



Bochner-Kaehler metrics and connections of Ricci type

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Abstract

We apply the results from [CS] about special symplectic geometries to the case of Bochner-Kaehler metrics. We obtain a (local) classification of these based on the orbit types of the adjoint action in $\mathfrak{su}(n, 1)$. The relation between Sasaki and Bochner-Kaehler metrics in cone and transversal metrics constructions is discussed. The connection of the special symplectic and Weyl connections is outlined. The duality between the Ricci-type and Bochner-Kaehler metrics is shown.

Keywords: Bochner-Kähler metric, Sasaki metric, Ricci type connection, Weyl structure.

1 Bochner-Kaehler metrics

The curvature tensor of the Levi-Civita connection of a Kaehler metric g decomposes (under the action of $\mathfrak{u}(n)$) into its Ricci and Bochner part ([Bo]). The metric is said to be Bochner-Kaehler, iff the Bochner part of its curvature tensor vanishes.

A remarkable relationship was revealed among following types of geometric structures in the article [CS]: manifolds with a connection of Ricci type, manifolds with a connection with the special symplectic holonomy, pseudo-Riemannian Bochner-Kähler structures, manifolds with a Bochner-bi-Lagrangian connection. All these geometric objects are instantons of the same construction, and they are called special symplectic geometries. The word "symplectic" comes from the fact that they all carry a symplectic connection; special stands for the common special type of the curvature of the connection: let (M, ω) be a symplectic manifold. Then the curvature of the special symplectic geometries is of the form

$$R_h(X, Y) = 2\omega(X, Y)h + X \circ (hY) - Y \circ (hX), \quad (1)$$

where $\mathfrak{h} \subset \mathfrak{sp}(n, \mathbb{R})$ (or $\mathfrak{sp}(n, \mathbb{C})$) is a Lie algebra, $h \in \mathfrak{h}$, $\circ : S^2(TM) \rightarrow \mathfrak{h}$, is an \mathfrak{h} -equivariant product with special properties (see [CS]). For Bochner-Kaehler structures the special form of curvature translates as follows: let (M, g, J, ω) be a Kaehler structure on a manifold M . That is J is the orthogonal complex structure which is parallel with respect to the Levi-Civita connection of g , and the Kaehler form ω is defined by $\omega(x, y) = g(x, Jy)$. The Kaehler structure is Bochner-Kaehler

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iff the curvature of the Levi-civita connection of the metric has the above form, where $\mathfrak{h} = \mathfrak{u}(n)$, and \circ is given as:

$$(X \circ Y)Z = \omega(X, Z)Y + \omega(Y, Z)X + \omega(JX, Z)JY + \omega(JY, Z)JX + \omega(JX, Y)JZ. \quad (2)$$

That is iff the curvature is of the form

$$\begin{aligned} R_\rho(X, Y) &= 2g(X, JY)\rho + 2g(X, \rho Y)J + (\rho Y \wedge JX) \\ &\quad - (\rho X \wedge JY) + (X \wedge J\rho Y) - (Y \wedge J\rho X), \end{aligned} \quad (3)$$

where $(X \wedge Y)Z = g(X, Z)Y - g(Y, Z)X$.

2 Kaehler and Sasaki manifolds

The following considerations are motivated by the lecture given by Krzysztof Galicki at Winter School of Geometry and Physics, Srní, Czech republic, 2004. ([BG]).

Sasaki metric. One of the possible (equivalent) definition of the *Sasakian manifold* C is that it is a Riemannian manifold with the metric g , on which there exists a unit length Killing vector field ξ such that the curvature tensor R of the Levi-Civita connection ∇ of g satisfies:

$$R(X, \xi)Y = g(\xi, Y)X - g(X, Y)\xi.$$

The one form λ dual to ξ defines a contact distribution $\mathfrak{D} = \{X \in TC \mid \lambda(X) = g(X, \xi) = 0\}$. The vector field ξ is called the characteristic vector field of the contact distribution \mathfrak{D} .

Transversal Kaehler metric. Further consider J defined by $J(X) = -\nabla_X \xi$. It is an automorphism of the tangent bundle TC and its restriction to \mathfrak{D} gives rise to a complex structure J on \mathfrak{D} . Then $(\nabla_X J)(Y) = 0$ for $X, Y \in \mathfrak{D}$ and thus there is a so called *transversal Kaehler structure* on \mathfrak{D} . The Kaehler structure then factorizes to the set of leaves of the foliation generated by ξ (the characteristic foliation) if this is locally an orbifold. See [BG] for details.

Conversely, the transversal Kaehler structure on a compact distribution \mathfrak{D} (given as $\mathfrak{D} = \{X \mid \lambda(X) = 0, \lambda \in \Omega^1(C)\}$) on a manifold C translates to a Sasakian structure on C : given a metric $g_{\mathfrak{D}}$ on \mathfrak{D} with a parallel complex structure J on \mathfrak{D} , and a transversal symmetry ξ of \mathfrak{D} , one extends $g_{\mathfrak{D}}$ to the whole of TC with $g(X, Y) = g_{\mathfrak{D}}(X, Y)$ for $X, Y \in \mathfrak{D}$, $g(\xi, \xi) = 1$ and $g(\xi, X) = 0$.

Unlike a general Kaehler metric, any Bochner-Kaehler metric can be realized (locally) as the transverse metric of an appropriate Sasaki metric. The Theorem B from [CS] (see also 3.4) says, that given a simply connected manifold M with a Bochner-Kaehler metric g there is always a principal T -bundle $\pi : C \rightarrow M$, where T is a one-dimensional Lie group, and this bundle carries a connection whose curvature equals -2ω , where ω is the Kaehler form corresponding to g . The horizontal distribution of the connection yields a contact distribution on C . Thus we have following:

Proposition 2.1 *Let M be a $2n$ -dimensional (real) manifold with Bochner-Kaehler metric g , J be the corresponding complex structure. Then there exists a Sasaki manifold such that the set of leaves of the characteristic foliation is isomorphic (together with from the Sasaki one induced structure) to some cover of M (with the Bochner-Kaehler structure induced from M). \square*

Cone metric. On the other hand, a manifold C is Sasakian if and only if the "cone metric" $(t^2 \cdot g + (dt)^2)$ on $C \times \mathbb{R}_+$ is Kaehler, where the complex structure J' on the cone is the extension of J such that

$$J'(\xi) = t\partial_t, \quad J'(\partial_t) = -\frac{1}{t}\xi, \quad (4)$$

$\xi \in \mathfrak{X}(\widehat{C})$ being the lift of the characteristic vector field ξ on C .

Following the ideas from [BG], there arise questions what happens if we require the transversal metric to be Bochner-Kaehler. What special has to be the Sasaki metric on C ? Will then the "cone metric" be Bochner-Kaehler?

Let us state a technical lemma about Bochner-Kaehler manifolds.

Lemma 2.2 *Let M be a manifold with a Bochner-Kaehler metric g and let X_0 be a non-vanishing vector field on M such that the curvature R of the Levi-Civita connection satisfies $R(X_0, JX_0) \equiv 0$. Then M is flat.*

Proof. The curvature of the Levi-Civita connection of the Bochner-Kaehler metric is of the form R_ρ , $\rho \in \mathfrak{u}(n)$ (see (3)), the vector field X_0 can be normed to the unit length and we can write

$$0 = R(X_0, JX_0) = -2\rho + 2g(X_0, \rho JX_0)J - 2X_0 \wedge \rho X_0 - 2JX_0 \wedge \rho JX_0,$$

that is

$$\rho = g(X_0, \rho JX_0)J - X_0 \wedge \rho X_0 - JX_0 \wedge \rho JX_0. \quad (5)$$

Applying ρ to X_0 we get

$$\begin{aligned} \rho X_0 &= g(X_0, \rho JX_0)JX_0 - g(X_0, X_0)\rho X_0 + g(\rho X_0, X_0)X_0 - g(JX_0, X_0)\rho JX_0 + g(\rho JX_0, X_0)JX_0 \\ &= 2g(X_0, \rho JX_0)JX_0 - \rho X_0, \end{aligned}$$

and we have

$$\rho X_0 = g(X_0, \rho JX_0)JX_0. \quad (6)$$

that means $\rho X_0 = cJX_0$ for a real valued function c on M . Substituing back to (6) we get

$$cJX_0 = g(X_0, J(cJX_0))J = -cJX_0,$$

and $c = 0$. That is $\rho X_0 = 0$, and $0 = \rho JX_0 = J\rho X_0$, and the formula (5) implies $\rho = 0$, that is the curvature vanishes. \square

Proposition 2.3 *Let M be a $2n$ -dimensional (real) manifold with Bochner-Kaehler metric g , J be the corresponding complex structure, further let C be the Sasakian manifold from the theorem (2.1) with the dimension $2n+1$. Then the manifold $\widehat{C} = C \times \mathbb{R}_+$ with the complex structure defined by (4) is Bochner Kaehler if and only if M is locally isomorphic (as the Kaehler structure) to the complex projective space $\mathbb{C}P^n$. The cone \widehat{C} is then a flat manifold.*

Proof. The existence of the given manifold is just consequence of the proposition [?] and the cone construction. We will write \bar{g} and \hat{g} for the metrics on C and \hat{C} respectively, and $\bar{\nabla}$, $\hat{\nabla}$ for the corresponding Levi-Civita connection. The curvature of κ gives us

$$[\bar{X}, \bar{Y}] = \bar{[X, Y]} - 2\omega(X, Y)\xi,$$

for $X, Y \in \mathfrak{X}(M)$ and since $[\xi, X] = 0$ there is $\bar{\nabla}_\xi X = \bar{\nabla}_X \xi$. That gives for the torsion of the connection $\bar{\nabla}$

$$\begin{aligned} 0 = T(\bar{X}, \bar{Y}) &= \bar{\nabla}_{\bar{X}} \bar{Y} - \bar{\nabla}_{\bar{Y}} \bar{X} - [\bar{X}, \bar{Y}] = \bar{\nabla}_{\bar{X}} \bar{Y} - \bar{\nabla}_{\bar{Y}} \bar{X} - \bar{[X, Y]} + 2\omega(X, Y)\xi \\ &= \bar{\nabla}_{\bar{X}} \bar{Y} - \bar{\nabla}_{\bar{Y}} \bar{X} - \bar{\nabla}_X \bar{Y} + \bar{\nabla}_Y \bar{X} + 2\omega(X, Y)\xi \end{aligned}$$

On the other hand

$$\begin{aligned} 0 = \xi \bar{g}(\bar{X}, \bar{Y}) &= \bar{g}(\bar{\nabla}_\xi \bar{X}, \bar{Y}) + \bar{g}(\bar{X}, \bar{\nabla}_\xi \bar{Y}) \\ &= X(\bar{g}(\xi, \bar{Y})) - \bar{g}(\xi, \bar{\nabla}_X \bar{Y}) + \bar{Y}(\bar{g}(\bar{X}, \xi)) - \bar{g}(\bar{\nabla}_Y \bar{X}, \xi) \\ &= -\bar{g}(\xi, \bar{\nabla}_X \bar{Y} + \bar{\nabla}_Y \bar{X}). \end{aligned}$$

We conclude that $\bar{\nabla}_X \bar{Y} + \bar{\nabla}_Y \bar{X} = 0$ and

$$\bar{\nabla}_X \bar{Y} = \bar{\nabla}_X \bar{Y} - \omega(X, Y)\xi. \quad (7)$$

Further

$$0 = \bar{X} \bar{g}(\xi, \xi) = 2\bar{g}(\bar{\nabla}_X \xi, \xi),$$

and

$$\begin{aligned} 0 = \bar{X} \bar{g}(\xi, \bar{Y}) &= \bar{g}(\bar{\nabla}_X \xi, \bar{Y}) + \bar{g}(\xi, \bar{\nabla}_X \bar{Y}) \\ &= \bar{g}(\bar{\nabla}_X \xi, \bar{Y}) - \omega(X, Y) \\ &= g(\pi(\bar{\nabla}_X \xi), Y) + g(JX, Y). \end{aligned}$$

That is

$$\bar{\nabla}_X \bar{X} = \bar{\nabla}_X \xi = -JX. \quad (8)$$

With the like-wise computation one gets $\bar{\nabla}_\xi \xi = 0$.

Similarly one gets for $X, Y \in \mathfrak{X}(C)$

$$\hat{\nabla}_{\hat{X}} \hat{Y} = \widehat{\bar{\nabla}_X Y} - tg(X, Y)\partial_t, \quad \text{especially } \hat{\nabla}_{\hat{\xi}} \hat{\xi} = -t\partial_t,$$

and

$$\hat{\nabla}_{\hat{X}} \partial_t = \hat{\nabla}_{\partial_t} \hat{X} = \frac{1}{t} \hat{X}, \quad \text{especially } \hat{\nabla}_{\partial_t} \hat{\xi} = \frac{1}{t} \hat{\xi} \quad \text{and} \quad \hat{\nabla}_{\partial_t} \partial_t = 0.$$

There is the canonical projection $\pi : \hat{C} = C \times \mathbb{R}_+ \rightarrow C \rightarrow M = C/T$. From the above two equations we see, that the fibres of this projection sit totally geodesically in \hat{C} and the metric on the fiber gives the "true" 2-dimensional cone. That is the restriction of the metric to the fibers is flat

and we have $\widehat{R}(\partial_t, J(\partial_t))\widehat{\xi} = 0$ and $\widehat{R}(\partial_t, J(\partial_t))\partial_t$. It remains to compute $\widehat{R}(\partial_t, J(\partial_t))$ on the lifts of the vectors in TM . We will write \widehat{X} for a lift of a vector $X \in TM$ with respect to the projection π .

$$\begin{aligned}\widehat{R}(\partial_t, J(\partial_t))\widehat{X} &= -\frac{1}{t} \left(\widehat{\nabla}_{\partial_t} \widehat{\nabla}_{\widehat{\xi}} \widehat{X} - \widehat{\nabla}_{\widehat{\xi}} \widehat{\nabla}_{\partial_t} \widehat{X} - \widehat{\nabla}_{[\partial_t, \widehat{\xi}]} \widehat{X} \right) \\ &= -\frac{1}{t} \left(\widehat{\nabla}_{\partial_t} (-\widehat{JX}) - \widehat{\nabla}_{\widehat{\xi}} \left(\frac{1}{t} \widehat{X} \right) \right) \\ &= -\frac{1}{t} \left(-\frac{1}{t} \widehat{JX} + \frac{1}{t} \widehat{JX} \right) = 0.\end{aligned}$$

Consequently $\widehat{R}(\partial_t, J(\partial_t)) = 0$ and according to the previous lemma 2.2 the metric on \widehat{C} is flat.

There is the following known relation between the curvature \overline{R} of the metric \overline{g} on C and \widehat{R} , the curvature of the Levi-Civita connection $\widehat{\nabla}$ of the cone metric on \widehat{C} :

$$\widehat{R}(\widehat{X}, \widehat{Y})\widehat{Z} = \overline{R}(X, Y)Z + \overline{g}(X, Z)Y - \overline{g}(Y, Z)X,$$

for $X, Y, Z \in T(C)$, which can be checked with an easy computation exercise.

This shows, that C is a manifold with constant sectional curvature $K = 1$ (see [KN]), that is locally a unit sphere.

Now the vector field ξ on C is from the construction a Killing one, and has the constant unit length that is the leaves of the foliation are the circles coming from the natural \mathbb{C} action on the sphere (at least locally; the unit length of the Killing field excludes other circle actions). Then the resulting factor space, that is M , is locally isomorphic to the complex projective space. \square

3 General construction

Let us quickly review the construction from [CS], which gives rise to all special symplectic geometries. All manifolds with special symplectic connection are locally isomorphic to the factor manifold of the oriented projectivization of the cone $\mathcal{C} = Ad_G x \subset \mathfrak{g}$, where x is an appropriate element in the parabolic 2-gradable Lie algebra \mathfrak{g} , where we factor along the flow of the convenient vector field. The special symplectic connection is then induced on the factor from one of the components of the Maurer-Cartan form on \mathfrak{g} , which decomposes due to the 2-grading.

Some of the standard notions from the theory of contact structures are used without definitions. The reader can find them and all the proofs of the theorems stated in this section, in [CS].

Symplectic algebra as subalgebra of a 2-graded algebra. Let V be a vector space (either real or complex) with a symplectic form ω . Let $\mathfrak{h} \subset \mathfrak{sp}(V, \omega) = \{h \in \text{End}(V) | \omega(x, y) + \omega(x, hy) = 0 \text{ for all } x, y \in V\}$ such, that there exists an \mathfrak{h} -equivariant map $\circ : S^2(V) \rightarrow \mathfrak{h}$ and an $\text{ad}_{\mathfrak{h}}$ -invariant inner product $(,)$ which satisfy the following identities:

$$\begin{aligned}(h, x \circ y) &= \omega(hx, y) = \omega(hy, x) \\ (x \circ y)z - (x \circ z)y &= 2\omega(y, z)x - \omega(x, y)z + \omega(x, z)y,\end{aligned}$$

for all $x, y, z \in V$ and $h \in \mathfrak{h}$.

Then there exists a unique simple Lie algebra \mathfrak{g} with a 2-grading of the parabolic type, that is

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2,$$

where \mathfrak{g}^{-2} and \mathfrak{g}^2 are one-dimensional. The grading corresponds to \mathfrak{h} in the following sense:

$$\mathfrak{g}^{ev} := \mathfrak{g}^{-2} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^2 \cong \mathfrak{sl}_{\alpha_0} \oplus \mathfrak{h} \quad \text{and} \quad \mathfrak{g}^{odd} := \mathfrak{g}^{-1} \oplus \mathfrak{g}^1 \cong \mathbb{F}^2 \otimes V \quad \text{as a } \mathfrak{g}^{ev}\text{-module,}$$

where \mathfrak{g}^2 , resp. \mathfrak{g}^{-2} , are root spaces of a long root α_0 , resp $-\alpha_0$, and \mathfrak{sl}_{α_0} is the Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{F})$ generated by the root spaces and the corresponding coroot H_{α_0} which lies in \mathfrak{g}^0 . We will also write $\mathfrak{p} = \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2$ for the parabolic subalgebra of \mathfrak{g} and $\mathfrak{p}_0 := \mathfrak{h} \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2$. Let further P and P_0 be corresponding connected subgroups of G .

Further we fix a non-zero \mathbb{F} -bilinear area form $a \in \Lambda^2(\mathbb{F}^2)^*$. There is a canonical $\mathfrak{sl}(2, \mathbb{F})$ -equivariant isomorphism

$$S^2(\mathbb{F}^2) \longrightarrow \mathfrak{sl}(2, \mathbb{F}), \quad (ef) \cdot g := a(e, g)f + a(f, g)e \quad \text{for all } e, f, g \in \mathbb{F}^2, \quad (9)$$

and under this isomorphism, the Lie bracket on $\mathfrak{sl}(2, \mathbb{F})$ is given by

$$[ef, gh] = a(e, g)fh + a(e, h)fg + a(f, g)eh + a(f, h)eg. \quad (10)$$

Thus, if we fix a basis $e_+, e_- \in \mathbb{F}^2$ with $a(e_+, e_-) = 1$, then we have the identifications

$$H_{\alpha_0} = -e_+e_-, \quad \mathfrak{g}^{\pm 2} = \mathbb{F}e_{\pm}^2, \quad \mathfrak{g}^{\pm 1} = e_{\pm} \otimes V.$$

The cone in 2-gradable algebra and its projectivization. Using the Cartan-Killing form (up to the multiple) we identify \mathfrak{g} and \mathfrak{g}^* , and we define the root cone $\hat{\mathcal{C}}$ and its (oriented) projectivization \mathcal{C} as follows:

$$\hat{\mathcal{C}} := G \cdot e_+^2 \subset \mathfrak{g} \cong \mathfrak{g}^*, \quad \mathcal{C} := p(\hat{\mathcal{C}}) \subset \mathbb{P}^o(\mathfrak{g}) \cong \mathbb{P}^o(\mathfrak{g}^*),$$

where $\mathbb{P}^o(\mathfrak{g})$ is the set of *oriented* lines in \mathfrak{g} , i.e. $\mathbb{P}^o \cong S^d$ if $\mathbb{F} = \mathbb{R}$, and $\mathbb{P}^o \cong \mathbb{C}\mathbb{P}^d$ if $\mathbb{F} = \mathbb{C}$, where $d = \dim \mathfrak{g} - 1$, and where $p : \mathfrak{g} \setminus 0 \rightarrow \mathbb{P}^o(\mathfrak{g})$ is the principal \mathbb{R}^+ -bundle (\mathbb{C}^* -bundle, respectively) defined by the canonical projection. Thus, the restriction $p : \hat{\mathcal{C}} \rightarrow \mathcal{C}$ is a principal bundle as well.

Contact structure on the projectivized cone. Being a coadjoint orbit, $\hat{\mathcal{C}}$ carries a canonical G -invariant symplectic structure Ω . Moreover, the *Euler vector field* defined by

$$E_0 \in \mathfrak{X}(\hat{\mathcal{C}}), \quad (E_0)_v := v$$

generates the principal action of p and satisfies $\mathfrak{L}_{E_0}(\Omega) = \Omega$, so that the distribution

$$\mathcal{D} = dp(E_0^{\perp \Omega}) \subset T\mathcal{C} \quad (11)$$

yields a G -invariant contact distribution on \mathcal{C} , see [CS], Proposition 3.2.

The cone as homogeneous space. Let $\lambda = \iota_{E_0}(\Omega)$. Then we define the bundle \mathfrak{R} :

$$\mathfrak{R} := \{(\lambda, \hat{\xi}) \in \hat{\mathcal{C}} \times T\hat{\mathcal{C}} \subset T^*\mathcal{C} \times T\hat{\mathcal{C}} \mid \lambda(dp(\hat{\xi})) = 1\}.$$

Let P and P_0 be subgroups of G corresponding to the subalgebras \mathfrak{p} and \mathfrak{p}_0 of \mathfrak{g} .

Lemma 3.1 *As homogeneous spaces, we have $\mathcal{C} = G/P$, $\hat{\mathcal{C}} = G/P_0$, and $\mathfrak{R} = G/H$.*

Proof. See [CS] □

Transversal symmetry defines the geometry

For each $a \in \mathfrak{g}$ we define the vector fields $a^* \in \mathfrak{X}(\mathcal{C})$ and $\hat{a}^* \in \mathfrak{X}(\hat{\mathcal{C}})$ corresponding to the infinitesimal action of a , i.e.

$$(a^*)_{[v]} := \left. \frac{d}{dt} \right|_{t=0} (\exp(ta) \cdot [v]) \quad \text{and} \quad (\hat{a}^*)_v := \left. \frac{d}{dt} \right|_{t=0} (\exp(ta) \cdot v). \quad (12)$$

Note that a^* is a contact symmetry (with respect to the canonical contact distribution on $\hat{\mathcal{C}}$), and \hat{a}^* is its Hamiltonian lift. Let

$$\hat{\mathcal{C}}_a := \{\lambda \in \hat{\mathcal{C}} \mid \lambda(a^*) \in \mathbb{R}^+(\in \mathbb{C}^*, \text{ respectively})\} \quad \text{and} \quad \mathcal{C}_a := p(\hat{\mathcal{C}}_a) \subset \mathcal{C}, \quad (13)$$

so that $p : \hat{\mathcal{C}}_a \rightarrow \mathcal{C}_a$ is a principal \mathbb{R}^+ -bundle (\mathbb{C}^* -bundle, respectively) and the restriction of a^* to \mathcal{C}_a is a positively transversal contact symmetry. Then there exists a unique section λ of the bundle $p : \hat{\mathcal{C}}_a \rightarrow \mathcal{C}_a$ such that $\lambda(a^*) = 1$ and therefore, we obtain the section

$$\sigma_a : \mathcal{C}_a \longrightarrow \mathfrak{X} = \mathbb{G}/\mathbb{H}, \quad \sigma_a(u) := (\lambda(u), \hat{a}^*(u)) \in \mathfrak{X}. \quad (14)$$

Let $\pi : \mathbb{G} \rightarrow \mathbb{G}/\mathbb{H} = \mathfrak{X}$ be the canonical projection, and let $\Gamma_a := \pi^{-1}(\sigma_a(\mathcal{C}_a)) \subset \mathbb{G}$. The restriction $\pi : \Gamma_a \rightarrow \sigma_a(\mathcal{C}_a) \cong \mathcal{C}_a$ is then a principal \mathbb{H} -bundle.

Theorem 3.2 *Let $a \in \mathfrak{g}$ be such that $\mathcal{C}_a \subset \mathcal{C}$ from (13) is non-empty, define $a^* \in \mathfrak{X}(\mathcal{C})$ and $\hat{a}^* \in \mathfrak{X}(\hat{\mathcal{C}})$ as in (12), and let $\pi : \Gamma_a \rightarrow \mathcal{C}_a$ with $\Gamma_a \subset \mathbb{G}$ be the principal \mathbb{H} -bundle from above. Then there are functions $\rho : \Gamma_a \rightarrow \mathfrak{h}$, $u : \Gamma_a \rightarrow V$, $f : \Gamma_a \rightarrow \mathbb{F}$ such that*

$$\text{Ad}_{g^{-1}}(a) = \frac{1}{2}e_-^2 + \rho + e_+ \otimes u + \frac{1}{2}fe_+^2 \quad (15)$$

for all $g \in \Gamma_a$.

The restriction of the $\mu_{\mathfrak{h}} + \mu_{-1} + \mu_{-2}$ part of the Maurer-Cartan form to Γ_a yields a pointwise linear isomorphism $T\Gamma_a \rightarrow \mathfrak{h} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{-2}$, and we can further decompose it as

$$\mu_{\mathfrak{h}} + \mu_{-1} + \mu_{-2} = -2\kappa \left(\frac{1}{2}e_-^2 + \rho \right) + e_- \otimes \theta + \eta, \quad \kappa \in \Omega^1(\Gamma_a), \quad \theta \in \Omega^1(\Gamma_a) \otimes V, \quad \eta \in \Omega^1(\Gamma_a) \otimes \mathfrak{h}.$$

Theorem 3.3 *Let $a \in \mathfrak{g}$ and $\mathcal{C}_a \subset \mathcal{C}$ as before. Let $U \subset \mathcal{C}_a$ be a regular open subset, i.e. the local quotient $M_U := \Gamma_a^{\text{loc}} \setminus U$ is a manifold, where*

$$\Gamma_a := \exp(\mathbb{F}a) \subset \mathbb{G}.$$

Let $\omega \in \Omega^2(M)$ be the unique symplectic form on M_U , such that $\pi^(\omega) = -2d(E_0 \lrcorner \Omega)$. Then M_U carries a canonical special symplectic connection associated to \mathfrak{g} , and the (local) principal Γ_a -bundle $\pi : U \rightarrow M$ admits a connection $\kappa \in \Omega^1(U)$ whose curvature is given by $d\kappa = \pi^*(\omega)$.*

Proof. Sketch: The connections forms of the desired connections are projections of the forms η and κ on Γ_a over U to the corresponding H -bundle over M_U . \square

Conversely, any manifold with special symplectic connection comes in this way (locally). Namely, there is the following theorem (Theorem B from [CS]):

Theorem 3.4 *Let (M, ω) be a symplectic manifold with a special symplectic connection of class C^4 , and let \mathfrak{g} be the Lie algebra associated to the special symplectic condition as described at the beginning of this section.*

- i) *Then there is a principal \hat{T} -bundle $\hat{M} \rightarrow M$, where \hat{T} is a one dimensional Lie group which is not necessarily connected, and this bundle carries a principal connection with curvature ω .*
- ii) *Let $T \subset \hat{T}$ be the identity component. Then there is an $a \in \mathfrak{g}$ such that $T \equiv T_a \subset G$, and a T_a -equivariant local diffeomorphism $\hat{i} : \hat{M} \rightarrow C_a$ which for each sufficiently small open subset $V \subset \hat{M}$ induces a connection preserving diffeomorphism $\iota : T^{loc}\mathcal{V} \rightarrow T^{loc}U = M_U$, where $U := \hat{i}(V) \subset C_a$ and M_U carries the connection from 3.3.*

\square

4 Construction of Bochner-Kaehler metrics

4.1 A little of linear algebra

Let us first recall some facts from the linear algebra. Let V be a complex $(n+1)$ -dimensional vector space, h a hermitian form of the signature $(n, 1)$ on V . The (real) Lie algebra $\mathfrak{u}(n, 1)$ is defined as follows:

$$\mathfrak{u}(n, 1) := \{A \in \text{End}(V) | h(Av, w) + h(v, Aw) = 0, v, w \in V\}.$$

It is the Lie algebra of the Lie group

$$U(n, 1) = \{A \in \text{Aut}(V) | h(Av, Aw) = h(v, w), v, w \in V\}.$$

Take matrices with the determinant 1 in $U(n, 1)$ or traceless matrices in $\mathfrak{u}(n, 1)$ to get something special.

Example 4.1 *We consider the standard hermitian form on the complex space \mathbb{C}^{n+1} of signature $(n, 1)$ ($h(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n x_i \bar{y}_i - x_{n+1} \bar{y}_{n+1}$, for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^{n+1}$). Then as a matrix algebra $\mathfrak{su}(n, 1)$ can be written as*

$$\mathfrak{su}(n, 1) = \left\{ \left(\begin{array}{c|c} A & \mathbf{v} \\ \hline \mathbf{v}^* & -\text{tr} A \end{array} \right), A \in \mathfrak{u}(n), \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{C}^n \right\}.$$

The elements of the bundle Γ_a from the general construction in the previous section, can be described with the structure functions from (3.2) as follows:

$$\begin{pmatrix} \rho - \frac{1}{n+2}(\text{tr} \rho)\mathbb{I}_n & u & u \\ -u^* & -\frac{1}{2}(\text{tr} \rho - i(f+1)) & \frac{i}{2}(f-1) \\ u^* & \frac{i}{2}(1-f) & -\frac{1}{2}(\text{tr} \rho + i(f+1)) \end{pmatrix}. \quad (16)$$

The grading of $\mathfrak{su}(n, 1) = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2 \equiv \mathbb{R}e_-^2 \oplus e_- \otimes V \oplus (\mathfrak{h} \oplus \mathbb{R}e_+e_-) \oplus e_+ \otimes V \oplus \mathbb{R}e_+^2$ is given as follows (all the matrices are $(n+1) \times (n+1)$ ones):

$$e_{\pm}^2 = \left(\begin{array}{c|cc} \mathbf{O} & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ \hline 0 \dots 0 & i & \pm i \\ 0 \dots 0 & \mp i & -i \end{array} \right), \quad H_{\alpha_0} = e_+e_- = \left(\begin{array}{c|cc} \mathbf{O} & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ \hline 0 \dots 0 & 0 & 1 \\ 0 \dots 0 & 1 & 0 \end{array} \right),$$

$$\mathfrak{h} = \left\{ \left(\begin{array}{c|cc} \mathbf{A} & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ \hline 0 \dots 0 & -\frac{1}{2} \text{tr } A & 0 \\ 0 \dots 0 & 0 & -\frac{1}{2} \text{tr } A \end{array} \right), A \in \mathfrak{u}(n-1) \right\},$$

$$\mathfrak{g}^{\pm 1} = \left\{ \left(\begin{array}{c|cc} \mathbf{O} & \mathbf{v} & \pm \mathbf{v} \\ \hline -\mathbf{v}^* & 0 & 1 \\ \pm \mathbf{v}^* & 1 & 0 \end{array} \right), \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} \in \mathbb{C}^{n-1} \right\}.$$

The action of the algebra $\mathfrak{h} = \mathfrak{u}(n-1)$ on V is then given as the adjoint matrix action and one easily computes that for $\rho \in \mathfrak{u}(n-1)$ there is

$$\rho \cdot u = \rho u + \frac{1}{2} \text{tr}(\rho)u \quad (17)$$

The hermitian form h is uniquely determined either by its real part g (real valued symmetric bilinear form on V) or by its imaginary part ω , the antisymmetric real valued form on V . ($\omega(x, y) = g(x, Jy)$, where J is the complex structure on V).

Lemma 4.2 *There is a $U(n, 1)$ -equivariant map $m : V \rightarrow \mathfrak{u}(n, 1) : x \mapsto x \wedge Jx$, where*

$$(x \wedge Jx)z = g(x, z)Jx - g(Jx, z)x,$$

and the $U(n, 1)$ action on $\mathfrak{u}(n, 1)$ is given by ad representation.

Proof. The morphism $(x \wedge Jx)$ is in $\mathfrak{u}(n, 1)$:

$$\begin{aligned} g((x \wedge Jx)y, z) + g(y, (x \wedge Jx)z) &= g(g(x, y)Jx - g(Jx, y)x, z) + g(y, g(x, z)Jx - g(Jx, z)x) \\ &= g(x, y)g(Jx, z) - g(Jx, y)g(x, z) + g(y, Jx)g(x, z) - \\ &\quad - g(y, x)g(Jx, z) \\ &= 0, \end{aligned}$$

for any $x, y, z \in V$. For the $A \in U(n, 1) \subset GL(n+1, \mathbb{C})$ there is:

$$\begin{aligned} (\text{Ad}_A \circ m(x))y &= A \circ m(x) \circ A^{-1}(y) = A \circ (g(x, A^{-1})Jx - g(Jx, A^{-1})x) \\ &= g(x, A^{-1}y)AJx - g(Jx, A^{-1}y)Ax = g(Ax, y)AJx - g(AJx, y)Ax \\ &= g(Ax, y)JAx - g(JAx, y)Ax \\ &= (m \circ \text{Ad}_A(x))(y), \end{aligned}$$

where we have used the invariance of g with respect to the morphisms from $U(n, 1)$. \square

Remark 4.3 *i) The image of the morphism m are from the definition rank one morphisms.*

ii) The value of the morphism $x \wedge Jx$ on a vector z is actually $i\overline{h(x, z)}x$, but we stick to write it in the form $g(x, z)Jx - g(Jx, z)x$, which comes from the morphism $x \wedge y$: $(x \wedge y)z = g(x, z)y - g(y, z)x$.

Lemma 4.4 *The morphism m is not injective: $x \wedge Jx$ and $y \wedge Jy$ determine the same element in $\mathfrak{u}(n, 1)$ if and only if $x = e^{ik}y$, $k \in \mathbb{R}$, $x, y \in V$.*

Proof. If $x \wedge Jx$ and $y \wedge Jy$ determine the same morphism of V then x and y lie on the same complex line, that is $y = ax + bJx$, $a, b \in \mathbb{R}$. Then there is

$$y \wedge Jy = (ax + bJx) \wedge (-bx + aJx) = (a^2 + b^2)x \wedge Jx, \quad (18)$$

that is x and y differ by a multiple of a complex unit. \square

Lemma 4.5 *A morphism $x \wedge Jx$ lies in $\mathfrak{su}(n, 1)$ iff $g(x, x) = 0$.*

Proof. The trace of the rank one morphism is equal to the eigenvalue of a non-zero eigenvector which lies in the image line. For the morphism $x \wedge Jx$ we take the eigenvector x . Then

$$(x \wedge Jx)x = g(x, x)Jx - g(Jx, x)x = g(x, x)Jx = ig(x, x)x, \quad (19)$$

the eigenvalue of the eigenvector x is $ig(x, x)$, which gives the result. \square

Lemma 4.6 *There are two orbits of the adjoint action of $SU(n, 1)$ on rank 1 matrices in $\mathfrak{su}(n, 1)$. For a given vector $x \in V$, the morphism $x \wedge Jx$ lies in one of the orbits, the morphism $-x \wedge Jx$ in the other one.*

Proof. Any rank 1 morphism in $\mathfrak{su}(n, 1)$ has all eigenvalues equal zero (if it would have an eigenvector with non-zero eigenvalue, it would have to have at least one other, to be traceless; then it would not be of rank 1). Then its canonical Jordan normal form has exactly one block of size 2. According to the $SU(n, 1)$ -orbits classification of $\mathfrak{su}(n, 1)$, see Lemma 4.10, the morphism belongs to some of the type 2 orbit.

Denote $D = x \wedge Jx$. Then $Jx \wedge J(Jx) = D$ and $ax \wedge aJx = a^2J$, $a \in \mathbb{R}$, that is the morphism $-x \wedge Jx$ cannot be written in the form $y \wedge Jy$ for any $y \in V$.

\square

4.2 Geometric interpretation of \widehat{C} and C , the Lie algebra cone and its projectivization

Let us follow the construction for $G = SU(n, 1)$. There is $\widehat{C} = Ad_{SU(n,1)}(x) \subset \mathfrak{su}(n, 1)$, where x is some maximal root element. The roots in $\mathfrak{su}(n, 1)$ have the same length and any rank 1 matrix in $\mathfrak{su}(n, 1)$ is a maximal root element.

Lemma 4.7 *The cone $C \subset \mathfrak{su}(n, 1)$ is isomorphic to $\mathbb{C}^n \setminus \{0, \dots, 0\}$*

Proof. Since any maximal root element in $\mathfrak{su}(n, 1)$ is a rank 1 matrix, then the cone \widehat{C} is equal to one of the two orbits of the adjoint action on the rank 1 matrices (according to the orbit of the generating element). According to the lemmas 4.4 and 4.5 we have

$$\widehat{C} \cong \{x \in \mathbb{C}^{n+1} | g(x, x) = 0\} / S^1.$$

We describe the isomorphism explicitly. Let us choose the basis e_1, \dots, e_{n+1} of \mathbb{C}^{n+1} in which the metric g has the standard form $(e^1)^2 + \dots + (e^n)^2 - (e^{n+1})^2$. We can choose a unique representative \mathbf{y} in each orbit of S^1 of the zero length vectors such that $\mathbf{y} = (x_1, \dots, x_{n+1}) = (\mathbf{x}, x_{n+1}) \in \mathbb{C}^{n+1}$ with $x_{n+1} \in \mathbb{R}^+$ with respect to the basis (e_i) . If we denote $\|\cdot\|$ the standard metric $(e^1)^2 + \dots + (e^n)^2$ in \mathbb{C}^n we can write the representative in the form $(\mathbf{x}, \|\mathbf{x}\|)$, where \mathbf{x} is an arbitrary vector in \mathbb{C}^n . Hence \widehat{C} is isomorphic to $\mathbb{C}^n \setminus \{0, \dots, 0\}$. \square

Lemma 4.8 *The projectivized cone $C \subset \mathbb{P}_o(\mathfrak{su}(n, 1))$ is isomorphic to S^{2n-1} .*

Proof. Consider the surjective map $\pi : \widehat{C} \rightarrow \mathbb{C}^n \setminus \{0, \dots, 0\}$, $(\mathbf{x}, \|\mathbf{x}\|) \mapsto (\frac{x_1}{\|\mathbf{x}\|}, \dots, \frac{x_n}{\|\mathbf{x}\|})$, which maps the elements in \widehat{C} which differ by a real multiple to the same point on the sphere S^{2n-1} . \square

4.2.1 The module structure of the cone

Being a homogeneous space (see (3.1)), the cone C is a $SU(n, 1)$, resp. $\mathfrak{su}(n, 1)$ -module.

Let us notice, that the identification of the cone C with the sphere S^{2n-1} is subject to the choice of the standard hermitian form on \mathbb{C}^{n+1} , respectively to the choice of the normal base with respect to it. We have to have this in mind when considering different block forms of the matrices in $\mathfrak{su}(n, 1)$.

The proof of the lemma (4.10) shows that if a matrix of the form $\begin{pmatrix} A & 0 \\ 0 & -\text{tr}(A) \end{pmatrix}$ is in $\mathfrak{su}(n, 1)$, then $A \in \mathfrak{u}(n)$ (with respect to some orthonormal basis of the given hermitian form).

The one-parametric subgroup generated by A are then matrices of the form

$$\begin{pmatrix} G & 0 \\ 0 & \frac{1}{\det(G)} \end{pmatrix}, \text{ where } G \in U(n). \text{ As we have seen above, a point in our sphere } S^{2n-1} \text{ corresponds}$$

to a class of null vectors in \mathbb{C}^{n+1} which differ by a complex multiple. According to the lemma 4.2 the group $SU(n, 1)$ acts on these classes in a standard way and we get

$$G \cdot \mathbf{x} \sim \begin{pmatrix} G & 0 \\ 0 & \frac{1}{\det(G)} \end{pmatrix} (\mathbf{x}, 1) = \left(G\mathbf{x}, \frac{1}{\det(G)} \right) \sim \det(G)G\mathbf{x} \quad (20)$$

For the action of the matrices in $\mathfrak{u}(n)$ of the above form on the tangent bundle of the sphere we get then:

$$A \cdot \mathbf{x} = (\text{tr}(A)E + A)\mathbf{x}, \quad (21)$$

which is in accordance with the action of $\mathfrak{u}(n-1)$ on \mathbb{C}^{n-1} from (17), if we represent A in the matrix form from (16).

Thus we get in fact the structure of $U(n)$ and $\mathfrak{u}(n)$ -module on C regarded as a sphere S^{2n-1} .

The action of the whole group $SU(n,1)$ or the whole algebra $\mathfrak{su}(n,1)$ respectively is then non-linear (as a $\mathbb{C}^n \rightarrow \mathbb{C}^n$ mapping).

Lemma 4.9 *The canonical symplectic form on $\widehat{C} \subset \mathfrak{g}^*$ corresponds under our identification to the standard symplectic form on \mathbb{C}^n , the one form $\lambda = \iota_{E_0}(\Omega)$ is then $\lambda = z_i d\bar{z}_i + \bar{z}_i dz_i$ in the complex coordinates z_i on \mathbb{C}^n .*

Proof. The Euler vector field on \mathbb{C}^n is $E_0 = z_i \frac{\partial}{\partial z_i}$. The form $\lambda = z_i d\bar{z}_i + \bar{z}_i dz_i$ is then the only $U(n)$ -invariant one-form α on \mathbb{C}^n satisfying $\mathcal{L}_{E_0}\alpha = \alpha$ and $\alpha(E_0) = 1$. \square

Up to now, the construction was common for all the Bochner-Kaehler geometries. The choice of a transversal symmetry of the canonical contact distribution actually determines the geometry. Let $A \in \mathfrak{su}(n,1)$ and let us consider the vector field $\xi v = \frac{\partial}{\partial t}|_0 \text{Ad}(\exp(tA))v$ on \widehat{C} . This vector field is a contact symmetry with respect to the distribution $\widehat{\mathcal{D}}$ on \widehat{C} and thus it determines a section of $\widehat{C} \rightarrow C$ that is a contact form on C (the identification of \mathfrak{g} and \mathfrak{g}^* gives an identification of $C \subset \mathfrak{su}(n,1)$ and $C^* \subset \mathfrak{su}(n,1)^*$).

The section $\lambda : C \rightarrow \widehat{C}$ is given by the equation $\lambda(\xi) = 1$. The image of C in $\widehat{C} = \mathbb{C}^n - \{0\}$ is then a hyperplane, which we will call Σ_A . The tangent space of Σ_A is then characterized by $T_v(C) = \{X \in \mathbb{C}^n | \omega(X, iA \cdot v) = 0\}$ and consequently there is

$$\mathcal{D}_v = \{X_v | g(X, iv) = 0 = g(X, iA \cdot v)\}. \quad (22)$$

The projectivized cone C and the CR -sphere. According to the lemma (3.1), the projectivized cone C is a homogeneous space $SU(n,1)/P$, where P is a parabolic subgroup of $SU(n,1)$, corresponding to the subalgebra $\mathfrak{u}(n) \oplus \mathbb{C}^{n-1} \oplus e_+^2$ of $\mathfrak{su}(n,1)$, see (4.1). As we have seen in the previous lemma, it is isomorphic to the sphere S^{2n-1} . The adjoint action of $SU(n,1)$ on \widehat{C} corresponds to the standard action of $SU(n,1)$ on the null-vectors (with respect to the standard hermitian form of the signature $(n,1)$) in \mathbb{C}^{n+1} , and thus as a homogeneous space it is exactly the CR -sphere (see for example [ČS]).

The underlying structure connected with this space is the canonic CR -distribution \mathcal{D} on the sphere:

$$\widehat{\mathcal{D}}_v = \{x \in \mathbb{C}^n | (x, iv) = 0, (x, v) = 0\},$$

where $(,)$ denotes the standard metric in \mathbb{C}^n .

4.3 Classification of Bochner-Kaehler metrics

All Bochner-Kaehler manifolds come from the mentioned construction for the Lie algebra $\mathfrak{su}(n, 1)$ and the resulting manifolds are isomorphic if we take in the course of construction matrices A_0 lying on the same adjoint orbit of $SU(n, 1)$ in $\mathfrak{su}(n, 1)$. Thus we can classify the Bochner-Kaehler manifolds according to which orbit of the action induces the given manifold.

There are four types of orbits of the adjoint action of $GL(n + 1, \mathbb{C})$ on $\mathfrak{su}(n, 1)$. Three types of these orbits are $SU(n, 1)$ orbits as well, the fourth one splits into two $SU(n, 1)$ orbits. We describe the orbit types according to the Jordan blocks of the matrices in the orbits.

Lemma 4.10 *There are five types of orbits of the adjoint action of the $SU(n, 1)$ on $\mathfrak{su}(n, 1)$. If we represent a morphism in the $\mathfrak{su}(n, 1)$ with a matrix A , than the orbit types look as follows:*

1. *The matrix is diagonalizable and its eigenvalues are purely imaginary.*
2. *The eigenvalues of matrices in the orbit are pure imaginary and there is just one Jordan block of the dimension 2 (there are n eigenvectors). There exists an eigenvector e and a root vector f , both in the block, such that*
 - 2a. $(e, f) = i$.
 - 2b. $(e, f) = -i$.
3. *The eigenvalues of matrices in the orbit are pure imaginary and there is just one Jordan block of the dimension 3 (there are $n - 1$ eigenvectors).*
4. *There are $n - 1$ pure imaginary eigenvalues corresponding to $n - 1$ eigenvectors and two eigenvalues $\lambda = \lambda_1 + i\lambda_2$ and $\mu = \mu_1 + i\mu_2$*

Proof. It is easy to see that if λ is an eigenvalue of a $\mathfrak{su}(p, q)$ -morphism then $-\bar{\lambda}$ is the eigenvalue of the morphism as well. There exists no nullplane in V , that is there are no two null-vectors x, y in V such that also $h(x, y) := (x, y) = 0$.

Let us suppose that the $\mathfrak{su}(n, 1)$ -morphism A has a Jordan block corresponding to an eigenvalue λ with non-zero real part, that is of size at least 2. Then there are $x, y \in V$ such that $Ax = \lambda x$, $Ay = \lambda y + x$, $(x, x) = 0$. Then

$$0 = (Ax, y) + (x, Ay) = (\lambda x, y) + (x, \lambda y + x) = \lambda(x, y) + \bar{\lambda}(x, y) + (x, x) = 2 \operatorname{Re} \lambda(x, y),$$

that is $(x, y) = 0$. Further

$$0 = (Ay, y) + (y, Ay) = (\lambda y + x, y) + (y, \lambda y + x) = \lambda(y, y) + (x, y) + \bar{\lambda}(y, y) + (y, x) = 2 \operatorname{Re} \lambda(y, y),$$

which implies $(y, y) = 0$ and we get a null-plane, which is a contradiction. All Jordan blocks corresponding to the eigenvalues with non-zero real part have the size one.

Further if a $\mathfrak{su}(n, 1)$ morphism has an eigenvalue with non-zero real part, then it has to have at least two eigenvalues which are not in $i\mathbb{R}$, otherwise the trace could not be zero. Let $\lambda = \lambda_1 + i\lambda_2$ and $\mu = \mu_1 + i\mu_2$, $\lambda_1, \mu_1, \lambda_2, \mu_2 \in \mathbb{R}$, $\lambda_1 \neq 0 \neq \mu_1$ be eigenvalues of a $\mathfrak{su}(n, 1)$ morphism A , with non-zero real parts. Let x resp. $y \in V$ be the corresponding eigenvectors. Then

$$0 = (Ax, x) + (x, Ax) = 2(\lambda x, x) = 2\lambda_1(x, x)$$

The eigenvector x is then a null-vector. With the same argument is y a null-vector too.

$$\begin{aligned} 0 &= (Ax, y) + (x, Ay) = (\lambda x, y) + (x, \mu y) = \lambda(x, y) + \bar{\mu}(x, y) \\ &= (\lambda + \bar{\mu})(x, y), \end{aligned}$$

and $\lambda = -\bar{\mu}$ otherwise x and y would generate a nullplane.

A $\mathfrak{su}(n, 1)$ morphism has consequently either exactly two eigenvalues λ, μ with non-zero real parts and in this case $\lambda = -\bar{\mu}$, or all its eigenvalues are pure imaginary (or zero). Let V_λ and $V_{-\bar{\lambda}}$ be the eigenspaces corresponding to the eigenvalues with non-zero real parts. The space $V_\lambda \oplus V_{-\bar{\lambda}}$ is A -invariant and so is the space $W := (V_\lambda \oplus V_{-\bar{\lambda}})^\perp$: let $x \in V_\lambda \oplus V_{-\bar{\lambda}}$ and $y \in W$ and then

$$(Ay, x) = -(y, Ax) = 0. \quad (23)$$

The restriction of the form h to $V_\lambda \oplus V_{-\bar{\lambda}}$ has the signature $(1, 1)$, therefore is h on W positive definite, that means $A|_W \in \mathfrak{u}(W, h)$ and $A|_W$ is diagonalizable (with eigenvalues in $i\mathbb{R}$). The matrix form of the morphism A is then $A = \text{diag}(\lambda_1 + i\lambda_2, -\lambda_1 + i\lambda_2, i\lambda_3, \dots, i\lambda_{n+1})$ in an appropriate basis e_1, \dots, e_{n+1} such that

$$(e_1, e_1) = (e_2, e_2) = 0, \quad (e_1, e_2) = 1, \quad (e_1, e_i) = (e_2, e_i) = 0 \quad \forall i \geq 3, \quad (e_i, e_j) = \delta_i^j \quad \forall i, j \geq 3.$$

Any such two morphisms lie evidently on the same $U(n, 1)$ orbit, that is also on the same $SU(n, 1)$ orbit.

Let us check how big can be the Jordan blocks of the Jordan normal forms of the morphism A with only pure imaginary (or zero) eigenvalues.

1. There exists a Jordan block of of the size at least 4. Let $i\lambda, \lambda \in \mathbb{R}$ be on the diagonal. The morphism $i\lambda$ is an unitary one, and so is the morphism $B := A - i\lambda$. The morphism B is then

unitary and in its Jordan normal form has the block of the form $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. And let $e_1, e_2, e_3,$

e_4 be the root vectors corresponding to the block, that is $Be_4 = e_3, Be_3 = e_2, Be_2 = e_1, Be_1 = 0$. Then we have

$$\begin{aligned} 0 &= (Be_1, e_2) + (e_1, Be_2) = 0 + (e_1, e_1) \implies (e_1, e_1) = 0, \\ 0 &= (Be_1, e_3) + (e_1, Be_3) = 0 + (e_1, e_2) \implies (e_1, e_2) = 0, \\ 0 &= (Be_2, e_3) + (e_2, Be_3) = (e_1, e_3) + (e_2, e_2) \implies (e_1, e_3) = -(e_2, e_2), \\ 0 &= (Be_3, e_3) + (e_3, Be_3) = (e_2, e_3) + (e_3, e_2) = 2 \text{Re}(e_2, e_3) \implies (e_2, e_3) \in i\mathbb{R} \\ &\quad \text{(it will be used in the point 2)} \\ 0 &= (Be_1, e_4) + (e_1, Be_4) = 0 + (e_1, e_3) \implies (e_1, e_3) = 0 \implies (e_2, e_2), \end{aligned}$$

and we would get a null-plane (e_1, e_2) . Thus there is no morphism in $\mathfrak{u}(n, 1)$ with the Jordan block of the size greater then 3 (and so is no such morphism in $\mathfrak{su}(n, 1)$).

2. Let there be a Jordan block of size 3. As in the first point we get vectors e_1, e_2, e_3 with $(e_1, e_1) = (e_1, e_2) = 0 \xrightarrow{\text{sign. } (n, 1)} (e_2, e_2) > 0$ and we may choose e_2 such that $(e_2, e_2) = 1$, then $(e_1, e_3) = -1$ and $(e_2, e_3) \in i\mathbb{R}$.

Now let us consider the transformation $f_1 := ae_1$, $f_2 := ae_2 + be_1$, $f_3 := ae_3 + be_2 + e_1$, $a, b, c \in \mathbb{C}$. Then we have still $Bf_1 = 0$, $Bf_2 = f_1$, and $Bf_3 = f_2$. After and easy computation one can choose a, b and c so that

$$(f_1, f_1) = (f_3, f_3) = 0, \quad (f_1, f_3) = -1, \quad (f_2, f_j) = \delta_2^j, \quad \text{for } j = 1, 2, 3$$

Now the space $\text{span}(e_1, e_2, e_3)$ is A -invariant and the restriction of h to this space has signature $(2, 1)$. It follows that the space $W := \text{span}(e_1, e_2, e_3)^\perp$ is A -invariant as well, and h is positive definite on it, that is $A|_W$ is diagonalizable with unitary transformation of W and the eigenvalues lie in $i\mathbb{R}$.

Again, all such $\mathfrak{su}(n, 1)$ morphisms lie on the same $SU(n, 1)$ orbit.

3. There exists a Jordan block of size 2. Let us suppose first, that there are at least two blocks of size 2. Let e_1, e_2, f_1, f_2 be corresponding linearly independent vectors such that $Ae_j = i\lambda_j e_j$ and $Af_j = i\lambda_j f_j + e_j$, $j = 1, 2$. Then

$$0 = (Ae_j, f_j) + (e_j, Af_j) = i\lambda_j(e_j, f_j) + \overline{i\lambda_j}(e_j, f_j)(e_j, e_j) = (e_j, e_j), \quad j = 1, 2.$$

Thus $(e_1, e_2) \neq 0$ according to the nullplane argument, and we can write

$$0 = (Ae_1, e_2) + (e_1, Ae_2) = i\lambda_1(e_1, e_2) + \overline{i\lambda_2}(e_1, e_2) = i(\lambda_1 - \lambda_2)(e_1, e_2),$$

which implies $\lambda_1 = \lambda_2 =: \lambda$. Then

$$0 = (Ae_1, f_2) + (e_1, Af_2) = i\lambda(e_1, f_2) + \overline{i\lambda}(e_1, f_2) + (e_1, e_2),$$

and we get the contradiction. Thus there is just one block of size 2. Let $e_1, f_1, e_3, \dots, e_{n+1}$ be a basis such that $Af_1 = i\lambda_1 f_1 + e_1$, $Ae_j = i\lambda_j e_j$, $\lambda_j \in \mathbb{R}$ for all $j = 1, \dots, n+1$. Then there is

$$0 = (Ae_1, e_j) + (e_1, Ae_j) = i\lambda_1(f_1, e_j) + (e_1, e_j) + \overline{i\lambda_j}(f_1, e_j) \quad (24)$$

$$= i(\lambda_1 - \lambda_j)(f_1, e_j) + (e_1, e_j), \quad (25)$$

$$0 = (Ae_1, e_j) + (e_1, Ae_j) = i\lambda_1(e_1, e_j) + \overline{i\lambda_j}(e_1, e_j) = i(\lambda_1 - \lambda_j)(e_1, e_j). \quad (26)$$

Then (25) and (26) imply according to the excluded third principle $(e_1, e_j) = 0$ for $j = 1, 3, \dots, n+1$. Since h is non-degenerate we have $(e_1, f_1) \neq 0$ and

$$0 = (Af_1, f_1) + (f_1, Af_1) = 2\text{Re}(Af_1, f_1) = 2\text{Re}(i\lambda_1(f_1, f_1) + (e_1, f_1)) = 2\text{Re}(e_1, f_1),$$

that means $(e_1, f_1) \in i\mathbb{R} \setminus 0$. Without loss of generality we can suppose that $(e_1, f_1) = \epsilon i$, $\epsilon = \pm 1$. If we consider the transformation $e_2 := f_1 + ce_1$, then $Ae_2 = i\lambda_1 e_2 + e_1$ and we can choose $c \in \mathbb{C}$ so that $(e_2, e_2) = 1$. Then again as in the previous points we can choose a basis e_1, \dots, e_{n+1} (e_1 and e_2 are already given) such that the morphism A is in its canonical Jordan form with exactly one Jordan block of size 2 in the basis, and for the basis vectors there is:

$$\begin{aligned} (e_1, e_1) &= (e_2, e_2) = 0, (e_1, e_2) = \epsilon i, \epsilon \in \{\pm 1\}, (e_1, e_j) = (e_2, e_j) = 0, \quad \text{for } j \geq 3 \\ (e_j, e_k) &= \delta_j^k \quad \text{for } j, k \geq 3. \end{aligned}$$

There are two orbits of the $SU(n, 1)$ of the morphisms of this type according to ϵ , that is according to the scalar product of e_1 and e_2 in the canonical basis. The number ϵ is evidently a $SU(n, 1)$ invariant.

4. The morphism A is diagonalizable. Again, we can choose a basis e_1, \dots, e_{n+1} in which the morphism has the diagonal form and

$$(e_1, e_1) = -1, \quad (e_1, e_j) = 0, \forall j \geq 2, \quad (e_j, e_k) = \delta_j^k \quad \forall j, k \geq 2,$$

and all such morphism lie on one $SU(n, 1)$ orbit.

In low dimensional cases ($\dim V \leq 2$) we have then less orbits types. For $\dim V = 3$ the morphism with the Jordan block of size 3 has all the eigenvalues zero. \square

Characteristic polynomial of the metrics. The characteristic polynomial determines all invariants of the adjoint orbit of the matrix $A \in \mathfrak{g}^1$. Thus the different types of adjoint orbits correspond to different types of characteristic polynomials (distinguished according to their roots) and we get invariants of the equivalent classes of Bochner-Kaehler metric.

The characteristic polynomial p_A of the matrices in Γ_A is according to (16):

$$\det \begin{pmatrix} \rho - \frac{1}{n+2}(\text{tr } \rho)\mathbb{I}_n & u & u \\ -u^* & -\frac{1}{2}(\text{tr } \rho - i(f+1)) & \frac{i}{2}(f-1) \\ u^* & \frac{i}{2}(1-f) & -\frac{1}{2}(\text{tr } \rho + i(f+1)) \end{pmatrix} - t\mathbb{I}_{n+2} \quad (27)$$

$$= \det(\rho - \frac{1}{n+2}(\text{tr } \rho) - t\mathbb{I}_n)(t^2 + (\text{tr } \rho)t + f + \frac{1}{4}(\text{tr } \rho)^2) + u^* \text{Cof}(\rho - \frac{1}{n+2}(\text{tr } \rho) - t\mathbb{I}_n)u, \quad (28)$$

where $\text{Cof}(X)$ means the cofactor matrix of X .

This is in accordance with the Bryant's result (see [Br])² on the orbits of the diagonalizable matrices with pure imaginary eigenvalues.

4.4 One "nice" type of Bochner-Kaehler metrics

We are now going to describe the first one of the five mentioned types of Bochner-Kaehler metrics in more detail. Namely let us investigate those metrics which come from the construction if we take in the course of it the matrix generating transversal symmetry to be diagonalizable with all eigenvalues pure imaginary (or zero). These are the matrices which acts as linear morphism ($U(n)$ -morphpism actually) on the sphere S^{2n-1} regarded as a projectivized cone C .

Any diagonalizable matrix $\hat{A} \in \mathfrak{su}(n, 1)$ can be written in the form $\text{diag}(i\lambda_1, \dots, i\lambda_n, -i\sum_{i=1}^n \lambda_i)$, where $A = \text{diag}(i\lambda_1, \dots, i\lambda_n)$ is in $\mathfrak{u}(n)$.

Then according to (21), A acts on the tangent bundle of the sphere:

$$A \cdot (x_1, \dots, x_n) = (i(\lambda_1 + \sigma)x_1, \dots, i(\lambda_n + \sigma)x_n) \in TC, \quad \sigma = \sum_{i=1}^n \lambda_i.$$

The action of a matrix A on C^n thus corresponds to the multiplication with the matrix A' , where $A' = A + \text{tr}(A)I$, I being the identity matrix.

Let $A_0 \in \mathfrak{su}(n, 1)$, $A_0 = \begin{pmatrix} A_0 & 0 \\ 0 & -\text{tr}(A_0) \end{pmatrix}$ be of the above diagonal form such that $\xi_0 v = A_0 \cdot v$ defines a transversal symmetry on C_0 , a non-empty open subset of C , that is $\xi_0 v \notin \mathcal{D}_v$ on C_0 . This

¹Let $\mathfrak{g} \subset \text{End}(V)$ be an irreducible representation of the Lie algebra \mathfrak{g} . Let $\varphi : \mathfrak{g} \rightarrow \mathbb{R}(t)$, such that $\varphi(\text{Ad}_g x) = \varphi(x)$ for all $x, g \in \mathfrak{g}$ and $\text{gr}(\varphi(x)) \leq n$. Then $\varphi(x)$ is a constant multiple of the characteristic polynomial of x .

²The functions ρ, u, f from 3.3 correspond to Bryant's functions S, T, U as follows: $\rho = iS, u = -T, U = -f$.

symmetry then defines a section λ of the line bundle $\widehat{C} \rightarrow C$. Then $\lambda(C) = \Sigma_A \subset \widehat{C}$,

$$\begin{aligned}
\Sigma_A &= \{(x \in \mathbb{C}^n | \lambda(\xi_0(x)) = 1\} \\
&= \{(x \in \mathbb{C}^n | (\xi_0(x), Jx) = 1\} \\
&= \{(x \in \mathbb{C}^n | (A_0 \cdot x, Jx) = 1\} \\
&= \{x \in \mathbb{C}^n | (x, -(A_0 + \text{tr}(A_0))Jx) = 1\} \\
&= \{x \in \mathbb{C}^n | \sum_{i=1}^n (\lambda_i + \sigma) |x_i|^2 = 1\}.
\end{aligned} \tag{29}$$

Since the globalization brings to life some new demanding questions, we keep on working locally only. Consider U a regular open subset of C with respect to ξ_0 , that is there is a submersion $\pi_U : U \rightarrow M_U$ onto some manifold M_U , the set of leaves of the foliation generated on U by ξ . The whole of C can be covered by regular subsets. We write $M_U = U/T$.

Our goal is to determine the Bochner-Kaehler connection on M_U which is induced there according to the general construction of special symplectic geometries. The Bochner-Kaehler connections are with one-to-one correspondence with the Bochner-Kaehler metrics which are further in one-to-one correspondence with the pair consisting of the fundamental form of the Kaehler structure and the complex structure on M_U .

There is the unique symplectic (which turns to the fundamental Kaehler one with the complex structure on M_U) form ω_U on M_U such that the pull-back of this form to $U \times \mathbb{R} \subset \widehat{C} \subset \mathbb{C}^n$ is the canonical symplectic form on \mathbb{C}^n (this is the form that comes with the above identifications from the Cartan-Killing form on $\mathfrak{su}(n, 1)$).

Lemma 4.11 *The complex projective space $\mathbb{C}P^n$ comes from our construction for $\mathfrak{g} = \mathfrak{su}(n + 1, 1)$ and*

$$A = \begin{pmatrix} -\frac{i}{2(n+2)} & & & \\ & \ddots & & \\ & & -\frac{i}{2(n+2)} & \\ & & & \frac{i(n+1)}{2(n+2)} \end{pmatrix}.$$

Proof. As we have already mentioned, the canonical CR -distribution on S is the structure which determines The matrix A acts on the sphere S^{2n-1} according to (21) as linear map given by the matrix

$$\begin{pmatrix} -\frac{i}{2} & & & \\ & \ddots & & \\ & & & -\frac{i}{2} \end{pmatrix},$$

and thus the contact symmetry ξ is given as $\xi(p) = -\frac{1}{2}ip$. The hyperplane Σ_A is a $2n + 1$ -dimensional sphere in \mathbb{C}^{n+1} , so that it coincides with the projectivization C of the Lie algebra cone and the Kaehler form is the standard form on \mathbb{C}^n . We get actually the cone construction from the theorem 2.3.

If we consider the matrix A in the form of 16, we see that the structure functions are $\rho = -\frac{1}{2}J$, $f = 1$, $u = 0$. \square

The generating metric on the contact distribution. The Bochner-Kaehler structure on the sphere S^{2n-1} lifts to the structure on Σ_A . The canonical complex structure on the contact distribution on C , that is canonical CR -distribution on the sphere lifts to the complex structure on the distribution $\mathfrak{D}_\Sigma \subset T\Sigma_A$, that is to the complex structure on the contact distribution on the section Σ_A of $\widehat{C} \rightarrow C$:

$$J_M(X) = JX - (X, A_0p)p - \frac{(X, p)}{|p|^2}Jp, \quad (30)$$

for vectors $X, Y \in T_p\Sigma$, and $|p|^2 = (p, p)$. This gives us then the metric on \mathfrak{D}_Σ

$$g(X, Y) = \omega(J_M X, Y) = (J_M X, JY) = (X, Y) - \frac{(X, p)(Y, p)}{|p|^2}. \quad (31)$$

This metric factors to the metric on $M_U \cong T\Sigma$, which is, according to the construction, Bochner-Kaehler. This can be confirmed also with the direct computation. We can view the metric (31) as a degenerated metric on the whole Σ . The corresponding Levi-Civita connection is then

$$(\nabla_X Y)_p = \nabla_X^0 Y + g(X, Y)\eta + (X, A_0 JY)p, \quad (32)$$

where $X, Y, Z \in \mathfrak{X}(\Sigma_A)$ ∇^0 is a flat connection in \mathbb{C}^n , $\eta = A_0 Jp - |A_0 p|^2 p$, $p \in \Sigma_A$. For the Levi-Civita connection of the metric on M_U we have then

$$\overline{\nabla_X Y} = \nabla_{\overline{X}} \overline{Y} - \frac{1}{2}\omega(X, Y)\xi_0 + \alpha(X)\overline{J_M Y} + \alpha(Y)\overline{J_M X},$$

where $X, Y \in \mathfrak{X}(M)$, \overline{X} is a lift of a vector field X on $\mathfrak{X}(M)$ to a vector field on Σ_A ($\overline{X} \in \mathfrak{D}_\Sigma$), and $\alpha(X) := g(\overline{X}, \xi_0) = (\overline{X}, \xi_0)$.

The Bochner-Kaehler form of the curvature of the metric.. Let us define the mapping $\rho : TM_U \rightarrow TM_U$ as

$$\overline{\rho X} := \nabla_{\xi_0} \overline{X} = \nabla_{\overline{X}} \xi_0 = A_0 \overline{X} + g(\overline{X}, \xi_0)\eta + g(\overline{X}, A_0^2 Jp)p. \quad (33)$$

Lemma 4.12 *The mapping ρ from (33) is in $\mathfrak{u}(n)$.*

Proof. Let us show first, that ρ is well-defined, that is $g_0(\rho \overline{X}, Jp) = 0$:

$$\begin{aligned} (\overline{\rho X}) &= g(\rho X, Jp) = g(\nabla_{\xi_0} \overline{X}, Jp) = \xi_0 \underbrace{g(\overline{X}, Jp)}_{=0} g(\overline{X}, \nabla_{\xi_0} Jp) \\ &= 0 - g(\overline{X}, \nabla_{\xi_0}^0 Jp + g(\xi_0, Jp) + g(\xi_0, -A_0 p)p) \\ &= -g(\overline{X}, A_0 Jp + (A_0 p, Jp)A_0 Jp) \\ &= -g(\overline{X}, A_0 Jp - \underbrace{(p, A_0 Jp)}_{=1} A_0 Jp) \\ &= -g(\overline{X}, A_0 Jp - A_0 Jp) = 0 \end{aligned}$$

Next we show $\rho J_M = J_M \rho$:

$$\begin{aligned}
\overline{\rho J_M X} &= A_0 J_M X + (\overline{J_M X}, A_0 \rho) \eta + (\overline{J_M X}, A_0^2 J p) p \\
&= A_0 (J \overline{X} - (\overline{X}, A_0 p) p) - \frac{1}{|p|^2} (\overline{X}, p) J p + (J \overline{X} - (\overline{X}, A_0 p) p) - \frac{1}{|p|^2} (\overline{X}, p) J p, A_0 p) \eta \\
&\quad + (J \overline{X} - (\overline{X}, A_0 p) p) - \frac{1}{|p|^2} (\overline{X}, p) J p, A_0^2 J p) p \\
&= A_0 J \overline{X} - (\overline{X}, A_0 p) A_0 p - \frac{1}{|p|^2} (\overline{X}, p) A_0 J p - \underbrace{(\overline{X}, A_0 p)}_{=0} \eta - (\overline{X}, A_0 p) \underbrace{(p, A_0 p)}_{=0} \eta \\
&\quad + (J \overline{X}, A_0^2 J \overline{X}) p + (\overline{X}, A_0 p) \underbrace{(A_0 p, J A_0 p)}_{=0} p \frac{1}{|p|^2} (\overline{X}, p) (A_0 J p, A_0 J p) p \\
&= A_0 J \overline{X} - (\overline{X}, A_0 p) + \frac{1}{|p|^2} (\overline{X}, p) \underbrace{(-A_0 J p + \eta + |A_0 p|^2)}_{=0} + (\overline{X}, A_0^2 p) p \\
&= A_0 J \overline{X} - (\overline{X}, A_0 p) A_0 p + (\overline{X}, A_0^2 p) p.
\end{aligned}$$

For the $J_M \rho$ we have

$$\begin{aligned}
\overline{J_M \rho X} &= J \overline{\rho X} - (\overline{\rho X}, A_0 p) p - \frac{1}{|p|^2} (\overline{\rho X}, p) J p \\
&= J (A_0 \overline{X} + (\overline{X}, \xi_0) \eta + (\overline{X}, A_0^2 J p) p) - (A_0 \overline{X} + (\overline{X}, \xi_0) \eta + (\overline{X}, A_0^2 J p) p, A_0 p) p \\
&\quad - \frac{1}{|p|^2} (A_0 \overline{X} + (\overline{X}, \xi_0) \eta + (\overline{X}, A_0^2 J p) p, p) J p \\
&= A_0 J \overline{X} + (\overline{X}, \xi_0) J \eta + (\overline{X}, A_0^2 J p) J p + (\overline{X}, A_0^2 p) p \\
&\quad - (\overline{X}, \xi_0) \left(\underbrace{(A_0 J p, A_0 p)}_{=0} - |A_0 p|^2 \underbrace{(p, A_0 p)}_{=0} \right) + \frac{1}{|p|^2} (\overline{X}, A_0 p) J p \\
&\quad - \frac{1}{|p|^2} (\overline{X}, A_0 p) \left(\underbrace{(A_0 J p, p)}_{=1} - |A_0 p|^2 |p|^2 \right) J p - \frac{1}{|p|^2} (\overline{X}, A_0^2 J p) |p|^2 J p \\
&= A_0 J \overline{X} + (\overline{X}, A_0 p) \left(-A_0 p - |A_0 p|^2 J p + \frac{1}{|p|^2} \left(J p - \frac{1}{|p|^2} (1 - |A_0 p|^2 |p|^2) \right) J p \right) \\
&\quad + (\overline{X}, A_0^2 p) p \\
&= A_0 J \overline{X} - (\overline{X}, A_0 p) A_0 p + (\overline{X}, A_0^2 p) p
\end{aligned}$$

Finally we show $g(\rho X, Y) = (A_0 \overline{X}, \overline{Y})$, which implies $g(\rho X, Y) = -g(\rho Y, X)$

$$\begin{aligned}
g(\rho X, Y) &= g(\overline{\rho X}, \overline{Y}) = g(A_0 \overline{X} + g(\overline{X}, \xi_0) \eta + g(\overline{X}, A_0^2 J p) p, \overline{Y}) \\
&= g(A_0 \overline{X} + g(\overline{X}, \xi_0) A_0 J p, \overline{Y}) \\
&= (A_0 \overline{X}, \overline{Y}) - \frac{1}{|p|^2} (A_0 \overline{X}, p) (\overline{Y}, p) + (\overline{X}, \xi_0) \left(\underbrace{(A_0 J p, \overline{Y})}_{=0} - \frac{1}{|p|^2} \underbrace{(A_0 J p, p)}_{=1} (\overline{Y}, p) \right) \\
&= (A_0 \overline{X}, \overline{Y}) + \frac{1}{|p|^2} (\overline{X}, A_0 p) (\overline{Y}, p) - \frac{1}{|p|^2} (\overline{X}, A_0 p) (\overline{Y}, p) \\
&= (A_0 \overline{X}, \overline{Y})
\end{aligned}$$

□

With the help of the map ρ , we can express the curvature of the Levi-Civita connection on M_U in the Bochner-Kaehler form (3).

Proposition 4.13 *Let M_U be a Bochner-Kaehler manifold which comes from the general construction in [CS] for $\mathfrak{h} = \mathfrak{u}(n-1)$, $\mathfrak{g} = \mathfrak{su}(n,1)$, and $A_0 \in \mathfrak{su}(n,1)$. Then the curvature R of the Bochner-Kaehler metric on M_U is given by*

$$R = R_{\frac{1}{2}\rho + \frac{1}{4}|A_0 p|^2 J}$$

Proof. Let ∇, R be Levi-Civita connection and curvature of the Bochner Kaehler metric induced on M_U , and $\bar{\nabla}, \bar{R}$ the ones of the metric g on \mathfrak{D}_Σ , $\alpha(X) = g(\bar{X}, \xi_0)$ as before. There is

$$\begin{aligned} (\nabla_X \alpha)(Y) &= X\alpha(Y) - \alpha(\nabla_X Y) = \bar{X}g(\bar{Y}, \xi_0) - g(\bar{\nabla}_X \bar{Y}, \xi_0) \\ &= g(\bar{\nabla}_X \bar{Y}, \xi_0) + g(\bar{Y}, \bar{\nabla}_X \xi_0) - g(\bar{\nabla}_X \bar{Y} - \frac{1}{2}\omega(X, Y)\xi_0 + \alpha(X)\bar{J}_M \bar{Y} + \alpha(Y)\bar{J}_M \bar{X}, \xi_0) \\ &= g(\bar{Y}, \bar{\rho}\bar{X}) + \frac{1}{2}\omega(X, Y)g(\xi_0, \xi_0) - \alpha(X)g(\bar{J}_M \bar{Y}, \xi_0) - \alpha(Y)g(\bar{J}_M \bar{X}, \xi_0) \\ &= g(\bar{\rho}\bar{X}, \bar{Y}) - g(X, J_M Y)|A_0 p|^2 - \alpha(X)\alpha(J_M Y) - \alpha(Y)\alpha(J_M X) \end{aligned}$$

Further

$$\begin{aligned} d\alpha(X, Y) &= (\nabla_X \alpha)(Y) - (\nabla_Y \alpha)(X) \\ &= g(\bar{\rho}\bar{X}, \bar{Y}) - g(\bar{\rho}\bar{Y}, \bar{X}) - g(\bar{X}, \bar{J}_M \bar{Y})|A_0 p|^2 + g(\bar{Y}, \bar{J}_M \bar{X})|A_0 p|^2 \\ &\quad - \alpha(X)\alpha(J_M Y) + \alpha(Y)\alpha(J_M X) - \alpha(Y)\alpha(J_M X) + \alpha(X)\alpha(J_M Y) \\ &= -2g(\bar{X}, \bar{\rho}\bar{Y}) - 2g(\bar{X}, \bar{J}_M \bar{Y})|A_0 p|^2. \end{aligned}$$

For the curvature tensor of the metric g we get then

$$\begin{aligned} \overline{R(X, Y)Z} &= R(\bar{X}, \bar{Y})\bar{Z} + (-2g(\bar{X}, \bar{\rho}\bar{Y}) - 2g(\bar{X}, \bar{J}_M \bar{Y})|A_0 p|^2) \bar{J}_M \bar{Z} \\ &\quad + (\alpha(Y)g(\bar{X}, \bar{Z}) - \alpha(X)g(\bar{Y}, \bar{Z})) \xi_0 + g(\bar{Y}, \bar{J}_M \bar{Z})\bar{\rho}\bar{X} - g(\bar{X}, \bar{J}_M \bar{Z})\bar{\rho}\bar{Y} \\ &\quad + \alpha(J_M Z) (\alpha(X)\bar{J}_M \bar{Y} - \alpha(Y)\bar{J}_M \bar{X}) + \alpha(Z) (\alpha(X)\bar{Y} - \alpha(Y)\bar{X}) \\ &\quad + \alpha(Z) (\alpha(J_M X)\bar{J}_M \bar{Y} - \alpha(J_M Y)\bar{J}_M \bar{X}) \\ &\quad + (-g(\bar{X}, \bar{\rho}\bar{Z}) - g(\bar{X}, \bar{J}_M \bar{Z})|A_0 p|^2 - \alpha(X)\alpha(J_M Z) - \alpha(Z)\alpha(J_M X)) \bar{J}_M \bar{Y} \\ &\quad - (-g(\bar{Y}, \bar{\rho}\bar{Z}) - g(\bar{Y}, \bar{J}_M \bar{Z})|A_0 p|^2 - \alpha(Y)\alpha(J_M Z) - \alpha(Z)\alpha(J_M Y)) \bar{J}_M \bar{X} \\ &\quad - 2g(\bar{X}, \bar{J}_M \bar{Y})\bar{\rho}\bar{Z} \\ &= R(\bar{X}, \bar{Y})\bar{Z} - 2(g(\bar{X}, \bar{\rho}\bar{Y}) + g(\bar{X}, \bar{J}_M \bar{Y})|A_0 p|^2) \bar{J}_M \bar{Z} \\ &\quad + (\alpha(Y)g(\bar{X}, \bar{Z}) - \alpha(X)g(\bar{Y}, \bar{Z})) \xi_0 - g(\bar{J}_M \bar{Y}, \bar{Z})\bar{\rho}\bar{X} + g(\bar{J}_M \bar{X}, \bar{Z})\bar{\rho}\bar{Y} \\ &\quad + \alpha(Z) (\alpha(X)\bar{Y} - \alpha(Y)\bar{X}) \\ &\quad + (g(\bar{\rho}\bar{X}, \bar{Z}) + g(\bar{J}_M \bar{X}, \bar{Z})|A_0 p|^2) \bar{J}_M \bar{Y} - (g(\bar{\rho}\bar{Y}, \bar{Z}) + g(\bar{J}_M \bar{Y}, \bar{Z})|A_0 p|^2) \bar{J}_M \bar{X} \\ &\quad - 2g(\bar{X}, \bar{J}_M \bar{Y})\bar{\rho}\bar{Z} \end{aligned}$$

$$\begin{aligned}
&= R(\bar{X}, \bar{Y})\bar{Z} - 2(g(\bar{X}, \bar{\rho}Y) + g(\bar{X}, \bar{J}_M Y)|A_0 p|^2)\bar{J}_M Z \\
&\quad + (\alpha(Y)g(\bar{X}, \bar{Z}) - \alpha(X)g(\bar{Y}, \bar{Z}))\xi_0 + (\bar{\rho}X \wedge \bar{J}_M Y)\bar{Z} - (\bar{\rho}Y \wedge \bar{J}_M X)\bar{Z} \\
&\quad + \alpha(Z)(\alpha(X)\bar{Y} - \alpha(Y)\bar{X}) + |A_0 p|^2(\bar{J}_M X \wedge \bar{J}_M Y)\bar{Z} - 2g(\bar{X}, \bar{J}_M Y)\bar{\rho}Z \\
&= (\bar{Y} \wedge A_0 \bar{J}\bar{X} - \bar{X} \wedge A_0 \bar{J}\bar{Y} + |A_0 p|^2 \bar{X} \wedge \bar{Y})\bar{Z} \\
&\quad - (g(X, Z)Y - g(Y, Z)X, A_0^2 p)p - 2(g(\bar{X}, \bar{\rho}Y) + g(\bar{X}, \bar{J}_M Y)|A_0 p|^2)\bar{J}_M Z \\
&\quad + (\alpha(Y)g(\bar{X}, \bar{Z}) - \alpha(X)g(\bar{Y}, \bar{Z}))\xi_0 + (\bar{\rho}X \wedge \bar{J}_M Y - \bar{\rho}Y \wedge \bar{J}_M X)\bar{Z} \\
&\quad + \alpha(Z)(\alpha(X)\bar{Y} - \alpha(Y)\bar{X}) + |A_0 p|^2(\bar{J}_M X \wedge \bar{J}_M Y)\bar{Z} - 2g(\bar{X}, \bar{J}_M Y)\bar{\rho}Z \\
&= (\bar{Y} \wedge A_0 \bar{J}\bar{X} - \bar{X} \wedge A_0 \bar{J}\bar{Y} + |A_0 p|^2 \bar{X} \wedge \bar{Y})\bar{Z} + (g(Y, Z)\bar{X} - g(X, Z)\bar{Y}, A_0^2 p)p \\
&\quad - 2(g(\bar{X}, \bar{\rho}Y) + g(\bar{X}, \bar{J}_M Y)|A_0 p|^2)\bar{J}_M Z + \alpha(Y)(\bar{X} \wedge \xi_0)\bar{Z} - \alpha(X)(\bar{Y} \wedge \xi_0)\bar{Z} \\
&\quad + (\bar{\rho}X \wedge \bar{J}_M Y - \bar{\rho}Y \wedge \bar{J}_M X)\bar{Z} + |A_0 p|^2(\bar{J}_M X \wedge \bar{J}_M Y)\bar{Z} - 2g(\bar{X}, \bar{J}_M Y)\bar{\rho}Z \\
&= (\bar{Y} \wedge A_0 \bar{J}\bar{X} - \bar{X} \wedge A_0 \bar{J}\bar{Y} + |A_0 p|^2 \bar{X} \wedge \bar{Y})\bar{Z} + (g(Y, Z)\bar{X} - g(X, Z)\bar{Y}, A_0^2 p)p \\
&\quad - 2(g(\bar{X}, \bar{\rho}Y) + g(\bar{X}, \bar{J}_M Y)|A_0 p|^2)\bar{J}_M Z + \alpha(Y)(\bar{X} \wedge \xi_0)\bar{Z} - \alpha(X)(\bar{Y} \wedge \xi_0)\bar{Z} \\
&\quad + (\bar{\rho}X \wedge \bar{J}_M Y - \bar{\rho}Y \wedge \bar{J}_M X)\bar{Z} + |A_0 p|^2(\bar{J}_M X \wedge \bar{J}_M Y)\bar{Z} - 2g(\bar{X}, \bar{J}_M Y)\bar{\rho}Z \\
&= (\bar{Y} \wedge A_0 \bar{J}\bar{X} - \bar{X} \wedge A_0 \bar{J}\bar{Y} + |A_0 p|^2 \bar{X} \wedge \bar{Y})\bar{Z} + (\bar{Y} \wedge (X, A_0^2 p)p)\bar{Z} - (\bar{X} \wedge (\bar{Y}, A_0^2 p)p)\bar{Z} \\
&\quad - 2(g(\bar{X}, \bar{\rho}Y) + g(\bar{X}, \bar{J}_M Y)|A_0 p|^2)\bar{J}_M Z + (\alpha(Y)\bar{X} \wedge \xi_0 - \alpha(X)\bar{Y} \wedge \xi_0)\bar{Z} \\
&\quad + (\bar{\rho}X \wedge \bar{J}_M Y - \bar{\rho}Y \wedge \bar{J}_M X)\bar{Z} + |A_0 p|^2(\bar{J}_M X \wedge \bar{J}_M Y)\bar{Z} \\
&\quad - 2g(\bar{X}, \bar{J}_M Y)\bar{\rho}Z \\
&= \bar{Y} \wedge (A_0 \bar{J}\bar{X} + (\bar{X}, A_0^2 p)p - \alpha(X)\xi_0)\bar{Z} - \bar{X} \wedge (A_0 \bar{J}\bar{Y} + (\bar{Y}, A_0^2 p)p - \alpha(Y)\xi_0)\bar{Z} \\
&\quad + |A_0 p|^2((\bar{X} \wedge \bar{Y} + \bar{J}_M X \wedge \bar{J}_M Y)\bar{Z} - 2g(\bar{X}, \bar{J}_M Y)\bar{J}_M Z) - 2g(\bar{X}, \bar{\rho}Y)\bar{J}_M Z \\
&\quad + (\bar{\rho}X \wedge \bar{J}_M Y - \bar{\rho}Y \wedge \bar{J}_M X)\bar{Z} - 2g(\bar{X}, \bar{J}_M Y)\bar{\rho}Z \\
&= (\bar{Y} \wedge \bar{J}_M \rho X)\bar{Z} - (\bar{X} \wedge \bar{J}_M \rho Y)\bar{Z} + |A_0 p|^2((\bar{X} \wedge \bar{Y} + \bar{J}_M X \wedge \bar{J}_M Y)\bar{Z} - 2g(\bar{X}, \bar{J}_M Y)\bar{J}_M Z) \\
&\quad - 2g(\bar{X}, \bar{\rho}Y)\bar{J}_M Z + (\bar{\rho}X \wedge \bar{J}_M Y - \bar{\rho}Y \wedge \bar{J}_M X)\bar{Z} - 2g(\bar{X}, \bar{J}_M Y)\bar{\rho}Z \\
&= (\bar{Y} \wedge \bar{J}_M \rho X - \bar{X} \wedge \bar{J}_M \rho Y + \bar{\rho}X \wedge \bar{J}_M Y - \bar{\rho}Y \wedge \bar{J}_M X - 2g(\bar{X}, \bar{\rho}Y)\bar{J}_M - 2g(\bar{X}, \bar{J}_M Y)\bar{\rho})\bar{Z} \\
&\quad + |A_0 p|^2((\bar{X} \wedge \bar{Y}) + \bar{J}_M X \wedge \bar{J}_M Y - 2g(\bar{X}, \bar{J}_M Y)\bar{J}_M)\bar{Z},
\end{aligned}$$

that is

$$\begin{aligned}
R(X, Y) &= Y \wedge (J_M \rho X) - X \wedge (J_M \rho Y) + \rho X \wedge J_M Y - \rho Y \wedge J_M X - 2g(X, \rho Y)J_M \\
&\quad - 2g(X, J_M Y)\rho + |A_0 \rho|^2(\bar{X} \wedge \bar{Y} + J_M X \wedge J_M Y - 2g(X, J_M Y)J_M)
\end{aligned}$$

□

Example 4.14 For the complex projective space we get the following characteristic polynomial $p(t)$ for the class of the Bochner-Kaehler metrics with the constant holomorphic curvature equal 1:

$$p(t) = \left(t + \frac{i}{2(n+2)}\right)^{n+1} \left(t - \frac{i(n+1)}{2(n+2)}\right),$$

which corresponds to the Bryant's one (see section 4.1.1. in [Br]).

5 Bochner tower

There is a question if one can embed (locally) the Bochner-Kaehler manifold into a Bochner-Kaehler manifold of two (real) dimensions greater.

Following the geometric interpretation of the construction of Bochner-Kaehler metrics, we would like to have some embedding $\mathfrak{su}(n, 1) \hookrightarrow \mathfrak{su}(n+1, 1)$, which would induce the embedding of corresponding Bochner-Kaehler manifolds.

Consider the embedding

$$G \mapsto \begin{pmatrix} 1 & 0 \\ 0 & G \end{pmatrix} \quad (34)$$

of the Lie group $SU(n, 1)$ into $SU(n+1, 1)$ and the corresponding embedding

$$A \mapsto \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} := B \text{ of the Lie algebras.}$$

This embedding yields the embedding of corresponding cones (and their projectivizations) in the Lie algebras, we have $C_{\mathfrak{su}(n,1)} \cong S^{2n-1} \hookrightarrow S^{2n+1} \cong C_{\mathfrak{su}(n+1,1)}$. This embedding is evidently $\mathfrak{su}(n, 1)$ -equivariant ($A \in \mathfrak{su}(n, 1)$ acts on $C_{\mathfrak{su}(n,1)}$ according to the action (21), B acts on the embedded cone according to (21) for $\mathfrak{su}(n+1, 1)$).

A matrix $A \in \mathfrak{su}(n, 1)$ acts on \mathbb{C} the same way as the matrix

$$D_{\lambda_0} = \left(\begin{array}{c|ccc} \lambda_0 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & & \\ 0 & & & \end{array} \begin{array}{c} A - \frac{\lambda_0}{n+1} E_n \end{array} \right) \quad (35)$$

on the image of $C_{\mathfrak{su}(n,1)} \subset C_{\mathfrak{su}(n+1,1)}$ under the described embedding.

Consequently we get the theorem

Theorem 5.1 *For any $\lambda_0 \in \mathbb{R}$ and any $A \in \mathfrak{su}(n, 1)$ the Bochner-Kaehler manifold corresponding to A (that is T/Σ_A , see 3.3) can be embedded totally geodesically into the Bochner-Kaehler manifold $T/\Sigma_{D_{\lambda_0}}$, where D_{λ_0} is given above.*

Proof. The manifold $\Sigma_{D_{\lambda_0}}$ is given by the equations (29). Then the vector $(0, v) \in \mathbb{C}^{n+1}$, $v \in \mathbb{C}^n$ lies in $\Sigma_{D_{\lambda_0}}$ evidently iff the vector v lies in Σ_A . The contact distribution $\mathcal{D}_{\Sigma_{D_{\lambda_0}}}$ on $\Sigma_{D_{\lambda_0}}$ is given by the equation (22) and apparently $\mathcal{D}_{\Sigma_{D_{\lambda_0}}} \cap (0, \mathbb{C}^n) = (0, \mathcal{D}_{\Sigma_A})$. Moreover the action (21) shows that for a $v \in \Sigma_A$ there is $(0, \xi_A(v)) = \xi_{D_{\lambda_0}}$. Thus the factor manifold Σ_A/T_{ξ_A} is embedded into the manifold $\Sigma_{D_{\lambda_0}}/T_{D_{\lambda_0}}$.

The lift of the Levi-Civita connection of the Bochner-Kaehler to the contact distribution D on $\Sigma_{D_{\lambda_0}}$ preserves the set of vectors of the form $(0, v)$, $v \in \mathbb{C}^n$ and thus it preserves also the distribution $(0, \Sigma_A)$. Thus the described embedding is a totally geodesic one. \square

Remark 5.2 *The previous statement can be reformulated as: Bochner-Kaehler manifold can be totally geodesically embedded into the one-parametric class of Bochner-Kaehler manifolds of the (complex) dimension one higher.*

Example 5.3 *Let us compute what the given embedding yields for the complex projective space $\mathbb{C}P^{n-1}$. As we have seen, the complex projective space comes from the construction for $\mathfrak{g} = \mathfrak{su}(n, 1)$ and*

$$A = \begin{pmatrix} -\frac{i}{2(n+1)} & & & \\ & \ddots & & \\ & & -\frac{i}{2(n+1)} & \\ & & & \frac{in}{2(n+1)} \end{pmatrix}. \text{ Choosing } \lambda_0 = -\frac{i}{2(n+2)} \text{ we get } \mathbb{C}P^{n-1} \text{ we get}$$

$$D_0 = \begin{pmatrix} -\frac{i}{2(n+2)} & & & \\ & \ddots & & \\ & & -\frac{i}{2(n+2)} & \\ & & & \frac{i(n+1)}{2(n+2)} \end{pmatrix},$$

and the Bochner-Kaehler manifold corresponding to this matrix is $\mathbb{C}P^n$.

6 Bochner-Kähler and Ricci-type connections duality

In this section we describe the duality between the manifolds with the Bochner-Kähler metrics of type 1. (see 4.10) and Ricci flat connections.

Recall the general construction from the section 3. So far we were interested in the case with $\mathfrak{g}_1 = \mathfrak{su}(n, 1)$. If we consider the construction for the parabolic 2-gradable algebra $\mathfrak{g}_2 := \mathfrak{sp}(n, \mathbb{R})$, we get a manifold with the connection of Ricci type. Recall the two standard embeddings of $\mathfrak{u}(n+1)$, first into $\mathfrak{su}(n+1, 1)$ (that was described in the previous section), second into $\mathfrak{sp}(n+1, \mathbb{R})$.

Theorem 6.1 *Consider the action of the Lie algebras $\mathfrak{g}_1 = \mathfrak{su}(n+1, 1)$ and $\mathfrak{g}_2 = \mathfrak{sp}(n+1, \mathbb{R})$ on the projectivized cones \hat{C}_1, \hat{C}_2 . Then the following are equivalent*

- i) *For $a_i \in \mathfrak{g}_i$ the actions of $T_{a_i} \subset G_i$ on \hat{C}_i are conjugate for $i=1,2$.*
- ii) *$a_i \in \mathfrak{u}(n+1)$, where $\mathfrak{u}(n+1) \subset \mathfrak{g}_i$ for $i = 1,2$ via the two standard embeddings.*

Proof. We have already computed the action of $\mathfrak{u}(n+1)$ on C_1 (see (21)). Observe, that the diagonalizable matrices in $\mathfrak{su}(n+1, 1)$ are the only matrices, which act on C_1 (which is isomorphic to the sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$) in the standard way (as on the vectors in \mathbb{C}^n). As for the action on C_2 we have to go quickly through the general construction (Section 3) for $\mathfrak{g} = \mathfrak{sp}(n+1)$. We have $S^2(\mathbb{R}^{2n+2}) \cong \mathfrak{sp}(n+1)$ ($(x \circ y)(z) = \omega(x, z)y + \omega(y, z)x$, for $x, y, z \in \mathbb{R}^{2n+2}$). You can prove with an easy computation as in the 4.2, that the map $x \mapsto x^2$ is the $Sp(n+1)$ -module homomorphism of the space $\mathbb{R}^{2n+2}/\mathbb{Z}_2$ and $S^2(\mathbb{R}^{2n+2})$, where the image of the morphism corresponds to rank-one elements in $\mathfrak{sp}(n+1)$. The action of $\mathfrak{sp}(n+1)$ on $\hat{C}_2 \cong \mathbb{R}^{2n+2}/\mathbb{Z}_2$ is just a standard one, and thus the action of $\mathfrak{u}(n+1) \hookrightarrow \mathfrak{sp}(n+1)$ on $C_2 = \mathbb{P}_0(\hat{C}_2) \cong \mathbb{R}P^{2n+2}$ is just a standard action on the real projective space. This is locally the same as the action of $\mathfrak{u}(n+1) \hookrightarrow \mathfrak{su}(n+1, 1)$ on the sphere C_1 . \square

Thus we get the following theorem:

Theorem 6.2

- i) Let (M, ω, ∇) be a symplectic manifold with a connection of Ricci type, and suppose that the corresponding element $A \in \mathfrak{sp}(n+1, \mathbb{R})$ from 3.4 is conjugate to an element of $\mathfrak{u}(n+1) \subset \mathfrak{sp}(n+1, \mathbb{R})$. Then M carries a canonical Bochner-Kähler metric whose Kähler form is given by ω .
- ii) Conversely, let (M, J, ω) be a Bochner-Kähler metric such that the element $a \in \mathfrak{su}(n+1, 1)$ from 3.4 is conjugate to an element of $\mathfrak{u}(n+1) \subset \mathfrak{su}(n+1, 1)$. Then (M, ω) carries a canonical connection of Ricci-type.

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