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NATURAL DIFFERENTIAL OPERATORS AND GRAPH COMPLEXES

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Abstract. We show how the machine invented by S. Merkulov [19, 20, 22] can be used to study and classify natural operators in differential geometry. We also give an interpretation of graph complexes arising in this context in terms of representation theory. As application, we prove several results on classification of natural operators acting on vector fields and connections.

Introduction

This work started in an attempt to understand S. Merkulov's idea of "PROP profiles" [19, 22] and see if and how it may be used to investigate natural structures in geometry. It turned out that classifications of these geometric structures in many interesting cases boiled down to calculations of the cohomology of certain graph complexes. More precisely, for a wide class of natural operators, the following principle holds.

Principle. *For a given type of natural differential operators, there exists a graph complex*

$$(\mathcal{G}\mathrm{r}^*, \delta) = (\mathcal{G}\mathrm{r}^0 \stackrel{\delta}{\to} \mathcal{G}\mathrm{r}^1 \stackrel{\delta}{\to} \mathcal{G}\mathrm{r}^2 \stackrel{\delta}{\to} \cdots)$$

such that, in stable ranges,

$$\{\textit{natural operators of a given type}\} \cong H^0(\mathcal{G}\mathrm{r}_*, \delta).$$

Stability means that the dimension of the underlying manifold is bigger than some constant explicitly determined by the type of natural operators. For example, for multilinear natural operators $TM^{\times d} \to TM$ from the d-fold product of the tangent bundle into itself the stability means that $\dim(M) \geq d$. In smaller dimensions, "exotic" operations described in [5] occur.

In all cases we studied, the corresponding graph complex appeared to be acyclic in positive dimensions, so the cohomology describing natural operators was the only nontrivial piece of the cohomology of $(\mathcal{G}\mathrm{r}_*, \delta)$. Standard philosophy of strongly homotopy structures [13] suggests that the graph complex $(\mathcal{G}\mathrm{r}^*, \delta)$ describes *stable strongly homotopy operators* of a given type.

Graph complexes arising in the Principle are in fact isomorphic to subspaces of fixed elements in suitable Chevalley-Eilenberg complexes, so, formally speaking, we claim that a certain Chevalley-Eilenberg cohomology is the cohomology of some graph complex. Instances of this

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phenomenon were systematically used by M. Kontsevich in his seminal paper [11]. The details of operadic graph complexes were then written down by J. Conant [2], J. Conant and K. Vogtmann [3, 4], M. Mulase and M. Penkava [23], M. Penkava [25], and M. Penkava and A. Schwarz [26]. What makes the Principle exciting is the miraculous fact that the corresponding graph complexes are of the type studied during the "renaissance of operads" and powerful methods developed in this period culminating in [16, 18, 21] apply.

Another way to view the proposed method is as a formalization of the "abstract tensor calculus" attributed to R. Penrose. When we studied differential geometry in kindergarten, many of us, trying to avoid being swamped by dozens of indices, draw simple pictures consisting of nodes representing tensors (which resembled little insects) and lines joining legs of these insects symbolizing contraction of indices. We attempt to put this kindergarten approach on a solid footing.

Thus the purpose of this paper is two-fold. The first one is to set up principles of abstract tensor calculus as a useful language for 'stable' geometric objects. This will be done in Sections 1–4. The logical continuation should be translating textbooks on differential geometry into this language, because all basic properties of fundamental objects (vector fields, forms, currents, connections and their torsions and curvatures) are of stable nature.

We then show, in Sections 5–7, how results on graph complexes may give explicit classifications of natural operators in stable ranges. As an example we derive from a rather deep result of [15] a characterization of operators on vector fields (Theorem 5.1 and its Corollary 5.3). As another application we prove that all natural operators on linear connections and vector fields, with values in vector fields, are freely generated by compositions of covariant derivatives and Lie brackets, and by traces of these compositions – see Theorems 7.2 and 7.6, and their Corollaries 7.3 and 7.7, in conjunction with Theorems 6.2 and 6.3. It is interesting that in the concrete geometric situations studied in this paper, the graph complexes popping out are variants of the insertion operad of [1].

The article is supplemented by an appendix in which we explain the relation between invariant tensors and graphs. We believe that the appendix, which can be read independently, will help to understand the constructions of Sections 3 and 4.

The theory of invariant operators sketched out in this paper leads to directed, not necessarily connected or simply-connected, graphs. A similar theory can be formulated also for symplectic manifolds, where the corresponding graph complexes would be those appearing in the context of anti-modular operads (modular versions of anticyclic operads, see [17, Definition 5.20]). Something very close to a symplectic version of our theory has in fact already been worked out in [28].

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1. Natural operators

Informally, a natural differential operator is a recipe that constructs from a geometric object another one, in a natural fashion, and which is locally a function of coordinates and their derivatives.

1.1. **Example.** Let M be a n-dimensional smooth manifold. The classical *Lie bracket* $X, Y \mapsto [X, Y]$ is a natural operation that constructs from two vector fields on M a third one.

Given a local coordinate system $(x^1, \ldots, x^n)$ on M, the vector fields X and Y are locally expressions

$$X = \sum_{1 \leq i \leq n} X^i \frac{\partial}{\partial x^i},\ Y = \sum_{1 \leq i \leq n} Y^i \frac{\partial}{\partial x^i},$$

where X^i, Y^i are smooth functions on M. If we define $X^i_j := \partial X^i / \partial x^j$ and $Y^i_j := \partial Y^i / \partial x^j$, $1 \leq i, j \leq n$, then the Lie bracket is locally given by the formula

$$[X, Y] = \sum_{1 \leq i,j \leq n} \left(X^j Y^i_j - Y^j X^i_j \right) \frac{\partial}{\partial x^i}.$$

In the rest of the paper, we use Einstein's convention assuming summations over repeated indices. In this context, indices $i, j, k, \ldots$ will always be natural numbers between 1 and the dimension of the underlying manifold, which will typically be denoted n.

1.2. **Example.** The *covariant derivative* $(\Gamma, X, Y) \mapsto \nabla_X Y$ is a natural operator that constructs from a linear connection Γ and a two vector fields X and Y, a vector field $\nabla_X Y$. In local coordinates,

$$\nabla_X Y = \left(\Gamma^i_{jk} X^j Y^k + X^j Y^i_j \right) \frac{\partial}{\partial x^i}, \tag{1}$$

where Γ^i_{jk} are Christoffel symbols.

Natural operations can be composed into more complicated ones. Examples of these 'composed' operations are the *torsion*

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$$

and *curvature*

$$R(X,Y)Z := \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$$

of the linear connection Γ.

1.3. **Example.** Let X be a vector field and ω a 1-form on M. Denote by $\omega(X) \in C^\infty(M)$ the evaluation of the form ω on X. Then $(X,\omega) \mapsto \exp(\omega(X))$ defines a natural differential operator with values in smooth functions. Clearly, the exponential can be replaced by an arbitrary smooth function $\varphi : \mathbb{R} \to \mathbb{R}$, giving rise to a natural operator $\mathfrak{O}_\varphi(X,\omega) := \varphi(\omega(X))$.

1.4. **Example.** 'Randomly' generated local formulas need not lead to natural operators. As we will see later, neither

$$O_1(X,Y) = X^1_3 Y^4 \frac{\partial}{\partial x^2} \ \text{ nor } \ O_2(X,Y) = X^j Y^i_j \frac{\partial}{\partial x^i}$$

behaves properly under coordinate changes, so they do not give rise to vector-field valued natural operators.

We may summarize the above examples by saying that a natural differential operator is a recipe given locally as a smooth function in coordinates and their derivatives, such that the local formula is invariant under coordinate changes. After this motivation, we give precise definitions of geometric objects and operators between them. Our exposition follows [8], see also [10].

Let us say first what we mean by a geometric object. Denote by $\mathtt{Man}_n$ the category of n-dimensional manifolds and open embeddings. Let $\mathtt{Fib}_n$ be the category of smooth fiber bundles over n-dimensional manifolds with morphisms differentiable maps covering morphisms of their bases in $\mathtt{Man}_n$.

1.5. **Definition.** A *natural bundle* is a functor $\mathfrak{B} : \mathtt{Man}_n \to \mathtt{Fib}_n$ such that for each $M \in \mathtt{Man}_n$, $\mathfrak{B}(M)$ is a bundle over M. Moreover, $\mathfrak{B}(M')$ is the restriction of $\mathfrak{B}(M)$ for each open submanifold $M' \subset M$, the morphism $\mathfrak{B}(M') \to \mathfrak{B}(M)$ induced by $M' \hookrightarrow M$ being the inclusion $\mathfrak{B}(M') \hookrightarrow \mathfrak{B}(M)$.

Let us recall a structure theorem for natural bundles due to Krupka, Palais and Terng [12, 24, 27]. For each $s \geq 1$ we denote by $\mathrm{GL}^{(s)}_n$ the group of s-jets of local diffeomorphisms $\mathbb{R}^n \to \mathbb{R}^n$ at 0, so that $\mathrm{GL}^{(1)}_n$ is the ordinary general linear group GL_n of linear invertible maps $A : \mathbb{R}^n \to \mathbb{R}^n$. Let $Fr^{(s)}(M)$ be the bundle of s-jets of frames on M whose fiber over $z \in M$ consist of s-jets of local diffeomorphisms of neighborhoods of $0 \in \mathbb{R}^n$ with neighborhoods of $z \in M$. It is clear that $Fr^{(s)}(M)$ is a principal $\mathrm{GL}^{(s)}_n$-bundle and that $Fr^{(1)}(M)$ is the ordinary GL_n-bundle of frames $Fr(M)$.

1.6. **Theorem** (Krupka, Palais, Terng). *For each natural bundle $\mathfrak{B}$, there exists $l \geq 1$ and a manifold $\mathcal{B}$ with a smooth $\mathrm{GL}_n^{(l)}$-action such that there is a functorial isomorphism*

$$\mathfrak{B}(M) \cong Fr^{(l)}(M) \times_{\mathrm{GL}_n^{(l)}} \mathcal{B} := (Fr^{(l)}(M) \times \mathcal{B})/\mathrm{GL}_n^{(l)}. \tag{2}$$

Conversely, each smooth $\mathrm{GL}_n^{(l)}$-manifold $\mathcal{B}$ induces, via (2), a natural bundle $\mathfrak{B}$. We will call $\mathcal{B}$ the *fiber* of the natural bundle $\mathfrak{B}$. If the action of $\mathrm{GL}_n^{(l)}$ on $\mathcal{B}$ does not reduce to an action of the quotient $\mathrm{GL}_n^{(l-1)}$ we say that $\mathfrak{B}$ has *order* l.

1.7. **Example.** Vector fields are sections of the *tangent bundle* $T(M)$. The fiber of this bundle is $\mathbb{R}^n$, with the standard action of GL_n. The description

$$T(M) \cong Fr(M) \times_{\mathrm{GL}_n} \mathbb{R}^n$$

is classical.

1.8. **Example.** De Rham m-forms are sections of the bundle $\Omega^m(M)$ whose fiber is the space of anti-symmetric m-linear maps $Lin(\bigwedge^m(\mathbb{R}^n), \mathbb{R})$, with the obvious induced GL_n-action. The presentation

$$\Omega^m(M) \cong Fr(M) \times_{\mathrm{GL}_n} Lin(\textstyle\bigwedge^m(\mathbb{R}^n), \mathbb{R})$$

is also classical.

A particular case is $\Omega^0(M) \cong Fr(M) \times_{\mathrm{GL}_n} \mathbb{R} \cong M \times \mathbb{R}$, the bundle whose sections are smooth functions. We will denote this natural bundle by $\mathbb{R}$, believing there will be no confusion with the symbol for the reals.

1.9. **Example.** Linear connections are sections of the *bundle of connections* $Con(M)$ [10, Section 17.7] which we recall below. Let us first describe the group $\mathrm{GL}_n^{(2)}$. Its elements are expressions of the form $A = A_1 + A_2$, where $A_1 : \mathbb{R}^n \to \mathbb{R}^n$ is a linear invertible map and A_2 is a linear map from the symmetric product $\mathbb{R}^n \odot \mathbb{R}^n$ to $\mathbb{R}^n$. The multiplication in $\mathrm{GL}_n^{(2)}$ is given by

$$(A_1 + A_2)(B_1 + B_2) := A_1(B_1) + A_1(B_2) + A_2(B_1, B_1).$$

The unit of $\mathrm{GL}_n^{(2)}$ is $id_{\mathbb{R}^n} + 0$ and the inverse is given by the formula

$$(A_1 + A_2)^{-1} = A_1^{-1} - A_1^{-1}(A_2(A_1^{-1}, A_1^{-1})).$$

Let $\mathcal{C}$ be the space of linear maps $Lin(\mathbb{R}^n \otimes \mathbb{R}^n, \mathbb{R}^n)$, with the left action of $\mathrm{GL}_n^{(2)}$ given as

$$(Af)(u \otimes v) := A_1 f(A_1^{-1}(u), A_1^{-1}(v)) - A_2(A_1^{-1}(u), A_1^{-1}(v)), \tag{3}$$

for $f \in Lin(\mathbb{R}^n \otimes \mathbb{R}^n, \mathbb{R}^n)$, $A = A_1 + A_2 \in \mathrm{GL}_n^{(2)}$ and $u, v \in \mathbb{R}^n$. The bundle of connections is then the order 2 natural bundle represented as

$$Con(M) := Fr^{(2)}(M) \times_{\mathrm{GL}_n^{(2)}} \mathcal{C}.$$

Observe that, while the action of $\mathrm{GL}_n^{(2)}$ on the vector space $\mathcal{C}$ is not linear, the restricted action of $\mathrm{GL}_n \subset \mathrm{GL}_n^{(2)}$ on $\mathcal{C}$ is the standard action of the general linear group on the space of bilinear maps.

For $k \geq 0$ we denote by $\mathfrak{B}^{(k)}$ the bundle of k-jets of local sections of the natural bundle $\mathfrak{B}$ so that $\mathfrak{B}^{(0)} = \mathfrak{B}$. If $\mathfrak{B}$ is represented as in (2), then

$$\mathfrak{B}^{(k)}(M) \cong Fr^{(k+l)}(M) \times_{\mathrm{GL}_n^{(k+l)}} \mathcal{B}^{(k)},$$

where $\mathcal{B}^{(k)}$ is the space of k-jets of local diffeomorphisms $\mathbb{R}^n \to \mathcal{B}$ defined in a neighborhood of $0 \in \mathbb{R}^n$.

1.10. **Definition.** Let $\mathfrak{F}$ and $\mathfrak{G}$ be natural bundles. A (finite order) *natural differential operator* $\mathfrak{O} : \mathfrak{F} \to \mathfrak{G}$ is a natural transformation (denoted by the same symbol) $\mathfrak{O} : \mathfrak{F}^{(k)} \to \mathfrak{G}$, for some $k \geq 1$. We denote the space of all natural differential operators $\mathfrak{F} \to \mathfrak{G}$ by $\mathfrak{Nat}(\mathfrak{F}, \mathfrak{G})$.

If $\mathfrak{F}$ and $\mathfrak{G}$ are natural bundles of order $\leq l$, with fibers $\mathcal{F}$ and $\mathcal{G}$, respectively, then each natural operator in the above definition is induced by an $\mathrm{GL}_n^{(k+l)}$-equivariant map $O : \mathcal{F}^{(k)} \to \mathcal{G}$, for some $k \geq 0$. Conversely, such an equivariant map induces a natural operator $\mathfrak{O} : \mathfrak{F} \to \mathfrak{G}$. This means that the study of natural operators between natural bundles is reduced to the study of equivariant maps. The procedure described above is therefore called the *IT reduction* (from invariant-theoretic).

From this moment on, we impose the following assumptions on natural bundles $\mathfrak{F}$, $\mathfrak{G}$ an operators $\mathfrak{O} : \mathfrak{F} \to \mathfrak{G}$ between them.

- **A1** The fibers $\mathcal{F}$ and $\mathcal{G}$ of the bundles $\mathfrak{F}$ and $\mathfrak{G}$ are vector spaces and the restricted actions of $\mathrm{GL}_n \subset \mathrm{GL}_n^{(l)}$ on $\mathcal{F}$ and $\mathcal{G}$ are rational linear representations,
- **A2** the action of $\mathrm{GL}_n^{(l)}$ on the fiber $\mathcal{G}$ of $\mathfrak{G}$ is linear, and
- **A3** we consider only *polynomial* differential operators for which the induced map of the fibers $O : \mathcal{F}^{(k)} \to \mathcal{G}$ is a polynomial map.

Observe that we do not require the action of the *full* group $\mathrm{GL}_n^{(l)}$ on the fiber of $\mathfrak{F}$ to be linear. Assumption A2 is needed for the cohomology in Theorem 2.2 in Section 2 to be well-defined, assumptions A1 and A3 are necessary to relate this cohomology to a graph complex.

Polynomiality A3 rules out operators as $\mathfrak{O}_\varphi$ from Example 1.3. There is probably no systematic way how to study operators of this type – imagine that φ is an arbitrary, not even real analytic, smooth function. Clearly most if not all "natural" natural operators considered in differential geometry are polynomial, so assumption A3 seems to be justified. As argued in [10, Section 24] and as we will also see later in Remarks 5.2 and 7.1, in some situations the operators possess a certain homogeneity which automatically implies polynomiality.

1.11. **Example.** Given natural bundles $\mathfrak{B}'$ and $\mathfrak{B}''$ with fibers $\mathcal{B}'$ resp. $\mathcal{B}''$, there is an obviously defined natural bundle $\mathfrak{B}' \times \mathfrak{B}''$ with fiber $\mathcal{B}' \times \mathcal{B}''$. With this notation, the Lie bracket is a natural operator

$$[-,-] : T \times T \to T$$

and the covariant derivative an operator

$$\nabla : Con \times T \times T \to T,$$

where T is the tangent space functor and Con the bundle of connections recalled in Example 1.9. The corresponding equivariant maps of fibers can be easily read off from local formulas given in Examples 1.1 and 1.2.

1.12. **Example.** The operator $\mathfrak{O}_\varphi : T \times \Omega^1 \to C^\infty$ from Example 1.3 is induced by the GL_n-equivariant map $O_\varphi : \mathbb{R}^n \times \mathbb{R}^{n*} \to \mathbb{R}$ given by $o_\varphi(v, \alpha) := \varphi(\alpha(v))$. Clearly, $\mathfrak{O}_\varphi$ satisfies A3 if and only if $\varphi : \mathbb{R} \to \mathbb{R}$ is a polynomial.

2. Natural operators and cohomology

We start this section by a brief recollection of two classical constructions. For a Lie algebra $\mathfrak{h}$ and a $\mathfrak{h}$-module W, the *Chevalley-Eilenberg* cohomology $H^*(\mathfrak{h}, W)$ of $\mathfrak{h}$ with coefficients in W is the cohomology of the cochain complex $(C^*(\mathfrak{h}, W), \delta_{CE})$ defined by

$$C^m(\mathfrak{h}, W) := Lin({\textstyle\bigwedge^m} \mathfrak{h}, W),\ m \geq 0,$$

with δ_{CE} the sum $\delta_{CE} = \delta_1 + \delta_2$, where

$$(\delta_1 f)(h_1, \ldots, h_{m+1}) := \sum_{1 \leq i \leq m+1} (-1)^{i+1} \cdot h_i f(h_1, \ldots, \hat{h}_i, \ldots, h_{m+1}) \text{ and} \tag{4}$$

$$(\delta_2 f)(h_1, \ldots, h_{m+1}) := \sum_{1 \leq i < j \leq m+1} (-1)^{i+j} \cdot f([h_i, h_j], h_1, \ldots, \hat{h}_i, \ldots, \hat{h}_j, \ldots, h_{m+1}), \tag{5}$$

for $f \in C^m(\mathfrak{h}, W)$, $h_1, \ldots, h_{m+1} \in \mathfrak{h}$ and $\hat{}$ denoting the omission. If $m = 0$, the summation in the right hand side of (5) runs over the empty set, so we put $(\delta_2 f)(h) := 0$ for $f \in C^0(\mathfrak{h}, W)$.

The second notion we need to recall is the semidirect product of groups. Assume that G and H are Lie groups, with G acting on H by homomorphisms. One then defines the *semidirect product* $G \rtimes H$ as the Cartesian product $G \times H$ with the multiplication

$$(g_1, h_1)(g_2, h_2) := (g_1 g_2, g_2^{-1}(h_1) h_2),\ g_1, g_2 \in G,\ h_1, h_2 \in H.$$

Both G and H are subgroups of $G \rtimes H$ and their union $G \cup H$ generates $G \rtimes H$. Let us close this introductory part by formulating a proposition that ties the above two constructions together.

If W is a left $G \rtimes H$-module, the inclusion $H \subset G \rtimes H$ induces a left H-action on W which in turn induces an infinitesimal action of $\mathfrak{h}$ on W. One may therefore consider the cochain complex $(C^*(\mathfrak{h}, W), \delta_{CE})$.

Since G acts by homomorphisms, the unit of H is G-fixed, so there is an induced action of G on the Lie algebra $\mathfrak{h}$ of H. The group G acts also on W, via the inclusion $G \subset G \rtimes H$. These two actions give rise, in the usual way, to an action of G on $C^*(\mathfrak{h}, W)$. Let us denote $C^*_G(\mathfrak{h}, W)$ the subspace of G-fixed elements of $C^*(\mathfrak{h}, W)$. We have the following:

2.1. **Proposition.** *The subspace of fixed elements $C^*_G(\mathfrak{h}, W) \subset C^*(\mathfrak{h}, W)$ is δ_{CE}-closed, so it makes sense to consider the cohomology $H^*_G(\mathfrak{h}, W) := H^*(C^*_G(\mathfrak{h}, W), \delta_{CE})$. For H connected, there is an isomorphism*

$$H^0_G(\mathfrak{h}, W) \cong W^{G \rtimes H}, \tag{6}$$

where $W^{G\rtimes H}$ denotes, as usual, the space of $G \rtimes H$-fixed elements in W.

Proof. We leave a direct verification of the δ_{CE}-closeness of $C^*_G(\mathfrak{h}, W)$ as a simple exercise to the reader. It is equally easy to see that $H^0_G(\mathfrak{h}, W)$ consists of elements of W which are simultaneously G-fixed and $\mathfrak{h}$-invariant. If H is connected, the exponential map is an epimorphism, thus $\mathfrak{h}$-invariant elements in W are precisely those which are H-fixed. This, along with the fact that $G \cup H$ generates $G \rtimes H$, gives (6). □

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In Section 1 we recalled that natural differential operators $\mathfrak{O} \in \mathfrak{Nat}(\mathfrak{F}, \mathfrak{G})$ between natural bundles of order $\leq l$ with fibers $\mathcal{F}$ resp. $\mathcal{G}$, correspond to $\mathrm{GL}^{(k+l)}_n$-equivariant maps $O : \mathcal{F}^{(k)} \to \mathcal{G}$ with some $k \geq 0$. This can be expressed by the isomorphism:

$$\mathfrak{Nat}(\mathfrak{F}, \mathfrak{G}) \cong \bigcup_{k\geq 0} Map_{\mathrm{GL}^{(k+l)}_n}(\mathcal{F}^{(k)}, \mathcal{G}), \tag{7}$$

where $Map_{\mathrm{GL}^{(k+l)}_n}(\mathcal{F}^{(k)}, \mathcal{G})$ is the space of polynomial $\mathrm{GL}^{(k+l)}_n$-equivariant maps $\mathcal{F}^{(k)} \to \mathcal{G}$ – see assumption A3 on page 6. The space $Map(\mathcal{F}^{(k)}, \mathcal{G})$ of all polynomial maps has the standard $\mathrm{GL}^{(k+l)}_n$-action induced from the actions on $\mathcal{F}^{(k)}$ and $\mathcal{G}$. As usual, the space of equivariant maps is the fixed subspace

$$Map_{\mathrm{GL}^{(k+l)}_n}(\mathcal{F}^{(k)}, \mathcal{G}) = Map(\mathcal{F}^{(k)}, \mathcal{G})^{\mathrm{GL}^{(k+l)}_n}.$$

Let us see how Proposition 2.1 describes these spaces.

The crucial observation is that $\mathrm{GL}^{(s)}_n$ is, for each $s \geq 1$, a semidirect product [10, Section 13]. If $(\mathbb{R}^n)^{\odot r}$ denotes the rth symmetric power of $\mathbb{R}^n$, $r \geq 1$, then elements of $\mathrm{GL}^{(s)}_n$ are expressions

$$A = A_1 + A_2 + A_3 + \cdots + A_s,\ \ A_i \in Lin\left((\mathbb{R}^n)^{\odot i}, \mathbb{R}^n\right),\ \ 1 \leq i \leq s,$$

such that $A_1 : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. The space $Lin\left((\mathbb{R}^n)^{\odot i}, \mathbb{R}^n\right)$ is of course canonically isomorphic to the space $Sym\left((\mathbb{R}^n)^{\otimes i}, \mathbb{R}^n\right)$ of symmetric multilinear maps. We will identify these two spaces in the sequel. We leave as an exercise to write formulas for the product and inverse; for $s = 2$ it was done in Example 1.9.

Denote by $\mathrm{NGL}_n^{(s)}$ the prounipotent radical of $\mathrm{GL}_n^{(s)}$,

$$\mathrm{NGL}_n^{(s)} = \{A = A_1 + A_2 + A_3 + \cdots + A_s \in \mathrm{NGL}_n^{(s)};\ A_1 = id\}.$$

Then $\mathrm{GL}_n^{(s)}$ is the semidirect product

$$\mathrm{GL}_n^{(s)} = \mathrm{GL}_n \rtimes \mathrm{NGL}_n^{(s)},$$

with GL_n acting on $\mathrm{NGL}_n^{(s)}$ by adjunction. Denote finally $\mathfrak{ngl}_n^{(s)}$ the Lie algebra of $\mathrm{NGL}_n^{(s)}$,

$$\mathfrak{ngl}_n^{(s)} = \{a = a_2 + a_3 + \cdots + a_s;\ \ a_i \in \mathit{Sym}\left((\mathbb{R}^n)^{\otimes i}, \mathbb{R}^n\right),\ 2 \le i \le s\}. \tag{8}$$

Assume that the action of $\mathrm{GL}_n^{(l)}$ on the fiber $\mathcal{G}$ of $\mathfrak{G}$ is linear. Then $\mathit{Map}(\mathcal{F}^{(k)}, \mathcal{G})$ is a linear representation of $\mathrm{GL}_n^{(k+l)}$ and Proposition 2.1 applied to $G = \mathrm{GL}_n$, $H = \mathrm{NGL}_n^{(k+l)}$ and $W = \mathit{Map}(\mathcal{F}^{(k)}, \mathcal{G})$ gives

$$\mathit{Map}_{\mathrm{GL}_n^{(k+l)}}(\mathcal{F}^{(k)}, \mathcal{G}) \cong H^0_{\mathrm{GL}_n}\left(\mathfrak{ngl}_n^{(k+l)}, \mathit{Map}(\mathcal{F}^{(k)}, \mathcal{G})\right). \tag{9}$$

For each $k \ge 0$, the inclusion $\mathit{Map}(\mathcal{F}^{(k)}, \mathcal{G}) \hookrightarrow \mathit{Map}(\mathcal{F}^{(k+1)}, \mathcal{G})$ together with the projection $\mathfrak{ngl}_n^{(k+l+1)} \to \mathfrak{ngl}_n^{(k+l)}$ induces a GL_n-invariant inclusion

$$C^*(\mathfrak{ngl}_n^{(k+l)}, \mathit{Map}\left(\mathcal{F}^{(k)}, \mathcal{G}\right)) \hookrightarrow C^*\left(\mathfrak{ngl}_n^{(k+l+1)}, \mathit{Map}(\mathcal{F}^{(k+1)}, \mathcal{G})\right)$$

which commutes with the differentials. Let us denote

$$C^*\left(\mathfrak{ngl}_n^{(\infty)}, \mathit{Map}(\mathcal{F}^{(\infty)}, \mathcal{G})\right) := \bigcup_{k\ge 0}\bigcup_{l \ge 1} C^*\left(\mathfrak{ngl}_n^{(k+l)}, \mathit{Map}(\mathcal{F}^{(k)}, \mathcal{G})\right) \tag{10}$$

and $C^*_{\mathrm{GL}_n}\left(\mathfrak{ngl}_n^{(\infty)}, \mathit{Map}(\mathcal{F}^{(\infty)}, \mathcal{G})\right)$ the GL_n-stable subspace of $C^*\left(\mathfrak{ngl}_n^{(\infty)}, \mathit{Map}(\mathcal{F}^{(\infty)}, \mathcal{G})\right)$. Finally, let

$$H^*_{\mathrm{GL}_n^{(\infty)}}\left(\mathfrak{ngl}_n^{(\infty)}, \mathit{Map}(\mathcal{F}^{(\infty)}, \mathcal{G})\right) := H^*(C^*_{\mathrm{GL}_n}\left(\mathfrak{ngl}_n^{(\infty)}, \mathit{Map}(\mathcal{F}^{(\infty)}, \mathcal{G})), \delta_{CE}\right).$$

Then (7) together with (9) and the fact that cohomology commutes with direct limits implies:

2.2. **Theorem.** *Let $\mathfrak{F}$ and $\mathfrak{G}$ be natural bundles with fibers $\mathcal{F}$ resp. $\mathcal{G}$ of orders $\le l$. Suppose that the action of $\mathrm{GL}_n^{(l)}$ on $\mathcal{G}$ is linear. Then, under the above notation*

$$\mathfrak{Nat}(\mathfrak{F}, \mathfrak{G}) \cong H^0_{\mathrm{GL}_n}\left(\mathfrak{ngl}_n^{(\infty)}, \mathit{Map}(\mathcal{F}^{(\infty)}, \mathcal{G})\right). \tag{11}$$

In the following sections we show that, in many interesting cases, the cohomology in the right hand side of (11) is the cohomology of a certain graph complex.

3. Natural operators and graphs

We are going to describe natural differential operators by certain spaces spanned by graphs. Roughly speaking, graphs, viewed as contraction schemes for indices, will encode elementary GL_n-invariant tensors in (10). Our approach is based on a translation of the Invariant Tensor Theorem into the graph language which we formulate in the Appendix. Although this translation is probably known to experts, we were not able to find a suitable reference providing all necessary details. We believe that the Appendix will help to understand the following sections.

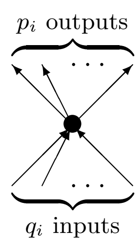
Suppose that $\mathfrak{B}$ is a natural bundle satisfying A1 on page 6, so that the induced action of $\mathrm{GL}_n \subset \mathrm{GL}_n^{(l)}$ on the fiber $\mathcal{B}$ is rational linear. According to standard facts of the representation theory of GL_n recalled, for instance, in [8, § 1.4], an equivalent assumptions is that, as a GL_n-module, $\mathcal{B}$ is the direct sum of GL_n-modules

$$\mathcal{B} = \bigoplus_{1 \leq i \leq b} \mathcal{B}_i, \tag{12}$$

where $\mathcal{B}_i$ is, for each $1 \leq i \leq b$, either the space $\mathit{Lin}(\mathbb{R}^{n\otimes q_i}, \mathbb{R}^{n\otimes p_i})$ for some $p_i, q_i \geq 0$, with the standard GL_n-action, or a subspace of this space consisting of maps whose inputs and/or outputs have a specific symmetry, which can for example be expressed by a Young diagram.

In other words, $\mathcal{B}_i$ are spaces of multilinear maps whose coordinates are tensors $T^{a_1,\ldots,a_{p_i}}_{b_1,\ldots,b_{q_i}}$ with q_i input indices and p_i output indices, which may or may not enjoy some kind of symmetry. We will graphically represent these tensors as corollas with q_i-inputs and p_i outputs:

(13)



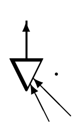
.

Instead of the dot $\bullet$ we may sometime use different symbols for the node, such as ∇, $\blacksquare$, $\circ$, &c.

3.1. **Example.** The fiber of the tangent bundle T is $\mathbb{R}^n = \mathit{Lin}(\mathbb{R}^{n\otimes 0}, \mathbb{R}^{n\otimes 1})$, so one has in (12) $b = 1$, $p_1 = 1$, $q_0 = 0$. Elements of the fiber of T are tensors X^a symbolized by

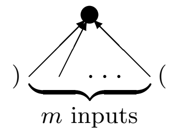
.

The fiber $\mathcal{C}$ of the connection bundle Con (see Example 1.9) is $\mathit{Lin}(\mathbb{R}^{n\otimes 2}, \mathbb{R}^{n\otimes 1})$, therefore $b = 1$, $p_1 = 1$ and $q_1 = 2$. Elements of $\mathcal{C}$ are GL_n-tensors (Christoffel symbols) Γ^a_{bc} represented by



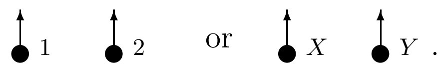
.

An example with a(n anti-)symmetry is the bundle Ω^m of de Rham m-forms, $m \geq 0$. Its fiber is the space $Lin(\bigwedge^m \mathbb{R}^n, \mathbb{R}^{n\otimes 0}) = Lin(\bigwedge^m \mathbb{R}^n, \mathbb{R})$ of anti-symmetric tensors $\omega_{)b_1,\ldots,b_m(}$ which in our pictorial language will be represented as



with inverted braces indicating antisymmetry.

Sometimes we will need decorations of nodes. For example, the product bundle $T \times T$ has fiber $\mathbb{R}^n \times \mathbb{R}^n$ generated by tensors X^a, Y^a which will be denoted



Let again $\mathfrak{B}$ be a natural bundle with fiber $\mathcal{B}$ decomposed as in (12). It is easy to see that the fiber $\mathcal{B}^{(k)}$ of the k-jet bundle $\mathfrak{B}^{(k)}$ decomposes, as a GL_n-module, into

$$\mathcal{B}^{(k)} = \bigoplus_{1 \leq i \leq b} \mathcal{B}_i^{(k)},$$

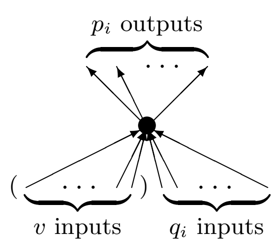
where

$$\mathcal{B}_i^{(k)} = \bigoplus_{0 \leq v \leq k} Sym(\mathbb{R}^{n\otimes v}, \mathbb{R}) \otimes \mathcal{B}_i. \tag{14}$$

This means that if elements of $\mathcal{B}_i$ are tensors $T^{a_1,\ldots,a_{p_i}}_{b_1,\ldots,b_{q_i}}$, then elements of $\mathcal{B}_i^{(k)}$ are tensors

$$_{(s_1,\ldots,s_v)}T^{a_1,\ldots,a_{p_i}}_{b_1,\ldots,b_{q_i}},\ v \leq k,$$

with braces indicating the symmetry in $(s_1, \ldots, s_v)$. In terms of pictures this amounts to adding new symmetric inputs to corollas (13), so elements of $\mathcal{B}_i^{(k)}$ will be symbolized by

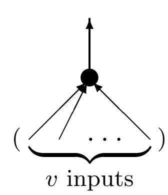
(15) 

$, v \leq k.$

3.2. **Example.** The fiber of the kth tangent bundle $T^{(k)}$ is the space of tensors

$$X^a_{(s_1,\ldots,s_v)} := \frac{\partial^u X^a}{\partial x^{s_1} \cdots \partial x^{s_v}},\ v \leq k, \tag{16}$$

which we draw as

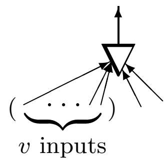
(17) 

$, v \leq k.$

The fiber of the bundle $Con^{(k)}$ is the space of tensors

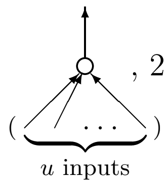
$$_{(s_1,\ldots,s_v)}\Gamma^a_{bc} := \frac{\partial^u \Gamma^a_{bc}}{\partial x^{s_1} \cdots \partial x^{s_v}},\ v \leq k,$$

which we depict as

(18) 

$, v \leq k.$

As follows from (8), $\mathfrak{ngl}_n^{(k+l)} = \bigoplus_{2\leq u\leq k+l} Sym(\mathbb{R}^{n\otimes u}, \mathbb{R}^n)$. Therefore $\mathfrak{ngl}_n^{(k+l)}$ is the space of symmetric tensors $\varphi^b_{(s_1,\ldots,s_u)}$, $2 \leq u \leq k+l$, or in pictures,

(19) 

$, 2 \leq u \leq k+l.$

In what follows, white corollas (19) will *always denote* elements of $\mathfrak{ngl}_n^{(k+l)}$ for some $k+l \geq 2$.

– – – – –

In the rest of this section we construct a graded space $\mathcal{G}\mathrm{r}^*_{\mathfrak{F},\mathfrak{G}}$ spanned by graphs representing GL_n-invariant cochains in $C^*_{\mathrm{GL}_n}\left(\mathfrak{ngl}_n^{(\infty)}, Map(\mathcal{F}^{(\infty)}, \mathcal{G})\right)$. Since

$$C^0_{\mathrm{GL}_n}\left(\mathfrak{ngl}_n^{(\infty)}, Map(\mathcal{F}^{(\infty)}, \mathcal{G})\right) = Map_{\mathrm{GL}_n}(\mathcal{F}^{(\infty)}, \mathcal{G}),$$

this in particular means that we describe equivariant maps from $Map_{\mathrm{GL}_n}(\mathcal{F}^{(\infty)}, \mathcal{G})$ by linear combinations of graphs. The differentials will be studied in the next section.

Suppose that the natural bundles $\mathfrak{F}$ and $\mathfrak{G}$ satisfy assumption A1 on page 6, and see what can be said about the space $C^m_{\mathrm{GL}_n}\left(\mathfrak{ngl}_n^{(k+l)}, Map(\mathcal{F}^{(k)}, \mathcal{G})\right)$ of GL_n-equivariant polynomial maps from $Map(\mathfrak{ngl}_n^{(k+l)} \times \mathcal{F}^{(k)}, \mathcal{G})$ that are m-homogeneous and antisymmetric in $\mathfrak{ngl}_n^{(k+l)}$. By the polynomiality assumption A3,

$$C^m_{\mathrm{GL}_n}\left(\mathfrak{ngl}_n^{(k+l)}, Map(\mathcal{F}^{(k)}, \mathcal{G})\right) \cong \bigoplus_{t\geq 0} Lin_{\mathrm{GL}_n}\left(\bigwedge\nolimits^m \mathfrak{ngl}_n^{(k+l)} \otimes {\mathcal{F}^{(k)}}^{\otimes t}, \mathcal{G}\right), \tag{20}$$

where $Lin_{\mathrm{GL}_n}(-,-)$ denotes the space of GL_n-equivariant linear maps.

Let us decompose the fibers $\mathcal{F}$ and $\mathcal{G}$ of natural bundles $\mathfrak{F}$ and $\mathfrak{G}$ into the direct sum (12),

$$\mathcal{F} = \bigoplus_{1\leq i\leq f} \mathcal{F}_i \text{ and } \mathcal{G} = \bigoplus_{1\leq i\leq g} \mathcal{G}_i.$$

By (14), the components of the fiber $\mathcal{F}^{(k)}$ of the k-jet bundle $\mathfrak{F}^{(k)}$, $k \geq 0$, are the direct sums

$$\mathcal{F}_i^{(k)} = \bigoplus_{0\leq v\leq k} \mathcal{F}_i^{[v]},\ 1 \leq i \leq f,$$

with $\mathcal{F}_i^{[v]} := Sym(\mathbb{R}^{n\otimes v}, \mathbb{R}^n) \otimes \mathcal{F}_i$. Using the above decompositions and description (8) of $\mathfrak{ngl}_n^{(k+l)}$, one can rewrite the right hand side of (20) into

$$\bigoplus_{t\geq 0} \bigoplus_{S(k,l,t)} Lin_{\mathrm{GL}_n}\left(\bigwedge_{1\leq i\leq m} Sym(\mathbb{R}^{n\otimes u_i}, \mathbb{R}^n) \otimes \bigotimes_{1\leq s\leq t} \mathcal{F}_{i_s}^{[v_s]}, \mathcal{G}_i\right), \tag{21}$$

where $S(k,l,t)$ is the set of integers $u_1, \ldots, u_m$, $i_1, \ldots, i_t$, $v_1, \ldots, v_t$ and i such that

$$2 \leq u_1, \ldots, u_m \leq k+l,\ 1 \leq i_1, \ldots, i_t \leq f,\ 0 \leq v_1, \ldots, v_t \leq k \ \text{ and } 1 \leq i \leq g.$$

Let us fix a multiindex $\omega = (u_1, \ldots, u_m, i_1, \ldots, i_t, v_1, \ldots, v_t, i) \in S(k,l,t)$. By our assumptions, the space $\mathcal{F}_{i_s}^{[v_s]}$ is, for each $1 \leq s \leq t$, isomorphic to the space $Lin_{\mathfrak{I}_s}^{\mathfrak{O}_s}(\mathbb{R}^{n\otimes(v_s+q_{i_s})}, \mathbb{R}^{n\otimes p_{i_s}})$ of linear maps having a symmetry specified by subsets $\mathfrak{I}_s \subset \mathbf{k}[\Sigma_{v_s+q_{i_s}}]$, $\mathfrak{O}_s \subset \mathbf{k}[\Sigma_{p_{i_s}}]$, see Remark 8.6 of the Appendix for the notation. Similarly, $\mathcal{G}_i \cong Lin_{\mathfrak{I}}^{\mathfrak{O}}(\mathbb{R}^{n\otimes c}, \mathbb{R}^{n\otimes d})$, for some $c, d \geq 0$ and subsets $\mathfrak{I} \subset \mathbf{k}[\Sigma_c]$, $\mathfrak{O} \subset \mathbf{k}[\Sigma_d]$. The expression

$$Lin_{\mathrm{GL}_n}\left(\bigwedge_{1\leq i\leq m} Sym(\mathbb{R}^{n\otimes u_i}, \mathbb{R}^n) \otimes \bigotimes_{1\leq s\leq t} \mathcal{F}_{i_s}^{[v_s]}, \mathcal{G}_i\right) \tag{22}$$

in (21) is therefore isomorphic to

$$Lin_{\mathrm{GL}_n}\left(\bigwedge_{1\leq i\leq m} Sym(\mathbb{R}^{n\otimes u_i}, \mathbb{R}^n) \otimes \bigotimes_{1\leq s\leq t} Lin_{\mathfrak{I}_s}^{\mathfrak{O}_s}(\mathbb{R}^{n\otimes(v_s+q_{i_s})}, \mathbb{R}^{n\otimes p_{i_s}}), Lin_{\mathfrak{I}}^{\mathfrak{O}}(\mathbb{R}^{n\otimes c}, \mathbb{R}^{n\otimes d})\right). \tag{23}$$

Let us remark that in all applications discussed in this paper, we will always have $p_{i_s} = 1$ for $1 \leq s \leq t$, $c = 0$ and $d = 1$.

Observe that (23) is the space in (81) of the Appendix, with an appropriate choice of the parameters, which in this case is $r := t - m$, and

$$\begin{aligned} h_i &:= u_i && \text{for } 1 \leq i \leq m, \text{ and} \\ h_i := v_s + q_{i_s},\ \mathfrak{I}_i &:= \mathfrak{I}_s,\ \mathfrak{O}_i := \mathfrak{O}_s && \text{for } i = s+m,\ 1 \leq s \leq t, \end{aligned}$$

therefore the methods developed in the Appendix apply. We believe that the reader can tolerate a certain incompatibility between the notation used in this section and the notation of the Appendix – the alphabet does not have enough letters to avoid notational conflicts.

By Proposition 8.10 and Remark 81 of the Appendix, the space (23) is related to the space $\mathcal{G}\mathrm{r}_\omega^m$ spanned by graphs with vertices of three types:

1st type: t 'black' vertices (15) with $p_i := p_{i_s}$, $q_i := q_{i_s}$ and $v := v_s$, representing tensors in $\mathcal{F}_{j_s}^{[v_s]}$, $1 \leq s \leq t$,

2nd type: one vertex (13) with $p_i := c$ and $q_i := d$ called the *anchor*, representing tensors in the dual $\mathcal{G}_i^*$ of $\mathcal{G}_i$, and

3rd type: m 'white' vertices (19) with $u := u_i$ representing generators of the Lie algebra $\mathfrak{ngl}_n^{(k+l)}$, $1 \leq i \leq m$.

Our graphs are directed and oriented, where an *orientation* is, by definition, an equivalence class of linear orders of the set of white vertices, modulo the relation identifying orders that differ by an even number of transpositions. If the orientations of two graphs G' and G'' differ by an odd number of transpositions, we put $G' = -G''$ in $\mathcal{G}\mathrm{r}^m_\omega$. Our notion of orientation is not the traditional one but resembles orientations in various graph complexes [17, § II.5.5].

Let us emphasize that the graphs spanning $\mathcal{G}\mathrm{r}^m_\omega$ are not required to be connected, and multiple edges and loops are allowed . The vertices above are Merkulov's *genes* [22]. The unique vertex of the 2nd type marks the place where we evaluate the composition along the graph at an element of $\mathcal{G}^*$, which explains the dualization in the definition of this vertex.

Proposition 8.10 (or its obvious extension mentioned in Remark 8.12), combined with the isomorphism between (22) and (23), gives an epimorphism

$$R^m_{n,\omega} : \mathcal{G}\mathrm{r}^m_\omega \twoheadrightarrow Lin_{\mathrm{GL}_n}\left(\bigwedge_{1\leq i\leq m} Sym(\mathbb{R}^{n\otimes u_i}, \mathbb{R}^n) \otimes \bigotimes_{1\leq s\leq t} \mathcal{F}^{[v_s]}_{i_s}, \mathcal{G}_i\right) \tag{24}$$

which is, by Proposition 8.11, a monomorphism if $n + m \geq$ the number of edges of graphs in $\mathcal{G}\mathrm{r}^m_\omega$.

The central result of this section, Theorem 3.3 below, uses the limit

$$\mathcal{G}\mathrm{r}^m_{\mathfrak{F},\mathfrak{G}} := \bigcup_{k\geq 0}\bigcup_{l\geq 1}\bigoplus_{t\geq 0}\bigoplus_{\omega\in S(k,l,t)} \mathcal{G}\mathrm{r}^m_\omega. \tag{25}$$

The space $\mathcal{G}\mathrm{r}^m_{\mathfrak{F},\mathfrak{G}}$ is spanned by graphs with an arbitrary number of the 1st type vertices with an arbitrary $v \geq 0$ in (15), one 2nd type vertex representing tensors in $\mathcal{G}^*_i$ for $1 \leq i \leq g$, and m 3rd type vertices with an arbitrary $u \geq 2$ in (19).

3.3. **Theorem.** *The epimorphisms $R^m_{n,\omega}$ in (24) assemble, for each $m \geq 0$, into a surjection*

$$R^m_n : \mathcal{G}\mathrm{r}^m_{\mathfrak{F},\mathfrak{G}} \twoheadrightarrow C^m_{\mathrm{GL}_n}\left(\mathfrak{ngl}^{(\infty)}_n, Map(\mathcal{F}^{(\infty)}, \mathcal{G})\right). \tag{26}$$

The restriction

$$R^m_n(e) : \mathcal{G}\mathrm{r}^m_{\mathfrak{F},\mathfrak{G}}(e) \to C^m_{\mathrm{GL}_n}\left(\mathfrak{ngl}^{(\infty)}_n, Map(\mathcal{F}^{(\infty)}, \mathcal{G})\right)$$

of the map R^m_n to the subspace $\mathcal{G}\mathrm{r}^m_{\mathfrak{F},\mathfrak{G}}(e) \subset \mathcal{G}\mathrm{r}^m_{\mathfrak{F},\mathfrak{G}}$ spanned by graphs with $\leq e$ edges, is a monomorphism whenever $n = \dim(M) \geq e - m$.

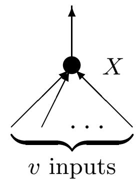
Proof. The maps $R^m_{n,\omega}$ of (24) assemble, for each $k \geq 0$ and $l \geq 1$, into an epimorphism

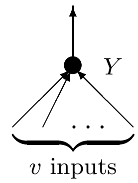
$$R^m_{n,k,l} := \bigoplus_{t\geq 0}\bigoplus_{\omega\in S(k,l,t)} R^m_{n,\omega} : \bigoplus_{t\geq 0}\bigoplus_{\omega\in S(k,l,t)} \mathcal{G}\mathrm{r}^m_\omega \twoheadrightarrow \bigoplus_{t\geq 0} Lin_{\mathrm{GL}_n}\left(\bigwedge^m \mathfrak{ngl}^{(k+l)}_n \otimes \mathcal{F}^{(k)^{\otimes t}}, \mathcal{G}\right).$$

Recalling (10), (20), and the definition (25) of the graph complex $\mathcal{G}\mathrm{r}^m_{\mathfrak{F},\mathfrak{G}}$, we conclude that $R^m_n := \bigoplus_{k\geq 0}\bigoplus_{l\geq 1} R^m_{n,k,l}$ is the desired surjection (26). The second part of the theorem follows from Proposition 8.11 applied to the constituents $R^m_{n,\omega}$ of R^m_n. □

[November 11, 2007]

3.4. **Example.** Let us discuss the case $\mathfrak{F} = T \times T$ and $\mathfrak{G} = T$, where T is the tangent bundle functor. Graphs spanning the vector space $\mathcal{G}\mathrm{r}^m_{T\times T,T}$ have finite number of the 1st type vertices (17)



and/or


, $v \geq 0$,

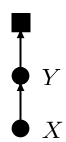
marking the places where to insert tensors $X^a_{(s_1,\ldots,s_v)}$ and $Y^a_{(s_1,\ldots,s_v)}$ of the fiber of $(T \times T)^{(\infty)}$. The unique vertex

of the 2nd type is the place to insert a tensor of the fiber $\mathbb{R}^{n*}$ of T^*. There of course will also be m vertices (19) of the 3rd type for generators of $\mathfrak{ngl}^{(\infty)}_n$.

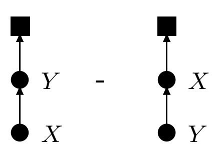
Observe that we omitted braces indicating the symmetry because inputs of all vertices are symmetric and no confusion may occur. Let us inspect how $\mathcal{G}\mathrm{r}^0_{T\times T,T}$ describes GL_n-equivariant maps in

$$Map_{\mathrm{GL}_n}\left((\mathbb{R}^n \times \mathbb{R}^n)^{(\infty)}, \mathbb{R}^n\right) = C^0_{\mathrm{GL}_n}\left(\mathfrak{ngl}^{(\infty)}_n, Map((\mathbb{R}^n \times \mathbb{R}^n)^{(\infty)}, \mathbb{R}^n)\right).$$

The graph



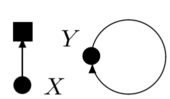
describes the equivariant map that sends an element $(X^a, X^a_b, Y^a, Y^a_b) \in (\mathbb{R}^n \times \mathbb{R}^n)^{(1)}$ into the element $(X^j Y^a_j) \in \mathbb{R}^n$. It is precisely the map O_2 considered in Example 1.4. The linear combination



represents the local formula

$$(X^a, X^a_b, Y^a, Y^a_b) \mapsto (X^j Y^a_j - Y^j X^a_j)$$

for the Lie bracket $[X, Y]$ of two vector fields. We allow also graphs as

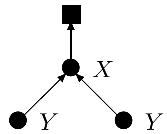


,

which represents the map

$$(X^a, X^a_b, Y^a, Y^a_b) \mapsto (X^a Y^i_i)$$

involving the trace Y^i_i of Y. We leave as an exercise to identify the map defined by



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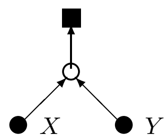
An example of a degree 1 cochain in $C^1_{\mathrm{GL}_n}\left(\mathfrak{ngl}^{(2)}_n, Map(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)\right)$ is provided by

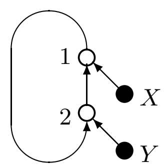


,

which defines the GL_n-equivariant 1-cochain

$$(\varphi^a_{bc}, X^a, Y^a) \mapsto (\varphi^a_{ij} X^i Y^j).$$

As explained in Remark 8.14 of the Appendix, for degrees ≥ 2 our interpretation of graphs involves the antisymmetrization in white vertices. For instance, the graph



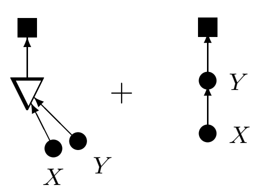
represents the cochain in $C^2_{\mathrm{GL}_n}(\mathfrak{ngl}^{(2)}_n, Map(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}))$ given by

$$(\varphi^a_{bc}, \psi^a_{bc}, X^a, Y^a) \mapsto (\varphi^i_{jk}\psi^j_{il} - \psi^i_{jk}\varphi^j_{il}) X^k Y^l.$$

The reason why the expected traditional $\frac{1}{2!}$-factor is missing is explained in Remark 4.5.

3.5. **Example.** In this example we express local formulas for the covariant derivative, torsion and curvature in terms of graphs. The covariant derivative is the operator $\nabla : Con \times T^{\times 2} \to T$ locally given by the graph

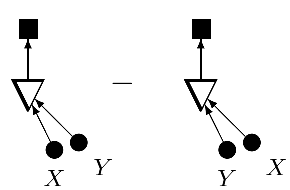
(27) $\nabla_X Y :$



,

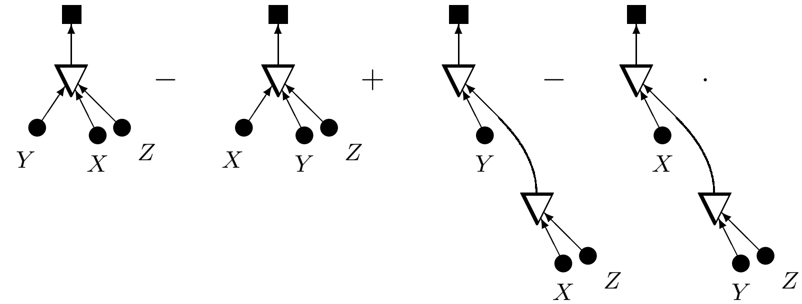
which is a graphical form of formula (1). The torsion $T : Con \times T^{\times 2} \to T$ is given by

$T(X,Y):$

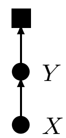
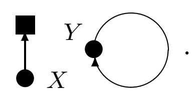


and the curvature $R : Con \times T^{\times 3} \to T$ as

$R(X,Y)Z$:



3.6. **Example.** This example shows that the map R_n^m from Theorem 3.3 need not be a monomorphism below the 'stable range.' Consider again the two graphs from Example 3.4:

$G_1 :=$  and $G_2 :=$  .

The number of edges of both graphs is 2. As we already saw, G_1 represents the formula

$$G_1 : \sum_{1 \le i,j \le n} X^j \frac{\partial Y^i}{\partial x^j} \frac{\partial}{\partial x^i}$$

and G_2 the formula

$$G_2 : \sum_{1 \le i,j \le n} \frac{\partial Y^j}{\partial x^j} X^i \frac{\partial}{\partial x^i}.$$

For $n = 1$ both formulas give the same result, namely

$$X \frac{\partial Y}{\partial x} \frac{\partial}{\partial x},$$

therefore $R_1^0(G_1) = R_1^0(G_2)$. For $n \ge 2$ one clearly has $R_n^0(G_1) \ne R_n^0(G_2)$.

4. The differential

In this section we express the restriction of the Chevalley-Eilenberg differential onto the subcomplex $C^*_{\mathrm{GL}_n}\left(\mathfrak{ngl}_n^{(\infty)}, Map(\mathcal{F}^{(\infty)}, \mathcal{G})\right)$ of GL_n-equivariant cochains in terms of graph complexes. This very straightforward interpretation of the graph complex differential was suggested by Anton Alekseev.

Let us describe first the bracket in the limit $\mathfrak{ngl}_n^{(\infty)} = \varinjlim \mathfrak{ngl}_n^{(s)}$ of Lie algebras $\mathfrak{ngl}_n^{(s)}$ recalled in (8). If finite sums $a = a_2 + a_3 + a_4 + \cdots$ and $b = b_2 + b_3 + b_4 + \cdots$ are elements of $\mathfrak{ngl}_n^{(\infty)}$, $a_u, b_u \in Sym((\mathbb{R}^n)^{\otimes u}, \mathbb{R}^n)$, $u \ge 2$, then $[a,b] = [a,b]_3 + [a,b]_4 + \cdots$ (no quadratic term) with

$$[a,b]_u = \sum_{s+t=u+1} \sum_{1 \le i \le s} \big(S(a_s \circ_i b_t) - S(b_s \circ_i a_t)\big),$$

where $S(-)$ denotes the symmetrization (see Remark 4.5) of a linear map $\mathbb{R}^{n\otimes u} \to \mathbb{R}^n$, $a_s \circ_i b_t$ is the insertion of b_t into the ith slot of a_s and $b_s \circ_i a_t$ has the similar obvious meaning. For $v_1, \ldots, v_u \in \mathbb{R}^n$ we easily get

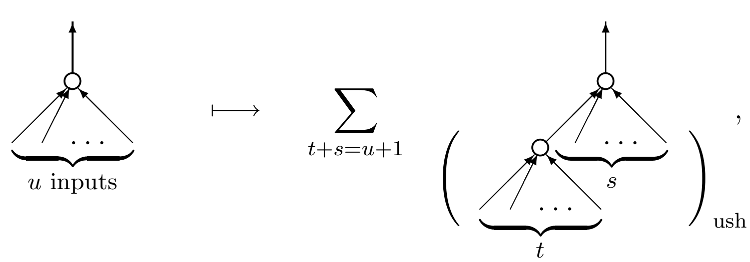
$$[a,b]_u(v_1,\ldots,v_u) = \sum_{s+t=u+1} \frac{s!t!}{u!} \sum_{\sigma} \big\{ a_s(b_t(v_{\sigma(1)},\ldots,v_{\sigma(t)}), v_{\sigma(t+1)},\ldots,v_{\sigma(u)}) - \\ - b_s(a_t(v_{\sigma(1)},\ldots,v_{\sigma(t)}), v_{\sigma(t+1)},\ldots,v_{\sigma(u)}) \big\}, \tag{28}$$

where σ runs over all $(t, s-1)$-unshuffles σ, i.e. permutations $\sigma \in \Sigma_u$ such that $\sigma(1) < \ldots < \sigma(t)$, $\sigma(t+1) < \ldots < \sigma(u)$.

4.1. **Remark.** In the rest of the paper, we will consider $\mathfrak{ngl}_n^{(\infty)}$ with the *modified* Lie bracket, given by formula (28) *without* the $\frac{s!t!}{u!}$-coefficients. Since this modified Lie algebra is isomorphic to the original one, via the isomorphism $a_s \mapsto s! \cdot a_s$, for $a_s \in \mathit{Sym}\big((\mathbb{R}^n)^{\otimes s}, \mathbb{R}^n\big)$, $s \geq 2$, our modification is purely conventional. The advantage of this modified bracket is that the corresponding replacement rule (29) is a linear combination of graphs without fractional coefficients.

To help the reader to appreciate the idea of the differential, we start with an informal definition. A precise formula including signs and orientations is given in (32). At the beginning of Section 2 we decomposed the CE-differential into the sum $\delta_{CE} = \delta_1 + \delta_2$. Let us analyze the action of the second piece δ_2 first. A graph G representing a GL_n-invariant m-cochain has m white vertices that mark the places where to insert elements of $\mathfrak{ngl}_n^{(\infty)}$. Let us label, for $m \geq 1$, these white vertices by $\ell \in \{1,\ldots,m\}$ and denote the vertex labelled ℓ by w_ℓ. If $m = 0$, there are no white vertices and no labelling is necessary.

The effect of the differential δ_2 on the graph G is, by the definition recalled in (5), the following. For each $\ell \in \{1,\ldots,m\}$ insert to the vertex w_ℓ the element $[h_i, h_j]$ and to the remaining white vertices elements $h_1, \ldots, \hat{h}_i, \ldots, \hat{h}_j, \ldots, h_{m+1}$, make the summation over all $1 \leq i < j \leq m+1$ and antisymmetrize in $h_1, \ldots, h_{m+1}$. Denote the resulting $(m+1)$-cochain by G_ℓ. Then $\delta_2(G) = \varepsilon_1 \cdot G_1 + \cdots + \varepsilon_1 \cdot G_m$, where $\varepsilon_1, \ldots, \varepsilon_m \in \{-1, +1\}$ are appropriate signs. A moment's reflection reveals that G_ℓ is obtained by replacing the vertex w_ℓ by:

(29)


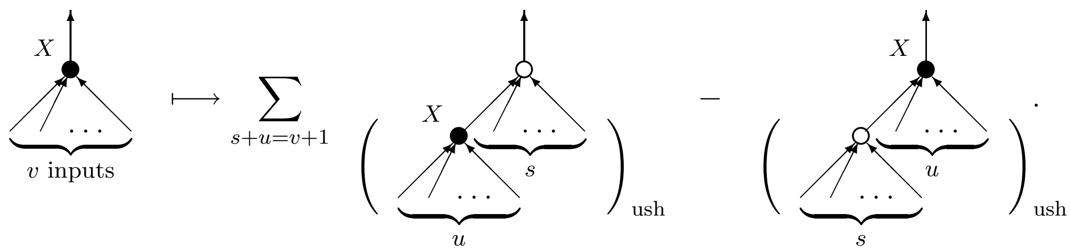
where the braces $(-)_{\mathrm{ush}}$ indicate that the summation over all $(t, s-1)$-unshuffles of the inputs has been performed. This is precisely the formula for the generators of the *homological vector* *field* introduced by Merkulov [20, 22]. One also recognizes (29) as the graphical representation of the axioms of L_∞-algebras as given in [13, page 160].

A similar analysis shows that δ_1 acts by replacing each vertex of type 1 or 2 by the pictorial representation of the action of $\mathfrak{ngl}_n^{(\infty)}$ on tensors corresponding to this vertex. We will show instances of these 'pictorial presentations' in the following two examples.

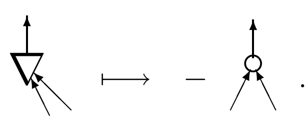
4.2. **Example.** Consider a symmetric map $\xi : \mathbb{R}^{n\otimes v} \to \mathbb{R}^n$ representing an element in the fiber of the k-th tangent space $T^{(k)}$ with coordinates $X^a_{(s_1,\ldots,s_v)}$ (see (16) of Example 3.2). The action of an element $a = a_2 + a_3 + a_4 + \cdots \in \mathfrak{ngl}_n^{(\infty)}$ on ξ is given by $a\xi = (a\xi)_{u+1} + (a\xi)_{u+2} + \cdots$, where

$$(a\xi)_v = \sum_{s+u=v+1} \left(\sum_{1\le i\le s} S(a_s \circ_i \xi) - \sum_{1\le i\le v} S(\xi \circ_i a_s) \right).$$

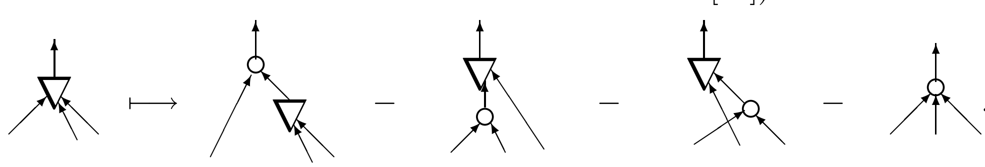
Removing fractional coefficients by modifying the $\mathfrak{ngl}_n^{(\infty)}$-action (compare Remark 4.1), one can graphically express the above rule by the following polarization of (29):

(30) 

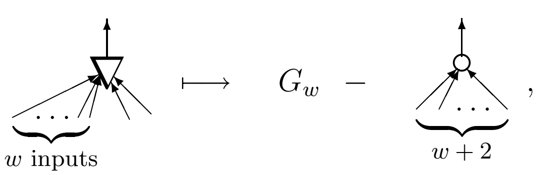
4.3. **Example.** Let us write explicitly two initial replacement rules for the connection and its derivatives. The first one is the infinitesimal version of (3):



The next one is a graphical form of an equation that can be found in [10, Section 17.7] (but notice a different convention for covariant derivatives used in [10]):



We are not going to give a general formula. For our purposes, it will be enough to know that it is of the form

(31) 

where G_w is a linear combination of 2-vertex trees with one vertex (18), with $v < w$, and one vertex (19) with $u < w + 2$.

Let us write a formal definition of the graph differential. For each oriented graph $G \in \mathcal{G}\mathrm{r}^m_{\mathfrak{F},\mathfrak{G}}$ we define $\delta(G) \in \mathcal{G}\mathrm{r}^{m+1}_{\mathfrak{F},\mathfrak{G}}$ as the sum over the set $Vert(G)$ of vertices of G,

$$\delta(G) = \sum_{v \in Vert(G)} \varepsilon_v \cdot \delta_v(G), \tag{32}$$

where δ_v is the replacement of the vertex v determined by the type of v and geometric data as explained above. The signs ε_v and the orientations of the graphs in $\delta_v(G)$ are determined in the following way.

(i) The operation δ_v replaces a 1st or 2nd type vertex v by a linear combination of graphs containing precisely one white vertex. The orientation of the graphs in $\delta_v(G)$ is given by the unique linear order such that this new white vertex is the minimal element and the relative order of the remaining white vertices is unchanged. The sign ε_v is $+1$. Symbolically

$$\delta_v(\circ < \cdots < \circ) = +1 \cdot (\delta_v(\bullet) < \circ < \cdots < \circ). \tag{33}$$

(ii) Let v be a white vertex. We may assume that, after changing the sign of the graph G if necessary, v is the minimal element in an order determining the orientation. The orientation of graphs in $\delta_v(G)$ is then given by requiring that the lower left white vertex in the right hand side of (29) is the minimal one, the upper right white vertex of (29) is the next one, and that the relative order of the remaining white vertices is unchanged. The sign ε_v is again $+1$. Symbolically,

$$\delta_v(\circ < \cdots < \circ) = +1 \cdot (\delta_v(\circ) < \circ < \cdots < \circ).$$

We leave as a simple exercise to derive from the rule (ii) that, if the white vertex v is the ith element of a linear order determining the orientation of G, for some $1 \leq i \leq m$, the orientations of graphs in $\delta_v(G)$ are symbolically expressed as

$$\delta_v(\circ < \cdots < \circ) = (-1)^{i+1} \cdot (\underbrace{\circ < \cdots < \circ}_{i-1} < \delta_v(\circ) < \underbrace{\circ < \cdots < \circ}_{m-i}). \tag{34}$$

Let us emphasize that the applications in this paper use only the initial part $\delta : \mathcal{G}\mathrm{r}^0_{\mathfrak{F},\mathfrak{G}} \to \mathcal{G}\mathrm{r}^1_{\mathfrak{F},\mathfrak{G}}$ of the differential, so we do not need to pay much attention to signs and orientations. Since the graphs spanning $\mathcal{G}\mathrm{r}^0_{\mathfrak{F},\mathfrak{G}}$ (resp. $\mathcal{G}\mathrm{r}^1_{\mathfrak{F},\mathfrak{G}}$) have no white vertices (resp. one white vertex), the orientation issue is trivial and all ε_v's in (32) are $+1$.

4.4. **Theorem.** *The object $\mathcal{G}\mathrm{r}^*_{\mathfrak{F},\mathfrak{G}} = (\mathcal{G}\mathrm{r}^*_{\mathfrak{F},\mathfrak{G}}, \delta)$ is a cochain complex and the maps R^m_n in (26) assemble into a cochain map*

$$R^*_n : (\mathcal{G}\mathrm{r}^*_{\mathfrak{F},\mathfrak{G}}, \delta) \to \left(C^*_{\mathrm{GL}_n}\left(\mathfrak{ngl}^{(\infty)}_n, Map(\mathcal{F}^{(\infty)}, \mathcal{G})\right), \delta_{CE}\right).$$

Proof. Using the antisymmetry of f, one can rewrite equations (4) and (5) into

$$\begin{aligned}(\delta_1 f)(h_1,\ldots,h_{m+1}) &= \frac{1}{m!}\operatorname{Ant}\big(h_1 f(h_2,\ldots,h_{m+1})\big) \quad \text{and}\\ (\delta_2 f)(h_1,\ldots,h_{m+1}) &= \frac{1}{2(m-1)!}\operatorname{Ant}\big(f([h_2,h_1],h_3,\ldots,h_{m+1})\big),\end{aligned}$$

where $\operatorname{Ant}(-)$ denotes the antisymmetrization, see Remark 4.5 below. The inverted order of h_1 and h_2 in the bracket $[h_2,h_1]$ in the second line reflects our ordering of the white vertices in the right hand side of (29). If the multilinear map f itself is an antisymmetrization $\operatorname{Ant}(F)$ of a map F, one can further rewrite the above displays into

$$(\delta_1 f)(h_1,\ldots,h_{m+1}) = \operatorname{Ant}\big(h_1 F(h_2,\ldots,h_{m+1})\big) \quad \text{and} \tag{35}$$

$$(\delta_2 f)(h_1,\ldots,h_{m+1}) = \operatorname{Ant}\Big(\sum_{1\le i\le m}(-1)^{i+1}F(h_1,\ldots,h_{i-1},[h_{i+1},h_i],h_{i+2},\ldots,h_{m+1})\Big). \tag{36}$$

After this preparation, we prove that R^*_n is a chain map by verifying that $(R^{m+1}_n\circ\delta)(G) = (\delta_{CE}\circ R^m_n)(G)$ for each graph G generating $\mathcal{G}\mathrm{r}^m_{\mathfrak{F},\mathfrak{G}}$. After choosing a linear order of white vertices of G compatible with its orientation, an appropriate version of the 'state sum' (68) gives a multilinear map F such that $R^m_n(G) = \operatorname{Ant}(F)$, see Remark 8.14 of the Appendix.

It is not difficult to see that R^*_n translates the part of the differential $\delta(G)$ in (32) given by the summation over the 1st and 2nd type vertices into formula (35) for $\delta_1(f)$ and the part of $\delta(G)$ given by the summation over the white vertices to formula (36) for $\delta_2(f)$. This fact is also reflected by the obvious similarity between formulas (35) and (36) for the Chevalley-Eilenberg differential and symbolic formulas (33) and (34) for the graph differential.

The condition $\delta^2 = 0$ can be verified directly using the fact that the local replacement rules used in (32) are duals of Lie algebra actions and checking that the orientations were defined in such a way that the signs combine properly. One may, however, proceed also as follows.

Since both the domain and target of the map R^*_n, as well as R^*_n itself, are defined in terms of "standard representations," R^m_n makes sense for an arbitrary natural n. Let $G\in\mathcal{G}\mathrm{r}^m_{\mathfrak{F},\mathfrak{G}}$. By the finitary nature of objects involved, there exists $e\ge 0$ such that all graphs that constitute $\delta^2(G)\in\mathcal{G}\mathrm{r}^{m+2}_{\mathfrak{F},\mathfrak{G}}$ have $\le e$ edges. Choose $n\ge e-m-2$. We already know that R^*_n commutes with the differentials, therefore

$$R^{m+2}_n(\delta^2(G)) = \delta^2_{CE}(R^m_n(G)) = 0.$$

By the second part of Theorem 3.3 this implies that $\delta^2(G)=0$. □

4.5. **Remark.** In this paper, the *antisymmetrization* of an element x of some (say) right Σ_k-module, $k\ge 1$, is given by the formula

$$\operatorname{Ant}(x) := \sum_{\sigma\in\Sigma_k}\operatorname{sgn}(\sigma)\cdot x\sigma,$$

without the traditional $\frac{1}{k!}$-factor. This convention is forced by the standard definition of the Lie algebra associated to an associative algebra $(A, \cdot)$ – the bracket $[a', a''] := a' \cdot a'' - a'' \cdot a'$, $a', a'' \in A$, does not involve the $\frac{1}{2!}$-factor. On the other hand, we define the *symmetrization* of x as above by the expected formula

$$S(x) := \frac{1}{k!} \sum_{\sigma \in \Sigma_k} x\sigma.$$

4.6. **Remark.** Applications of our theory will often be based on a suitable choice of a subspace of $\mathfrak{Nat}(\mathfrak{F}, \mathfrak{G})$, together with the corresponding subcomplex of the graph complex $\mathfrak{Gr}^*_{\mathfrak{F},\mathfrak{G}}$. These subobjects, denoted for the purposes of this remark by $\underline{\mathfrak{Nat}}(\mathfrak{F}, \mathfrak{G})$ and $\underline{\mathfrak{Gr}}^*_{\mathfrak{F},\mathfrak{G}} = (\underline{\mathfrak{Gr}}^*_{\mathfrak{F},\mathfrak{G}}, \underline{\delta})$, will be chosen so that the number of edges of graphs spanning $\underline{\mathfrak{Gr}}^m_{\mathfrak{F},\mathfrak{G}}$ will be, for each $m \geq 0$, bounded by $C + m$, where C is a fixed constant.

An example is the subcomplex $\mathfrak{Gr}^*_\bullet(d)$ of the graph complex $\mathfrak{Gr}^*_{T^{\times d},T}$, introduced in Section 5, that describes d-multilinear operators on vector fields. Graphs spanning $\mathfrak{Gr}^*_\bullet(d)$ have precisely $d + m$ edges, so $C = d$ for this subcomplex. Another example is the subcomplex $\mathfrak{Gr}^*_{\bullet\nabla}(d)$ of $\mathfrak{Gr}^*_{Con \times T^{\times d},T}$ describing 'connected' d-multilinear operators used in Section 7. Each degree m graph spanning this subcomplex has at most $2d + m - 1$ edges, i.e. $C = 2d - 1$ in this case. The third example is the complex $\mathfrak{Gr}^*_{\bullet\nabla\circlearrowleft}(d)$ introduced on page 25 describing 'connected' operators in $\mathfrak{Nat}(Con \times T^{\otimes d}, \mathbb{R})$. For this complex, $C := 2d$.

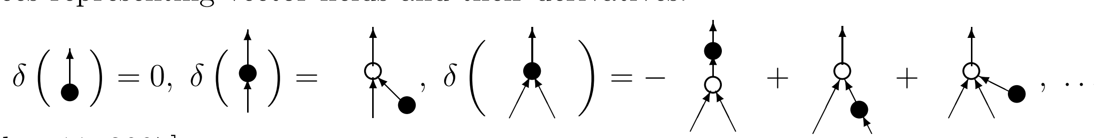
Let $(\underline{\mathfrak{Gr}}^*_{\mathfrak{F},\mathfrak{G}}, \underline{\delta})$, $\underline{\mathfrak{Nat}}(\mathfrak{F}, \mathfrak{G})$ and the constant C be as above. By Theorem 2.2 combined with Theorem 4.4, the restriction $\underline{R}^*_n$ of R^*_n induces the map

$$H^0(\underline{R}^*_n) : H^0(\underline{\mathfrak{Gr}}^*_{\mathfrak{F},\mathfrak{G}}, \underline{\delta}) \to \underline{\mathfrak{Nat}}(\mathfrak{F}, \mathfrak{G}) \tag{37}$$

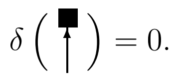
which is an isomorphism in *stable dimensions*. By this we mean that the dimension n of the underlying manifold M is $\geq C$. If this happens, then the map $\underline{R}^*_n$ is, by Proposition 8.11, a chain isomorphism, so $H^0(\underline{R}^*_m)$ is an isomorphism, too. If the dimension of M is less than the stable dimension, one cannot say anything about the induced map $H^0(\underline{R}^*_m)$, although the chain map $\underline{R}^*_n$ is still a chain epimorphism.

4.7. **Example.** In this example we prove a baby version of Theorem 5.1. Namely, we show that the only natural bilinear operations on vector fields on manifolds of dimensions ≥ 2 are scalar multiples of the Lie bracket.

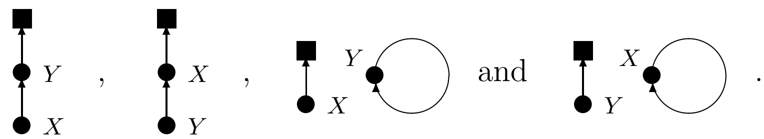
It will be convenient to have ready some initial cases of formula (30) for the replacement rule of vertices representing vector fields and their derivatives:



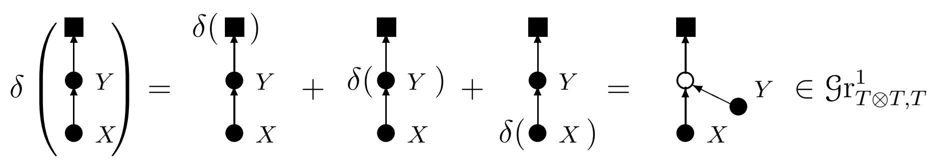
It is also clear that

(38) 

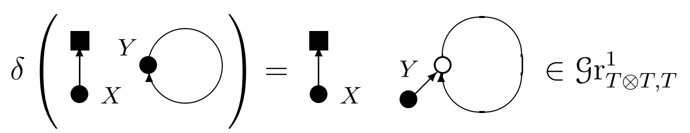
Let us denote by $\mathcal{G}\mathrm{r}^*_{T\otimes T,T} \subset \mathcal{G}\mathrm{r}^*_{T\times T,T}$ the subcomplex describing bilinear operators. Its degree 0 part $\mathcal{G}\mathrm{r}^0_{T\otimes T,T}$ is spanned by



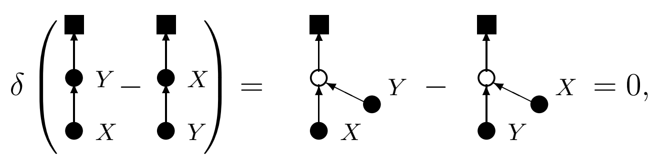
One easily calculates the differential of the leftmost term:

(39) 

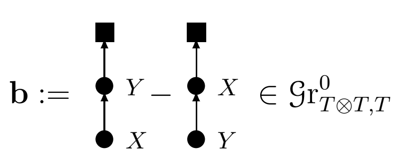
and similarly one gets



The formula for the differential of the remaining two generators of $\mathcal{G}\mathrm{r}^0_{T\otimes T,T}$ is obtained by interchanging $X \leftrightarrow Y$ in the previous two displays. One clearly has



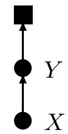
because the inputs of white vertices are symmetric. It is easy to verify, using elementary linear algebra, that the element

(40) 

representing the Lie bracket in fact spans all cochains in $\mathcal{G}\mathrm{r}^0_{T\otimes T,T}$. We conclude that $H^0(\mathcal{G}\mathrm{r}^*_{T\otimes T,T}, \delta)$ is one-dimensional, generated by the bracket $[X, Y]$. The complex $\mathcal{G}\mathrm{r}^*_{T\otimes T,T}$ clearly fits into the scheme discussed in Remark 4.6 (with $C = 2$), which proves the statement in the first paragraph of this example.

[November 11, 2007]

4.8. **Example.** We close this section by an example suggested by the referee which will further illuminate the meaning of the graph differential. The graph

(41) 

represents the local expression

$$\left(X^i\frac{\partial}{\partial x^i}, Y^i\frac{\partial}{\partial x^i}\right) \longmapsto X^i\frac{\partial Y^j}{\partial x^i}\frac{\partial}{\partial x^j}. \tag{42}$$

If $\{y^i\}$ is a different set of coordinates, then X and Y transforms to

$$X^i\frac{\partial y^s}{\partial x^i}\frac{\partial}{\partial y^s} \ \text{ and } \ Y^j\frac{\partial y^r}{\partial x^j}\frac{\partial}{\partial y^r},$$

respectively. Having this transformed X act on the transformed Y gives

$$X^i\frac{\partial y^s}{\partial x^i}\frac{\partial}{\partial y^s}\left(Y^j\frac{\partial y^r}{\partial x^j}\right)\frac{\partial}{\partial y^r} = X^i\frac{\partial y^s}{\partial x^i}\frac{\partial Y^j}{\partial y^s}\frac{\partial y^r}{\partial x^j}\frac{\partial}{\partial y^r} + X^i\frac{\partial y^s}{\partial x^i}Y^j\frac{\partial}{\partial y^s}\left(\frac{\partial y^r}{\partial x^j}\right)\frac{\partial}{\partial y^r}.$$

The first term in the right hand side is equal to the expression in (42) under change-of-coordinates, so the second term represents the extent to which this expression is not invariant. It is equal to

$$X^iY^j\frac{\partial^2 y^r}{\partial x^i\partial x^j}\frac{\partial}{\partial y^r},$$

which translates directly to the formula (39) for the differential of (41) in the graph complex.

5. Operations on vector fields

In this section we consider differential operators acting on a finite number of vector fields $X, Y, Z, \ldots$ with values in vector fields, that is, operators in $\mathfrak{Nat}(T^{\times\infty}, T) := \bigcup_{d\geq 0}\mathfrak{Nat}(T^{\times d}, T)$ The first statement of this section is:

5.1. **Theorem.** *Let M be a smooth manifold and d a natural number such that $\dim(M) \geq d$. Then each d-multilinear natural operator from vector fields to vector fields is a sum of iterations of the Lie bracket containing each of d variables precisely once, and all relations between these expressions follow from the Jacobi identity and antisymmetry.*

In particular, there are precisely $(d-1)!$ linearly independent operators of the above type.

Theorem 5.1 is an obvious consequence of Proposition 5.6 below and the formula for the dimension of the kth piece of the operad $\mathcal{L}ie$ for Lie algebras that can be found for example in [7, Example 3.1.12]. Theorem 5.1 describes *multilinear* operators and does not cover operators as

$$\mathfrak{O}(X, Y, Z) := [X, Y] + [X, [X, Z]]$$

but can easily be extended to cover also these cases. Since all operators are assumed to be polynomial, they decompose into the sum of their homogeneous parts. For instance, the operator in the above display is the sum of the homogeneity-2 part $[X, Y]$ and the homogeneity-3 part $[X, [X, Z]]$.

5.2. **Remark.** As indicated above, each operator $\mathfrak{O} \in \mathfrak{Nat}(T^{\times\infty}, T)$ decomposes into the sum of its homogeneous parts. Let us explain this phenomenon in more detail. The local formula O for the operator $\mathfrak{O}$ is the sum

$$O = O_1 + \cdots + O_r,$$

where O_d is the part of O consisting of terms with precisely d occurrences of the vector field variables. The action of the structure group $\mathrm{GL}_n^{(\infty)}$ on the typical fiber of the prolongation of $T^{\times\infty}$ is linear, which is expressed by the manifest linearity of the replacement rule (30) in the vector field variable. This implies that the map O is $\mathrm{GL}_n^{(\infty)}$-equivariant if and only if each of its homogeneous components O_d is $\mathrm{GL}_n^{(\infty)}$-equivariant, $1 \leq d \leq r$. Therefore

$$\mathfrak{O} = \mathfrak{O}_1 + \cdots + \mathfrak{O}_r,$$

where $\mathfrak{O}_d$ is the operator defined by the local formula O_d, $1 \leq d \leq r$.

We conclude that to classify operators of the above type, it suffices to classify homogeneous operators. It is a standard fact that each homogeneous operator of degree d is either d-multilinear or a sum of operators obtained from d-multilinear operators by repeating one or more of their variables. We will call this procedure the *depolarization* of multilinear operators.

Theorem 5.1 therefore implies the following.

5.3. **Corollary.** *Let M be a smooth manifold. Each natural differential operator from vector fields on M to vector fields on M whose all components are of homogeneity $\leq \dim(M)$ is a sum of iterations of the Lie bracket. All relations between these iterations follow from the Jacobi identity and antisymmetry.*

In Example 4.7 we studied in detail the graph complex $\mathcal{G}\mathrm{r}^*_{T\otimes T,T}$ describing bilinear operators. Bearing this example in mind, we introduce $\mathcal{G}\mathrm{r}^*_\bullet(d) = \mathcal{G}\mathrm{r}^*_{T^{\otimes d},T} \subset \mathcal{G}\mathrm{r}^*_{T^{\times d},T}$, the subcomplex describing d-multilinear operators. Its degree m component is spanned by graphs with d vertices of the first type labelled by $X_1, \ldots, X_d$, m white vertices of the third type and one 2nd type vertex which we call the *anchor*. Observe that $\mathcal{G}\mathrm{r}^*_\bullet(d)$ is precisely the graph complex $\mathcal{G}\mathrm{r}^*_{\bullet(b)\nabla(c)}$ of Corollary 8.13 with $b := d$ and $c := 0$. The collection $\mathcal{G}\mathrm{r}^0_\bullet = \{\mathcal{G}\mathrm{r}^0_\bullet(d)\}_{d\geq 1}$ of degree 0 subspaces admits two types of operations.

(i) For graphs $G' \in \mathcal{G}\mathrm{r}^0_\bullet(u)$, $G'' \in \mathcal{G}\mathrm{r}^0_\bullet(v)$ and $1 \leq i \leq u$, one has the $\circ_i$-product $G' \circ_i G'' \in \mathcal{G}\mathrm{r}^0_\bullet(u + v - 1)$ given by the following straightforward extension of the Chapoton-Livernet vertex insertion [1, § 1.5] to non-simply connected graphs.

Assume that $X'_1, \ldots, X'_u$ are the black vertices of G', $X''_1, \ldots, X''_v$ the black vertices of G'' and $In(X'_i)$ the set of inputs of X'_i in G'. Then

$$G' \circ_i G'' := \sum_{f : In(X'_i) \to \{X''_1, \ldots, X''_v\}} G' \circ_i^f G'' \in \mathcal{G}\mathrm{r}^0_\bullet(u+v-1),$$

where $G' \circ_i^f G'' \in \mathcal{G}\mathrm{r}^0_\bullet(u+v-1)$ is the graph obtained by replacing the vertex X'_i of G' by G'' and grafting the inputs of X'_i on black vertices of G'' following f.

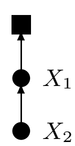
In more detail, one starts by cutting off the anchor $\blacksquare$ of G'' and grafts the resulting free edge on the vertex of G' immediately above X'_i. Then one grafts each input edge e of X'_i on the vertex $f(e)$ of G''. Finally, one changes the labels $X'_1, \ldots, X'_{i-1}, X''_1, \ldots, X''_v, X'_{i+1}, \ldots, X'_u$ of the black vertices of the graph obtained in this way into $X_1, \ldots, X_{u+v-1}$.

(ii) The second operation is the right action of the symmetric group: for each $G \in \mathcal{G}\mathrm{r}^0_\bullet(d)$ and a permutation $\sigma \in \Sigma_d$, one has $G\sigma \in \mathcal{G}\mathrm{r}^0_\bullet(d)$ given by permuting the labels $X_1, \ldots, X_d$ of the black vertices of G according to σ.

5.4. **Proposition.** *The collection* $\mathcal{G}\mathrm{r}^0_\bullet = \{\mathcal{G}\mathrm{r}^0_\bullet(d)\}_{d \geq 1}$ *with the above operations is an operad with unit* $\blacksquare\!\!\!\bullet \in \mathcal{G}\mathrm{r}^0_\bullet(1)$ [17]. *The operad structure of* $\mathcal{G}\mathrm{r}^0_\bullet$ *restricts to* $H^0(\mathcal{G}\mathrm{r}^*_\bullet, \delta) = Ker(\delta : \mathcal{G}\mathrm{r}^0_\bullet \to \mathcal{G}\mathrm{r}^1_\bullet)$.

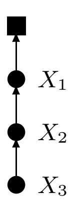
Proof. The operad axioms for the operations in (i) and (ii) above are verified directly, compare also [1, § 1.5]. The simplest way to see that the operad structure of $\mathcal{G}\mathrm{r}^0_\bullet$ restricts to the kernel of δ is to extend the operations (i) and (ii), in the obvious manner, to the graded collection $\mathcal{G}\mathrm{r}^*_\bullet$, making $(\mathcal{G}\mathrm{r}^*_\bullet, \delta)$ a dg-operad. This, in particular, means that δ is a derivation with respect to these extended $\circ_i$-operations, which implies the second part of the proposition. $\square$

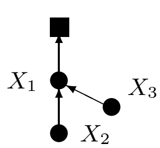
5.5. **Example.** An instructive example of the vertex insertion can be found in [1, § 1.5]. We present here a simpler one, taken from the proof of [1, Theorem 1.9]. Let $\mathbf{p}$ be the graph

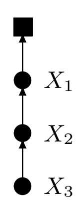


$\in \mathcal{G}\mathrm{r}^0_\bullet(2)$.

Then one has

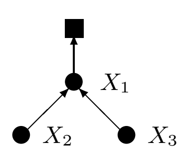
$\mathbf{p} \circ_1 \mathbf{p} =$


$+$


$\in \mathcal{G}\mathrm{r}^0_\bullet(3)$ and $\mathbf{p} \circ_2 \mathbf{p} =$


$\in \mathcal{G}\mathrm{r}^0_\bullet(3)$.

The above display implies that the associator $Ass(\mathbf{p}) := \mathbf{p} \circ_1 \mathbf{p} - \mathbf{p} \circ_2 \mathbf{p}$ equals



and is therefore symmetric in X_2 and X_3. This, by definition, means that $\mathbf{p}$ represents a pre-Lie multiplication [1, § 1.1]. We will see below that $\mathcal{G}\mathrm{r}^0_\bullet$ is indeed closely related to the operad $p\mathcal{L}ie$ for pre-Lie algebras.

Let $\tau \in \Sigma_2$ be the generator. By standard properties of pre-Lie algebras [1, Proposition 1.2], the antisymmetrization $\mathbf{p}(\tau - 1\!\!1)$ of the element $\mathbf{p}$ from Example 5.5 is a Lie bracket. Observe that $\mathbf{p}(\tau - 1\!\!1)$ equals the element $\mathbf{b}$ introduced in (40).

5.6. **Proposition.** *The 0th cohomology $H^0(\mathcal{G}\mathrm{r}^*_\bullet(d), \delta)$ is, for each $d \geq 2$, generated by the Lie bracket $\mathbf{b} = \mathbf{p}(\tau - 1\!\!1) \in H^0(\mathcal{G}\mathrm{r}^*_\bullet(2), \delta)$, by iterating operations (i) and (ii) above. There are no relations between these iterations other than those following from the Jacobi identity and antisymmetry.*

A compact formulation of Proposition 5.6 is that the operad $H^0(\mathcal{G}\mathrm{r}^*_\bullet, \delta) = \{H^0(\mathcal{G}\mathrm{r}^*_\bullet(d), \delta)\}_{d\geq 1}$ is isomorphic to the operad $\mathcal{L}ie = \{\mathcal{L}ie(d)\}_{d\geq 1}$ for Lie algebras [17, Example II.3.34], via an isomorphism that sends the generator $\beta \in \mathcal{L}ie(2)$ of $\mathcal{L}ie$ into $\mathbf{b} \in \mathcal{G}\mathrm{r}^0_\bullet(2)$. Graphs spanning $\mathcal{G}\mathrm{r}^0_\bullet(d)$ have d edges which explains the stability condition $\dim(M) \geq d$ in Theorem 5.1. The rest of this section is devoted to a proof of its main result.

Proof of Proposition 5.6. It is clear from formulas (29), (30) and (38) that the differential preserves connected components of underlying graphs. Therefore, for each $d \geq 1$, $\mathcal{G}\mathrm{r}^*_\bullet(d)$ is the direct sum

$$\mathcal{G}\mathrm{r}^*_\bullet(d) = \bigoplus_{c\geq 1} \mathcal{G}\mathrm{r}^*_{\bullet c}(d),$$

where $\mathcal{G}\mathrm{r}^*_{\bullet c}(d)$ denotes the subcomplex spanned by graphs with c connected components. In particular, $\mathcal{G}\mathrm{r}^*_{\bullet 1}(d)$ is the subcomplex of connected graphs. It is easy to see that $\mathcal{G}\mathrm{r}^0_{\bullet 1}$ is a suboperad of $\mathcal{G}\mathrm{r}^*_\bullet$.

As the Lie bracket represented by $\mathbf{b} \in \mathcal{G}\mathrm{r}^0_{\bullet 1}(2)$ is antisymmetric and satisfies the Jacobi identity, the rule $F(\beta) := \mathbf{b}$, where $\beta \in \mathcal{L}ie(2)$ is the generator, defines an operad homomorphism $F : \mathcal{L}ie \to \mathcal{G}\mathrm{r}^0_{\bullet 1}$. Since the Lie bracket and its iterations are natural operators, $Im(F) \subset Ker(\delta : \mathcal{G}\mathrm{r}^0_{\bullet 1} \to \mathcal{G}\mathrm{r}^1_{\bullet 1})$. Proposition 5.6 will clearly be established if we prove that

(i) the operad map $F : \mathcal{L}ie \to \mathcal{G}\mathrm{r}^0_{\bullet 1}$ induces an isomorphism $\mathcal{L}ie \cong H^0(\mathcal{G}\mathrm{r}^*_{\bullet 1}, \delta)$, and

(ii) $H^0(\mathcal{G}\mathrm{r}^*_{\bullet c}(d), \delta) = 0$, for each $c \geq 2$, $d \geq 1$.

Part (i) is highly nontrivial, but it in fact has already been proved in [15]. Indeed, the operad $\mathcal{G}\mathrm{r}^0_{\bullet 1}$ is precisely the operad $p\mathcal{L}ie$ describing pre-Lie algebras [1] and $F : \mathcal{L}ie \to \mathcal{G}\mathrm{r}^0_{\bullet 1}$ corresponds, under the identification $\mathcal{G}\mathrm{r}^0_{\bullet 1} \cong p\mathcal{L}ie$, to the inclusion $\iota : \mathcal{L}ie \hookrightarrow p\mathcal{L}ie$ induced by the antisymmetrization of the pre-Lie product. The dg operad rpL^* of [15] coincides, in degrees 0 and 1, with the complex $\mathcal{G}\mathrm{r}^*_{\bullet 1}$ and the isomorphism in (i) is isomorphism (2) of [15].

Let us prove (ii). For each $m \geq 0$, $d \geq 1$, consider the span $\mathcal{G}\mathrm{r}^m_{\bullet\circlearrowleft}(d)$ of connected graphs with d vertices $X_1, \ldots, X_d$ of type 1, m 'white' vertices of type 3 and no vertex of type 2. The direct sum

$$\mathcal{G}\mathrm{r}^*_{\bullet\circlearrowleft}(d) = \bigoplus_{m \geq 0} \mathcal{G}\mathrm{r}^m_{\bullet\circlearrowleft}(d)$$

is a cochain complex, with the differential defined in the same way as the differential in $\mathcal{G}\mathrm{r}^*_{\bullet}(d)$ and denoted again by δ. We claim that, for each $c \geq 2$ and $d \geq 1$, there is an isomorphism of cochain complexes

$$\mathcal{G}\mathrm{r}^*_{\bullet c}(d) \cong \bigoplus_{i_1+\cdots+i_c=d} \mathcal{G}\mathrm{r}^*_{\bullet 1}(i_1) \otimes \big(\mathcal{G}\mathrm{r}^*_{\bullet\circlearrowleft}(i_2) \odot \cdots \odot \mathcal{G}\mathrm{r}^*_{\bullet\circlearrowleft}(i_c)\big) \tag{43}$$

where $\odot$ as usual denotes the symmetric product. To prove this isomorphism, observe that each graph $G \in \mathcal{G}\mathrm{r}^*_{\bullet c}(d)$ decomposes into the disjoint union

$$G = G_1 \sqcup G_2 \sqcup \cdots \sqcup G_c, \tag{44}$$

of its connected components. Precisely one of these components contains the unique type 2 vertex $\stackrel{\blacksquare}{\uparrow}$, let us assume it is G_1. Then $G_1 \in \mathcal{G}\mathrm{r}^*_{\bullet 1}(i_1)$ and $G_s \in \mathcal{G}\mathrm{r}^*_{\bullet\circlearrowleft}(i_s)$ for $2 \leq s \leq c$, with some $i_1 + \cdots + i_c = d$. Decomposition (44) is clearly unique up to the order of $G_2, \ldots, G_c$ and is preserved by the differential. This proves (43). By Künneth and Mashke's theorems, (ii) follows from $H^0(\mathcal{G}\mathrm{r}^*_{\bullet\circlearrowleft}(d), \delta) = 0$, $d \geq 1$, which is the same as showing that

$$\text{the map } \delta : \mathcal{G}\mathrm{r}^0_{\bullet\circlearrowleft}(d) \to \mathcal{G}\mathrm{r}^1_{\bullet\circlearrowleft}(d) \text{ is a monomorphism for each } d \geq 1. \tag{45}$$

Let us inspect the structure of $\mathcal{G}\mathrm{r}^*_{\bullet\circlearrowleft}(d)$. It is clear from simple graph combinatorics that each graph in $\mathcal{G}\mathrm{r}^m_{\bullet\circlearrowleft}(d)$ has genus 1, therefore it contains a unique wheel. Denote $\mathcal{G}\mathrm{r}^m_{\bullet\circlearrowleft}(d,w) \subset \mathcal{G}\mathrm{r}^m_{\bullet\circlearrowleft}(d)$ the subspace spanned by graphs that have precisely w vertices (of either type) on the wheel, $w \geq 0$. It is obvious from (29) and (30) that

$$\delta(\mathcal{G}\mathrm{r}^m_{\bullet\circlearrowleft}(d,w)) \subset \mathcal{G}\mathrm{r}^{m+1}_{\bullet\circlearrowleft}(d,w) \oplus \mathcal{G}\mathrm{r}^{m+1}_{\bullet\circlearrowleft}(d,w+1),\ d \geq 1,\ w \geq 0;$$

see also Figure 1. Let us denote by δ^0 the component of δ that preserves the number of vertices on the wheel and δ^1 the component that raises it by one. We claim that in order to prove (45), it is enough to verify that

$$\text{the map } \delta^0 : \mathcal{G}\mathrm{r}^0_{\bullet\circlearrowleft}(d) \to \mathcal{G}\mathrm{r}^1_{\bullet\circlearrowleft}(d) \text{ is a monomorphism for each } d \geq 1. \tag{46}$$

The spaces $\mathcal{G}\mathrm{r}^m_{\bullet\circlearrowleft}(d,p)$ form a bicomplex $(\mathcal{G}\mathrm{r}^{*,*}_{\bullet\circlearrowleft}(d), \delta)$ with $\mathcal{G}\mathrm{r}^{p,q}_{\bullet\circlearrowleft}(d) := \mathcal{G}\mathrm{r}^{p+q}_{\bullet\circlearrowleft}(d,p)$ and δ the sum $\delta^0 + \delta^1$, where $\delta^0 : \mathcal{G}\mathrm{r}^{*,*}_{\bullet\circlearrowleft}(d) \to \mathcal{G}\mathrm{r}^{*,*+1}_{\bullet\circlearrowleft}(d)$ and $\delta^0 : \mathcal{G}\mathrm{r}^{*,*}_{\bullet\circlearrowleft}(d) \to \mathcal{G}\mathrm{r}^{*+1,*}_{\bullet\circlearrowleft}(d)$ are defined above. Condition (46) then implies (45) via a standard spectral sequence argument. The only subtlety is that our bicomplex is not a first quadrant one, thus the convergence of the related spectral sequence has to be checked. We therefore decided to prove the implication (46) $\Longrightarrow$ (45) by the following elementary calculation.

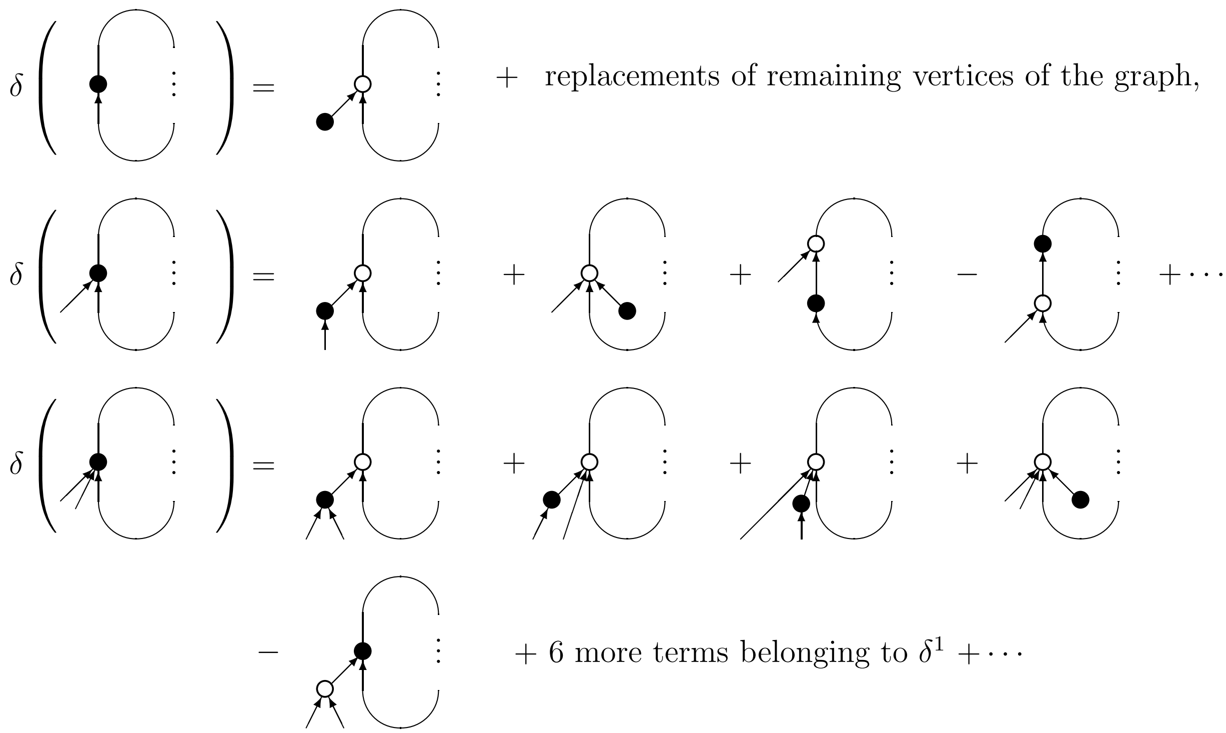


FIGURE 1. Action of δ on $\mathcal{G}r^0_{\bullet\circlearrowleft}$ – the replacement rule for a type 1 vertex on the wheel.

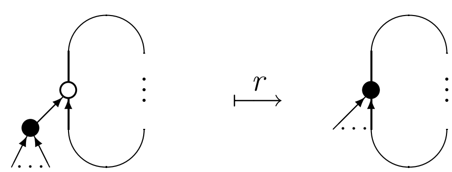


FIGURE 2. The map $r : \overline{\mathcal{G}r}^1_{\bullet\circlearrowleft}(d, w) \to \mathcal{G}r^0_{\bullet\circlearrowleft}(d, w)$ contracts the unique edge connecting the binary white vertex on the wheel with a black vertex outside the wheel.

Suppose that (45) does not hold and let $x \in \mathcal{G}r^0_{\bullet\circlearrowleft}(d)$ be such that $\delta(x) = 0$ while $x \neq 0$. There exists a decomposition $x = x_a + x_{a+1} + \cdots + x_{a+s}$ with $x_w \in \mathcal{G}r^0_{\bullet\circlearrowleft}(d, w)$ for $a \leq w \leq a + s$ in which $x_a \neq 0$. Since $\delta^0(x_a)$ is the component of $\delta(x)$ in $\mathcal{G}r^1_{\bullet\circlearrowleft}(d, a)$, $\delta^0(x_a) = 0$. Then (46) implies $x_a = 0$, a contradiction.

Denote by $\overline{\mathcal{G}r}^1_{\bullet\circlearrowleft}(d, w) \subset \mathcal{G}r^1_{\bullet\circlearrowleft}(d, w)$ the subspace spanned by graphs with one binary white vertex on the wheel, as in the left graph in Figure 2. Both $\overline{\mathcal{G}r}^1_{\bullet\circlearrowleft}(d, w)$ and $\mathcal{G}r^1_{\bullet\circlearrowleft}(d, w)$ have canonical bases provided by isomorphism classes of graphs, therefore one has a canonical projection $\pi : \mathcal{G}r^1_{\bullet\circlearrowleft}(d, w) \to \overline{\mathcal{G}r}^1_{\bullet\circlearrowleft}(d, w)$. In addition to the projection, there is a second map $r : \overline{\mathcal{G}r}^1_{\bullet\circlearrowleft}(d, w) \to \mathcal{G}r^0_{\bullet\circlearrowleft}(d, w)$ whose definition is clear from Figure 2.

Let $G \in \mathcal{G}r^0_{\bullet\circlearrowleft}(d, w)$ be a graph. Observe that $\mathcal{G}r^0_{\bullet\circlearrowleft}(d, 0) = 0$, we may therefore assume $w \geq 1$. Recall that the differential $\delta(G)$ is the sum (32) of local replacements $\delta_v(G)$ over $v \in \mathit{Vert}(G)$. Let $\mathit{Vert}_{\circlearrowleft}(G) \subset \mathit{Vert}(G)$ be the subset of vertices on the wheel. For $v \in \mathit{Vert}_{\circlearrowleft}(G)$, the contribution

$\delta_v(G)$ contains precisely one graph in $\mathcal{G}\mathrm{r}^1_{\bullet\circlearrowright}(d,w)$ with the binary white vertex – see again Figure 1. Denote this graph $\overline{\delta}^0_v(G)$ and define

$$\overline{\delta}^0(G) := \sum_{v \in Vert_{\circlearrowright}(G)} \overline{\delta}^0_v(G).$$

It is clear that $Im(\overline{\delta}^0) \subset \overline{\mathcal{G}\mathrm{r}}^1_{\bullet\circlearrowright}(d,w)$, $\overline{\delta}^0 = \pi \circ \delta^0$ and $r \circ \overline{\delta}^0 = w \cdot id$. Combining these facts, we obtain

$$r \circ \pi \circ \delta^0 = w \cdot id,$$

which implies (46) and finishes the proof. □

We believe that one can even show that the complex $(\mathcal{G}\mathrm{r}^*_{\bullet\circlearrowright}(d), \delta)$ used in the above proof is acyclic in all dimensions. Let us close this section by formulating the following interesting consequence of the proof of Proposition 5.6.

5.7. **Corollary.** *In stable dimensions, there are no nontrivial differential operators from vector fields to functions.*

Proof. It is clear that d-multilinear operators from vector fields to functions are described by the graph complex $\mathcal{G}\mathrm{r}^*_{\bullet\circlearrowright}(d)$ introduced in our proof of Proposition 5.6. Condition (45) implies that there are no nontrivial d-multilinear operators of this type. The corollary then follows from the standard (de)polarization trick. □

6. Structure of the space of natural operators

In Example 1.8 we considered the trivial natural bundle $\mathbb{R}$ whose sections are smooth functions. Let $\mathfrak{F}$ be another natural bundle. The space $\mathfrak{Nat}(\mathfrak{F}, \mathbb{R})$ of natural operators $\mathfrak{O} : \mathfrak{F} \to \mathbb{R}$ with the obvious 'pointwise' multiplication is clearly a commutative algebra, with unit $\mathbb{1}$ the operator that sends all sections of $\mathfrak{F}$ into the constant section $1 \in \mathbb{R}$. This indicates that spaces of natural operators may sometimes have a rich algebraic structure that can be used to simplify their classification.

6.1. **Definition.** We say that $\mathfrak{F}$ is a bundle with *connected replacement rules* if the replacement rules send a connected graph to a linear combination of connected graphs.

All natural bundles considered in this paper have connected replacement rules, and the author does not know any 'natural' natural operator that has not. We will see that the space of natural operators between bundles with connected replacement rules exhibits some freeness property. Before we formulate the first statement of this type, we introduce the following convention for graphs describing operators with values in functions.

The graph complex $\mathcal{G}r^*_{\mathfrak{F},\mathbb{R}}$ for operators in $\mathfrak{Nat}(\mathfrak{F},\mathbb{R})$ is spanned by graphs with vertices of the 1st type representing tensors in a prolongation of the fiber of $\mathfrak{F}$, vertices (18) of the third type and one 2nd type vertex $\blacksquare$ which in this case has no inputs and no outputs. Therefore $\blacksquare$ is an isolated vertex bearing no information and we discard it from the picture. With this convention, graphs spanning $\mathcal{G}r^*_{\mathfrak{F},\mathbb{R}}$ have vertices of the 1st and 3rd type only. The disjoint union of graphs spanning $\mathcal{G}r^*_{\mathfrak{F},\mathbb{R}}$ translates into the pointwise multiplication of the corresponding operators and the unit $\mathbb{1} \in \mathfrak{Nat}(\mathfrak{F},\mathbb{R})$ is represented by the 'exceptional' empty graph.

6.2. **Theorem.** *Let $\mathfrak{F}$ be a natural bundle with connected replacement rules. Then, in stable dimensions, the commutative unital algebra $\mathfrak{Nat}(\mathfrak{F},\mathbb{R})$ is free, generated by the subspace $\mathfrak{Nat}_1(\mathfrak{F},\mathbb{R})$ of natural operators represented by connected graphs. In other words,*

$$\mathfrak{Nat}(\mathfrak{F},\mathbb{R}) \cong \mathbb{R}[\mathfrak{Nat}_1(\mathfrak{F},\mathbb{R})],$$

where $\mathbb{R}[-]$ denotes the polynomial algebra functor.

Proof. Each graph spanning $\mathcal{G}r^*_{\mathfrak{F},\mathbb{R}}$ decomposes into the disjoint union of its connected components. The differential δ, by assumption, preserves this decomposition which is clearly unique up to the order of components. The proof is finished by recalling that the disjoint union of graphs expresses the pointwise multiplication of operators. $\square$

Let $\mathfrak{F},\mathfrak{G}$ be natural bundles. The pointwise multiplication makes the space $\mathfrak{Nat}(\mathfrak{F},\mathfrak{G})$ a unital module over the unital algebra $\mathfrak{Nat}(\mathfrak{F},\mathbb{R})$. We prove a structure theorem also for this space.

6.3. **Theorem.** *Suppose that both $\mathfrak{F}$ and $\mathfrak{G}$ are bundles with connected replacement rules. Then, in stable dimensions, $\mathfrak{Nat}(\mathfrak{F},\mathfrak{G})$ is the free $\mathfrak{Nat}(\mathfrak{F},\mathbb{R})$-module generated by the subspace $\mathfrak{Nat}_1(\mathfrak{F},\mathfrak{G})$ of operators represented by connected graphs,*

$$\mathfrak{Nat}(\mathfrak{F},\mathfrak{G}) \cong \mathfrak{Nat}_1(\mathfrak{F},\mathfrak{G}) \otimes \mathfrak{Nat}(\mathfrak{F},\mathbb{R}). \tag{47}$$

Proof. The proof is similar to the proof of Theorem 6.2. The graph complex $\mathcal{G}r^*_{\mathfrak{F},\mathfrak{G}}$ describing operators in $\mathfrak{Nat}(\mathfrak{F},\mathfrak{G})$ is spanned by graphs with vertices of the first and third types, and one vertex of the second type. Each such a graph is the disjoint union of its connected components as in (44) and the differential preserves this decomposition. Precisely one of these components contains the vertex of the third type thus representing an operator in $\mathfrak{Nat}_1(\mathfrak{F},\mathfrak{G})$. The remaining components describe operators from $\mathfrak{Nat}_1(\mathfrak{F},\mathbb{R})$ and assemble, via the pointwise multiplication, into an operator in $\mathfrak{Nat}(\mathfrak{F},\mathbb{R})$. $\square$

Theorems 6.2 and 6.3 imply that in order to classify operators in $\mathfrak{Nat}(\mathfrak{F},\mathfrak{G})$, it is enough to understand the 'connected' subspaces $\mathfrak{Nat}_1(\mathfrak{F},\mathbb{R})$ and $\mathfrak{Nat}_1(\mathfrak{F},\mathfrak{G})$. We will use this fact in the next section.

6.4. **Example.** In Section 5 we studied natural operators on vector fields with values in vector fields, that is, operators in $\mathfrak{Nat}(T^{\times\infty}, T) := \bigcup_{d\geq 0} \mathfrak{Nat}(T^{\times d}, T)$. We also considered operators with values in functions and proved, in Corollary 5.7, that there are no nontrivial operators of this type in stable dimensions.

This means that $\mathfrak{Nat}(T^{\times\infty}, \mathbb{R})$ is the trivial commutative algebra $\mathbb{R}$ and (47) reduces to the isomorphism $\mathfrak{Nat}(T^{\times\infty}, T) \cong \mathfrak{Nat}_1(T^{\times\infty}, T)$ which says that all operators from vector fields to vector fields live, in stable dimensions, on connected graphs.

7. Operators on connections and vector fields

We will consider operators acting on a linear connection Γ and a finite number of vector fields $X, Y, Z, \ldots$, with values in vector fields, such as the covariant derivative $\nabla_X Y$, torsion $T(X,Y)$ and curvature $R(X,Y)Z$ recalled in Example 1.2. By Theorems 6.2 and 6.3, the structure of the space $\mathfrak{Nat}(\mathit{Con} \times T^{\times\infty}, T) := \bigcup_{d\geq 0} \mathfrak{Nat}(\mathit{Con} \times T^{\times d}, T)$ of these operators is determined by the 'connected' subspaces $\mathfrak{Nat}_1(\mathit{Con} \times T^{\times\infty}, T)$ and $\mathfrak{Nat}_1(\mathit{Con} \times T^{\times\infty}, \mathbb{R})$. In this section we describe these spaces. The following remark should be compared to Remark 5.2 in Section 5.

7.1. **Remark.** The local formula O for a natural differential operator $\mathfrak{O}$ in $\mathfrak{Nat}(\mathit{Con} \times T^{\times\infty}, T)$ or in $\mathfrak{Nat}(\mathit{Con} \times T^{\times\infty}, \mathbb{R})$ decomposes into

$$O = \sum_{a,b\geq 0} O_{a,b} \ \ \text{(finite sum)}$$

where $O_{a,b}$ is the part of O containing precisely a ∇-variables and b vector field variables. For example, the local formula (1) for the covariant derivative represented by the graph in (27) is the sum $O_{1,2} + O_{0,2}$, where $O_{1,2}(X,Y,\Gamma) := \Gamma^i_{jk} X^j Y^k \frac{\partial}{\partial x^i}$ and $O_{0,2}(X,Y,\Gamma) := X^j Y^i_j \frac{\partial}{\partial x^i}$.

In contrast to Section 5, here the action of the structure group $\mathrm{GL}^{(\infty)}_n$ on the typical fiber is linear only in the vector-field variables – the non-linearity in the ∇-variables is manifested in the presence of the 'isolated' white vertex in the replacement rule (31). Nevertheless, one may still decompose

$$\mathfrak{O} = \mathfrak{O}_1 + \cdots + \mathfrak{O}_r,$$

with $\mathfrak{O}_k$ the operator represented by the local formula $O_d := \sum_{a\geq 0} O_{a,d}$, $1 \leq d \leq r$. Therefore *homogeneity* and *multilinearity* in this section always refer to the vector fields variables.

The first half of this section will be devoted to the study of the space $\mathfrak{Nat}_1(\mathit{Con} \times T^{\times\infty}, T)$, the space $\mathfrak{Nat}_1(\mathit{Con} \times T^{\times\infty}, \mathbb{R})$ will be addressed in the second half of this section. As in Section 5, we start with multilinear operators.

7.2. **Theorem.** *Let $d \geq 0$. On smooth manifolds of dimension $\geq 2d-1$, each d-multilinear operator in $\mathfrak{Nat}_1(Con \times T^{\otimes d}, T)$ is a linear combination of iterations of the covariant derivative and the Lie bracket which contains each of the vector fields $X_1, \ldots, X_d$ exactly once. All relations follow from the anticommutativity and the Jacobi identity of the Lie bracket.*

If g_d denotes the number of linearly independent operators of this type, the generating function $g(t) = \sum_{d\geq 1} \frac{1}{d!} g_d t^d$ is determined by the functional equation

$$e^{g(t)}\left(1 - t - g^2(t)\right) = 1. \tag{48}$$

Equation (48) can be expanded into inductive formula (54) from which one can calculate some initial values of g_k as $g_1 = 1$, $g_2 = 3$, $g_3 = 26$, &c. Theorem 7.2 will follow from Proposition 7.4 below. The depolarization of Theorem 7.2 is:

7.3. **Corollary.** *On a smooth manifold M, each operator from $\mathfrak{Nat}_1(Con \times T^{\times\infty}, T)$ whose all components are of homogeneity $\leq \frac{1}{2}(\dim(M)+1)$ is a linear combination of compositions of the covariant derivative and the Lie bracket. All relations between these compositions follow from the anticommutativity and the Jacobi identity of the Lie bracket.*

The central object of this section will be the subcomplex $\mathcal{G}\mathrm{r}^*_{\bullet\nabla 1}(d)$ of the graph complex $\mathcal{G}\mathrm{r}^*_{Con\times T^{\times d},T}$ describing 'connected' d-multilinear operators. Its degree m piece $\mathcal{G}\mathrm{r}^m_{\bullet\nabla 1}(d)$ is spanned by connected graphs with d vertices (17) labelled by $X_1, \ldots, X_d$, some number of vertices (18) labelled ∇, m white vertices (19) and one vertex . It is clear that $\mathcal{G}\mathrm{r}^*_{\bullet\nabla 1}(d)$ is precisely the subcomplex spanned by connected graphs, of the direct sum

$$\mathcal{G}\mathrm{r}^*_{\bullet\nabla}(d) := \bigoplus_{c\geq 0} \mathcal{G}\mathrm{r}^*_{\bullet(d)\nabla(c)},$$

where $\mathcal{G}\mathrm{r}^*_{\bullet(d)\nabla(c)}$ is the graph complex of Corollary 8.13. As in Proposition 5.4, one easily sees that the collection $\mathcal{G}\mathrm{r}^0_{\bullet\nabla 1} = \{\mathcal{G}\mathrm{r}^0_{\bullet\nabla 1}(d)\}_{d\geq 1}$ forms an operad. It is also not difficult to verify that each graph spanning $\mathcal{G}\mathrm{r}^m_{\bullet\nabla 1}(d)$ has at most $2d+m-1$ edges, which explains the stability condition in Theorem 7.2.

Let $\mathcal{P} = \{\mathcal{P}(d)\}_{d\geq 1}$ be the operad describing algebras with two independent operations – a bilinear product $\star$ satisfying no other conditions and a Lie bracket. Of course, $\mathcal{P}$ is the free product (= the coproduct in the category of operads, see [14, p. 137]) of the free operad $\Gamma(\star)$ generated by the bilinear operation $\star$ and the operad $\mathcal{L}ie$ for Lie algebras, $\mathcal{P} = \Gamma(\star) * \mathcal{L}ie$. Recall that we denoted by $\beta \in \mathcal{L}ie(2)$ the generator.

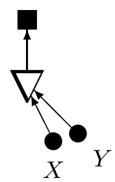
Define the operad homomorphism $F : \mathcal{P} \to \mathcal{G}\mathrm{r}^0_{\bullet\nabla 1}$ by $F(\beta) := \mathbf{b}$ and $F(\star) := \mathbf{c}$, where $\mathbf{b} \in \mathcal{G}\mathrm{r}^0_{\bullet\nabla 1}(2)$ is the graph (40) representing the Lie bracket and $\mathbf{c} \in \mathcal{G}\mathrm{r}^0_{\bullet\nabla 1}(2)$ the graph (27) for the covariant derivative. As in Section 5 we easily see that F is well-defined and that $Im(F) \subset Ker(\delta : \mathcal{G}\mathrm{r}^0_{\bullet\nabla 1} \to \mathcal{G}\mathrm{r}^1_{\bullet\nabla 1})$. Theorem 7.2 clearly follows from

7.4. **Proposition.** *The map* $F : \mathcal{P} \to \mathcal{G}r^0_{\bullet\nabla 1}$ *induces an isomorphism* $\mathcal{P} \cong H^0(\mathcal{G}r^0_{\bullet\nabla 1}, \delta)$. *The generating function* $p(t) := \sum_{d\geq 1} \frac{1}{d!}\dim(\mathcal{P}(d)) \cdot t^d$ *for the operad* $\mathcal{P}$ *satisfies (48).*

Proof. The map F embeds into the following diagram of operads and their homomorphisms:

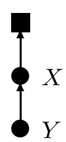
$$\begin{tikzcd} \mathcal{P} = \Gamma(\star) * \mathcal{L}ie \arrow[r, "F"] \arrow[drr, "id * \iota"] & \mathcal{G}r^0_{\bullet\nabla 1} \arrow[r, "A"] & \Gamma(\star) * p\mathcal{L}ie \arrow[d, "T"] \\ & & \Gamma(\star) * p\mathcal{L}ie. \end{tikzcd} \tag{49}$$

Let us define the remaining maps in (49). As in [1], one can show that the operad $\mathcal{G}r^0_{\bullet\nabla 1}$ is isomorphic to the operad $\Gamma(\star) * p\mathcal{L}ie$ governing structures consisting of a bilinear multiplication $\star$ and an independent pre-Lie product $\circ$. The map $A : \mathcal{G}r^0_{\bullet\nabla 1} \to \Gamma(\star) * p\mathcal{L}ie$ in (49) is the isomorphism that sends the graph



$\in \mathcal{G}r^0_{\bullet\nabla 1}(2)$

into $X \star Y \in \Gamma(\star)(2)$ and the graph



$\in \mathcal{G}r^0_{\bullet\nabla 1}(2)$

into $X \circ Y \in p\mathcal{L}ie(2)$. The map $T : \Gamma(\star) * p\mathcal{L}ie \to \Gamma(\star) * p\mathcal{L}ie$ is the 'twist'

$$T(X \star Y) := X \star Y - Y \circ X \ \text{ and } \ T(X \circ Y) := X \circ Y.$$

It is evident that the composition TAF coincides with the coproduct $id * \iota$ of the identity $id : \Gamma(\star) \to \Gamma(\star)$ and the map $\iota : \mathcal{L}ie \to p\mathcal{L}ie$ given by the antisymmetrization of the pre-Lie product

$$\iota([X,Y]) := Y \circ X - X \circ Y,$$

which is an inclusion by [15, Proposition 3.1]. This implies that $id * \iota$ is a monomorphism, therefore F is a monomorphism, too.

Now, to prove that F induces an isomorphism $\mathcal{P} \cong H^*(\mathcal{G}r^0_{\bullet\nabla 1}, \delta)$, it suffices to show that the dimensions of the spaces $H^0(\mathcal{G}r^0_{\bullet\nabla 1}(d), \delta)$ and $\mathcal{P}(d)$ are the same, for each $d \geq 1$. Our calculation of the dimension of $H^0(\mathcal{G}r^0_{\bullet\nabla 1}(d), \delta)$ will be based on the fact that $(\mathcal{G}r^*_{\bullet\nabla 1}(d), \delta)$ forms a bicomplex. For integers p, q denote by $\mathcal{G}r^{p,q}_{\bullet\nabla 1}(d)$ the subspace of $\mathcal{G}r^{p+q}_{\bullet\nabla 1}(d)$ spanned by graphs with precisely $-p$ ∇-vertices. It immediately follows from the replacement rules (29), (30) and (31) that $\delta = \delta' + \delta''$, where

$$\delta'(\mathcal{G}r^{p,q}_{\bullet\nabla 1}(d)) \subset \mathcal{G}r^{p+1,q}_{\bullet\nabla 1}(d) \ \text{ and } \ \delta''(\mathcal{G}r^{p,q}_{\bullet\nabla 1}(d)) \subset \mathcal{G}r^{p,q+1}_{\bullet\nabla 1}(d).$$

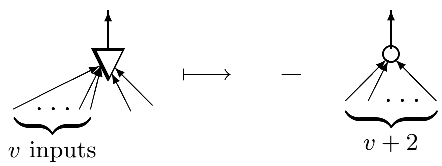
[November 11, 2007]

$$\begin{tikzcd}
\mathcal{G}r^{-2,2}_{\bullet\nabla 1}(3) \arrow[r, "\delta'"] & \mathcal{G}r^{-1,2}_{\bullet\nabla 1}(3) \arrow[r, "\delta'"] & \mathcal{G}r^{0,2}_{\bullet\nabla 1}(3) \\
 & \mathcal{G}r^{-1,1}_{\bullet\nabla 1}(3) \arrow[u, "\delta''"'] \arrow[r, "\delta'"] & \mathcal{G}r^{0,1}_{\bullet\nabla 1}(3) \arrow[u, "\delta''"'] \\
 & & \mathcal{G}r^{0,0}_{\bullet\nabla 1}(3) \arrow[u, "\delta''"']
\end{tikzcd}$$

FIGURE 3. The bicomplex $(\mathcal{G}r^{*,*}_{\bullet\nabla 1}(3), \delta' + \delta'')$.

It is also clear from simple graph combinatorics that the bicomplex $(\mathcal{G}r^{*,*}_{\bullet\nabla 1}(d), \delta)$ is bounded by the triangle $p = 0$, $p + q = 0$ and $q = d - 1$, see Figure 3.

The horizontal differential δ' in $\mathcal{G}r^{*}_{\bullet\nabla 1}(d)$ is easy to describe – it replaces ∇-vertices according the rule

(50) 

, $v \geq 0$,

and leaves other vertices unchanged.

7.5. **Remark.** At this point we make a digression and observe that $(\mathcal{G}r^{*}_{\bullet\nabla 1}(d), \delta')$ is a particular case of the following construction. For each collection $(U^*, \vartheta_U) = \{(U^*(s), \vartheta_U)\}_{s \geq 2}$ of right dg-Σ_s-modules $(U^*(s), \vartheta_U)$, one may consider the complex $\mathcal{G}r^{*}_{\bullet 1}[U](d) = (\mathcal{G}r^{*}_{\bullet 1}[U^*](d), \vartheta)$ spanned by connected graphs with d vertices (17) labelled $X_1, \ldots, X_d$, one vertex $\overset{\blacksquare}{\uparrow}$ and a finite number of vertices decorated by elements of U. The grading of $\mathcal{G}r^{*}_{\bullet 1}[U^*](d)$ is induced by the grading of U^* and the differential ϑ replaces U-decorated vertices, one at a time, by their ϑ_U-images and leaves other vertices unchanged. It is a standard fact [18] (see also [14, Theorem 21]) that the assignment $(U^*, \vartheta_U) \mapsto (\mathcal{G}r^{*}_{\bullet 1}[U^*](d), \vartheta)$ is a polynomial, hence exact, functor, so

$$H^*(\mathcal{G}r^{*}_{\bullet 1}[U^*](d), \vartheta) \cong \mathcal{G}r^{*}_{\bullet 1}[H^*(U, \vartheta_U)](d). \tag{51}$$

Let now $(E^*, \vartheta_E) = \{(E^*(s), \vartheta_E)\}_{s \geq 2}$ be such that $E^0(s)$ is spanned by symbols (18), with $v + 2 = s$, $E^1(s)$ by symbols (19) with $u = s$, and $E^m(s) = 0$ for $m \geq 2$. The differential ϑ_E is defined by replacement rule (50). More formally, $E^0(s) = \mathrm{Ind}^{\Sigma_s}_{\Sigma_{s-2}}(\mathbf{1}_{s-2})$ and $E^1(s) = \mathbf{1}_s$, where $\mathbf{1}_{s-2}$ (resp. $\mathbf{1}_s$) denotes the trivial representation of the symmetric group Σ_{s-2} (resp. Σ_s). The differential ϑ_E then sends the generator $1 \in \mathbf{1}_{s-2}$ into $-1 \in \mathbf{1}_s$. It is clear that, with this particular choice of the collection (E^*, ϑ_E),

$$(\mathcal{G}r^{*}_{\bullet\nabla 1}(d), \delta') \cong (\mathcal{G}r^{*}_{\bullet 1}[E^*](d), \vartheta). \tag{52}$$

Let us continue with the proof of Proposition 7.4. Equations (51) and (52) in Remark 7.5 imply that

$$H^*(\mathcal{G}r^*_{\bullet\nabla 1}(d), \delta') = \mathcal{G}r^*_{\bullet 1}[H^*(E, \vartheta_E)](d). \tag{53}$$

Since $\vartheta_E : E^0(s) \to E^1(s)$ is an epimorphism, the collection $H^*(E, \vartheta_E) = \{H^*(E(s), \vartheta_E)\}_{s\geq 2}$ is concentrated in degree 0 and $H^0(E(s), \vartheta_E)$ is the kernel of the map $\vartheta_E : E^0(s) \to E^1(s)$. We conclude that $\mathcal{G}r^*_{\bullet 1}[H^*(E, \vartheta_E)](d)$ is spanned by graphs with d vertices (17) labelled $X_1, \ldots, X_d$, one vertex and some number of vertices decorated by the collection $H^0(E, \vartheta_E) = \{H^0(E(s), \vartheta_E)\}_{s\geq 2}$.

In particular, the graded space $\mathcal{G}r^*_{\bullet 1}[H^*(E, \vartheta_E)](d)$ and hence, by (53), also the horizontal cohomology $H^*(\mathcal{G}r^*_{\bullet\nabla 1}(d), \delta')$, is concentrated in degree 0. This implies that the first term $(E_1^{p,q}, d_1) = (H^p(\mathcal{G}r^{*,q}_{\bullet\nabla 1}, \delta'), d_1)$ of the corresponding spectral sequence is supported by the diagonal $p + q = 0$, so this spectral sequence degenerates at this level and

$$\dim(H^0(\mathcal{G}r^*_{\bullet\nabla 1}(d), \delta)) = \dim(H^0(\mathcal{G}r^*_{\bullet\nabla 1}(d), \delta')) = \dim(\mathcal{G}r^0_{\bullet 1}[H^0(E, \vartheta_E)](d).$$

Denote the common value of the dimensions in the above display g_d. We claim that the sequence $\{g_d\}_{d\geq 1}$ satisfies the recursion:

$$\begin{aligned} \frac{g_{n+1}}{(n+1)!} \;=\; & \frac{g_n}{n!} + \frac{1}{2!}\sum_{i+j=n}\frac{g_i g_j}{i!j!} + \frac{1}{3!}\sum_{i+j+k=n}\frac{g_i g_j g_k}{i!j!k!} + \frac{1}{4!}\sum_{i+j+k+l=n}\frac{g_i g_j g_k g_l}{i!j!k!l!} + \cdots \\ & + \frac{2(2-1)-1}{2!}\sum_{i+j=n+1}\frac{g_i g_j}{i!j!} + \frac{3(3-1)-1}{3!}\sum_{i+j+k=n+1}\frac{g_i g_j g_k}{i!j!k!} + \cdots . \end{aligned} \tag{54}$$

This can be seen as follows. Graphs G spanning $\mathcal{G}r^0_{\bullet 1}[H^0(E, \vartheta_E)](d)$ are rooted trees with a distinguished vertex (= root). The vertex of G adjacent to the root might either be a vertex (17) or a vertex decorated by $H^0(E, \vartheta_E)$. The contribution from trees of the first type is reflected by the first line of (54), in which the coefficients $1, 1/2!, 1/3!, \ldots$ equal $\dim(\mathbb{1}_s)/s!$, $s \geq 1$, where $\mathbb{1}_s$ is the trivial representation of the symmetric group Σ_s spanned by the vertex (17) with $u = s$. The second line of (54) counts contributions from trees of the second type. The coefficients are $\dim(H^0(E(s), \vartheta_E))/s!$, $s \geq 2$. It is simple to assemble (54) into equation (48).

Let us show that the generating function $p(t) := \sum_{d\geq 1} \frac{1}{d!}\dim(\mathcal{P}(d)) \cdot t^d$ for the operad $\mathcal{P}$ also satisfies (48). Since $\mathcal{P}$ is, as the coproduct of quadratic Koszul operads, itself quadratic Koszul, one has the functional equation [7, Theorem 3.3.2]:

$$q(-p(t)) = -t. \tag{55}$$

relating p with the generating function $q(t) := \sum_{d\geq 1} \frac{1}{d!}\dim(\mathcal{Q}(d)) \cdot t^d$ of its quadratic dual $\mathcal{Q}$.

For convenience of the reader, we make a digression and briefly recall the definition of quadratic operads and their quadratic duals. Details can be found in [17, II.3.2] or in the original source [7].

An operad $\mathcal{A}$ is *quadratic* if it is the quotient $\Gamma(E)/(R)$ of the free operad $\Gamma(E)$ on the right Σ_2-module $E := \mathcal{A}(2)$ of arity-two operations of $\mathcal{A}$, modulo the operadic ideal (R) generated by some subspace $R \subset \Gamma(E)(3)$.

Each quadratic operad $\mathcal{A} = \Gamma(E)/(R)$ as above has its *quadratic dual* $\mathcal{A}^!$ [17, Definition II.3.37] defined as follows. Let us denote $E^\vee := E^* \otimes \mathrm{sgn}_2$ the linear dual of the right Σ_2-module E twisted by the signum representation. One then has a natural isomorphism $\Gamma(E^\vee)(3) \cong \Gamma(E)(3)^*$ of right Σ_3-modules. Let $R^\perp \subset \Gamma(E^\vee)(3)$ denote the annihilator of R in $\Gamma(E^\vee)(3) \cong \Gamma(E)(3)^*$. The quadratic dual of $\mathcal{A}$ is the quotient $\mathcal{A}^! := \Gamma(E^\vee)/(R^\perp)$.

To describe the quadratic dual $\mathcal{Q}$ of the operad $\mathcal{P}$ introduced on page 33 is an easy task. The operad $\mathcal{Q}$ governs algebras V with two bilinear operations, $\bullet$ and $*$, such that $\bullet$ is commutative associative, $*$ is 'nilpotent'

$$(a * b) * c = a * (b * c) = 0,\ a, b, c \in V,$$

and these two operations annihilate each other:

$$(a \bullet b) * c = a * (b \bullet c) = a \bullet (b * c) = 0,\ a, b, c \in V.$$

It is immediately obvious that

$$\dim(\mathcal{Q}(1)) = 1,\ \dim(\mathcal{Q}(2)) = 3 \ \text{ and } \ \dim(\mathcal{Q}(d)) = \dim(\mathcal{C}om(d)) = 1 \text{ for } d \geq 3,$$

where $\mathcal{C}om$ denotes the operad for commutative associative algebras. The generating function for $\mathcal{Q}$ therefore equals $q(t) = e^t - 1 + t^2$ and equation (55) gives

$$e^{-p(t)} - 1 + p(t)^2 = -t,$$

which is equivalent to (48). We proved that the generating functions $g(t)$ and $p(t)$ satisfy the same functional equation and, by definition, the same initial condition $p(0) = g(0) = 0$, therefore they coincide and $\dim(H^0(\mathcal{G}\mathrm{r}^0_{\bullet\nabla 1}(d), \delta)) = \dim(\mathcal{P}(d))$ for each $d \geq 1$. $\square$

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In the rest of this section we study operators in $\mathfrak{Nat}_1(Con \times T^{\times\infty}, \mathbb{R})$. Roughly speaking, we prove that all operators in this space are traces in the following sense. Let $\mathfrak{O} \in \mathfrak{Nat}(Con \times T^{\times\infty}, T)$ be an operator acting on vector fields $X_0, X_1, X_2, \ldots$ and a connection Γ. Suppose that $\mathfrak{O}$ is a linear order 0 differential operator in X_0. This means that the local formula $O(X_0, X_1, X_2, \ldots, \Gamma) \in \mathbb{R}$ for $\mathfrak{O}$ is a linear function of X_0 and does not contain derivatives of X_0. For such an operator we define $Tr_{X_0}(\mathfrak{O}) \in \mathfrak{Nat}(Con \times T^{\times\infty}, \mathbb{R})$ by the local formula

$$Tr_{X_0}(O)(X_1, X_2, \ldots, \Gamma) := \mathrm{Trace}(O(-, X_1, X_2, \ldots, \Gamma) : \mathbb{R}^n \to \mathbb{R}^n) \in \mathbb{R}.$$

It is easy to see that $Tr_{X_0}(\mathfrak{O})$ is well defined. Let us formulate a structure theorem for multilinear operators from $\mathfrak{Nat}_1(Con \times T^{\times\infty}, \mathbb{R})$.

7.6. **Theorem.** *Let $d \geq 0$. On smooth manifolds of dimension $\geq 2d$, each d-multilinear operator in $\mathfrak{Nat}_1(Con \times T^{\otimes d}, \mathbb{R})$ is the trace of a $(d+1)$-multilinear operator from $\mathfrak{Nat}_1(Con \times T^{\otimes(d+1)}, T)$.*

Theorem 7.6 will follow from Proposition 7.8 below. A depolarized version of Theorem 7.6 is:

7.7. **Corollary.** *On a smooth manifold M, each operator from $\mathfrak{Nat}_1(Con \times T^{\times\infty}, \mathbb{R})$ whose all components are of homogeneity $\leq \frac{1}{2}\dim(M)$ is a trace of an operator from $\mathfrak{Nat}_1(Con \times T^{\times\infty}, T)$.*

Denote by $\mathfrak{Gr}^*_{\bullet\nabla\circlearrowleft}(d)$ the graph complex describing operators in $\mathfrak{Nat}_1(Con \times T^{\otimes d}, \mathbb{R})$. The degree m-component of this complex is spanned by connected graphs with d vertices (17) labelled $X_1, \ldots, X_d$, some number of vertices (18) labelled ∇ and m white vertices (19). It is not difficult to see that the number of edges of graphs spanning $\mathfrak{Gr}^0_{\bullet\nabla\circlearrowleft}(d)$ is $\leq 2d$, which explains the stability assumption in Theorem 7.6.

We will also consider the subcomplex $\mathfrak{Gr}^*_{\bullet\nabla Tr}(d) \subset \mathfrak{Gr}^*_{\bullet\nabla 1}(d+1)$ of graphs describing operators in $\mathfrak{Nat}_1(Con \times T^{\otimes(d+1)}, T)$ for which the trace is defined. Clearly, the degree m component $\mathfrak{Gr}^m_{\bullet\nabla Tr}(d)$ of this subcomplex is spanned by connected graphs with one vertex $\overset{\downarrow}{\bullet}$ labelled X_0, one vertex $\overset{\blacksquare}{\uparrow}$, d vertices (17) labelled $X_1, \ldots, X_d$, a finite number of vertices (18) labelled ∇ and m white vertices (19). The trace is represented by the map map $Tr : \mathfrak{Gr}^*_{\bullet\nabla Tr}(d) \to \mathfrak{Gr}^*_{\bullet\nabla\circlearrowleft}(d)$ that removes the vertices $\overset{\downarrow}{\bullet}$ and $\overset{\blacksquare}{\uparrow}$ and connect the two loose edges created in this way by a directed wheel. It is clear that this map commutes with the differentials. We now establish Theorem 7.6 by proving the following.

7.8. **Proposition.** *The map $Tr : (\mathfrak{Gr}^*_{\bullet\nabla Tr}(d), \delta) \to (\mathfrak{Gr}^*_{\bullet\nabla\circlearrowleft}(d), \delta)$ induces an epimorphism of cohomology $H^0(\mathfrak{Gr}^*_{\bullet\nabla Tr}(d), \delta) \to H^0(\mathfrak{Gr}^*_{\bullet\nabla\circlearrowleft}(d), \delta)$.*

Proof. As in the proof of Proposition 7.4 we observe that both $(\mathfrak{Gr}^*_{\bullet\nabla Tr}(d), \delta)$ and $(\mathfrak{Gr}^*_{\bullet\nabla\circlearrowleft}(d), \delta)$ are bicomplexes, with $\mathfrak{Gr}^{p,q}_{\bullet\nabla Tr}(d)$ (resp. $(\mathfrak{Gr}^{p,q}_{\bullet\nabla\circlearrowleft}(d))$ spanned by graphs in $\mathfrak{Gr}^{p+q}_{\bullet\nabla Tr}(d)$ (resp. $(\mathfrak{Gr}^{p+q}_{\bullet\nabla\circlearrowleft}(d))$ with precisely $-p$ ∇-vertices. The differential in both complexes decomposes as $\delta = \delta' + \delta''$ where δ' (the 'horizontal part') raises the p-degree by one and preserves the q-degree, and δ'' (the 'vertical part') preserves the q-degree and raises the p-degree by one.

The map $Tr : (\mathfrak{Gr}^*_{\bullet\nabla Tr}(d), \delta) \to (\mathfrak{Gr}^*_{\bullet\nabla\circlearrowleft}(d), \delta)$ obviously preserves the bigradings, therefore it induces the map

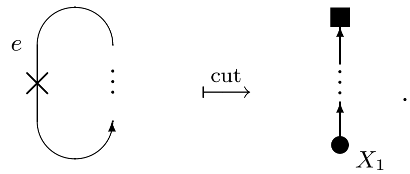
$$H^*(Tr, \delta') : H^*(\mathfrak{Gr}^*_{\bullet\nabla Tr}(d), \delta') \to H^*(\mathfrak{Gr}^*_{\bullet\nabla\circlearrowleft}(d), \delta') \tag{56}$$

of the horizontal cohomology. Using the same considerations as in the proof of Proposition 7.4, we identify this map with

$$Tr : \mathfrak{Gr}^*_{\bullet Tr}[H^*(E, \vartheta_E)](d) \to \mathfrak{Gr}^*_{\bullet\circlearrowleft}[H^*(E, \vartheta_E)](d), \tag{57}$$

where (E^*, ϑ_E) is the dg-collection introduced in Remark 7.5 and the graph complexes in (57) are defined analogously as the graph complex $\mathfrak{Gr}^*_{\bullet 1}[H^*(E, \vartheta_E)](d)$ used in the proof of Proposition 7.4.

Let us show that the map in (57) is an epimorphism. Consider a graph G in $\mathcal{G}r^*_{\bullet\circlearrowleft}[H^*(E,\vartheta_E)](d)$ and choose a directed edge e in the (unique) wheel of G. Let $\widehat{G}$ be the graph in $\mathcal{G}r^*_{\bullet Tr}[H^*(E,\vartheta_E)](d)$ obtained by cutting e in the middle and decorating the loose ends thus created by vertices and as in the following display:



Clearly $Tr(\widehat{G}) = G$ which proves that (57) is surjective.

So, we have two spectral sequences, $(E^{p,q}_*, d_*)$ and $(F^{p,q}_*, d_*)$, such that

$$(E^{p,q}_0, d_0) = (\mathcal{G}r^{p,q}_{\bullet\nabla Tr}(d), \delta),\ (F^{p,q}_0, d_0) = (\mathcal{G}r^{p,q}_{\bullet\nabla\circlearrowleft}(d), \delta),$$

and the map $Tr_* : (E^{p,q}_*, d_*) \to (F^{p,q}_*, d_*)$ induced by the trace map $Tr : \mathcal{G}r^*_{\bullet\nabla Tr}(d) \to \mathcal{G}r^*_{\bullet\nabla\circlearrowleft}(d)$. The map $Tr_1 : (E^{p,q}_1, d_1) \to (F^{p,q}_1, d_1)$ of the first levels of the spectral sequences is (56) and we identified this map with epimorphism (57). It is also clear that the first terms of both spectral sequences are supported by the diagonal $p + q = 0$, so these spectral sequences degenerate at this level. A standard argument then implies that the map $H^0(Tr) : H^0(\mathcal{G}r^*_{\bullet\nabla Tr}(d), \delta) \to H^0(\mathcal{G}r^*_{\bullet\nabla\circlearrowleft}(d), \delta)$ in Proposition 7.8 is an epimorphism. $\square$

8. Appendix: Invariant tensors and graph complexes

Recall that, for finite-dimensional $\mathbf{k}$-vector spaces U and W, one has canonical isomorphisms

$$Lin(U,W)^* \cong Lin(W,U),\ Lin(U,V) \cong U^* \otimes V \text{ and } (U \otimes W)^* \cong U^* \otimes V^*, \tag{58}$$

where $Lin(-,-)$ denotes the space of $\mathbf{k}$-linear maps, $(-)^*$ the linear dual and $\otimes$ the tensor product over $\mathbf{k}$. The first isomorphism in (58) is induced by the non-degenerate pairing

$$Lin(U,W) \otimes Lin(W,U) \to \mathbf{k}$$

that takes $f \otimes g \in Lin(U,W) \otimes Lin(W,U)$ into the trace of the composition $Tr(f \circ g)$, the remaining two isomorphisms are obvious. In this appendix, by a *canonical isomorphism* we will usually mean a composition of isomorphisms of the above types. Einstein's convention assuming summation over repeated (multi)indices is used throughout this section. We will also assume that the ground field $\mathbf{k}$ is of characteristic zero.

In what follows, V will be an n-dimensional $\mathbf{k}$-vector space and $\mathrm{GL}(V)$ the group of linear automorphisms of V. We start by considering the vector space $Lin(V^{\otimes k}, V^{\otimes l})$ of $\mathbf{k}$-linear maps $f : V^{\otimes k} \to V^{\otimes l}$, $k, l \geq 0$. Since both $V^{\otimes k}$ and $V^{\otimes l}$ are natural $\mathrm{GL}(V)$-modules, it makes sense to study the subspace $Lin_{\mathrm{GL}(V)}(V^{\otimes k}, V^{\otimes l}) \subset Lin(V^{\otimes k}, V^{\otimes l})$ of $\mathrm{GL}(V)$-equivariant maps.

As there are no $\mathrm{GL}(V)$-equivariant maps in $Lin(V^{\otimes k}, V^{\otimes l}) = 0$ if $k \neq l$ (see, for instance, [10, §24.3]), the only interesting case is $k = l$. For a permutation $\sigma \in \Sigma_k$, define the *elementary invariant tensor* $t_\sigma \in Lin(V^{\otimes k}, V^{\otimes k})$ as the map given by

$$t_\sigma(v_1 \otimes \cdots \otimes v_k) := v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}, \quad \text{for } v_1, \ldots, v_k \in V. \tag{59}$$

It is simple to verify that t_σ is $\mathrm{GL}(V)$-equivariant.

Invariant Tensor Theorem. *The space $Lin_{\mathrm{GL}(V)}(V^{\otimes k}, V^{\otimes k})$ is spanned by elementary invariant tensors t_σ, $\sigma \in \Sigma_k$. If $\dim(V) \geq k$, the tensors $\{t_\sigma\}_{\sigma \in \Sigma_k}$ are linearly independent.*

This form of the Invariant Tensor Theorem is a straightforward translation of [6, Theorem 2.1.4] describing invariant tensors in $V^{*\otimes k} \otimes V^{\otimes k}$ and remarks following this theorem, see also [10, Theorem 24.4]. The Invariant Tensor Theorem can be reformulated into saying that the map

$$\mathcal{R}_n : \mathbf{k}[\Sigma_k] \to Lin_{\mathrm{GL}(V)}(V^{\otimes k}, V^{\otimes k}) \tag{60}$$

from the group ring of Σ_k to the subspace of $\mathrm{GL}(V)$-equivariant maps given by $\mathcal{R}_n(\sigma) := t_\sigma$, $\sigma \in \Sigma_k$, is always an epimorphism and is an isomorphism for $n \geq k$ (recall n denoted the dimension of V).

The tensors $\{t_\sigma\}_{\sigma \in \Sigma_k}$ are not linearly independent if $\dim(V) < k$. For a subset $S \subset \{1, \ldots, k\}$ such that $\mathrm{card}(S) > \dim(V)$, denote by Σ_S the subgroup of Σ_k consisting of permutations that leave the complement $\{1, \ldots, k\} \setminus S$ fixed. It is simple to verify that then

$$\sum_{\sigma \in \Sigma_S} \mathrm{sgn}(\sigma) \cdot t_\sigma = 0 \tag{61}$$

in $Lin_{\mathrm{GL}(V)}(V^{\otimes k}, V^{\otimes k})$. By [6, II.1.3], all relations between the elementary invariant tensors are induced by the relations of the above type. In other words, the kernel of the map $\mathcal{R}_n$ in (60) is generated by the expressions

$$\sum_{\sigma \in \Sigma_S} \mathrm{sgn}(\sigma) \cdot \sigma \in \mathbf{k}[\Sigma_k],$$

where S and Σ_S are as above.

Observe that, with the convention used in (59) involving the inverses of σ in the right hand side, $\mathcal{R}_n$ is a ring homomorphism. In the following example we explain how the Invariant Tensor Theorem leads to graphs.

8.1. **Example.** Let us describe invariant tensors in $Lin\left(V^{\otimes 2} \otimes Lin(V^{\otimes 2}, V), V\right)$. The canonical identifications (58) determine a $\mathrm{GL}(V)$-equivariant isomorphism

$$\Phi : Lin\left(V^{\otimes 2} \otimes Lin(V^{\otimes 2}, V), V\right) \cong Lin(V^{\otimes 3}, V^{\otimes 3}).$$

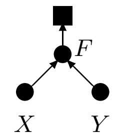
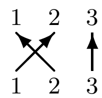
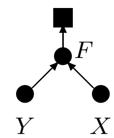
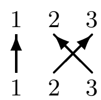
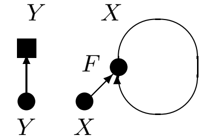
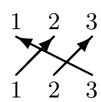
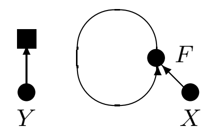
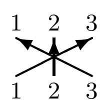
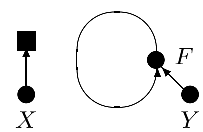
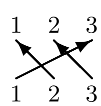
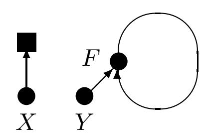
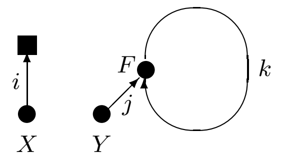
	$\Phi^{-1}(t_\sigma)$:	coordinate form:	graph:
$\sigma = identity$	$X \otimes Y \otimes F \mapsto F(X, Y)$	$X^j Y^k F^i_{jk} e_i$	
$\sigma =$ 	$X \otimes Y \otimes F \mapsto F(Y, X)$	$X^j Y^k F^i_{kj} e_i$	
$\sigma =$ 	$X \otimes Y \otimes F \mapsto Y \otimes Tr(F(X, -))$	$X^j Y^i F^k_{jk} e_i$	
$\sigma =$ 	$X \otimes Y \otimes F \mapsto Y \otimes Tr(F(-, X))$	$X^j Y^i F^k_{kj} e_i$	
$\sigma =$ 	$X \otimes Y \otimes F \mapsto X \otimes Tr(F(-, Y))$	$X^i Y^j F^k_{kj} e_i$	
$\sigma =$ 	$X \otimes Y \otimes F \mapsto X \otimes Tr(F(Y, -))$	$X^i Y^j F^k_{jk} e_i$	

FIGURE 4. Invariant tensors in $Lin(V^{\otimes 2} \otimes Lin(V^{\otimes 2}, V), V)$. The meaning of vertical braces on the right is explained in Example 8.3.

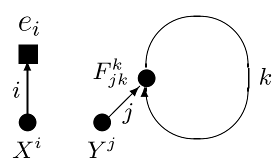
Applying the Invariant Tensor Theorem to $Lin(V^{\otimes 3}, V^{\otimes 3})$, one concludes that the subspace $Lin_{\mathrm{GL}(V)}(V^{\otimes 2} \otimes Lin(V^{\otimes 2}, V), V)$ is spanned by $\Phi^{-1}(t_\sigma)$, $\sigma \in \Sigma_3$, and that these generators are linearly independent if $\dim(V) \geq 3$. It is a simple exercise to calculate the tensors $\Phi^{-1}(t_\sigma)$ explicitly. The results are shown in the second column of the table in Figure 4 in which $X \otimes Y \otimes F$ is an element of $V^{\otimes 2} \otimes Lin(V^{\otimes 2}, V)$ and $Tr(-)$ the trace of a linear map $V \to V$.

Let us fix a basis $\{e_1, \ldots, e_n\}$ of V and write $X = X^a e_a$, $Y = Y^a e_a$ and $F(e_a, e_b) = F^c_{ab} e_c$, for some scalars $X^a, Y^a, F^c_{ab} \in \mathbf{k}$, $1 \leq a, b, c \leq n$. The corresponding coordinate forms of the elementary tensors are shown in the third column of the table. Observe that the expressions in this column are all possible *contractions of indices* of the tensors X, Y and F.

The contraction schemes for indices are encoded by the rightmost column as follows. Given a graph G from this column, decorate its edges by symbols i, j, k. For example, for the graph in the bottom right corner of the table, choose the decoration



To each vertex of this edge-decorated graph we assign the coordinates of the corresponding tensors with the names of indices determined by decorations of edges adjacent to this vertex. For example, to the F-vertex we assign F^k_{jk}, because its left ingoing edge is decorated by j and its right ingoing edge which happens to be the same as its outgoing edge, is decorated by k. The vertex, called *the anchor*, plays a special role. We assign to it the basis of V indexed by the decoration of its ingoing edge. We get



As the final step we take the product of the factors assigned to vertices and perform the summation over repeated indices. The result is

$$\sum_{1\leq i,j,k\leq n} X^i Y^j F^k_{jk} e_i.$$

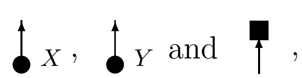
In this formula we made an exception from Einstein's convention and wrote the summation explicitly to emphasize the idea of the construction. A formal general definition of this process of interpreting graphs as contraction schemes is given below.

Let $\widehat{\mathfrak{G}\mathrm{r}}_{\mathrm{ex}}$ be the vector space spanned by the six graphs in the last column of the table; the hat indicates that the graphs are not oriented. The subscript "ex" is an abbreviation of "example," and distinguishes this graph complex from other objects with similar names used throughout the paper. The procedure described above gives an epimorphism

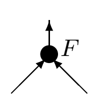
$$\widehat{\mathrm{R}}_n : \widehat{\mathfrak{G}\mathrm{r}}_{\mathrm{ex}} \to \mathit{Lin}_{\mathrm{GL}(V)}\big(V^{\otimes 2} \otimes \mathit{Lin}(V^{\otimes 2}, V), V\big) \tag{62}$$

which is an isomorphism if $n \geq 3$. The map $\widehat{R}_n$ defined in this way obviously does not depend on the choice of the basis $\{e_1, \ldots, e_n\}$ of V.

The space $\widehat{\mathfrak{G}\mathrm{r}}_{\mathrm{ex}}$ can also be defined as the span of all directed graphs with three unary vertices

(63) 

and one "planar" binary vertex

(64) 

whose planarity means that its inputs are linearly ordered. In pictures, this order is determined by reading the inputs from left to right.

[November 11, 2007]

Let us generalize calculations in Example 8.1 and describe $\mathrm{GL}(V)$-invariant elements in

$$Lin\left(Lin(V^{\otimes h_1}, V^{\otimes p_1}) \otimes \cdots \otimes Lin(V^{\otimes h_r}, V^{\otimes p_r}), Lin(V^{\otimes c}, V^{\otimes d})\right), \tag{65}$$

where $r, p_1, \ldots, p_r, h_1, \ldots, h_r, c$ and d are non-negative integers. The above space is canonically isomorphic to

$$V^{*\otimes p_1} \otimes V^{\otimes h_1} \otimes \cdots \otimes V^{*\otimes p_r} \otimes V^{\otimes h_r} \otimes V^{*\otimes c} \otimes V^{\otimes d},$$

which is in turn isomorphic to

$$V^{*\otimes(p_1+\cdots+p_r+c)} \otimes V^{\otimes(h_1+\cdots+h_r+d)}, \tag{66}$$

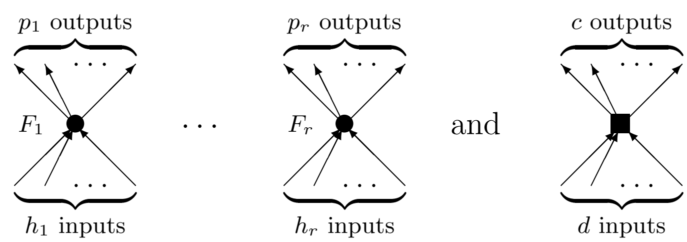
via the isomorphism that moves all V^*-factors to the left, without changing their relative order. By the last and first isomorphisms in (58), the space in (66) is isomorphic to

$$Lin(V^{\otimes(p_1+\cdots+p_r+c)}, V^{\otimes(h_1+\cdots+h_r+d)}).$$

We will denote the composite isomorphism between (65) and the space in the above display by Φ. Since all isomorphisms above are $\mathrm{GL}(V)$-equivariant, Φ is equivariant, too, thus the space (65) may contain nontrivial $\mathrm{GL}(V)$-equivariant maps only if

$$p_1 + \cdots + p_r + c = h_1 + \cdots + h_r + d. \tag{67}$$

Denote by $\widehat{\mathfrak{Gr}}$ the space spanned by all directed graphs with $r+1$ planar vertices



,

where planarity means that linear orders of the sets of input and output edges are specified. Observe that the number of edges of each graph spanning $\widehat{\mathfrak{Gr}}$ equals the common value of the sums in (67). For each graph $G \in \widehat{\mathfrak{Gr}}$ we define a $\mathrm{GL}(V)$-equivariant map $\widehat{\mathrm{R}}_n(G)$ in the space (65) as follows.

As in Example 8.1, choose a basis $(e_1, \ldots, e_n)$ of V and let $(e^1, \ldots, e^n)$ be the corresponding dual basis of V^*. For $F_i \in Lin(V^{\otimes h_i}, V^{\otimes p_i})$, $1 \leq i \leq r$, write

$$F_i = F_i{}^{a^i_1,\ldots,a^i_{p_i}}_{b^i_1,\ldots,b^i_{h_i}}\ e_{a_1} \otimes \cdots \otimes e_{a_{p_i}} \otimes e^{b_1} \otimes \cdots \otimes e^{b_{h_i}}$$

with some scalars $F_i{}^{a^i_1,\ldots,a^i_{p_i}}_{b^i_1,\ldots,b^i_{h_i}} \in \mathbf{k}$ or, more concisely, $F_i = F_i{}^{A^i}_{B^i}\, e_{A^i} \otimes e^{B^i}$, where A^i abbreviates the multiindex $(a^i_1, \ldots, a^i_{p_i})$, B^i the multiindex $(b^i_1, \ldots, b^i_{h_i})$, $e_{A^i} := e_{a_1} \otimes \cdots \otimes e_{a_{p_i}}$, $e^{B^i} := e^{b_1} \otimes \cdots \otimes e^{b_{h_i}}$ and, as everywhere in this paper, summations over repeated (multi)indices are assumed.

A *labelling* of a graph $G \in \widehat{\mathfrak{Gr}}$ is a function $\ell : Edg(G) \to \{1, \ldots, n\}$, where $Edg(G)$ denotes the set of edges of G. Let $Lab(G)$ be the set of all labellings of G. For $\ell \in Lab(G)$ and $1 \leq i \leq r$, define $A^i(\ell)$ to be the multiindex $(a^i_1, \ldots, a^i_{p_i})$ such that a^i_s equals $\ell(e)$, where e is the edge that starts at the s-th output of the vertex F_i, $1 \leq s \leq p_i$. Likewise, put $I(\ell) := (i_1, \ldots, i_c)$ with $i_t := \ell(e)$, where now e is the edge that starts at the t-th output of the $\blacksquare$-vertex, $1 \leq t \leq c$. Let $B^i(\ell)$ and $J(\ell)$ have similar obvious meanings, with 'inputs' taken instead of 'outputs.' For $F_1 \otimes \cdots \otimes F_r \in \mathit{Lin}(V^{\otimes h_1}, V^{\otimes p_1}) \otimes \cdots \otimes \mathit{Lin}(V^{\otimes h_r}, V^{\otimes p_r})$ define finally

$$\widehat{\mathrm{R}}_n(G)(F_1 \otimes \cdots \otimes F_r) := \sum_{\ell \in \mathit{Lab}(G)} F_1{}^{A^1(\ell)}_{B^1(\ell)} \otimes \cdots \otimes F_r{}^{A^r(\ell)}_{B^r(\ell)}\, e_{J(\ell)} \otimes e^{I(\ell)} \in \mathit{Lin}(V^{\otimes c}, V^{\otimes d}). \tag{68}$$

It is easy to check that $\widehat{\mathrm{R}}_n(G)$ is a $\mathrm{GL}(V)$-fixed element of the space (65). The nature of the summation in (68) is close to the *state sum model* for link invariants, see [9, Section I.8], with states being the values of labels of the edges of the graph.

8.2. **Proposition.** *Let $r, p_1, \ldots, p_r, h_1, \ldots, h_r, c$ and d be non-negative integers. Then the map*

$$\widehat{\mathrm{R}}_n : \widehat{\mathfrak{G}\mathrm{r}} \to \mathit{Lin}_{\mathrm{GL}(V)}\left(\mathit{Lin}(V^{\otimes h_1}, V^{\otimes p_1}) \otimes \cdots \otimes \mathit{Lin}(V^{\otimes h_r}, V^{\otimes p_r}), \mathit{Lin}(V^{\otimes c}, V^{\otimes d})\right)$$

defined by (68) is an epimorphism. If $n \geq e$, where e is the number of edges of graphs spanning $\widehat{\mathfrak{G}\mathrm{r}}$ and $n = \dim(V)$, $\widehat{\mathrm{R}}_n$ is also an isomorphism.

Observe that we do not need to assume (67) in Proposition 8.2. If (67) is not satisfied, then there are no $\mathrm{GL}(V)$-invariant elements in (65) and also the space $\widehat{\mathfrak{G}\mathrm{r}}$ is trivial, thus $\widehat{\mathrm{R}}_n$ is an isomorphism of trivial spaces.

Proof of Proposition 8.2. By the above observation, we may assume (67). Consider the diagram

$$\begin{tikzcd}
\mathbf{k}[\Sigma_k] \arrow[r, "\mathcal{R}_n"] & \mathit{Lin}_{\mathrm{GL}(V)}(V^{\otimes(p_1+\cdots+p_r+c)}, V^{\otimes(h_1+\cdots+h_r+d)}) \\
\widehat{\mathfrak{G}\mathrm{r}} \arrow[u, "\Psi"', "\cong"] \arrow[r, "\widehat{\mathrm{R}}_n"] & \mathit{Lin}_{\mathrm{GL}(V)}\left(\mathit{Lin}(V^{\otimes h_1}, V^{\otimes p_1}) \otimes \cdots \otimes \mathit{Lin}(V^{\otimes h_r}, V^{\otimes p_r}), \mathit{Lin}(V^{\otimes c}, V^{\otimes d})\right) \arrow[u, "\Phi"', "\cong"]
\end{tikzcd} \tag{69}$$

in which $\mathcal{R}_n$ is the map (60), $\widehat{\mathrm{R}}_n$ is defined in (68) and Φ is the composition of canonical isomorphisms and reshufflings of factors described on page 43 above. The map Ψ is defined as follows.

Let us denote, for the purposes of this proof only, by $\mathrm{Ou}(F_i)$ the linearly ordered set of outputs of the F_i-vertex, $1 \leq i \leq r$, and by $\mathrm{Ou}(\blacksquare)$ the linearly ordered set of outputs of $\blacksquare$. The set $\mathrm{Ou} := \mathrm{Ou}(F_1) \cup \cdots \cup \mathrm{Ou}(F_r) \cup \mathrm{Ou}(\blacksquare)$ is linearly ordered by requiring that

$$\mathrm{Ou}(F_1) < \cdots < \mathrm{Ou}(F_r) < \mathrm{Ou}(\blacksquare)$$

(we believe that the meaning of this shorthand is obvious). Let In be the linearly ordered set of inputs defined in the similar way. The orders define unique isomorphisms

$$\mathrm{Ou} \cong (1, \ldots, k) \text{ and } \mathrm{In} \cong (1, \ldots, k) \tag{70}$$

of ordered sets.

Since graphs spanning $\widehat{\mathfrak{G}\mathrm{r}}$ are determined by specifying how the outputs of vertices are connected to its inputs, there exists a one-to-one correspondence $G \leftrightarrow \varphi_G$ between graphs $G \in \widehat{\mathfrak{G}\mathrm{r}}$ and isomorphisms $\varphi_G : \mathrm{Ou} \stackrel{\cong}{\to} \mathrm{In}$. Given (70), such φ_G can be interpreted as an element of the symmetric group Σ_k. The map Ψ is then defined by $\Psi(G) := \varphi_G$.

It is simple to verify that the diagram (69) commutes, so the proposition follows from the Invariant Tensor Theorem. □

In the light of diagram (69), Proposition 8.2 may look just as a clumsy reformulation of the Invariant Tensor Theorem. Graphs become relevant when symmetries occur.

8.3. **Example.** Let $\mathit{Sym}(V^{\otimes 2}, V) \subset \mathit{Lin}(V^{\otimes 2}, V)$ be the subspace of symmetric bilinear maps, i.e. maps satisfying $f(v', v'') = f(v'', v')$ for $v', v'' \in V$. Let us explain how to use calculations of Example 8.1 to describe $\mathrm{GL}(V)$-equivariant maps in $\mathit{Lin}\left(V^{\otimes 2} \otimes \mathit{Sym}(V^{\otimes 2}, V), V\right)$.

The right Σ_2-action on $\mathit{Lin}(V^{\otimes 2}, V)$ given by permuting the inputs of bilinear maps is such that the space $\mathit{Sym}(V^{\otimes 2}, V)$ equals the subspace $\mathit{Lin}(V^{\otimes 2}, V)^{\Sigma_2}$ of Σ_2-fixed elements. This right Σ_2-action induces a left Σ_2-action on $\mathit{Lin}\left(V^{\otimes 2} \otimes \mathit{Lin}(V^{\otimes 2}, V), V\right)$ which commutes with the $\mathrm{GL}(V)$-action, therefore it restricts to a left Σ_2-action on the subspace $\mathit{Lin}_{\mathrm{GL}(V)}\left(V^{\otimes 2} \otimes \mathit{Lin}(V^{\otimes 2}, V), V\right)$ of $\mathrm{GL}(V)$-equivariant maps.

There is also a left Σ_2-action on the linear space $\widehat{\mathfrak{G}\mathrm{r}}_{\mathrm{ex}}$ interchanging the inputs of the F-vertices of generating graphs. It is simple to check that the map (62) of Example 8.1 is equivariant with respect to these two Σ_2-actions, hence it induces the map

$$\Sigma_2 \backslash \widehat{\mathrm{R}}_n : \Sigma_2 \backslash \widehat{\mathfrak{G}\mathrm{r}}_{\mathrm{ex}} \to \Sigma_2 \backslash \mathit{Lin}_{\mathrm{GL}(V)}\left(V^{\otimes 2} \otimes \mathit{Lin}(V^{\otimes 2}, V), V\right) \tag{71}$$

of left cosets. Observe that, by a standard duality argument,

$$\Sigma_2 \backslash \mathit{Lin}_{\mathrm{GL}(V)}\left(V^{\otimes 2} \otimes \mathit{Lin}(V^{\otimes 2}, V), V\right) \cong \mathit{Lin}_{\mathrm{GL}(V)}\left(V^{\otimes 2} \otimes \mathit{Sym}(V^{\otimes 2}, V), V\right). \tag{72}$$

Let us denote $\widehat{\mathfrak{G}\mathrm{r}}_{\mathrm{ex},\bullet} := \Sigma_2 \backslash \widehat{\mathfrak{G}\mathrm{r}}_{\mathrm{ex}}$. The bullet $\bullet$ in the subscript signalizes the presence of vertices with fully symmetric inputs. By definition, graphs $G', G'' \in \widehat{\mathfrak{G}\mathrm{r}}_{\mathrm{ex}}$ are identified in the quotient $\widehat{\mathfrak{G}\mathrm{r}}_{\mathrm{ex},\bullet}$ if they differ only by the order of inputs of the F-vertex. In Figure 4, this identification is indicated by vertical braces. We see that $\widehat{\mathfrak{G}\mathrm{r}}_{\mathrm{ex},\bullet}$ is again a space *spanned by graphs,* this time with no linear order on the inputs of the F-vertex. So we may *define* $\widehat{\mathfrak{G}\mathrm{r}}_{\mathrm{ex},\bullet}$ as the space spanned by directed graphs with vertices (63) and one binary (ordinary, non-planar) vertex (64). We conclude by interpreting (71) as the map

$$\widehat{\mathrm{R}}_n : \widehat{\mathfrak{G}\mathrm{r}}_{\mathrm{ex},\bullet} \to Lin_{\mathrm{GL}(V)}\left(V^{\otimes 2} \otimes Sym(V^{\otimes 2}, V), V\right). \tag{73}$$

It follows from the properties of the map (62) and the characteristic zero assumption that $\widehat{\mathrm{R}}_n$ is always an epimorphism and is an isomorphism if $n \geq 3$.

At this point we want to incorporate, by generalizing the pattern used in Example 8.3, symmetries into Proposition 8.2. Unfortunately, it turns out that treating the space (65) in full generality leads to a notational disaster. To keep the length of formulas within a reasonable limit, we decided to *assume from now on* that $p_1 = \cdots = p_r = 1$, $c = 0$ and $d = 1$. This means that we will restrict our attention to maps in

$$Lin\left(Lin(V^{\otimes h_1}, V) \otimes \cdots \otimes Lin(V^{\otimes h_r}, V), V\right). \tag{74}$$

For graphs this assumption implies that the vertices $F_1, \ldots, F_r$ have precisely one output, and that the anchor $\blacksquare$ has one input and no outputs. The number of inputs of F_i will be called the *arity* of F_i, $1 \leq i \leq r$. Condition (67) reduces to

$$r = h_1 + \cdots + h_r + 1$$

and one also sees that r equals the number of edges of the generating graphs.

The above generality is sufficient for the purposes of the present paper concerned with vector-field valued operators. A modification to the general case is straightforward but notationally challenging.

The space $Lin(V^{\otimes h}, V)$ admits, for each $h \geq 0$, a natural right Σ_h-action given by permuting inputs of multilinear maps. A *symmetry* of maps in $Lin(V^{\otimes h}, V)$ will be specified by a subset $\mathfrak{I} \subset \mathbf{k}[\Sigma_h]$. We then denote

$$Lin_{\mathfrak{I}}(V^{\otimes h}, V) := \left\{ f \in Lin(V^{\otimes h}, V);\ f\mathfrak{s} = 0 \text{ for each } \mathfrak{s} \in \mathfrak{I} \right\}.$$

For $\mathfrak{I}$ as above and a left Σ_h-module U, we will abbreviate by $\mathfrak{I}\backslash U$ the left coset $\mathfrak{I}U\backslash U$.

8.4. **Example.** Let $\mathfrak{I} := I_h \subset \mathbf{k}[\Sigma_h]$ be the augmentation ideal. Then $Lin_{I_h}(V^{\otimes h}, V)$ is the space of symmetric maps,

$$Lin_{I_h}(V^{\otimes h}, V) = Sym(V^{\otimes h}, V).$$

We leave as an exercise to describe in this language the spaces of *anti*symmetric maps.

8.5. **Example.** Let $h := v + 2$, $v \geq 0$, and let $\nabla \subset \mathbf{k}[\Sigma_h]$ be the image of the augmentation ideal I_v of $\mathbf{k}[\Sigma_v]$ in $\mathbf{k}[\Sigma_h]$ under the map of group rings induced by the inclusion $\Sigma_v \hookrightarrow \Sigma_v \times \Sigma_2 \hookrightarrow \Sigma_h$ that interprets permutations of $(1, \ldots, v)$ as permutations of $(1, \ldots, v, v+1, v+2)$ keeping the last two elements fixed. Then $Lin_\nabla(V^{\otimes h}, V)$ consists of multilinear maps $V^{\otimes(v+2)} \to V$ that are symmetric in the first v inputs.

8.6. **Remark.** It is clear how to generalize the above notion of symmetry to maps in the left Σ_p-right Σ_h-module $Lin(V^{\otimes h}, V^{\otimes p})$ for general $p, h \geq 0$. A symmetry of these maps will be specified by subsets $\mathfrak{I} \in \mathbf{k}[\Sigma_h]$ and $\mathfrak{O} \in \mathbf{k}[\Sigma_p]$, the corresponding subspaces will then be

$$Lin_{\mathfrak{I}}^{\mathfrak{O}}(V^{\otimes h}, V^{\otimes p}) := \left\{ f \in Lin(V^{\otimes h}, V^{\otimes p});\ f\mathfrak{s} = 0 = \mathfrak{t}f \text{ for each } \mathfrak{s} \in \mathfrak{I} \text{ and } \mathfrak{t} \in \mathfrak{O} \right\}.$$

Suppose we are given subsets $\mathfrak{I}_i \subset \mathbf{k}[\Sigma_{h_i}]$, $1 \leq i \leq r$. Our aim is to describe $\mathrm{GL}(V)$-invariant elements in the space

$$Lin\left(Lin_{\mathfrak{I}_1}(V^{\otimes h_1}, V) \otimes \cdots \otimes Lin_{\mathfrak{I}_r}(V^{\otimes h_r}, V), V\right). \tag{75}$$

Let

$$\mathfrak{I} := \mathfrak{I}_1 \cup \cdots \cup \mathfrak{I}_r \subset \mathbf{k}[\Sigma_{h_1} \times \cdots \times \Sigma_{h_r}],$$

where $\mathfrak{I}_i$ is, for $1 \leq i \leq r$, identified with its image in $\mathbf{k}[\Sigma_{h_1} \times \cdots \times \Sigma_{h_r}]$ under the map induced by the group inclusion $\Sigma_{h_i} \hookrightarrow \Sigma_{h_1} \times \cdots \times \Sigma_{h_r}$.

As in Example 8.3, we use the fact that, for $1 \leq i \leq r$, each $Lin(V^{\otimes h_i}, V)$ is a right Σ_{h_i}-space, hence the tensor product $Lin(V^{\otimes h_1}, V) \otimes \cdots \otimes Lin(V^{\otimes h_r}, V)$ has a natural right $\Sigma_{h_1} \times \cdots \times \Sigma_{h_r}$-action which induces a left $\Sigma_{h_1} \times \cdots \times \Sigma_{h_r}$-action on the space (74). This action restricts to the subspace of $\mathrm{GL}(V)$-equivariant maps.

There is also a left $\Sigma_{h_1} \times \cdots \times \Sigma_{h_r}$-action on the space $\widehat{\mathfrak{G}\mathrm{r}}$ given by permuting, in the obvious manner, the inputs of the vertices $F_1, \ldots, F_r$ of generating graphs. The map $\widehat{\mathrm{R}}_n$ of Proposition 8.2 is equivariant with respect to the above two actions and induces the map

$$\mathfrak{I}\backslash\widehat{\mathrm{R}}_n : \mathfrak{I}\backslash\widehat{\mathfrak{G}\mathrm{r}} \to \mathfrak{I}\backslash Lin_{\mathrm{GL}(V)}\left(Lin(V^{\otimes h_1}, V) \otimes \cdots \otimes Lin(V^{\otimes h_r}, V), V\right)$$

of left quotients. Denoting $\widehat{\mathfrak{G}\mathrm{r}}_{\mathfrak{I}} := \mathfrak{I}\backslash\widehat{\mathfrak{G}\mathrm{r}}$ and realizing that, by duality, the codomain of $\mathfrak{I}\backslash\widehat{\mathrm{R}}_n$ is isomorphic to the subspace of $\mathrm{GL}(V)$-fixed elements in (75), we obtain the map (denoted again $\widehat{\mathrm{R}}_n$)

$$\widehat{\mathrm{R}}_n : \widehat{\mathfrak{G}\mathrm{r}}_{\mathfrak{I}} \to Lin_{\mathrm{GL}(V)}\left(Lin_{\mathfrak{I}_1}(V^{\otimes h_1}, V) \otimes \cdots \otimes Lin_{\mathfrak{I}_r}(V^{\otimes h_r}, V), V\right) \tag{76}$$

which is, by Proposition 8.2, an epimorphism and is an isomorphism if $\dim(V) \geq r$.

8.7. **Remark.** As in Example 8.3, it turns out that the quotient $\widehat{\mathfrak{G}\mathrm{r}}_{\mathfrak{I}} = \mathfrak{I}\backslash\widehat{\mathfrak{G}\mathrm{r}}$ is a *space of graphs* though, for general symmetries, "space of graphs" means a free wheeled operad on a certain Σ-module [16]. In the cases relevant for our paper, we however remain in the realm of 'classical' graphs, as shown in the following example, see also the proof of Corollary 8.13.

8.8. **Example.** Suppose that, for some $1 \leq i \leq r$, $\mathfrak{I}_i$ equals the augmentation ideal I_{h_i} of $\mathbf{k}[\Sigma_{h_i}]$ as in Example 8.4. Then, in the quotient $\mathfrak{I}\backslash\widehat{\mathfrak{G}\mathrm{r}}$, one identifies graphs that differ by the order of inputs of the vertex F_i. In other words, modding out by $\mathfrak{I}_i \subset \mathfrak{I}$ erases the order of inputs of F_i, turning F_i into an ordinary (non-planar) vertex. If $\mathfrak{I}_i = \nabla$ as in Example 8.5, one gets a vertex (18) of arity $v+2$, $v \geq 0$, whose first v inputs are symmetric.

We still need one more level of generalization that will reflect the antisymmetry of the Chevalley-Eilenberg complex (10) in the Lie algebra variables. As a motivation for our construction, we offer the following continuation of Examples 8.1 and 8.3.

8.9. **Example.** We will consider the tensor product $V \otimes V$ as a left Σ_2-module, with the action $\tau(v' \otimes v'') := -(v'' \otimes v')$, for $v', v'' \in V$ and the generator $\tau \in \Sigma_2$. The subspace $(V \otimes V)^{\Sigma_2}$ of Σ_2-fixed elements is then precisely the second exterior power $\bigwedge^2 V$. This left action induces a $\mathrm{GL}(V)$-equivariant right Σ_2-action on the space $\mathit{Lin}\left(V^{\otimes 2} \otimes \mathit{Sym}(V^{\otimes 2}, V), V\right)$ such that

$$\mathit{Lin}\left(V^{\otimes 2} \otimes \mathit{Sym}(V^{\otimes 2}, V), V\right)/\Sigma_2 \cong \mathit{Lin}\left(\bigwedge^2 V \otimes \mathit{Sym}(V^{\otimes 2}, V), V\right).$$

The above isomorphism restricts to an isomorphism

$$\mathit{Lin}_{\mathrm{GL}(V)}\left(V^{\otimes 2} \otimes \mathit{Sym}(V^{\otimes 2}, V), V\right)/\Sigma_2 \cong \mathit{Lin}_{\mathrm{GL}(V)}\left(\bigwedge^2 V \otimes \mathit{Sym}(V^{\otimes 2}, V), V\right). \tag{77}$$

of the subspaces of $\mathrm{GL}(V)$-equivariant maps.

Likewise, $\widehat{\mathcal{G}\mathrm{r}}_{\mathrm{ex},\bullet}$ carries a right Σ_2-action that interchanges the labels X and Y of the -vertices of graphs in the last column of Figure 4 and multiplies the sign of the corresponding generator by -1. The map (73) is Σ_2-equivariant, therefore it induces the map

$$\widehat{\mathrm{R}}_n/\Sigma_2 : \widehat{\mathcal{G}\mathrm{r}}_{\mathrm{ex},\bullet}/\Sigma_2 \to \mathit{Lin}_{\mathrm{GL}(V)}\left(V^{\otimes 2} \otimes \mathit{Sym}(V^{\otimes 2}, V), V\right)/\Sigma_2.$$

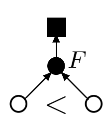
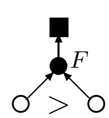
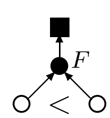
Let us denote $\mathcal{G}\mathrm{r}^2_{\mathrm{ex},\bullet} := \widehat{\mathcal{G}\mathrm{r}}_{\mathrm{ex},\bullet}/\Sigma_2$ and $\mathrm{R}^2_n := \widehat{\mathrm{R}}_n/\Sigma_2$. Using (77), one rewrites the above map as an epimorphism

$$\mathrm{R}^2_n : \mathcal{G}\mathrm{r}^2_{\mathrm{ex},\bullet} \twoheadrightarrow \mathit{Lin}_{\mathrm{GL}(V)}\left(\bigwedge^2 V \otimes \mathit{Sym}(V^{\otimes 2}, V), V\right)$$

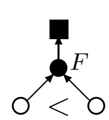
which is an isomorphism if $n \geq 3$.

The space $\mathcal{G}\mathrm{r}^2_{\mathrm{ex},\bullet}$ is isomorphic to the span of the set of directed, oriented graphs with one (non-planar) binary vertex F, an anchor , and two 'white' vertices . By an *orientation* we mean a linear order of white vertices. A graph with the opposite orientation is identified with the original one taken with the opposite sign. It is clear that, with $\mathcal{G}\mathrm{r}^2_{\mathrm{ex},\bullet}$ defined in this way, the map $\mathcal{G}\mathrm{r}^2_{\mathrm{ex},\bullet} \to \widehat{\mathcal{G}\mathrm{r}}_{\mathrm{ex},\bullet}/\Sigma_2$ that replaces the first (in the linear order given by the orientation) white vertex by the black vertex labelled by X, and the second white vertex by the black vertex labelled by Y, is an isomorphism.

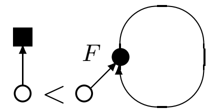
The symmetry of the inputs of the vertex F implies the following identities in $\mathcal{G}\mathrm{r}^2_{\mathrm{ex},\bullet}$:

 $= -$  $= -$  ,

from which one concludes that

 $= 0.$

Therefore $\mathcal{G}\mathrm{r}^2_{\mathrm{ex},\bullet}$ is in this case one-dimensional, spanned by the equivalence class of the oriented directed graph



.

In the notation of Figure 4, the above graph represents the map that sends $(X \wedge Y) \otimes F \in \bigwedge^2 V \otimes \mathit{Sym}(V^{\otimes 2}, V)$ into

$$X \otimes \mathit{Tr}(F(Y,-)) - Y \otimes \mathit{Tr}(F(X,-)) \in V.$$

Let us turn to our final task. We want to describe $\mathrm{GL}(V)$-invariant elements in the space

$$\mathit{Lin}\left(\bigwedge_{1\leq i\leq m} \mathit{Sym}(V^{\otimes h_i}, V) \otimes \bigotimes_{m+1\leq i\leq r} \mathit{Lin}_{\mathfrak{I}_i}(V^{\otimes h_i}, V), V\right) \tag{78}$$

where, as before, $r, h_1, \ldots, h_r$ are positive integers, $\mathfrak{I}_i \subset \mathbf{k}[\Sigma_{h_i}]$ for $m+1 \leq i \leq r$, and m is an integer such that $1 \leq m \leq r$. Having in mind the description of the space of symmetric multilinear maps given in Example 8.4, we extend the definition of $\mathfrak{I}_i$ also to $1 \leq i \leq m$, by putting $\mathfrak{I}_i := I_{h_i}$. The first step is to identify the exterior power $\bigwedge_{1\leq i\leq m} \mathit{Sym}(V^{\otimes h_i}, V)$ with the fixed point set of an action of a suitable finite group. This can be done as follows.

For $1 \leq w \leq m$, let $A(w) \subset \{1, \ldots, m\}$ be the subset $A(w) := \{1 \leq i \leq m;\ h_i = h_w\}$. Then

$$\{1, \ldots, m\} = \textstyle\bigcup_{1\leq w\leq m} A(w)$$

is a decomposition of $\{1, \ldots, m\}$ into not necessarily distinct subsets. Let $\widehat{\Sigma} \subset \Sigma_m$ be the subgroup of permutations of $\{1, \ldots, m\}$ preserving this decomposition.

The group $\widehat{\Sigma}$ acts on $\bigotimes_{1\leq i\leq m} \mathit{Sym}(V^{\otimes h_i}, V)$ by permuting the corresponding factors. If we consider this tensor product as a left $\widehat{\Sigma}$-module with this permutation action twisted by the signum representation, then

$$\bigwedge_{1\leq i\leq m} \mathit{Sym}(V^{\otimes h_i}, V) \cong \left(\bigotimes_{1\leq i\leq m} \mathit{Sym}(V^{\otimes h_i}, V)\right)^{\widehat{\Sigma}}.$$

The above left $\widehat{\Sigma}$-action on $\bigotimes_{1\leq i\leq m} \mathit{Sym}(V^{\otimes h_i}, V)$ induces a dual $\mathrm{GL}(V)$-equivariant right $\widehat{\Sigma}$-action on the space (78).

There is a right $\widehat{\Sigma}$-action on the quotient $\widehat{\mathcal{G}\mathrm{r}}_{\mathfrak{I}} = \mathfrak{I}\backslash\widehat{\mathcal{G}\mathrm{r}}$ defined as follows. For a graph $G \in \widehat{\mathcal{G}\mathrm{r}}$ representing an element $[G] \in \widehat{\mathcal{G}\mathrm{r}}_{\mathfrak{I}}$ and for $\sigma \in \widehat{\Sigma}$, let G^σ be the graph obtained from G by permuting the vertices $F_1, \ldots, F_m$ according to σ. We then put $[G]\sigma := \mathrm{sgn}(\sigma)[G^\sigma]$. Since, by the definition of $\widehat{\Sigma}$, σ may interchange only vertices with the same number of inputs and the same symmetry, our definition of G^σ makes sense.

It is simple to see that the map $\widehat{\mathrm{R}}_n$ in (76) is $\widehat{\Sigma}$-equivariant, giving rise to the map

$$\widehat{\mathrm{R}}_n/\widehat{\Sigma} : \widehat{\mathcal{G}\mathrm{r}}_{\mathfrak{I}}/\widehat{\Sigma} \to \mathit{Lin}_{\mathrm{GL}(V)}(\mathit{Lin}_{\mathfrak{I}_1}(V^{\otimes h_1}, V) \otimes \cdots \otimes \mathit{Lin}_{\mathfrak{I}_r}(V^{\otimes h_r}, V), V)/\widehat{\Sigma}$$

of right cosets. The codomain of $\widehat{\mathrm{R}}_n/\widehat{\Sigma}$ is easily seen to be isomorphic to the subspace of $\mathrm{GL}(V)$-equivariant elements in (78). The above calculations are summarized in the following proposition in which $\mathcal{G}\mathrm{r}^m_{\mathfrak{I}} := \widehat{\mathcal{G}\mathrm{r}}_{\mathfrak{I}}/\widehat{\Sigma}$ and $\mathrm{R}^m_n := \widehat{\mathrm{R}}_n/\widehat{\Sigma}$.

8.10. **Proposition.** *Let $r, h_1, \ldots, h_r$ be non-negative integers, $1 \leq m \leq r$, and $\mathfrak{I}_i \subset \mathbf{k}[\Sigma_{h_i}]$ for $m+1 \leq i \leq r$. Then the map*

$$\mathrm{R}^m_n : \mathcal{G}\mathrm{r}^m_{\mathfrak{I}} \to \mathit{Lin}_{\mathrm{GL}(V)}\left(\bigwedge_{1\leq i\leq m} \mathit{Sym}(V^{\otimes h_i}, V) \otimes \bigotimes_{m+1\leq i\leq r} \mathit{Lin}_{\mathfrak{I}_i}(V^{\otimes h_i}, V), V\right) \tag{79}$$

constructed above is an epimorphism. If, moreover, the dimension n of $V \geq$ the number of edges of graphs spanning $\mathcal{G}\mathrm{r}^m_{\mathfrak{I}}$, R^m_n is also an isomorphism.

The following result says that the presence of vertices with symmetric inputs miraculously *extends* the stability range. In applications, these vertices will represent the Lie algebra generators in the Chevalley-Eilenberg complex.

8.11. **Proposition.** *Suppose that $h_1, \ldots, h_m \geq 2$. If $n \geq e - m$, where n is the dimension of V and e the number of edges of graphs spanning $\mathcal{G}\mathrm{r}^m_{\mathfrak{I}}$, then the map R^m_n in Proposition 8.10 is an isomorphism.*

Proof. Let G be a graph spanning $\mathcal{G}\mathrm{r}^m_{\mathfrak{I}}$ and $S \subset \mathit{Edg}(G)$ a subset of edges of G such that $\mathrm{card}(S) > n$. For each permutation σ of elements of S, denote by G_σ the graph obtained by cutting the edges belonging to S in the middle and regluing them following the automorphism σ. The linear combination

$$\sum_{\sigma\in\Sigma_S} \mathrm{sgn}(\sigma)\cdot G_\sigma \in \mathcal{G}\mathrm{r}^m_{\mathfrak{I}} \tag{80}$$

is then a graph-ical representation of the expression in (61), thus the kernel of R^m_n is generated by expressions of this type. Since, by assumption, $\mathrm{card}(S) \leq n+m$ and $h_1, \ldots, h_m \geq 2$, the set S must necessarily contain two input edges of the *same* symmetric vertex of G. This implies that the sum (80) vanishes, because with each graph G_σ it contains the same graph with the opposite sign. This shows that the kernel of R^m_n is trivial. □

8.12. **Remark.** By an absolutely straightforward generalization of the above constructions, one can obtain versions of Proposition 8.10 and Proposition 8.11 describing the space

$$\mathit{Lin}_{\mathrm{GL}(V)}\left(\bigwedge_{1\leq i\leq m} \mathit{Sym}(V^{\otimes h_i}, V) \otimes \bigotimes_{m+1\leq i\leq r} \mathit{Lin}^{\mathfrak{O}_i}_{\mathfrak{I}_i}(V^{\otimes h_i}, V^{\otimes p_i}), \mathit{Lin}^{\mathfrak{O}}_{\mathfrak{I}}(V^{\otimes c}, V^{\otimes d})\right) \tag{81}$$

in terms of a space spanned by graphs. Since the notational aspects of such a generalization are horrendous, we must leave the details as an exercise to the reader.

[November 11, 2007]

We finish the appendix by a corollary tailored for the needs of this paper. For non-negative integers m, b and c, denote by $\mathfrak{Gr}^m_{\bullet(b)\nabla(c)}$ the space spanned by directed, oriented graphs with

(i) m unlabeled 'white' vertices with fully symmetric inputs and arities ≥ 2,
(ii) b 'black' labelled vertices with fully symmetric inputs and arities ≥ 0,
(iii) c labelled ∇-vertices, and
(iv) the anchor .

In item (iii), a ∇-vertex means a vertex with the symmetry described in Example 8.5, see also Example 8.8. As in Example 8.9, an *orientation* is given by a linear order on the set of white vertices. If G' and G'' are graphs in $\mathfrak{Gr}^m_{\bullet(b)\nabla(c)}$ whose orientations differ by an odd number of transpositions, then we identify $G' = -G''$ in $\mathfrak{Gr}^m_{\bullet(b)\nabla(c)}$.

8.13. **Corollary.** *For each non-negative integers m, b and c there exists a natural epimorphism*

$$\mathrm{R}^m_{\bullet(b)\nabla(c),n} : \mathfrak{Gr}^m_{\bullet(b)\nabla(c)} \twoheadrightarrow$$
$$\bigoplus_{\vec{h}\in\mathfrak{H}} Lin_{\mathrm{GL}(V)}\left(\bigwedge_{1\leq i\leq m} Sym(V^{\otimes h_i}, V) \otimes \bigotimes_{m+1\leq i\leq m+b} Sym(V^{\otimes h_i}, V) \bigotimes_{m+b+1\leq i\leq m+b+c} Lin_{\Delta}(V^{\otimes h_i}, V), V\right),$$

with the direct sum taken over the set $\mathfrak{H}$ of all multiindices $\vec{h} = (h_1, \ldots, h_{m+b+c})$ such that

$$h_1, \ldots, h_m \geq 2,\ h_{m+1}, \ldots, h_{m+b} \geq 0 \quad \textit{and} \quad h_{m+b+1}, \ldots, h_{m+b+c} \geq 2.$$

The map $\mathrm{R}^m_{\bullet(b)\nabla(c),n}$ is an isomorphism if $n = \dim(V) \geq b + c$.

Proof. The map $\mathrm{R}^m_{\bullet(b)\nabla(c),n}$ is constructed by assembling the maps R^m_n from Proposition 8.10 as follows. For a multiindex $\vec{h} = (h_1, \ldots, h_{m+b+c}) \in \mathfrak{H}$ as in the corollary take, in Proposition 8.10, $r := m + b + c$ and

$$\mathfrak{I}_i = \mathfrak{I}_i(\vec{h}) := \begin{cases} I_{h_i}, & \text{for } m+1 \leq i \leq m+b \text{ and} \\ \nabla, & \text{for } m+b+1 \leq i \leq r, \end{cases}$$

see Examples 8.4 and 8.5 for the notation. Let $\mathrm{R}^m_n(\vec{h})$ be the map (79) corresponding to the above choices and $\mathrm{R}^m_{\bullet(b)\nabla(c),n} := \bigoplus_{\vec{h}\in\mathfrak{H}} \mathrm{R}^m_n(\vec{h})$. We only need to show that the graph space $\mathfrak{Gr}^m_{\bullet(b),\nabla(c)}$ is isomorphic to the direct sum of the double quotients $\mathfrak{Gr}^m_{\mathfrak{I}(\vec{h})} = \mathfrak{I}(\vec{h})\backslash\widehat{\mathfrak{Gr}}/\widehat{\Sigma}$.

As we argued in Example 8.8, the left quotient $\widehat{\mathfrak{Gr}}_{\mathfrak{I}(\vec{h})} = \mathfrak{I}(\vec{h})\backslash\widehat{\mathfrak{Gr}}$ is spanned by directed graphs with r labelled vertices $F_1, \ldots, F_r$ such that the 1st type vertices $F_1, \ldots, F_m$ ('white' vertices) have fully symmetric inputs and arities $h_1, \ldots, h_m$, and the remaining vertices $F_{m+1}, \ldots, F_r$ are as in items (ii)–(iv) of the definition of $\mathfrak{Gr}^m_{\bullet(b)\nabla(c)}$ but with fixed arities $h_{m+1}, \ldots, h_r$.

Modding out $\widehat{\mathfrak{Gr}}_{\mathfrak{I}(\vec{h})}$ by $\widehat{\Sigma}$ identifies graphs that differ by a relabelling of white vertices of the same arity and the sign given by to the signum of this relabelling. This clearly means that the map

$$\mathcal{G}\mathrm{r}^m_{\bullet(b),\nabla(c)} \to \bigoplus_{\vec{h}\in\mathfrak{H}} \mathcal{G}\mathrm{r}^m_{\mathfrak{I}(\vec{h})} = \bigoplus_{\vec{h}\in\mathfrak{H}} \widehat{\mathcal{G}\mathrm{r}}_{\mathfrak{I}(\vec{h})}/\widehat{\Sigma}$$

that assigns to the first (in the linear order given by the orientation) white vertex of graphs generating $\mathcal{G}\mathrm{r}^m_{\bullet(b),\nabla(c)}$ label F_1, to the second white vertex label F_2, etc., is an isomorphism. By simple combinatorics, graphs spanning $\mathcal{G}\mathrm{r}^m_{\bullet(b),\nabla(c)}$ have precisely $m+b+c$ edges which completes the proof of the corollary. □

8.14. **Remark.** Proposition 8.10 and its Corollary 8.13 was obtained by applying the double-coset reduction $\mathfrak{I}\backslash -/\widehat{\Sigma}$ and standard duality to the map $\widehat{\mathrm{R}}_n$ of Proposition 8.2. Backtracking all the constructions involved, one can see that, in Corollary 8.13, the invariant linear map $\mathrm{R}^m_{\bullet(b)\nabla(c),n}(G)$ corresponding to a graph $G \in \mathcal{G}\mathrm{r}^m_{\bullet(b)\nabla(c)}$ is given by the 'state sum' (68) *antisymmetrized* in the white vertices.

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[November 11, 2007]

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