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On weighted critical imbeddings of Sobolev spaces

D. E. Edmunds, H. Hudzik and M. Krbec*

Abstract

Our concern in this paper lies with two aspects of weighted exponential spaces connected with their role of target spaces for critical imbeddings of Sobolev spaces. We characterize weights which do not change an exponential space up to equivalence of norms. Specifically, we first prove that $L_{\exp t^\alpha}(\chi_B) = L_{\exp t^\alpha}(\rho)$ if and only if $\rho^q \in L_q$ with some $q > 1$. Second, we consider the Sobolev space $W^1_N(\Omega)$, where Ω a bounded domain in $\mathbb{R}^N$ with a sufficiently smooth boundary, and its imbedding into a weighted exponential Orlicz space $L_{\exp t^{p'}}(\Omega, \rho)$, where ρ is a radial and non-increasing weight function. We show that there exists no effective weighted improvement of the standard target $L_{\exp t^{N'}}(\Omega) = L_{\exp t^{N'}}(\Omega, \chi_\Omega)$ in the sense that if $W^1_N(\Omega)$ is imbedded into $L_{\exp t^{p'}}(\Omega, \rho)$, then $L_{\exp t^{p'}}(\Omega, \rho)$ and $L\ _{expt^{N'}}(\Omega)$ coincide up to equivalence of the norms; that is, we show that there exists no effective improvement of the standard target space. The same holds for critical cases of higher-order Sobolev spaces and even Besov and Lizorkin-Triebel spaces.

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1 Introduction

The critical imbeddings of Sobolev spaces have attracted a lot of attention in recent years. Various settings of the problem have been studied in many papers by means of real and Fourier analysis. We refer to [Tri] for a fairly comprehensive list of references. The celebrated critical (or limiting) imbedding theorem [Tru], [Po], states in its basic form that the Sobolev space $W_p^m = W_p^m(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a sufficiently smooth boundary, $1 < p < \infty$, and $mp = N$, is imbedded into the Orlicz space $L_\Phi(\Omega)$ with $\Phi(t) = \exp t^{N/(N-m)} - 1$. One of the possible methods of the proof is to expand Φ into its Taylor series, taking into account the definition of the Luxemburg norm and showing that

$$\|u|L_q\| \le cq^{1-m/N}\|u|W_p^m\|, \qquad q \text{ large.} \tag{1.1}$$

This yields

$$W_p^m \hookrightarrow L_{\exp t^{N/(N-m)}} \tag{1.2}$$

since the folklore general fact is that $f \in L_{\exp t^\beta}$ if and only if

$$\|f|L_k\| \le ck^{1/\beta} \tag{1.3}$$

(see e.g. [Tru], [Tri]). Let us note that here and in the following we use the usual shorter notation $L_{\exp t^\beta}$ for L_Ψ if $\Psi(t) = \exp t^\beta - 1$.

Let us observe that condition (1.3) is equivalent to

$$\sup \frac{f^*(t)}{(\log 1/t)^{1/\beta}} \le c, \tag{1.4}$$

where f^* is the non-increasing rearrangement of f and the supremum is taken over some neighbourhood of the origin (see e.g. [BS] and refinements in [EK2]).

Moreover, it is not difficult to show that

$$\lim_{k\to\infty} \frac{\|f|L_k\|}{k^{1/\beta}} = 0$$

if $f \in E_{\exp t^\beta}$, the closure of $C_0^\infty(\Omega)$ in $L_{\exp t^\beta}$.

The condition (1.1) expresses a control of the blow up of $\|u|L_q\|$ when q tends to ∞ (it is well known that $W_p^m(\Omega)$ contains essentially unbounded functions when $mp = N$). Let us observe that the above-mentioned extrapolation results can also be expressed in the language of suitable extrapolation methods (see [M] for the definitions of the Δ and Σ methods).

There are far fewer results on the weighted limiting imbeddings; they are the subject e.g. of [KScho], [BuS]. The paper [KScho] gives necessary and sufficient conditions for critical imbeddings with weights of the special type $|x|^\alpha \log(1/|x|)^\beta$; it is based on estimates for the Riesz transform, and one of the estimates established there also gave some motivation for our considerations here concerning more general weights.

Let Ω be a bounded domain in $\mathbb{R}^N$ with a sufficiently smooth boundary $\partial\Omega$. It is enough to assume such a regularity of $\partial\Omega$ that guarantees standard Sobolev imbeddings; in particular, Lipschitz domains are admissible. A *weight* or a *weight function* will be an integrable a.e. positive function (on Ω). If ρ is a weight and Φ is a Young function, then one can consider the weighted modular

$$m(\Phi, \rho, f) = \int_\Omega \Phi(|f(x)|)\rho(x)\, dx$$

and the resulting *weighted Orlicz space* $L_\Phi(\rho) = L_\Phi(\Omega, \rho)$ (we shall mostly omit the symbol for the domain). Of course, we have $L_\Phi = L_\Phi(\chi_\Omega)$. Let us observe that the integrability of ρ ensures the very reasonable property of $L_\Phi(\rho)$ that constant functions belong to this space.

Similarly as in (1.3) we have $L_{\exp t^\beta}(\rho)$ if and only if

$$\|f|L_k(\rho)\| \le ck^{1/\beta}. \tag{1.5}$$

Our concern in this paper lies with the weighted variant of (1.1), that is,

$$\|u|L_q(\rho)\| \le cq^{1-m/N}\|u|W_p^m\|, \qquad q \text{ large}, \tag{1.6}$$

implying

$$W_p^m \hookrightarrow L_{\exp t^{N/(N-m)}}(\rho). \tag{1.7}$$

Let us explain an immediate motivation for that. Suppose for a while that ρ is a weight in Ω, belonging to some $L^{1+\varepsilon}$ with $\varepsilon > 0$, and that $\rho \ge 1$ a.e. in Ω. Then plainly $L_\Phi(\rho) \hookrightarrow L_\Phi$ for any Young function Φ. Let us make a simple estimate using just Hölder's inequality. Assume that $f \in L_{\exp}(\Omega)$.

Then

$$\left(\int_\Omega |f(x)|^k \rho(x)\,dx\right)^{1/k} \le \left(\|f|L_{k(1+\varepsilon)'}\|\,\|\rho|L_{1+\varepsilon}\|\right)^{1/k}$$
$$\le c\,(k(1+\varepsilon)')^{1-\frac{1}{k(1+\varepsilon)'}}\;\max(1,\|\rho|L_{1+\varepsilon}\|),$$

where $(1+\varepsilon)'$ denotes the exponent conjugate to $1+\varepsilon$. Hence

$$\sup_k \frac{\|f|L_k(\rho)\|}{k} \le \widetilde{c}\sup_k \frac{\|f|L_k\|}{k}$$

with a suitable $\widetilde{c}$ independent of f so that f belongs to the weighted exponential space generated by the modular

$$\int_\Omega \Phi(f(x))\rho(x)\,dx,$$

where Φ is any Young function equivalent to exp at infinity (the extrapolation characterization of the exponential space carries over easily to this weighted case). Passing to powers of f we get the appropriate estimate for exponential spaces L_{expt^β} ($\beta > 1$) and in particular for $\beta = N/(N-m)$. One can naturally ask whether this yields an improvement of the critical imbedding theorem.

Let us note that Edmunds and Triebel (see e.g. [Tri, Chapter II/13]) established the following theorem (we state here the simplified version for W^1_N): If, for some $\varepsilon_0 \in (0,1)$,

$$\left(\int_0^{\varepsilon_0}\left(\frac{f^*(t)\kappa(t)}{\log(1/t)}\right)^N \frac{dt}{t}\right)^{1/N} \le c\|f|W^1_N\|$$

for all $f \in W^1_N$ and some c independent of f, where κ is a non-increasing function on $(0,\varepsilon_0)$, then $\kappa \in L_\infty$. When omitting κ on the left hand side, we get the norm in the Brezis-Wainger space BW_N (the Lorentz-Zygmund space $L^{\infty,N;-1}$—see [BR]; in other terms a special case of a Orlicz-Lorentz space; it is known (see again e.g. [BR] or [Tri]) that $BW_N \hookrightarrow L_{\exp t^{N'}}$). Hence the theorem says that there is no imbedding improvement in the framework of weighted Lorentz-Zygmund spaces with non-increasing weights and this raises the question how the situation changes in the scale of exponential target spaces.

2 Unbounded non-effective weights

In accordance with [HK] a weight ρ will be called *non-effective* (with respect to $L_{\exp t^\beta}$) if $L_{\exp t^\beta}(\rho) = L_\Phi(\chi_\Omega)$.

First we establish a theorem on non-effective weights for exponential Orlicz spaces (cf. [HK]) using direct means.

Here and everywhere in the following we shall consider for simplicity the ball $B \subset \mathbb{R}^N$, centered at the origin, $|B| = 1$ and a weight $\rho \geq 1$ a.e. on B. Let us observe that assertions in this section hold for any subset of $\mathbb{R}^N$ with a bounded measure.

The modular on $L_{\exp t^\alpha}(\rho) = L_{\exp t^\alpha}(B, \rho)$, $(\alpha \geq 1)$ is given by

$$m(\exp t^\alpha, \rho, f) = \int_B (\exp |f(x)|^\alpha - 1)\rho(x)\, dx.$$

Hence $f \in L_{\exp t^\alpha}(\rho)$ iff there is λ such that $m\left(\exp t^\alpha, \rho, f/\lambda\right) \leq 1$, that is,

$$\int_B \exp(|f(x)|/\lambda)^\alpha \rho(x)\, dx \leq 1 + \rho(B), \tag{2.1}$$

where $\rho(B) = \int_B \rho(x)\, dx$. The infimum of all λ such that (2.1) holds is the Luxemburg norm of f in $L_{\exp t^\alpha}(\rho)$, denoted by $\|f\|_{L_{\exp t^\alpha}(\rho)}$.

Similarly we introduce the weighted Orlicz space $L_\Phi(\rho) = L_\Phi(B, \rho)$ for a general Young function Φ.

Lemma 2.1. *Let $\rho \in L^{1/(1-\varepsilon)}(B)$ for some $\varepsilon \in (0,1)$, $\rho \geq 1$ on B, and $\alpha \geq 1$. Then $L_{\exp t^\alpha}(B, \chi_B) \hookrightarrow L_{\exp t^\alpha}(B, \rho)$.*

Proof. Without loss of generality we can suppose that $\|\rho|L^{1/(1-\varepsilon)}\| \leq 1$ (one can work with a multiple of exp instead of exp because it gives the same space both in the weighted and non-weighted case (with equivalent norms). In accordance with (2.1) we wish to prove that

$$\int_B \exp\left[\left(\frac{|f(x)|}{\lambda}\right)^\alpha + \log \rho(x)\right] dx \leq 1 + \rho(B)$$

for sufficiently large λ, whenever $\|f\|_{L_{\exp t^\alpha}(\chi_B)}$ is small enough. By convexity of exp and the assumption $\rho \geq 1$ a.e. in B,

$$\begin{aligned}
&\int_B \exp\left[\frac{|f(x)|^\alpha}{\lambda} + \log\rho(x)\right] dx \\
&\qquad = \int_B \exp\left[\frac{\varepsilon|f(x)|^\alpha}{\lambda\varepsilon} + (1-\varepsilon)\log\left(\rho(x)^{1/(1-\varepsilon)}\right)\right] dx \\
&\qquad \leq \varepsilon \int_B \exp\left[\frac{|f(x)|^\alpha}{\lambda\varepsilon}\right] dx + (1-\varepsilon)\int_B \exp\log\left[\rho(x)^{1/(1-\varepsilon)}\right] dx \\
&\qquad \leq \varepsilon(1+|B|) + (1-\varepsilon)|B| \leq \varepsilon + |B| \\
&\qquad \leq 1 + \rho(B).
\end{aligned}$$

□

Lemma 2.2. *Let $\rho \geq 1$, $\rho \in L_1(B)$, and $\alpha \geq 1$. If $L_{\exp t^\alpha}(B, \chi_B) \hookrightarrow L_{\exp t^\alpha}(B, \rho)$, then there exists $\varepsilon > 0$ such that $\rho \in L^{1+\varepsilon}(B)$.*

Proof. If $\rho \in L_1(B)$ and $\rho \geq 1$, then $\left(\rho^\delta\right)^{1/\alpha} \in L_1(B)$ for $0 < \delta \leq 1$. But then also $\left[\log(\rho^\delta)\right]^{1/\alpha} \in L_{\exp}(\chi_B)$. Indeed,

$$\begin{aligned}
\int_B \exp\left(\frac{\left[\log(\rho(x)^\delta)\right]^{1/\alpha}}{\lambda}\right)^\alpha dx &= \int_B \exp\left(\frac{\log(\rho(x)^\delta}{\lambda^\alpha}\right) dx \\
&= \int_B \exp\left(\log\rho(x)^{\delta/\lambda^\alpha}\right) dx \\
&= \int_B \rho(x)^{\delta/\lambda^\alpha}\, dx \\
&\leq \rho(B)
\end{aligned}$$

provided $\delta \leq \lambda^\alpha$, that is, $\lambda \geq \delta^{1/\alpha}$.

Hence $\left[\log\rho^\delta\right]^{1/\alpha} \in L_{\exp t^\alpha}(\rho)$ according to our assumption, in other words, $m\left(\exp t^\alpha, \rho, [\log\rho^\delta]^{1/\alpha}\right) \le 1$ for sufficiently large μ. This gives

$$\begin{aligned}1+\rho(B) &\ge \int_B \exp\left[\frac{\left[\log(\rho(x)^\delta\right]^{1/\alpha}}{\mu}\right]^\alpha \rho(x)\,dx \\ &= \int_B \exp\left[\frac{\log\rho(x)^\delta}{\mu^\alpha}\right]\rho(x)\,dx \\ &= \int_B \exp\left[\log\rho(x)^{\delta/\mu^\alpha}\right]\rho(x)\,dx \\ &= \int_B \rho(x)^{1+\delta/\mu^\alpha}\,dx\end{aligned}$$

for sufficiently large μ. But this means that $\rho \in L^{1+\delta/\mu^\alpha}(B)$. □

By Lemmas 2.1 and 2.2 we have the following

Corollary 2.3. *Let $\rho \ge 1$, $\rho \in L_1(B)$ and $\alpha \ge 1$. Then $L_{\exp t^\alpha}(B, \chi_B) = L_{\exp t^\alpha}(B,\rho)$ if and only if there exists $\varepsilon > 0$ such that $\rho \in L^{1+\varepsilon}(B)$.*

3 The case of radial weights

Let ρ be a radial and non-increasing weight on B, and suppose that $\rho \ge 1$ a.e. in B. We will prove that there is no effective weighted improvement of the well-known critical imbedding (1.7) of W_p^m ($mp = N$) into the Orlicz exponential space $L_{\exp t^{N/(N-p)}}(\chi_B)$.

Here we shall work with the non-increasing rearrangements. Recall that for a function f, measurable on some set $\Omega \subset \mathbb{R}^N$, its *non-increasing rearrangement* f^* is defined as

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \le t\},$$

where

$$\mu_f(\lambda) = |\{x \in \Omega : |f(x)| > \lambda\}|, \qquad \lambda > 0.$$

Equivalently,

$$f^*(t) = \sup_{|A|=t} \inf_{x\in A} |f(x)|.$$

Of course, extending f by zero to the whole of $\mathbb{R}^N$ one arrives at the same f^*. Plainly, the N-dimensional Lebesgue measure of the support of f is equal to the one-dimensional Lebesgue measure of $\operatorname{supp} f^*$.

We shall not go into the details of the critical imbeddings theory and refer e.g. to [Tri, Chapter II].

Now we are in position to prove the claim on weighted imbeddings. For simplicity we restrict ourselves to spaces of first order—the claim for higher order derivatives can be proved in a similar manner.

Theorem 3.1. *Let ρ be a radial and non-increasing weight on B and assume that $W^1_N(B) \hookrightarrow L_{\exp t^{N'}}(B,\rho)$. Then ρ is non-effective, that is, $L_{\exp t^{N'}}(B,\rho) = L_{\exp t^{N'}}(B,\chi_B)$.*

Proof. We shall use a suitable "extremal function". For given $j \in \mathbb{N}$ there exists a non-increasing function f_j which is the non-increasing rearrangement of a function $F_j \in W^1_N$, whose norm is equivalent to 1 (independently of j), and

$$f_j(t) \geq j^{1/N'} \qquad \text{for } t \sim 2^{-jN}, \tag{3.1}$$

where the equivalence is independent of j (we write $t \sim 2^{-jN}$ for the equivalence independent of the changing parameters, i.e. $c_1 \leq 2^{jN}t \leq c_2$ with c_1 and c_2 independent of j). An appropriate F_j can be constructed as follows (cf. [Tri, Chapter II/13]). Let $\varphi \in C^\infty_0(B)$ be spherically symmetric, non-increasing, $\varphi \geq 0$ in B, and positive on a set of positive measure (a "hat function"), and denote $\varphi_i(x) = \varphi(2^{i-1}x)$. Put

$$F_j(x) = j^{-1/N} \sum_{i=1}^{j} \varphi_i(x). \tag{3.2}$$

The above-claimed properties of F_j an f_j can be proved easily (note the minor technical fact that $f_j(t) \geq \text{const.}\, j^{1/N'}$ for an arbitrary φ with the above properties; one can consider sufficiently large φ in order that this inequality holds with the constant equal to 1). Hence we have

$$\left(\int_0^{2^{-jN}} f_j(t)^{k/N'} \rho(t)\, dt\right)^{1/k} \geq j^{1/N'} \left(\int_0^{2^{-jN}} \rho(t)\, dt\right)^{1/k}$$

for every $k \in \mathbb{N}$.

Suppose that $\rho \notin \bigcup_{q>1} L^q$. Since

$$\bigcup_{q>1} L^q = \bigcup_{q>1} L^{q,\infty}$$

(it follows e.g. from the Marcinkiewicz interpolation theorem or by a simple direct calculation) we have $\rho \notin \bigcup_{q>1} L^{q,\infty}$. This implies that for given $q > 1$ there exist infinitely many j and constants $A_j \to \infty$ such that

$$\rho(2^{-jN}) \geq A_j 2^{jN/q}.$$

Hence

$$\begin{aligned}\left(\int_0^{2^{-jN}} f_j(t)^{k/N'} \rho(t)\, dt\right)^{1/k} &\geq j^{1/N'} 2^{-jN/k} A_j^{1/k} 2^{jN/(qk)}\\ &\geq j^{1/N'} 2^{jN/(qk)-jN/k} A_j^{1/k}.\end{aligned}$$

We have

$$\frac{jN}{qk} - \frac{jN}{k} = \frac{jN}{k}\left(\frac{1}{q} - 1\right).$$

For every k let us choose $q = q_k$ such that

$$\frac{1}{k}\left(\frac{1}{q_k} - 1\right) \sim -1.$$

Let us fix arbitrary k. Since $A_j \to \infty$ we have $A_j^{1/k} \to \infty$ as $j \to \infty$ and the asymptotic estimate

$$\|F_j | L_k(\rho)\| \leq \text{const.}\, k^{1/N'}$$

for the control of the blowup of the L_k norms of F_j cannot hold (the functions F_j have the W_N^1 norms equivalent to 1). This is a contradiction and theorem is proved. □

The construction of the extremal functions f_j is similar for all Sobolev (and even for general spaces of Besov and Lizorkin-Triebel type) in the critical case. For instance, considering $W_p^{N/p}$, $B_{pq}^{N/p}$ or $F_{pq}^{N/p}$ one replaces the functions in (3.1) by

$$F_j(x) = j^{-1/p} \sum_{i=1}^{j} \varphi_i(x).$$

Then we have

Corollary 3.2. *Let ρ be a radial and non-increasing weight, $1 < p < \infty$, and assume that $W_p^{N/p}(B) \hookrightarrow L_{\exp t^{p'}}(B, \rho)$. Then ρ is non-effective, that is, $L_{\exp t^{p'}}(B, \rho) = L_{\exp t^{p'}}(B, \chi_B)$.*

The same claim holds for the spaces $B_{pq}^{N/p}$ $(0 < p < \infty,\ 1 < q \leq \infty)$ or $F_{pq}^{N/p}$ $(1 < p < \infty,\ 0 < q \leq \infty)$.

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Authors' addresses:

David E. Edmunds
Department of Mathematics
University of Sussex
Falmer BN1 9QH
Brighton
U.K.
e-mail: `davideedmunds@aol.com`

Henryk Hudzik
Faculty of Mathematics and Computer Science
Adam Mickiewicz Universty
ul. Umultowska 87, 61-61 Poznań
Poland
e-mail: `hudzik@amu.edu.pl`

Miroslav Krbec
Institute of Mathematics, v.v.i.
Academy of Sciences of the Czech Republic
Žitná 25, CZ-115 67 Prague 1
Czech Republic
e-mail: `krbecm@math.cas.cz`