi \in H, \\ &v_0 \in H, \int_\Omega d_2 \nabla v_0 \nabla \psi - (b_{21} u_0 + b_{22} v_0)\psi \, dx = 0 \text{ for all } \psi \in H. \end{aligned} \tag{32}$$

Due to Lemma 3.2, (32) is valid if and only if $\triangle u, \triangle v \in L^2(\Omega)$, (13) holds a.e. in $\Omega$ and (4) hold in sense of the functional from Remark 3.1. Let $[u, v] \in E_I(d_1, d_2)$. We want to show that $[u, v] \in E(d_1, d_2)$. According to Lemma 3.2, it is sufficient to prove that $\frac{\partial v}{\partial n} = 0$ on $\Gamma_U$, that means $\left[\frac{\partial v}{\partial n}, \zeta\right] = 0$ for any $\zeta \in H$ such that $\zeta = 0$ on $\partial\Omega \backslash \Gamma_U$ (see Remark 3.1). Let us consider such an arbitrary fixed $\zeta$. Denote by $W^{\frac{1}{2},2}(\Gamma_U)$ the space of traces of all functions from $W^{12}(\Omega)$ (see e.g. [9]). Under the assumption that $\Gamma_U$ is a smooth manifold with a smooth boundary, we have $W^{\frac{1}{2}2}(\Gamma_U) = \overline{\mathcal{D}(\Gamma_U)}$ (closure of the set of smooth functions with a compact support in $\Gamma_U$). For $\varphi \in H$, let us denote by $T\varphi$ its trace. The traces of functions from $H$ lie in $W^{\frac{1}{2}2}(\partial\Omega)$ and therefore there exist $\omega_n \in \mathcal{D}(\Gamma_U)$, $\omega_n \to T\zeta$ in $W^{\frac{1}{2}2}(\Gamma_U)$. We can extend $\omega_n$ onto the whole boundary $\partial\Omega$ by

$$\tilde{\omega}_n := \begin{cases} \omega_n & \text{on } \Gamma_U \\ 0 & \text{on } \partial\Omega \setminus \Gamma_U. \end{cases}$$

Denoting this $\tilde{\omega}_n$ by $\omega_n$ again, we have $\omega_n \in W^{\frac{1}{2}2}(\partial\Omega)$. There is a linear continuous extension mapping $R : W^{\frac{1}{2}2}(\partial\Omega) \to W^{12}(\Omega)$ (see [9], Theorem 5.7) such that $TR\varphi = \varphi$ on $\partial\Omega$ for all $\varphi \in W^{\frac{1}{2}2}(\partial\Omega)$. Setting $\zeta_n = R\omega_n + \zeta - RT\zeta$, we get $\zeta_n \in H$, $\zeta_n \to \zeta$. Since, $v_0$ is smooth and $T\zeta_n = \omega_n$ on $\Gamma_U$, $\omega_n \in \mathcal{D}(\Gamma_U)$, for any $n$ there is $\varepsilon_n > 0$ such that $v_0 \pm \varepsilon_n \zeta_n \in K_{v_0} = \{\phi \in H;\ \phi \geq 0 \text{ on } \Gamma_U\}$ Let us set $\varphi = u_0$, $\psi = v + v_0 \pm \varepsilon_n \zeta_n$ in (18) ($v + v_0 \pm \varepsilon_n \zeta_n > 0$ on $\Gamma_U$, i.e. $\psi \in K_v$) and $\varphi = u$, $\psi = v$ in (32). Subtracting the first expressions obtained in this way from (18) and (32), then the second expressions we get

$$\int_\Omega d_2 \nabla v \nabla \zeta_n - (b_{21} u + b_{22} v)\zeta_n \, dx = 0.$$

Using the Green Formula and the second equation from (13), we get $\left[\frac{\partial v}{\partial n}, \zeta_n\right] = 0$. The limiting process gives $\left[\frac{\partial v}{\partial n}, \zeta\right] = 0$. Since $\zeta \in H$ was arbitrary such that $\zeta = 0$ on $\partial\Omega \setminus \Gamma_U$, we have $\dfrac{\partial v}{\partial n} = 0$ on $\Gamma_U$ in the sense of the functional (see Remark 3.1). Hence, $[u, v] \in E(d_1, d_2)$. In the same way we can treat the case $d = d_0 \in C_p \cap C_q$ with $p$, $q$ from in (19) (ii). $\square$

**Remark 4.6** Note that the assertion of Lemma 4.5 is proved in [8], Lemma 2.2 without any assumption that $\Gamma_U$ is a smooth manifold, but under the following stronger version of the positivity assumptions (19) :

There is a small $\varepsilon > 0$ such that one of the following conditions is fulfilled.

$$\left[\begin{array}{l} \text{(i)} \ \sum_{i=p}^{p+m_p-1} \alpha_i e_i \geq \varepsilon > 0 \quad \text{on } \Gamma_U \ \text{ for some } \ \{\alpha_i\}_{i=p}^{p+m_p-1}; \\ \text{(ii)} \ \sum_{i=p}^{q+m_q-1} \widetilde{\alpha}_i e_i \geq \varepsilon > 0 \quad \text{on } \Gamma_U \ \text{ for some } \ \{\widetilde{\alpha}_i\}_{i=p}^{q+m_q-1}. \end{array}\right. \tag{33}$$

**Lemma 4.7** *Assume that* $\Gamma_U \subset \partial\Omega$ *is a smooth manifold in* $\mathbb{R}^{\mathbb{N}-1}$ *with the smooth boundary and let (19)* (i) *or (19)* (ii) *hold with some* $p$. *Then there is an open set* $\mathcal{U} \subset \mathbb{R}^2_+$ *which contains the arc* $C_E \cap C_p$ *or the point* $d_0 = C_p \cap C_q$, *respectively, such that* $E_I(d) = \{0\}$ *for all* $d \in \mathcal{U} \cap D_S$.

**Proof** Lemma 2.3 in [8], states that under the assumptions (33)(i) or (33)(ii) for any $d_0 \in C_p \cap C_E$ or for $d_0 = C_p \cap C_q$ there exists a neighbourhood $\mathcal{U}(d_0)$ such that $E_I(d) = \{0\}$ for all $d \in \mathcal{U}(d_0) \cap D_S$. The same assertion can be proved exactly in the same way under the assumption (19) *(i)* or (19) *(ii)* , respectively, if $\Gamma_U$ is a smooth manifold with a boundary. It is sufficient to use our Lemma 4.5 instead of Lemma 2.2 in [8]. Hence, the assertion of Lemma 4.7 holds with $\mathcal{U} = \cup_{d \in C_p \cap C_E} \mathcal{U}(d)$ or $\mathcal{U}(d_0)$ in the case of the assumption (19) *(i)* or (19) *(ii)* , respectively.

**Lemma 4.8** *Let all the assumptions of Lemma 4.7 hold and let* $\mathcal{U}$ *be from that Lemma. Then*

$$\deg(I - T_0(d, 0), B_r, 0) = 0$$

*for all* $r > 0$ *and all* $d \in \mathcal{U} \cap D_S$.

**Proof** is the same as the proof of Lemma 2.4 in [8], but by using our stronger Lemma 4.7 instead of Lemma 2.3 from [8]. $\square$

**Lemma 4.9** *Let (6) hold. If* $d = [d_1, d_2] \in \mathbb{R}^2_+$, $d_1 > \dfrac{b_{11}}{k_1}$ *then* $E_I(d_1, d_2) = \{0\}$.

**Proof.** Let $d_1 > \dfrac{b_{11}}{k_1}$ hold and let $u$, $v$ be a solution of the problem (26). It is easy to see that $\dfrac{1}{k_1}$ is the largest eigenvalue of the operator $A$. In particular,

$\dfrac{d_1}{b_{11}}$ is no eigenvalue of $A$ for $d_1 > \dfrac{b_{11}}{k_1}$ and therefore $d_1 I - b_{11} A$ is invertible. Expressing $u$ from the first equation of (26) and substituting it into the second inequality we obtain

$$u = b_{12}[d_1 I - b_{11}A]^{-1}Av, \tag{34}$$

$$\langle d_2 v + Sv, \varphi - v\rangle \geq 0 \ \text{ for all } \ \varphi \in K_v, \tag{35}$$

where

$$Sv := -b_{21}b_{12}A[d_1 I - b_{11}A]^{-1}Av - b_{22}Av.$$

The operator $S : H \to H$ is linear, continuous, compact, symmetric and positive.
It follows from the assumptions (7) (i) and Lemma 3.2, that if $u$, $v$ satisfy (26) with some $[d_1, d_2] \in \mathbb{R}^2_+$ then $0 \in K_v$. Hence, choosing $\varphi = 0$ in (35) we get

$$d_2\|v\|^2 + \langle Sv, v\rangle \leq 0. \tag{36}$$

The left hand side of (36) is nonnegative and the inequality (36) holds only if $v = 0$. Due to (34), we have $u = 0$. Hence, $E_I(d_1, d_2) = \{0\}$ if $d_1 > \frac{b_{11}}{k_1}$. □

**Lemma 4.10** (See [8], Lemma 2.5). *Let (1), (6) hold. Then for any* $\delta > 0$ *there is* $M > 0$ *such that*

$$\deg(I - T_0(d, 0), B_r, 0) = 1$$

*for all* $d = [d_1, d_2]$, $d_1 > M$, $d_2 > \delta$.

The following Rabinowitz type global bifurcation theorem is a particular case of a more abstract version [15], Theorem 7, cf. [2], Theorem 2.4.

**Theorem 4.11** *Assume that* $X$ *is a Hilbert space and* $\Lambda$ *is an interval (not necessarily closed or bounded) with* $[s_1, s_2] \subseteq \Lambda$. *Let* $F : \Lambda \times X \to X$ *be a nonlinear, continuous, compact mapping such that* $F(s, 0) = 0$ *for all* $s \in \Lambda$. *Hence, for any* $s$, $U = 0$ *is a solution of the equation*

$$U - F(s, U) = 0. \tag{37}$$

*Let us assume that there are* $\varepsilon, r > 0$ *such that*

$$\text{if } \ s \in \Lambda \cap ([s_1 - \varepsilon, s_1] \cup [s_2, s_2 + \varepsilon]),\ U \in \overline{B}_r, (37) \text{ holds then } U = 0 \tag{38}$$

*and*

$$\deg(I - F(s_1,\cdot), B_r, 0) \neq \deg(I - F(s_2,\cdot), B_r, 0).$$

*Then there is a connected set* $C \subseteq \Lambda \times (X \setminus \{0\})$ *of nontrivial solutions of (37) with the foll with* $X = V \times V$, $\Lambda = \mathbb{R}_+$, $F(s,U) = T(\sigma(s),U)$, $s_1 = s^0$, $s_2 = \widehat{s}$. *owing two properties:*

*(i)* $\overline{C}$ *contains a solution of the form* $(s,0)$ *with* $s \in [s_1, s_2]$.

*(ii) At least one of the following conditions is fulfilled.*

*1.* $C$ *is unbounded.*

*2.* $\overline{C}$ *contains a point of the form* $(s,0)$ *with* $s \in \overline{\Lambda} \setminus (s_1 - \varepsilon, s_2 + \varepsilon)$.

*3.* $\overline{C}$ *contains a point of the form* $(s,U)$ *where* $U \neq 0$ *and* $s \in \partial\Lambda$.

**Proof of Theorem 3.4.** We will show that if $\mathcal{U}$ is the open set from the assertion of Lemma 4.7 then all assumptions of Theorem 4.11 are fulfilled with $X = V \times V$, $\Lambda = \mathbb{R}_+$, $F(s,U) = T(\sigma(s),U)$, $s_1 = s^0$, $s_2 = \widehat{s}$ for $s^0$, $\widehat{s}$ satisfying the assumptions of Theorem 3.4. According to Remark 4.3, the problem (17) is equivalent to (30). It is clear that $U = 0$ satisfies the equation (30) for all $s \in \mathbb{R}_+$. It follows from Lemmas 4.7, 4.9 and Lemma 4.4 that the parameters $s^0$, $\widehat{s}$ satisfying the assumptions of Theorem 3.4 are not bifurcation points of (30), i.e. the assumption (38) is fulfilled. Further, we shall show that

$$\deg(I - T(\sigma(s^0),\cdot), B_r, 0) \neq \deg(I - T(\sigma(\widehat{s}),\cdot), B_r, 0), \tag{39}$$

for $r$ small enough. It follows from (24), Lemmas 4.7, 4.9 and the homotopy invariance property of the Leray-Schauder degree that for small $r > 0$ and for all $s$ satisfying $\sigma_1(s) > \dfrac{b_{11}}{k_1}$ or $\sigma(s) \in \mathcal{U} \cap D_S$ we get

$$\deg(I - T(\sigma(s),\cdot), B_r, 0) = \deg(I - T_0(\sigma(s),\cdot), B_r, 0) \tag{40}$$

(in particular, the degrees are defined). Hence it is sufficient to show that

$$\deg(I - T_0(\sigma(s^0),\cdot), B_r, 0) \neq \deg(I - T_0(\sigma(\widehat{s}),\cdot), B_r, 0) \tag{41}$$

instead of (39). Due to Lemma 4.9, for $s^0 : \sigma_1(s^0) > \dfrac{b_{11}}{k_1}$ the equation

$$U - T_0(\sigma(s^0), U) = 0 \tag{42}$$

has only trivial solution. Consequently, it follows from the homotopy invariance of the Leray-Schauder degree and Lemma 4.10 that

$$\deg(I - T_0(\sigma(s^0),\cdot), B_r, 0) = 1. \tag{43}$$

Let $\mathcal{U}$ be the open set from the assertion of Lemma 4.7 . Using Lemma 4.8 and the assumption $\sigma(s) \in \mathcal{U}$ we get

$$\deg(I - T_0(\sigma(\widehat{s}), \cdot), B_r, 0) = 0. \tag{44}$$

So, it follows from (40), (43), (44) that (41) holds, i.e. we have (39) for $r > 0$ small enough. Hence, we can apply Theorem 4.11, where $\Lambda = \mathbb{R}_+$, $X = V \times V$, $F(s, U) = T(\sigma(s), U)$, $s_1 = s^0$, $s_2 = \widehat{s}$. According to Theorem 4.11, there is a connected set $\mathcal{F} \subseteq S$ such that $\overline{\mathcal{F}}$ contains a point of the form $[s_B; 0, 0]$ with $s_B \in (s^0, \widehat{s})$. In particular, $s_B$ is a bifurcation point of the problem (17). Due to Lemma 4.4 and Lemma 4.9, it must be $\sigma_1(s_B) \leq \dfrac{b_{11}}{k_1}$. In particular, $s_B$ is a bifurcation point of the problem (17). Also, the set $\mathcal{F}$ is either unbounded or $\overline{\mathcal{F}}$ contains a point of the type $[s; 0, 0]$ with $s \in \overline{\mathbb{R}}_+ \setminus [s^0, \widehat{s}]$ or a point of the type $[0; u, v]$, $|||u||| + |||v||| \neq 0$. This completes the proof of Theorem 3.4. □

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