On the semi-Browder spectrum

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Abstract. An operator in a Banach space is called upper (lower) semi-Browder if it is upper (lower) semi-Fredholm and has a finite ascent (descent). We extend this notion to n-tuples of commuting operators and show that this notion defines a joint spectrum. Further we study relations between semi-Browder and (essentially) semi-regular operators.

Denote by $\mathcal{L}(X)$ the algebra of all bounded linear operators in a complex Banach space X and by I the identity operator in X. For T in $\mathcal{L}(X)$ denote by $N(T) = \{x \in X : Tx = 0\}$ and $R(T) = \{Tx : x \in X\}$ its kernel and range, respectively. Denote further $R^\infty(T) = \bigcap_{k=0}^\infty R(T^k)$ and $N^\infty(T) = \bigcup_{k=0}^\infty N(T^k)$.

The sets of all upper (lower) semi-Fredholm operators in X will be denoted by $\Phi_+(X)$ and $\Phi_-(X)$. Recall that $T \in \Phi_+(X)$ if and only if $\dim N(T) < \infty$ and $R(T)$ is closed; $T \in \Phi_-(X)$ if and only if $\operatorname{codim} R(T) < \infty$ (then $R(T)$ is closed automatically). The ascent and descent of T are defined by $a(T) = \min\{n : N(T^n) = N(T^{n+1})\}$ and $d(T) = \min\{n : R(T^n) = R(T^{n+1})\}$.

We say that an operator $T \in \mathcal{L}(X)$ is upper (lower) semi-Browder if it is upper (lower) semi-Fredholm and has a finite ascent (descent). The set of all upper (lower) semi-Browder operators in X will be denoted by $\mathcal{B}_+(X)$ and $\mathcal{B}_-(X)$. Semi-Browder operators were studied by many authors, see e.g. [4], [12], [14], [18], [20], [21], [22], [24]. The name was introduced in [6].

We extend the notion of semi-Browder operators to n-tuples of commuting operators. We discuss the lower semi-Browder case; the upper case is dual.

Let $T = (T_1, ..., T_n)$ be an n-tuple of mutually commuting operators in a Banach space X. We use the standard multiindex notation. Denote by $\mathbb{Z}_+$ the set of all non-negative integers. If $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}_+^n$ then denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $T^\alpha = T_1^{\alpha_1} \cdots T_n^{\alpha_n}$.

For $k = 0, 1, 2, ...$, denote $M_k(T) = R(T_1^k) + \cdots + R(T_n^k)$ and let $M'_k(T)$ be the smallest subspace of X containing the set $\bigcup\{R(T^\alpha) : \alpha \in Z_+^n \text{ and } |\alpha| = k\}$. Clearly $X = M_0(T) \supset M_1(T) \supset M_2(T) \supset \cdots$ and $X = M'_0(T) \supset M'_1(T) \supset M'_2(T) \supset \cdots$. Further

$$M'_{n(k-1)+1}(T) \subset M_k(T) \subset M'_k(T). \tag{1}$$

Indeed, if $\alpha = (\alpha_1, ..., \alpha_n) \in Z_+^n$ and $|\alpha| = n(k-1)+1$ then there exists i, $1 \le i \le n$ such that $\alpha_i \ge k$, so that $R(T^\alpha) \subset R(T_i^k) \subset M_k(T)$. This proves the first inclusion of (1) and the second inclusion is clear.

Denote $R^\infty(T) = \bigcap_{k=0}^\infty M_k(T) = \bigcap_{k=0}^\infty M'_k(T)$.

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If $M'_k(T) = M'_{k+1}(T)$ for some k then it is easy to see by induction that $M'_m(T) = M'_k(T)$ for every $m \geq k$, so that $R^\infty(T) = M'_k(T)$.

As usual we say that an n-tuple $T = (T_1, ..., T_n)$ of mutually commuting operators in X is lower semi-Fredholm ($T \in \Phi_-^{(n)}(X)$) if

$$\operatorname{codim} M_1(T) = \operatorname{codim}(R(T_1) + \cdots + R(T_n)) < \infty.$$

Clearly $T = (T_1, \ldots, T_n)$ is lower semi-Fredholm if and only if the operator $\hat{T} : X^n \to X$ defined by $\hat{T}(x_1, \ldots, x_n) = T_1x_1 + \cdots + T_nx_n$ is lower semi-Fredholm.

We say that $T = (T_1, \ldots, T_n)$ is semi-Browder if $\operatorname{codim} R^\infty(T) < \infty$. The set of all lower semi-Browder n-tuples will be denoted by $\mathcal{B}_-^{(n)}(X)$. Clearly $\Phi_-^{(n)}(X) \subset \mathcal{B}_-^{(n)}(X)$.

Define

$$\sigma_{\Phi_-}(T) = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : (T_1 - \lambda_1, \ldots, T_n - \lambda_n) \notin \Phi_-^{(n)}(X)\},$$

and

$$\sigma_{\mathcal{B}_-}(T) = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : (T_1 - \lambda_1, \ldots, T_n - \lambda_n) \notin \mathcal{B}_-^{(n)}(X)\}.$$

It is well known that σ_{Φ_-} satisfies the spectral mapping property [1]. In particular, $(T_1, ..., T_n) \in \Phi_-^{(n)}(X)$ if and only if $(T_1^k, ..., T_n^k) \in \Phi_-^{(n)}(X)$. Thus $\operatorname{codim} M_1(T) < \infty$ implies $\operatorname{codim} M_k(T) < \infty$ for every k.

Theorem 1. *Let $T = (T_1, ..., T_n)$ be an n-tuple of mutually commuting operators in a Banach space X. The following statements are equivalent:*
(a) $T \in \mathcal{B}_-^{(n)}(X)$.
(b) $T \in \Phi_-^{(n)}(X)$ *and there exists k such that* $M'_k(T) = M'_{k+1}(T)$.
(c) $T \in \Phi_-^{(n)}(X)$ *and there exists k such that* $M_k(T) = M_{k+1}(T)$.
(d) *There exists a subspace $Y \subset X$ invariant with respect to every T_i ($i = 1, \ldots, n$) such that $\operatorname{codim} Y < \infty$ and $T_1Y + \cdots + T_nY = Y$. It is possible to take $Y = R^\infty(T)$.*

Proof. (c) $\Rightarrow$ (b): Let $M_k(T) = M_{k+1}(T)$ for some k. Using the same argument as in the proof of (1) it is possible to show that $M'_{n(k-1)+1}(T) = M'_{n(k-1)+2}(T)$.

(b) $\Rightarrow$ (a): Let $M'_k(T) = M'_{k+1}(T)$ for some k. Then $M_k(T) \subset M'_k(T) = R^\infty(T)$. Further $T \in \Phi_-^{(n)}(X)$ implies $\operatorname{codim} M_k(T) < \infty$, so that $T \in \mathcal{B}_-^{(n)}(X)$.

(a) $\Rightarrow$ (d): Set $Y = R^\infty(T)$. Clearly Y is invariant with respect to T_i ($i = 1, \ldots, n$), $\operatorname{codim} Y < \infty$ and $Y = M_k(T) = M_{k+1}(T)$ for some k. If $y \in Y$ then for some $x_1, \ldots, x_n \in X$ we have

$$y = \sum_{i=1}^{n} T_i^{k+1} x_i = \sum_{i=1}^{n} T_i(T_i^k x_i) \in T_1Y + \cdots + T_nY.$$

(d) $\Rightarrow$ (c): Since $M_1(T) \supset M_1(T|_Y) = Y$ we have $\operatorname{codim} M_1(T) < \infty$ so that $T \in \Phi_-^{(n)}(X)$. Further $M'_1(T|_Y) = M'_0(T|_Y) = Y$ implies $R^\infty(T|_Y) = Y$ and $M_k(T) \supset M_k(T|_Y) \supset Y$ for every k. Thus the sequence $M_k(T)$ is constant for k big enough.

Corollary 2. *Let $T = (T_1, ..., T_n) \in \mathcal{B}_-^{(n)}(X)$. Then there exists $\epsilon > 0$ such that $(T_1 - \lambda_1, ..., T_n - \lambda_n) \in \mathcal{B}_-^{(n)}(X)$ for all complex numbers $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ with $\sum_{i=1}^n |\lambda_i| < \epsilon$. Moreover $\operatorname{codim} R^\infty(T_1 - \lambda_1, ..., T_n - \lambda_n) \leq \operatorname{codim} R^\infty(T_1, ..., T_n)$.*

Proof. Denote $Y = R^\infty(T)$. Then $\operatorname{codim} Y < \infty$ and $T_1Y + \cdots + T_nY = Y$. There exists $\epsilon > 0$ such that $(T_1-\lambda_1)Y+\cdots+(T_n-\lambda_n)Y = Y$ if $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, $\sum_{i=1}^n |\lambda_i| < \epsilon$, so that $R^\infty(T_1 - \lambda_1, ..., T_n - \lambda_n) \supset Y = R^\infty(T_1, ..., T_n)$.

Proposition 3. Suppose $T_1, ..., T_n, S_1, ..., S_n$ are mutually commuting operators in X such that $\sum_{i=1}^n T_iS_i = I$. Then $(T_1, ..., T_n) \in \mathcal{B}_-^{(n)}(X)$.

Proof. Clearly $M_1(T_1, ..., T_n) = X = M_0(T_1, ..., T_n)$ so that $(T_1, ..., T_n) \in \mathcal{B}_-^{(n)}(X)$.

Corollary 4. $\sigma_{\mathcal{B}_-}(T)$ is a compact subset of $\mathbb{C}^n$.

Proof. $\sigma_{\mathcal{B}_-}(T)$ is closed by Corollary 2. Further $\sigma_{\mathcal{B}_-}(T) \subset \sigma^{<T>}(T)$ where $< T >$ denotes the smallest closed subalgebra of $\mathcal{L}(X)$ containing $T_1, ..., T_n$ and the identity operator and $\sigma^{<T>}(T)$ denotes the spectrum in the commutative Banach algebra $< T >$. Thus $\sigma_{\mathcal{B}_-}(T)$ is bounded and hence compact.

Lemma 5. Let $T_1, ..., T_n, T_{n+1}$ be mutually commuting operators in a Banach space X. Suppose $\operatorname{codim} R^\infty(T_1, ..., T_n) = \infty$ and let $k \in \mathbb{N}$. Then there exists a complex number λ such that

$$\operatorname{codim} \left[R(T_1^k) + \cdots + R(T_n^k) + R((T_{n+1} - \lambda)^k)\right] \geq k. \tag{2}$$

Proof. Using condition (c) of Theorem 1 we can distinguish two cases:

(a) $(T_1, ..., T_n) \notin \Phi_-^{(n)}(X)$ so that $(0, ..., 0) \in \sigma_{\Phi_-}(T_1, ..., T_n)$. By the projection property for σ_{Φ_-} there exists $\lambda \in \mathbb{C}$ such that $(0, ..., 0, \lambda) \in \sigma_{\Phi_-}(T_1, ..., T_n, T_{n+1})$, i.e., $\operatorname{codim}[R(T_1^k) + \cdots + R(T_n^k) + R((T_{n+1} - \lambda)^k)] = \infty$. Hence we have (2).

(b) $\operatorname{codim} M_m(T) < \infty$ and $M_m(T) \neq M_{m+1}(T)$ for every $m \geq 1$ where $T = (T_1, ..., T_n)$.

Fix $k \in \mathbb{N}$. Then there exists $i, 1 \leq i \leq n$ such that $R(T_i^{k-1}) \not\subset M_k(T)$ (otherwise $M_{k-1}(T) = M_k(T)$). Denote $Y = X/M_k(T)$, so that $\dim Y < \infty$ and let $S : Y \mapsto Y$ be defined by $S(x + M_k(T)) = T_ix + M_k(T)$. Clearly $S^k = 0$ and $S^{k-1} \neq 0$.

Consider the operator $U : Y \mapsto Y$ defined by $U(x + M_k(T)) = T_{n+1}x + M_k(T)$. Clearly $US = SU$. Let Z be a subspace of Y satisfying $Z \oplus N(S^{k-1}) = Y$. In this decomposition U can be written as

$$U = \begin{pmatrix} U_{11} & 0 \\ U_{12} & U_{22} \end{pmatrix}.$$

Choose a complex number λ such that $U_{11} - \lambda$ is singular, i.e., there exists a non-zero $z \in Z$ with $(U - \lambda)z \in N(S^{k-1})$. Since $z \in N(S^k) \setminus N(S^{k-1})$ we have

$$S^{k-m}z \in N(S^m) \setminus N(S^{m-1}) \qquad (m = 1, \ldots, k).$$

Further

$$(U - \lambda)S^{k-m}z = S^{k-m}(U - \lambda)z \in S^{k-m}N(S^{k-1}) \subset N(S^{m-1}).$$

For $m = 1, \dots, k$ we have

$$\dim\bigl[N(S^m)/(U-\lambda)^m N(S^m)\bigr] = \dim N\bigl((U-\lambda)^m|_{N(S^m)}\bigr) \ge \dim N\bigl((U-\lambda)^m|_M\bigr),$$

where $M = N(S^{m-1}) \vee \{S^{k-m}z\}$ and $(U-\lambda)^m M \subset (U-\lambda)^{m-1}N(S^{m-1})$. Further

$$\begin{aligned}&\dim N\bigl((U-\lambda)^m|_M\bigr) = \dim\bigl[M/(U-\lambda)^m M\bigr]\\ \ge &\dim\bigl[M/(U-\lambda)^{m-1}N(S^{m-1})\bigr] = \dim\bigl[N(S^{m-1})/(U-\lambda)^{m-1}N(S^{m-1})\bigr] + 1,\end{aligned}$$

since $S^{k-m}z \notin N(S^{m-1})$. Thus, by induction,

$$\dim\bigl[N(S^m)/(U-\lambda)^m N(S^m)\bigr] \ge m \qquad (m = 1, \dots, k).$$

In particular $\dim(Y/(U-\lambda)^k Y) \ge k$. Consequently

$$\operatorname{codim}\bigl[R(T_1^k) + \cdots + R(T_n^k) + R((T_{n+1}-\lambda)^k)\bigr] \ge k.$$

Corollary 6. Let $T_1, ..., T_n, T_{n+1}$ be mutually commuting operators in a Banach space X. Suppose that $\operatorname{codim} R^\infty(T_1, ..., T_n) = \infty$. Then there exists $\lambda \in \mathbb{C}$ such that

$$\operatorname{codim} R^\infty(T_1, ..., T_n, T_{n+1} - \lambda) = \infty.$$

Proof. For each $k \ge 1$ we can find $\lambda_k \in \mathbb{C}$ such that

$$\begin{aligned}&\operatorname{codim} R^\infty(T_1, \dots, T_n, T_{n+1} - \lambda_k)\\ \ge&\operatorname{codim}[R(T_1^k) + \cdots + R(T_n^k) + R((T_{n+1}-\lambda_k)^k)] \ge k.\end{aligned}$$

Clearly $\lambda_k \in \sigma(T_{n+1})$ for every k. Thus we may assume (by passing to a subsequence, if necessary) that the sequence $\{\lambda_k\}$ is convergent, $\lambda_k \to \lambda \in \sigma(T_{n+1})$. We have

$$\lim_{k\to\infty} \operatorname{codim} R^\infty(T_1, \dots, T_n, T_{n+1} - \lambda_k) = \infty.$$

By Corollary 2 this implies that $\operatorname{codim} R^\infty(T_1, \dots, T_n, T_{n+1} - \lambda) = \infty$.

Corollary 7. If $T_1, ..., T_n, T_{n+1}$ be mutually commuting operators, then

$$\sigma_{\mathcal{B}_-}(T_1, ..., T_n) = P\sigma_{\mathcal{B}_-}(T_1, ..., T_{n+1}),$$

where $P : \mathbb{C}^{n+1} \mapsto \mathbb{C}^n$ is the projection onto the first n coordinates.

Proof. The inclusion $\subset$ was proved in Corollary 6. If $(T_1, ..., T_n) \in \mathcal{B}_-{}^{(n)}(X)$ then clearly

$$R^\infty(T_1, \dots, T_n, T_{n+1} - \lambda) \supset R^\infty(T_1, \dots, T_n),$$

so that $(T_1, \dots, T_n, T_{n+1} - \lambda) \in \mathcal{B}_-^{(n+1)}(X)$ for every $\lambda \in \mathbb{C}$. This proves the second inclusion.

Corollary 8. $\sigma_{\mathcal{B}_-}$ is a subspectrum in the sense of Żelazko [25]. Consequently, by [17], $\sigma_{\mathcal{B}_-}$ satisfies the spectral mapping property:

$$f\sigma_{\mathcal{B}_-}(T) = \sigma_{\mathcal{B}_-}f(T)$$

for every n-tuple $T = (T_1, ..., T_n)$ of mutually commuting operators and every m- tuple $f = (f_1, \dots, f_m)$ of functions analytic in a neighbourhood of the Taylor spectrum of $(T_1, ..., T_n)$.

The following lemma is a well–known stability result for semi-Fredholm operators.

Lemma 9. Let $T = (T_1, ..., T_n) \in \Phi_-^{(n)}(X)$. Then there exists $\epsilon > 0$ such that

$$\operatorname{codim} M_1(S) \le \operatorname{codim} M_1(T)$$

for every commuting n-tuple $S = (S_1, ..., S_n) \in \mathcal{L}(X)^n$ with $\sum_{i=1}^n \|S_i - T_i\| < \epsilon$.

The previous lemma enables to generalize the result of [12] for n-tuples of operators.

Theorem 10. Let $T = (T_1, ..., T_n) \in \mathcal{B}_-^{(n)}(X)$. Then there exists $\epsilon > 0$ such that $S \in \mathcal{B}_-^{(n)}(X)$ for every commuting n-tuple $S = (S_1, ..., S_n) \in \mathcal{L}(X)^n$ with $\sum_{i=1}^n \|S_i - T_i\| < \epsilon$.

Proof. Choose k such that $M_k(T) = R^\infty(T)$ and $\operatorname{codim} R^\infty(T) \le k$. Then $(T_1^{k+1}, \dots, T_n^{k+1}) \in \Phi_-^{(n)}(X)$. By the previous lemma there exists $\epsilon > 0$ with the following property: if $S = (S_1, ..., S_n)$ is a commuting n-tuple of operators in X with $\sum_{i=1}^n \|S_i - T_i\| < \epsilon$ then $(S_1^{k+1}, \dots, S_n^{k+1}) \in \Phi_-^{(n)}(X)$ and

$$\begin{aligned} &\operatorname{codim} M_1(S_1^{k+1}, ..., S_n^{k+1}) \le \operatorname{codim} M_1(T_1^{k+1}, \dots, T_n^{k+1}) \\ =&\operatorname{codim} M_{k+1}(T) = \operatorname{codim} R^\infty(T) \le k. \end{aligned}$$

Since $M_1(S) \supset M_2(S) \supset \cdots \supset M_{k+1}(S)$ and $\operatorname{codim} M_{k+1}(S) \le k$, there exists $j \le k$ such that $M_j(S) = M_{j+1}(S)$. Consequently $S \in \mathcal{B}_-^{(n)}(X)$.

From the general theory of joint spectrum it is easy to deduce the following consequences:

(a) The mapping $(T_1, \dots, T_n) \mapsto \sigma_{\mathcal{B}_-}(T_1, \dots, T_n)$ is upper semi-continuous. In particular, if $T_1 \in \mathcal{L}(X)$ and U is a neighbourhood of $\sigma_{\mathcal{B}_-}(T_1)$, then $\sigma_{\mathcal{B}_-}(S_1) \subset U$ for every operator S_1 close enough to T_1.

(b) $\sigma_{\mathcal{B}_-}$ is continuous on commuting elements, see [11], Theorem 1.9. More precisely, if $\{T_k\}_{k=1}^\infty \subset \mathcal{L}(X)$, $T \in \mathcal{L}(X)$, $\lim T_k = T$ and $T_k T = T T_k$, $k = 1, 2, \dots$, then $\lambda \in \sigma_{\mathcal{B}_-}(T)$ if and only if there exist $\lambda_k \in \sigma_{\mathcal{B}_-}(T_k)$ such that $\lambda_k \to \lambda$.

(c) Let $T, S \in \mathcal{L}(X)$, $TS = ST$. Then (cf. [11], Proposition 1.8)

$$\delta\big(\sigma_{\mathcal{B}_-}(T), \sigma_{\mathcal{B}_-}(S)\big) \le r_e(T - S),$$

where δ denotes the Hausdorff distance and r_e the essential spectral radius,

$$r_e(T) = \max\{|\lambda|, T - \lambda \quad \text{is not Fredholm}\} = \max\{|\lambda|, T - \lambda \notin \mathcal{B}_-(X)\},$$

see [7].

(d) Let $T, S \in \mathcal{L}(X)$, $TS = ST$. Then

$$TS \in \mathcal{B}_-(X) \Longleftrightarrow T, S \in \mathcal{B}_-(X),$$

see [6] and [16], Theorem 2.1.

(e) Let T and Q be commuting operators acting in X, let $T \in \mathcal{B}_-(X)$ and let Q be a quasinilpotent. Then $T + Q \in \mathcal{B}_-(X)$, see e.g. [11], Remark after Theorem 1.9, [18], Theorem 4.1 or [21], Corollary 2.

Analogously we can define the upper semi-Browder n-tuples. Let $T = (T_1, ..., T_n)$ be an n-tuple of mutually commuting operators in a Banach space X. We say that T is upper semi-Fredholm ($T \in \Phi_+^{(n)}(X)$) if the mapping $\tilde{T} : X \mapsto X^n$ defined by $\tilde{T}x = (T_1x, \dots, T_nx)$ is upper semi-Fredholm. We say that T is upper semi-Browder ($T \in \mathcal{B}_+^{(n)}(X)$) if $T \in \Phi_+^{(n)}(X)$ and $\dim N^\infty(T) < \infty$, where

$$N^\infty(T) = \bigcup_{k=1}^{\infty} \big[N(T_1^k) \cap \dots \cap N(T_n^k)\big].$$

Denote $T^* = (T_1^*, \dots, T_n^*) \in \mathcal{L}(X^*)^n$.

Theorem 11. Let $T = (T_1, ..., T_n)$ be an n-tuple of mutually commuting operators in a Banach space X. Then

$$T \in \mathcal{B}_-^{(n)}(X) \Longleftrightarrow T^* \in \mathcal{B}_+^{(n)}(X^*)$$

and

$$T \in \mathcal{B}_+^{(n)}(X) \Longleftrightarrow T^* \in \mathcal{B}_-^{(n)}(X^*).$$

Proof. The corresponding equivalences are well-known for semi-Fredholm n-tuples. Further it is easy to check that

$$N(T_1^k) \cap \dots \cap N(T_n^k) = {}^\perp\big[R(T_1^{*k}) + \dots + R(T_n^{*k})\big].$$

and

$$\big[R(T_1^k) + \dots + R(T_n^k)\big]^\perp = N(T_1^{*k}) \cap \dots \cap N(T_n^{*k}).$$

The statement of Theorem 11 is now an easy consequence of these identities.

For a commuting n-tuple $T = (T_1, ..., T_n) \in \mathcal{L}(X)^n$ we define the upper semi-Browder spectrum of T by

$$\sigma_{\mathcal{B}_+}(T) = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n, (T_1 - \lambda_1, \dots, T_n - \lambda_n) \notin \mathcal{B}_+^{(n)}(X)\}.$$

By the previous theorem it is easy to see that $\sigma_{\mathcal{B}_+}$ satisfies the same properties as $\sigma_{\mathcal{B}_-}$.

Define further the Browder spectrum $\sigma_{\mathcal{B}}$ of a commuting n-tuple $T = (T_1, ..., T_n)$ by

$$\sigma_{\mathcal{B}}(T) = \sigma_{\mathcal{B}_-}(T) \cup \sigma_{\mathcal{B}_+}(T).$$

For a single operator T_1 this definition coincides with the usual definition of the Browder spectrum of T_1 as the union of $\sigma_e(T_1)$ and the limit points of $\sigma(T_1)$, where $\sigma_e(T_1)$ denotes the essential spectrum of T_1, i.e.,

$$\sigma_e(T_1) = \{\lambda \in \mathbb{C}, T - \lambda \quad \text{is not Fredholm}\}$$

and $\sigma(T_1)$ denotes the ordinary spectrum of T_1. Again it is easy to see that $\sigma_{\mathcal{B}}$ satisfies all properties proved for $\sigma_{\mathcal{B}_-}$.

Remark. The possibility of extending the Browder spectrum to commuting n- tuples was proved in [3]. Our extension

$$\sigma_{\mathcal{B}}(T_1, ..., T_n) = \sigma_{\mathcal{B}_-}(T_1, ..., T_n) \cup \sigma_{\mathcal{B}_+}(T_1, ..., T_n)$$

exhibits similar properties as the spectrum

$$\sigma_b(T_1, ..., T_n) = \sigma_{Te}(T_1, ..., T_n) \cup \big(\sigma_T(T_1, ..., T_n)\big)'$$

defined there. (Here σ_t and σ_{te} denote the Taylor and and the essential Taylor spectrum and M' denotes the set of all limit points of a set M.) However these extensions differ for $n \geq 2$, an example will be given later.

The semi-Fredholm and semi-Browder operators are closely related with semi-regular and essentially semi-regular operators which (under various names) were intensively studied, see e. g. [5], [9], [10], [11], [13], [15], [16], [19] and [23]. An operator $T \in \mathcal{L}(X)$ is called semi-regular if it has closed range and $N(T) \subset R^\infty(T)$. T is essentially semi-regular if $R(T)$ is closed and $\dim\big[N(T)/(N(T) \cap R^\infty(T))\big] < \infty$.

From a number of equivalent properties of essentially semi-regular operators we point out the following Kato decomposition (see [16, Theorem 3.1], [19, Theorem 2.1]).

Proposition 12. An operator $T \in \mathcal{L}(X)$ is essentially semi-regular if and only if $R(T)$ is closed and there exist closed subspaces $X_1, X_2 \subset X$ invariant with respect to T such that $X = X_1 \oplus X_2$, $\dim X_1 < \infty$, $T|_{X_1}$ is nilpotent and $T|_{X_2}$ is semi-regular.

If $T \in \mathcal{L}(X)$ is a lower semi-Browder operator then the space X_2 in the Kato decomposition is determined uniquely and $X_2 = R^\infty(T)$. Thus $T|_{X_2}$ is onto. The analogous statement for n-tuples of commuting operator is not true.

Example. Denote by H the Hilbert space with an orthonormal basis $\{e_{i,j} : i,j \in Z, i \geq 0 \quad \text{or} \quad j \geq 0\} \cup \{e_{-1,-1}\}$. Define operators $T_1, T_2 \in \mathcal{L}(X)$ by

$$T_1 e_{i,j} = e_{i+1,j},$$
$$T_2 e_{i,j} = e_{i,j+1}.$$

We list some properties of the pair (T_1, T_2) :

(a) T_1 and T_2 are commuting isometries so that $(T_1, T_2) \in \mathcal{B}_+^{(n)}(X)$.

(b) Denote

$$Y = \vee\{e_{i,j} : i,j \in Z, i \geq 0 \quad \text{or} \quad j \geq 0\} = \{e_{-1,-1}\}^\perp.$$

Then $T_iY \subset Y \quad (i = 1, 2)$, $T_1Y + T_2Y = Y$ and $\operatorname{codim} Y = 1$. Thus $(T_1, T_2) \in \mathcal{B}_-^{(n)}(X)$.

(c) Denote by σ_t the Taylor spectrum. Then $(0, 0) \in \sigma_t(T_1, T_2)$. Indeed, $e_{-1,-1} \notin T_1H + T_2H$ so that $T_1H + T_2H \neq H$.

(d) $(0, 0)$ is a limit point of the Taylor spectrum of (T_1, T_2). Indeed, if $(0, 0)$ were an isolated point of $\sigma_t(T_1, T_2)$ then, using the Taylor functional calculus, it would be possible to decompose H as $H = H_1 \oplus H_2$ where $T_iH_j \subset H_j \quad (i, j = 1, 2)$, $\sigma_t(T_1|_{H_1}, T_2|_{H_1}) = \{0, 0\}$ and $\{0, 0\} \notin \sigma_t(T_1|_{H_2}, T_2|_{H_2})$. Since T_1 and T_2 are commuting isometries it would mean that the approximate point spectrum

$$\begin{aligned}&\sigma_\pi(T_1|_{H_1}, T_2|_{H_1}) \\ &\quad = \big\{(\lambda_1, \lambda_2) \in \mathbb{C}^2 : \inf\{\|(T_1 - \lambda_1)x\| + \|(T_2 - \lambda_2)x\|, x \in H_1, \|x\| = 1\} = 0\big\}\end{aligned}$$

is empty. Thus $H_1 = \{0\}$, a contradiction with the fact that

$$(0, 0) \in \sigma_t(T_1|_{H_1}, T_2|_{H_1}).$$

(e) We have

$$(0, 0) \in \sigma_t(T_1, T_2)' \subset \sigma_b(T_1, T_2)$$

and

$$(0, 0) \notin \sigma_{\mathcal{B}}(T_1, T_2) = \sigma_{\mathcal{B}_+}(T_1, T_2) \cup \sigma_{\mathcal{B}_-}(T_1, T_2).$$

Thus the joint spectra $\sigma_{\mathcal{B}}$ and σ_b are different.

(f) In the same way as in (d) it is possible to show that there is no (not necessarily orthogonal) decomposition $H = H_1 \oplus H_2$ such that $T_iH_j \subset H_j \quad (i, j = 1, 2)$, $T_1|_{H_1}$ and $T_2|_{H_1}$ are nilpotent and $T_1H_2 + T_2H_2 = H_2$. Thus there is no analogy to the Kato decomposition of a single semi-Browder operator.

Problem. Let $T = (T_1, \ldots, T_n)$ be a commuting n-tuple of operators in a Banach space X. Denote by σ_δ the defect spectrum of T, i. e.,

$$\sigma_\delta(T) = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : (T_1 - \lambda_1)X + \cdots + (T_n - \lambda_n)X \neq X\}.$$

Using Theorem 1 it is possible to obtain

$$\sigma_{\Phi_-}(T) \cup \sigma_\delta(T)' \subset \sigma_{\mathcal{B}}(T).$$

For $n = 1$ the opposite inclusion also takes place. It is an open problem whether $\sigma_{\Phi_-}(T) \cup \sigma_\delta(T)' = \sigma_{\mathcal{B}}(T)$ for $n \geq 2$.

Proposition 13. Let T be an essentially semi-regular operator on a Banach space X. Then $R^\infty(T)$ is closed, $TR^\infty(T) = R^\infty(T)$ and the operator $\tilde{T} : X/R^\infty(T) \mapsto X/R^\infty(T)$ induced by T is upper semi-Browder.

Proof. Set $M = R^\infty(T)$. Let $X = X_1 \oplus X_2$ be the Kato decomposition of T (see Proposition 12) and denote $T_i = T|_{X_i} \quad (i = 1, 2)$. Clearly $M = R^\infty(T_2) \subset X_2$. It is well-known that M is closed and $TM = M$, see e.g. [16], Lemma 1.4. Let $k \geq 1$ and

$x = x_1 \oplus x_2 \in X$ satisfy $T^k x \in M$. Then $T_2^k x_2 \in M$ so that $x_2 \in M$, see [16, Lemma 1.4]. Thus $x \in X_1 + M$ and $\dim N(\tilde{T}^k) \le \dim X_1$. Consequently $\dim N^\infty(\tilde{T}) \le \dim X_1 < \infty$.

Let $\pi : X \mapsto X/M$ be the canonical projection. As $M \subset R(T)$ and $R(\tilde{T}) = \{Tx + M, x \in X\} = \pi R(T)$, the range of $\tilde{T}$ is closed. Thus $\tilde{T}$ is upper semi-Browder.

Theorem 14. Let T be an operator on a Banach space X. Then the following conditions are equivalent:

(a) T is essentially semi-regular,

(b) there exists a closed subspace M of X such that $TM \subset M$, $T|_M$ is lower semi-Fredholm and the induced operator $\tilde{T} : X/M \mapsto X/M$ is upper semi-Fredholm,

(c) there exists a closed subspace M of X such that $TM \subset M$, $T|_M$ is lower semi-Browder and the induced operator $\tilde{T} : X/M \mapsto X/M$ is upper semi-Browder,

(d) there exists a closed subspace M of X such that $TM \subset M$, $T|_M$ is surjective and the induced operator $\tilde{T} : X/M \mapsto X/M$ is upper semi-Browder,

(e) there exists a closed subspace M of X such that $TM \subset M$, $T|_M$ is lower semi-Browder and the induced operator $\tilde{T} : X/M \mapsto X/M$ is bounded below.

Proof. By Proposition 13, $(a) \Rightarrow (d)$. The implications $(d) \Rightarrow (c) \Rightarrow (b)$ are straightforward.

$(b) \Rightarrow (a)$: First we show that $R(T)$ is closed. Let $\pi : X \mapsto X/M$ be the canonical projection. If $y \in R(T)$, $y = Tx$ for some $x \in X$, then $\pi y = Tx + M = \tilde{T}(x + M) \in R(\tilde{T})$, so that $R(T) \subset \pi^{-1} R(\tilde{T})$. Let $y \in X$ and $\pi y \in R(\tilde{T})$, i.e., $y + M = Tx + M$ for some $x \in X$. Then $y \in R(T) + M = R(T) + (F + TM) \subset R(T) + F$ for some finite dimensional subspace F of M. Thus $\pi^{-1}(R(\tilde{T})) \subset R(T) + F \subset \pi^{-1}(R(\tilde{T})) + F$. Further $\pi^{-1}(R(\tilde{T})) + F$ is closed since π is continuous, $R(\tilde{T})$ is closed and F finite dimensional. Hence $R(T) + F$ is closed, and so $R(T)$ is closed.

As $\pi N(T) \subset N(\tilde{T})$ and $\dim N(\tilde{T})$ is finite dimensional, there exists a finite dimensional subspace $G_1 \subset N(T)$ such that $N(T) \subset G_1 + N(T|_M)$. The operator $T|_M$ is lower semi-Fredholm and consequently essentially semi-regular, i.e., there exists a finite dimensional subspace G_2 of M such that $N(T|_M) \subset G_2 + R^\infty(T|_M)$. Thus

$$N(T) \subset G_1 + N(T|_M) \subset G_1 + G_2 + R^\infty(T|_M) \subset (G_1 + G_2) + R^\infty(T),$$

and T is essentially semi-regular.

$(a) \Rightarrow (e)$: Let $X = X_1 \oplus X_2$ be the Kato decomposition of T, i.e., $\dim X_1 < \infty$, $TX_1 \subset X_1$, $TX_2 \subset X_2$, $T|_{X_1}$ is nilpotent and $T_2 = T|_{X_2}$ is semi-regular. Set $M = X_1 \oplus R^\infty(T_2) = X_1 \oplus R^\infty(T)$. Clearly, M is closed and since $TR^\infty(T) = R^\infty(T)$, we have $T|_M$ is a lower semi-Browder operator.

Let $\tilde{T} : X/M \mapsto X/M$ be the operator induced by T. If $x = x_1 \oplus x_2$ satisfies $Tx \in M$ then $T_2 x_2 \in R^\infty(T_2)$, so that $x_2 \in R^\infty(T_2)$ and $x \in M$. Hence $N(\tilde{T}) = \{0\}$.

We show that $R(\tilde{T})$ is closed. Let $x, x_k \in X$ $(k = 1, 2, \ldots)$ and let $Tx_k + M \to x + M$ in the topology of X/M. Then $x \in \overline{R(T) + M} = R(T) + M$ since $M \subset R(T) + X_1$. Consequently $x + M \in R(\tilde{T})$. Hence $R(\tilde{T})$ is closed and $\tilde{T}$ is bounded below.

$(e) \Rightarrow (b)$: Clear.

It is well-known that if $T \in \mathcal{L}(X)$ is essentially semi-regular and K is compact operator commuting with T then $T + K$ is also essentially semi-regular [5], Theorem

5.9. Now we can prove a sharper result. Let us denote by

$$r_+(T) = \sup\{\epsilon \geq 0 : T - \lambda I \in \Phi_+(X) \quad \text{for} \quad |\lambda| < \epsilon\}$$

and

$$r_-(T) = \sup\{\epsilon \geq 0 : T - \lambda I \in \Phi_-(X) \quad \text{for} \quad |\lambda| < \epsilon\}$$

the semi-Fredholm radii of T. An operator $T \in \mathcal{L}(X)$ is upper (lower) semi-Fredholm if and only if $r_+(T) > 0$ $(r_-(T) > 0)$.

Lemma 15. Let A be an operator on a Banach space X and let M be a closed subspace of X such that $AM \subset M$. Then $r_e(A|_M) \leq r_e(A)$ and $r_e(\tilde{A}) \leq r_e(A)$ where $\tilde{A} : X/M \mapsto X/M$ is the operator induced by A.

Proof. Let $A \in \mathcal{L}(X)$ be a Fredholm operator and let $AM \subset M$. Then $R(A|_M)$ is closed (see [2], Lemma 4.3.1) and $\dim N(A|_M) \leq N(A) < \infty$. Thus $A|_M$ is upper semi-Fredholm. Further, $\operatorname{codim} R(\tilde{A}) \leq \operatorname{codim} R(A) < \infty$, and hence $\tilde{A}$ is lower semi-Fredholm.

The rest follows from the fact that upper and lower semi-Fredholm spectra contain the boundary of the essential spectrum [7].

Theorem 16. Let $T, S \in \mathcal{L}(X)$, $TS = ST$ and let T be essentially semi-regular. Let $\hat{T} = T|_{R^\infty(T)}$ and let $\tilde{T} : X/R^\infty(T) \mapsto X/R^\infty(T)$ be the operator induced by T. If $r_e(S) < \min\{r_-(\hat{T}), r_+(\tilde{T})\}$ then $T + S$ is essentially semi-regular.

Proof. By Theorem 14, $\hat{T} \in \Phi_-(X)$ and $\tilde{T} \in \Phi_+(X)$. As $TS = ST$, we have $SR^\infty(T) \subset R^\infty(T)$ and we can define the operators $\tilde{S} : X/R^\infty(T) \to X/R^\infty(T)$ and $\hat{S} = S|_{R^\infty(T)}$. Clearly, $\hat{T}\hat{S} = \hat{S}\hat{T}$ and $\tilde{T}\tilde{S} = \tilde{S}\tilde{T}$. By Lemma 15, $r_e(\hat{S}) \leq r_e(S) < r_-(\hat{T})$ and $r_e(\tilde{S}) \leq r_e(S) < r_+(\tilde{T})$. As in [11], Theorem 1.9 it is possible to deduce that $\hat{T} + \hat{S}$ is lower semi-Fredholm and $\tilde{T} + \tilde{S}$ is upper semi-Fredholm. By Theorem 14, $T + S$ is essentially semi-regular.

Corollary 17. Let T be an essentially semi-regular operator on a Banach space X, $S \in \mathcal{L}(X)$, $TS = ST$ and let S be a Riesz operator (i.e., $r_e(S) = 0$). Then $T + S$ is essentially semi-regular.

For $T \in \mathcal{L}(X)$ denote by

$$\sigma_\gamma(T) = \{\lambda \in \mathbb{C} : T - \lambda \quad \text{is not semi-regular}\}$$

and

$$\sigma_{\gamma e}(T) = \{\lambda \in \mathbb{C} : T - \lambda \quad \text{is not essentially semi-regular}\}.$$

The spectrum $\sigma_\gamma(T)$ and its essential version the set $\sigma_{\gamma e}(T)$ were studied (under various names) by many authors, see e.g., [9], [10], [11], [13], [15], [16], [19] and [23].

Corollary 18. Let $T \in \mathcal{L}(X)$. Then

$$\sigma_{\gamma e}(T) = \bigcap \sigma_\gamma(T + S)$$

where the intersection is taken over all Riesz operators in X commuting with T.

Proof. The inclusion $\supset$ follows from [19, Theorem 3.1]. The opposite inclusion follows from the previous corollary.

Theorem 19. Let X be an infinite dimensional Banach space and $S \in \mathcal{L}(X)$. Then the following conditions are equivalent:

(a) $\sigma_{\gamma e}(T+S) = \sigma_{\gamma e}(T)$ for every $T \in \mathcal{L}(X)$ commuting with S,
(b) S is a Riesz operator.

Proof. $(b) \Rightarrow (a)$: See Corollary 17.

$(a) \Rightarrow (b)$: Take $T = 0$. Then $\sigma_{\gamma e}(S) = \sigma_{\gamma e}(0) = \{0\}$. By [19], Corollary 3.4 or [16], Theorem 3.8, $\sigma_e(T) = \{0\}$ so that S is a Riesz operator.

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