SUBSCALAR OPERATORS AND GROWTH OF RESOLVENT

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Abstract. We construct a Banach space operator T which is not $\mathcal{E}(\mathbb{T})$-subscalar but $\|(T-z)^{-1}\| \le (|z|-1)^{-1}$ for $|z|>1$ and $m(T-z) \ge \text{const}\cdot(1-|z|)^3$ for $|z|<1$ (here m denotes the minimum modulus). This gives a negative answer to a variant of a problem of Laursen and Neumann. We also give a sufficient condition (in terms of growth of resolvent and of an analytic left inverse of $T-z$) implying that T is an $\mathcal{E}(\mathbb{T})$-subscalar operator. This condition is also necessary for Hilbert space operators.

1. Introduction

Generalized scalar operators are those Banach spaces operators possessing a C^∞-functional calculus. To be more specific, let $\mathcal{E}(\mathbb{C})$ denote the usual Fréchet algebra of all C^∞-functions on $\mathbb{C}$ with the topology of uniform convergence of derivatives of all orders on compact subsets of $\mathbb{C}$. Let X be a complex Banach space. A bounded linear operator $S \in B(X)$ is said [CF] to be an $\mathcal{E}(\mathbb{C})$-*scalar* (or *generalized scalar*) operator if there is a continuous algebra homomorphism $\Phi : \mathcal{E}(\mathbb{C}) \to B(X)$ for which $\Phi(1) = I$ and $\Phi(z) = S$. Here z denotes the identity function on $\mathbb{C}$. A bounded linear operator is $\mathcal{E}(\mathbb{C})$-*subscalar* if it is similar to the restriction of an $\mathcal{E}(\mathbb{C})$-scalar operator to one of its closed invariant subspaces. We refer to three books [CF], [EP] and [LN] for more information on $\mathcal{E}(\mathbb{C})$-scalar and $\mathcal{E}(\mathbb{C})$-subscalar operators.

The following statements are known to be equivalent (see [CF, LN]) :

(1) S is $\mathcal{E}(\mathbb{T})$-*scalar*, i.e., it has a continuous functional calculus on the Fréchet algebra $\mathcal{E}(\mathbb{T})$ of C^∞ functions on the unit circle $\mathbb{T}$;

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(2) S is $\mathcal{E}(\mathbb{C})$-scalar with spectrum $\sigma(S)$ in the unit circle $\mathbb{T}$;

(3) S is invertible, and there exist constants $C > 0$, $p \geq 0$ and $q \geq 0$ such that
$$\|S^n\| \leq Cn^p \qquad (n \in \mathbb{N})$$
and
$$\|S^{-n}\| \leq Cn^q \qquad (n \in \mathbb{N});$$

(4) $\sigma(S) \subset \mathbb{T}$ and there exist constants $C > 0$, $p \geq 0$ and $q \geq 0$ such that
$$\|(S-z)^{-1}\| \leq C\,(|z|-1)^{-p} \quad (|z|>1)$$
and
$$\|(S-z)^{-1}\| \leq C\,(1-|z|)^{-q} \quad (|z|<1).$$

The distinction between the growth of norms of positive and negative powers (and the resolvent growth inside and outside unit disc) will become apparent later on.

For $T \in B(X)$ we denote
$$m(T) = \inf\{\|Tx\| : x \in X, \|x\| = 1\}.$$

This quantity is called the *minimum modulus* of T ([GT]) or the *lower bound* of T ([LN]). It is easy to see that $m(T) > 0$ if and only if $T \in B(X)$ is one-to-one and with closed range. For invertible operators S we have $m(S) = \|S^{-1}\|^{-1}$.

The main question we consider in this note is the problem of intrinsic characterizations of $\mathcal{E}(\mathbb{T})$-subscalar operators (i.e. operators similar to a restriction of a $\mathcal{E}(\mathbb{T})$-scalar operator to an invariant subspace). Let $T \in B(X)$ be an $\mathcal{E}(\mathbb{T})$-subscalar operator. Using (3) for the invertible extension of T we obtain the existence of constants $C > 0$, $p \geq 0$ and $q \geq 0$ such that :
$$(P) \qquad \|T^n\| \leq Cn^p \text{ and } m(T^n)^{-1} \leq Cn^q.$$

It is natural to ask if the polynomial growth condition (P) above (in terms of norms and minimum moduli of iterates) characterizes $\mathcal{E}(\mathbb{T})$-subscalars operators (cf. K.B. Laursen and M.M. Neumann [LN, Problem 6.1.15] and M. Didas [Di]). This problem was also discussed in [MMN1, MMN2, MMN3, MMN4]. It was recently proved by the authors [BM2, BM1] that $\mathcal{E}(\mathbb{T})$-subscalars operators are indeed characterized by the polynomial growth condition (P).

Using the resolvent condition (4), it can be proved similarly that if $T \in B(X)$ is an $\mathcal{E}(\mathbb{T})$-subscalar operator then there exist constants $C > 0$, $p \geq 0$ and $q \geq 0$ such that

$$(R) \quad \|(T-z)^{-1}\| \leq \frac{C}{(|z|-1)^p} \; (|z| > 1) \text{ and } m(T-z) \geq C(1-|z|)^q \; (|z| < 1).$$

Note that if T is $\mathcal{E}(\mathbb{T})$-subscalar then $\sigma_{ap}(T)$, the approximate point spectrum of T given by

$$\sigma_{ap}(T) = \big\{\lambda \in \mathbb{C} : \inf\{\|(T-\lambda)x\| : \|x\| = 1\} = 0\big\},$$

is included in the unit circle. Moreover, either $\sigma(T)$ is included in the unit circle (and so T is $\mathcal{E}(\mathbb{T})$-scalar) or $\sigma(T) = \overline{\mathbb{D}}$, the closed unit disc.

Again it is natural to ask if the condition (R) implies the $\mathcal{E}(\mathbb{T})$-subscalarity of T. This is a variant of the open Problem 6.1.14 in [LN]. A characterization of $\mathcal{E}(\mathbb{T})$-subscalar operators in terms of the growth of the local resolvent of the adjoint has been given by Didas [Di].

The aim of this note is to show that the answer to the above problem is negative : there is a Banach space operator T satisfying condition (R) (with suitable p and q) which is not $\mathcal{E}(\mathbb{T})$-subscalar. We also give a sufficient condition (in terms of growth of resolvent and of an analytic left inverse of $T-z$) implying that T is an $\mathcal{E}(\mathbb{T})$-subscalar operator. This condition is also necessary for Hilbert space operators.

2. A Counterexample

Recall that an equivalent definition of decomposable operators is the following : $T \in B(X)$ is *decomposable* if for every open cover $\mathbb{C} = U \cup V$, there are closed invariant (for T) subspaces Y and Z of X such that $X = Y + Z$ and $\sigma(T \mid Y) \subset U$, $\sigma(T \mid Z) \subset V$. We refer for instance to [CF] and [LN]. An operator $T \in B(X)$ has *Bishop's property* (β) if, for every open set $U \subset \mathbb{C}$, the operator T_U defined by $T_U(f)(z) = (T-z)f(z)$ on the set $\mathcal{O}(U, X)$ of holomorphic functions from U into X is injective and has closed range. According to a result by E. Albrecht and J. Eschmeier [AE], $T \in B(X)$ is *subdecomposable* (i.e., T is similar to the restriction of a decomposable operator) if and only if T has Bishop's property (β).

Example 2.1. There exist a Banach space X and an operator $T \in B(X)$ such that

(i) $\|T\| \leq 1$, $\sigma_{ap}(T) = \mathbb{T}$ and $\sigma(T) = \overline{\mathbb{D}}$;
(ii) $\|(T-z)^{-1}\| \leq (|z|-1)^{-1} \qquad (|z| > 1)$,

(iii) there is a constant $C > 0$ such that

$$m(T - z) \geq C(1 - |z|)^3 \qquad (z \in \mathbb{D});$$

(iv) T is not $\mathcal{E}(\mathbb{T})$-subscalar ;

(v) T has Bishop's property (β).

The Construction. Let $X = c_0$ be the Banach space of all complex sequences converging to zero endowed with the supremum norm. We denote its standard basis by $e_1, e_2, \dots$. For $n \geq 1$ let

$$w_n = e^{\ln^2(n+2) - \ln^2(n+3)}.$$

Let $T \in B(X)$ be the weighted shift defined by $Te_n = w_n e_{n+1} \quad (n \geq 1)$.

The Proof. The proof of the properties of Example 2.1 will be obtained in several steps.

We first remark that $0 < w_n < 1$ for all n.

Claim 1. (w_n) is an increasing sequence and $\lim_{n\to\infty} w_n = 1$.

Proof. For each $n \geq 1$ there exists $x = x(n)$ such that $n + 2 \leq x \leq n + 3$ and

$$\ln^2(n+2) - \ln^2(n+3) = -2\frac{\ln x}{x}.$$

The function $g(x) = -2\frac{\ln x}{x}$ is increasing since $g'(x) = -2 \cdot \frac{1 - \ln x}{x^2} > 0 \quad (x > e)$. Therefore $\left(\ln^2(n+2) - \ln^2(n+3)\right)$ is an increasing sequence for $n \geq 1$ and

$$\lim_{n\to\infty} (\ln^2(n+2) - \ln^2(n+3)) = 0.$$

Hence (w_n) is an increasing sequence and $\lim_{n\to\infty} w_n = 1$. □

The previous Claim implies that $\|T\| \leq 1$. Therefore, for $|z| > 1$, we have

$$\left\|(T - z)^{-1}\right\| = \left\| -\frac{1}{z} \sum_{n \geq 0} \frac{1}{z^n} T^n \right\| \leq \frac{1}{|z| - 1}.$$

This proves (ii).

For $n \geq 1$ we have $T^n e_k = w_k w_{k+1} \cdots w_{k+n-1} e_{k+n} \quad (k \geq 1)$, and so

$$\begin{aligned} m(T^n) &= \inf_k w_k \cdots w_{k+n-1} \\ &= w_1 \cdots w_n \\ &= e^{\ln^2 3 - \ln^2 4} e^{\ln^2 4 - \ln^2 5} \cdots e^{\ln^2(n+2) - \ln^2(n+3)} \\ &= e^{\ln^2 3 - \ln^2(n+3)} = \frac{3^{\ln 3}}{(n+3)^{\ln(n+3)}}. \end{aligned}$$

Therefore T does not satisfy condition (P), and so [BM2] T is not $\mathcal{E}(\mathbb{T})$-subscalar. This proves (iv).

We also have $\lim_{n\to\infty} m(T^n)^{1/n} = 1$. Therefore [MZ] $\sigma_{ap}(T) \subset \{z : |z| = 1\}$. Since the spectrum of a weighted shift is circularly symmetric, we have in fact $\sigma_{ap}(T) = \{z : |z| = 1\}$. But $\partial\sigma(T) \subset \sigma_{ap}(T) \subset \sigma(T)$ and thus $\sigma(T)$ is either equal to $\overline{\mathbb{D}}$ or contained in $\mathbb{T}$. Since T is not invertible we have $\sigma(T) = \overline{\mathbb{D}}$. Another proof of the equality $\sigma(T) = \overline{\mathbb{D}}$ can be given using [Sh, Th. 4] and the fact that the spectral radius of T is one. This completes the proof of (i).

Note also that

$$\sum_n \frac{|\ln m(T^n)|}{n^2} < \infty,$$

so T satisfies the Beurling-type condition (B) (cf. [BM2]). Consequently, T has Bishop's property (β) (see [BM2]).

We prove now (iii).

Claim 2. $\lim_{n\to\infty} \frac{(1-w_n)^3}{w_{n+1}-w_n} = 0.$

Proof. Let $n \in \mathbb{N}$. Then there is an $x = x(n)$, $n+2 \le x \le n+3$ such that

$$\begin{aligned} w_{n+1} - w_n &= e^{\ln^2(n+3)-\ln^2(n+4)} - e^{\ln^2(n+2)-\ln^2(n+3)} \\ &= e^{\ln^2 x - \ln^2(x+1)} \left(\frac{2\ln x}{x} - \frac{2\ln(x+1)}{x+1} \right) \end{aligned}$$

and there is a $y = y(n)$, $x \le y \le x+1$ (i.e., $n+2 \le y \le n+4$) such that

$$w_{n+1} - w_n = -2e^{\ln^2 x - \ln^2(x+1)} \cdot \frac{1-\ln y}{y^2}.$$

Similarly, there is an $x' = x'(n)$, $n+2 \le x' \le n+3$ such that

$$\ln^2(n+2) - \ln^2(n+3) = -\frac{2\ln x'}{x'}.$$

We have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{(1-w_n)^3}{w_{n+1}-w_n} \\
&= \lim_{n\to\infty} \frac{\left(\frac{1-e^{\ln^2(n+2)-\ln^2(n+3)}}{\ln^2(n+2)-\ln^2(n+3)}\right)^3 \cdot \left(\ln^2(n+2)-\ln^2(n+3)\right)^3}{-2e^{\ln^2 x-\ln^2(x+1)}\frac{1-\ln y}{y^2}} \\
&= (-1)^3\left(-\frac{1}{2}\right)\lim_{n\to\infty} \frac{\left(\ln^2(n+2)-\ln^2(n+3)\right)^3}{\frac{1-\ln y}{y^2}} = \frac{1}{2}\lim_{n\to\infty}\frac{\left(\frac{-2\ln x'}{x'}\right)^3}{\frac{1-\ln y}{y^2}} \\
&= -4\lim_{n\to\infty}\frac{y^2}{x'^2}\cdot\lim_{n\to\infty}\frac{\ln^3 x'}{x'(1-\ln y)} = 0.
\end{aligned}$$

□

Claim 3. There is an $r>0$ such that $m(T-z)\geq(1-|z|)^3$ for all $z\in\mathbb{D}$, $|z|\geq r$.

Proof. Find n_0 such that

$$\frac{(1-w_n)^3}{w_{n+1}-w_n} < \frac{1}{16}$$

for all $n\geq n_0$. Find r, $1/2\leq r<1$, such that $r-(1-r)^3>w_{n_0}$.

Suppose on the contrary that there is a $\lambda\in\mathbb{D}$, $|\lambda|\geq r$ such that $m(T-\lambda)<(1-|\lambda|)^3$. Thus there exists $x=(x_i)\in X$ with $\|x\|=\max_i|x_i|=1$ and $\|(T-\lambda)x\|<(1-|\lambda|)^3$. Since

$$(T-\lambda)x=(-\lambda x_1, w_1x_1-\lambda x_2, w_2x_2-\lambda x_3,\cdots),$$

we have

$$|\lambda|\cdot|x_1|<(1-|\lambda|)^3$$

and

$$\sup_i|w_ix_i-\lambda x_{i+1}|<(1-|\lambda|)^3.$$

Without loss of generality we may assume that $\lambda>0$ and $x_i>0$ for all $i\geq1$. Indeed, replace λ by $|\lambda|$ and x_i by $|x_i|\quad(i\geq1)$. We have

$$\sup_i\big|w_i|x_i|-|\lambda|\cdot|x_{i+1}|\big|\leq\sup_i|w_ix_i-\lambda x_{i+1}|<(1-|\lambda|)^3.$$

Thus we may assume that there is a $\mu>r\geq1/2$ and $u=(u_i)\in X$ with $u_i\geq0\quad(i\in\mathbb{N})$, $\|u\|=\max_i u_i=1$ and

$$\mu\cdot u_1<(1-\mu)^3,\qquad \sup_i|w_iu_i-\mu u_{i+1}|<(1-\mu)^3. \tag{2.1}$$

We show that this is not possible. Write for short $a=(1-\mu)^3$.

Let $m \in \mathbb{N}$ satisfy $u_m = 1$ and $u_j < 1$ for all $j < m$.

We have $u_1 < \frac{(1-\mu)^3}{\mu} < 1$. Thus $m \geq 2$.

We show that $w_{m-1} \geq \mu - a$. Suppose on the contrary that $w_{m-1} < \mu - a$. By (2.1), we have

$$\begin{aligned} a > |w_{m-1}u_{m-1} - \mu u_m| &\geq \mu u_m - w_{m-1}u_{m-1} \\ &\geq \mu - (\mu - a)u_{m-1} \\ &= (\mu - a)(1 - u_{m-1}) + a \geq a, \end{aligned}$$

a contradiction. Hence

$$w_{m-1} \geq \mu - a. \tag{2.2}$$

We show now that $w_m \geq \mu + a$. Suppose on the contrary that $w_m < \mu + a$. Then $w_m - w_{m-1} \leq 2a$ and $1 - w_{m-1} \geq 1 - w_m \geq 1 - \mu - a$. Therefore we have

$$\frac{(1-w_m)^3}{w_m - w_{m-1}} \geq \frac{(1-\mu-a)^3}{2a} = \frac{\big(1-\mu-(1-\mu)^3\big)^3}{2(1-\mu)^3} \geq 1/16,$$

since $\mu \geq 1/2$ and $(1-\mu) - (1-\mu)^3 = (1-\mu)\mu(2-\mu) \geq \frac{1}{2}(1-\mu)$. Thus $m - 1 < n_0$, and so $\mu - a \geq r - (1-r)^3 > w_{n_0} \geq w_{m-1}$, a contradiction with (2.2). Hence

$$w_m \geq \mu + a. \tag{2.3}$$

Since $|w_m u_m - \mu u_{m+1}| < a$, we have $\mu u_{m+1} > w_m - a$, and so $u_{m+1} > \frac{w_m - a}{\mu} \geq 1$, a contradiction with the assumption that $\|u\| = 1$.

Hence $m(T - z) \geq (1 - |z|)^3$ for all $z \in \mathbb{D}$ with $|z| \geq r$. □

Since $m(T - z) > 0$ for all $z \in \mathbb{D}$ and the function

$$z \mapsto \frac{m(T-z)}{(1-|z|)^3}$$

is continuous, there is a constant $C > 0$ such that $m(T - z) \geq C \cdot (1 - |z|)^3$ for all $z \in \mathbb{D}$.

The proof of Example 2.1 is now complete.

Remarks 2.2. (a) Another proof of Bishop's property (β) for T can be given using [LN, 1.7.1].

(b) The fact that T has Beurling-type property (B) implies [BM2, Th. 4.5] that there exists a Banach space Y containing c_0 and an invertible operator $S \in B(Y)$ such that $T = S_{|X}$ and S satisfies

$$\sum_{n=-\infty}^{\infty} \frac{\log \|S^n\|}{1+n^2} < \infty.$$

Note that this condition implies [CF] that S is decomposable.

3. Sufficient conditions

We begin with the following sufficient condition.

Proposition 3.1. *Let $T \in B(X)$ be a Banach space operator satisfying*

$$\|(T-z)^{-1}\| \le C(|z|-1)^{-p} \qquad (|z|>1),$$

for some fixed constants $C > 0$ and $p \ge 0$. Suppose that there are $q \ge 0$ and an analytically dependant left inverse function $L : \mathbb{D} \to B(X)$ such that $L(z)(T-z) = I$ and

$$\|L(z)\| \le C(1-|z|)^{-q} \qquad (z \in \mathbb{D}).$$

Then T is $\mathcal{E}(\mathbb{T})$-subscalar.

We note that the growth condition on the analytically dependant left inverse function L implies that

$$\|x\| = \|L(z)(T-z)x\| \le C(1-|z|)^{-q}\|(T-z)x\|;$$

hence

$$m(T-z) \ge C^{-1}(1-|z|)^q.$$

Proof. A proof of this result can be given using Didas' criterion [Di] in terms of local resolvent of the adjoint of T. We give here a different proof.

It is a classical result (see [LN, Th. 1.5.12]) that the resolvent growth condition outside the closed unit disc implies a polynomial growth condition for the powers of $T:$ there is a constant $c > 0$ such that

$$\|T^n\| \le cn^p \qquad (n \in \mathbb{N}).$$

Write

$$L(z) = \sum_{i=0}^{\infty} L_i z^i \quad (z \in \mathbb{D}),$$

with $L_i \in B(X)$. Let $0 < r < 1$. By the Cauchy formula, for each $n \in \mathbb{N}$ we have

$$\|L_n\| \le \frac{\max\{\|L(z)\| : |z| \le r\}}{r^n} \le \frac{C}{r^n(1-r)^q}.$$

In particular, for $r = \frac{n}{n+q}$ (where the function $r \mapsto r^{-n}(1-r)^{-q}$ attains the minimum) we have

$$\|L_n\| \le C \cdot \left(\frac{n}{n+q}\right)^{-n} \left(1 - \frac{n}{n+q}\right)^{-q}.$$

We have

$$\begin{aligned} \lim_{n\to\infty} \left(\frac{n}{n+q}\right)^{-n} &= \lim_{n\to\infty} \left(1 - \frac{q}{n+q}\right)^{-n} \\ &= \lim_{n\to\infty} \left(1 - \frac{q}{n+q}\right)^{\frac{n+q}{q}\cdot\frac{-nq}{n+q}} = (e^{-1})^{-q} = e^q. \end{aligned}$$

Further, for $n \ge q$ we have

$$\left(1 - \frac{n}{n+q}\right)^{-q} = \left(\frac{q}{n+q}\right)^{-q} \le \left(\frac{q}{2n}\right)^{-q} = 2^q q^{-q} n^q.$$

Thus there is a constant $K > 0$ such that $\|L_n\| \le K \cdot n^q$ for all n.

We have

$$I = L(z)(T-z) = \sum_{i=0}^{\infty} L_i z^i (T-z) = L_0 T + \sum_{i=1}^{\infty} z^i (L_i T - L_{i-1})$$

for all $z \in \mathbb{D}$. Thus $L_0 T = I$ and $L_i T = L_{i-1}$ for all $i \ge 1$. Hence

$$L_n T^{n+1} = L_{n-1} T^n = \cdots = L_0 T = I.$$

Let $x \in X$, $\|x\| = 1$. Then

$$1 = \|x\| = \|L_{n-1} T^n x\| \le \|L_{n-1}\| \cdot \|T^n x\|.$$

Thus $\|T^n x\| \ge \|L_{n-1}\|^{-1}$, and so for some constant K' we have $m(T^n) \ge K' n^{-q}$ for all n. Hence T is $\mathcal{E}(\mathbb{T})$-subscalar by [BM2]. □

The next result gives an intrinsic characterization of $\mathcal{E}(\mathbb{T})$-subscalar operators on Hilbert spaces.

Theorem 3.2. *Let H be a Hilbert space and $T \in B(H)$. Then T is $\mathcal{E}(\mathbb{T})$-subscalar if and only if there are constants $C > 0$, $p \ge 0$, $q \ge 0$ and an analytic operator-valued function $L : \mathbb{D} \to B(H)$ such that*

- (i) $\|(T-z)^{-1}\| \le C(|z|-1)^{-p} \qquad (|z| > 1)$;
- (ii) $L(z)(T-z) = I \qquad (|z| < 1)$;
- (iii) $\|L(z)\| \le C(1-|z|)^{-q} \qquad (|z| < 1)$.

Proof. Suppose that T is a Hilbert space $\mathcal{E}(\mathbb{T})$-subscalar operator. According to [BM2, Th. 4.1], there are a Hilbert space K, constants $C' > 0$, $s \geq 0$ and an $\mathcal{E}(\mathbb{T})$-scalar extension $S \in B(K)$ such that $\sigma(S) = \sigma_{ap}(T) \subset \mathbb{T}$ and

$$\|S^m\| \leq C'|m|^s \qquad (m \in \mathbb{Z}, m \neq 0).$$

It is known that the power growth estimate $\|S^m\| \leq C'|m|^s$ implies that [LN, 1.5.12]

$$\|(S-z)^{-1}\| \leq C\big||z|-1\big|^{-s-1} \qquad (|z| \neq 1)$$

for a suitable constant $C > 0$. This implies

$$\|(T-z)^{-1}\| \leq C(|z|-1)^{-s-1} \qquad (|z| > 1).$$

We define $L : \mathbb{D} \mapsto B(H)$ by

$$L(z)x = P_H(S-z)^{-1}x \qquad (z \in \mathbb{D}, x \in H),$$

where $P_H \in B(K)$ is the orthogonal projection onto H.

Then L is analytic and we have

$$\|L(z)\| \leq \|(S-z)^{-1}\| \leq C(1-|z|)^{-s-1} \qquad (|z| < 1).$$

The equality $L(z)(T-z) = I$ on $\mathbb{D}$ follows from the equalities $(S-z)^{-1}(S-z) = I$ and $S_{|H} = T$.

The second implication follows from Proposition 3.1. ☐

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References

[AE] Albrecht, E. ; Eschmeier, J. Analytic functional models and local spectral theory. *Proc. London Math. Soc.* (3) 75 (1997), no. 2, 323–348.

[BM1] Badea, C. ; Müller, V. Invertible extensions and growth conditions, *C.R. Acad. Sci. Paris, Ser. I* 339(2004), 21-26.

[BM2] Badea, C. ; Müller, V. Growth conditions and inverse producing extensions, Preprint.

[GT] Gindler, H. A.; Taylor, A. E. The minimum modulus of a linear operator and its use in spectral theory, *Studia Math.* 22(1962/1963), 15–41.

[CF] Colojoară, I.; Foiaş, C. *Theory of generalized spectral operators*, Mathematics and its Applications, Vol. 9. Gordon and Breach, Science Publishers, New York-London-Paris, 1968.

[Di] Didas, M. $\mathcal{E}(\mathbb{T}^n)$-subscalar n-tuples and the Cesaro operator on H^p, *Ann. Univ. Sarav. Ser. Math.* 10 (2000), no. 2, pp. i–iii and 284–335.

[EP] Eschmeier, J.; Putinar, M. *Spectral decompositions and analytic sheaves*, London Mathematical Society Monographs. New Series, 10. The Clarendon Press, Oxford University Press, New York, 1996.

[LN] Laursen, K.B.; Neumann, M.M. *An introduction to local spectral theory*, London Mathematical Society Monographs. New Series, 20. The Clarendon Press, Oxford University Press, New York, 2000.

[MZ] Makai, E., Zemánek, J. The surjectivity radius, packing numbers and boundedness below, *Integral Equations Operator Theory* 6 (1983), 372–384.

[MMN1] Miller,T.L. ; Miller,V. ; Neumann, M.M. Growth conditions and decomposable extensions. in : *Trends in Banach spaces and operator theory* (Memphis, TN, 2001), 197–205, Contemp. Math., 321, Amer. Math. Soc., Providence, RI, 2003.

[MMN2] Miller,T.L.; Miller,V.; Neumann, M.M. Spectral subspaces of subscalar and related operators, *Proc. Amer. Math. Soc.* 132 (2004), no. 5, 1483–1493.

[MMN3] Miller,T.L. ; Miller,V. ; Neumann, M.M. Local spectral properties of weighted shifts, *J. Operator Theory* 51 (2004), no. 1, 71–88.

[MMN4] Miller,T.L.; Miller,V.; Neumann, M.M. Localization in the spectral theory of operators on Banach spaces, in: *Function spaces* (Edwardsville, IL, 2002), 247–262, Contemp. Math., 328, Amer. Math. Soc., Providence, RI, 2003.

[Sh] Shields, A.L. Weighted shift operators and analytic function theory, in *Topics in operator theory*, 49–128. Math. Surveys, 13, Amer. Math. Soc., Providence, R.I., 1974.

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